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Research article

Modified inertial subgradient extragradient algorithms for generalized equilibria systems with constraints of variational inequalities and fixed points

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Abstract: In this research, we studied modified inertial composite subgradient extragradient implicit rules for finding solutions of a system of generalized equilibrium problems with a common fixed-point problem and pseudomonotone variational inequality constraints. The suggested methods consisted of an inertial iterative algorithm, a hybrid deepest-descent technique, and a subgradient extragradient method. We proved that the constructed algorithms converge to a solution of the considered problem, which also solved some hierarchical variational inequality.

Keywords: system of generalized equilibrium problems; modified inertial subgradient extragradient algorithm; common fixed point; variational inequality
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1. Introduction

Throughout, assume H is a real Hilbert space with the inner product $\langle \cdot, \cdot \rangle$ and the norm $\| \cdot \|$. Assume $\emptyset \neq C \subset H$ is a closed and convex set. Let $S : C \to H$ be an operator and $\Theta : C \times C \to \mathbf{R}$ be a bifunction. Use $\mathrm{Fix}(S)$ to mean the fixed-point set of S.

Recall that the equilibrium problem (EP) is to search an equilibrium point in $\mathrm{EP}(\Theta)$, where

$$\mathrm{EP}(\Theta) = \{x \in C : \Theta(x, y) \geq 0,\ \forall y \in C\}.$$

Under the theory framework of equilibrium problems, there exists a unified way for exploring a broad number of problems originating in structural analysis, transportation, physics, optimization, finance, and economics [1,8,10,17,20,24–26,35]. In order to find an element in EP(Θ), one needs to make the hypotheses below:

(H1) $\Theta(v, v) = 0,\ \ \forall v \in C$;

(H2) $\Theta(w, v) + \Theta(v, w) \leq 0,\ \ \forall v, w \in C$;

(H3) $\lim_{\lambda\to 0^+} \Theta((1-\lambda)v + \lambda u, w) \leq \Theta(v, w),\ \ \forall u, v, w \in C$;

(H4) For every $v \in C$, $\Theta(v, \cdot)$ is convex and lower semicontinuous (l.s.c.).

In order to solve the equilibrium problems, in 1994, Blum and Oettli [1] obtained the following valuable lemma:

Lemma 1.1. *[1] Assume that $\Theta : C \times C \to \mathbf{R}$ fulfills the hypotheses (H1)–(H4). If $\forall x \in H$ and $\ell > 0$, let $T_\ell^\Theta : H \to C$ be an operator formulated below:*

$$T_\ell^\Theta(x) := \{y \in C : \Theta(y, z) + \frac{1}{\ell}\langle z - y, y - x\rangle \geq 0, \forall z \in C\}.$$

Then, (i) T_ℓ^Θ is single-valued and satisfies $\|T_\ell^\Theta v - T_\ell^\Theta w\|^2 \leq \langle T_\ell^\Theta v - T_\ell^\Theta w, v - w\rangle,\ \ \forall v, w \in H$; and (ii) $\mathrm{Fix}(T_\ell^\Theta) = \mathrm{EP}(\Theta)$, and $\mathrm{EP}(\Theta)$ is convex and closed.

In particular, in case of $\Theta(x, y) = \langle Ax, y - x\rangle,\ \forall x, y \in C$, and the EP reduces to the classical variational ineqiality problem (VIP) of seeking $x \in C$ such that

$$\langle Ax, y - x\rangle \geq 0,\ \ \forall y \in C.$$

The solution set of the VIP is denoted by $\mathrm{VI}(C, A)$.

An effective approach for settling EP and VIP is the Korpelevich's algorithm [15]. The Korpelevich extragradient technique has been adapted and applied extensively; see e.g., the modified extragradient method [11, 29, 34], subgradient extragradient method [3, 13, 16, 28, 31, 32], relaxed extragradient method [7], Tseng-type method [22,23,33], inertial extragradient method [14,27], and so on.

In 2010, Ceng and Yao [6] investigated the system generalized equilibrium problems (SGEP) of finding $(x, y) \in C \times C$ satisfying

$$\begin{cases} \Theta_1(x, u) + \langle B_1 y, u - x\rangle + \frac{1}{\alpha_1}\langle x - y, u - x\rangle \geq 0, & \forall u \in C, \\ \Theta_2(y, v) + \langle B_2 x, v - y\rangle + \frac{1}{\alpha_2}\langle y - x, v - y\rangle \geq 0, & \forall v \in C, \end{cases} \tag{1.1}$$

where $B_1, B_2 : H \to H$ are two nonlinear operators, $\Theta_1, \Theta_2 : C \times C \to \mathbf{R}$ are two bifunctions, and $\alpha_1, \alpha_2 > 0$ are two constants.

If $\Theta_1 = \Theta_2 = 0$, then the SGEP comes down to the generallized variational inequalities considered in [5]: Find $(x, y) \in C \times C$ satisfying

$$\begin{cases} \langle \alpha_1 B_1 y + x - y, u - x\rangle \geq 0, & \forall u \in C, \\ \langle \alpha_2 B_2 x + y - x, v - y\rangle \geq 0, & \forall v \in C, \end{cases}$$

with constants $\alpha_1, \alpha_2 > 0$.

To solve problem (1.1), the authors in [6] used a fixed point technique. In fact, the SGEP (1.1) can be transformed into the fixed-point problem.

Lemma 1.2. *[6] Suppose that the bifunctions $\Theta_1, \Theta_2 : C \times C \to \mathbf{R}$ satisfy the hypotheses (H1)–(H4) and $B_1, B_2 : H \to H$ are ρ-ism and σ-ism, respectively. Then, $(u^*, v^*) \in C \times C$ is a solution of SGEP (1.1) if and only if $u^\dagger \in \mathrm{Fix}(G)$, where $G := T^{\Theta_1}_{\alpha_1}(I - \alpha_1 B_1) T^{\Theta_2}_{\alpha_2}(I - \alpha_2 B_2)$ and $v^* = T^{\Theta_2}_{\alpha_2}(I - \alpha_2 B_2)u^*$ in which $\alpha_1 \in (0, 2\rho)$ and $\alpha_2 \in (0, 2\sigma)$.*

On the other hand, in 2018, Cai, Shehu, and Iyiola [2] proposed the modified viscosity implicit rule for solving EP and a fixed-point problem: for $x_1 \in C$, let $\{x_k\}$ be the sequence constructed below:

$$\begin{cases} u_k = \sigma_k x_k + (1 - \sigma_k) y_k, \\ v_k = P_C(u_k - \alpha_2 B_2 u_k), \\ y_k = P_C(v_k - \alpha_1 B_1 v_k), \\ x_{k+1} = P_C[\rho_k f(x_k) + (I - \rho_k \alpha F) S^k y_k], \quad \forall k \geq 1. \end{cases}$$

Under suitable conditions, Cai, Shehu, and Iyiola [2] proved $x_k \to u^* \in \mathrm{Fix}(S) \cap \mathrm{Fix}(G)$, which solves the hierarchical variational inequality (HVI):

$$\langle (\alpha F - f)u, v - u \rangle \geq 0, \ \forall v \in \mathrm{Fix}(S) \cap \mathrm{Fix}(G).$$

Moreover, Ceng and Shang [4] suggested an algorithm for solving the common fixed-point problem (CFPP) of finite nonexpansive mappings $\{S_r\}_{r=1}^N$, an asymptotically nonexpansive mapping S and VIP.

Algorithm 1.1. *[4] Let $x_1, x_0 \in H$ be arbitrary. Let $\gamma > 0$, $\ell \in (0,1)$, $\nu \in (0,1)$, and x_k be known. Calculate x_{k+1} via the following iterative steps:*
Step 1. *Set $p_k = S_k x_k + \varepsilon_k(S_k x_k - S_k x_{k-1})$ and calculate $y_k = P_C(p_k - \zeta_k A p_k)$, where ζ_k is the largest $\zeta \in \{\gamma, \gamma\ell, \gamma\ell^2, ...\}$ fulfilling $\zeta \|Ap_k - Ay_k\| \leq \nu \|p_k - y_k\|$.*
Step 2. *Compute $z_k = P_{C_k}(p_k - \zeta_k A y_k)$ where $C_k := \{y \in H : \langle p_k - \zeta_k A p_k - y_k, y_k - y \rangle \geq 0\}$.*
Step 3. *Compute $x_{k+1} = \rho_k f(x_k) + \sigma_k x_k + ((1 - \sigma_k)I - \rho_k \alpha F) S^k z_k$.*
Let $k := k + 1$ and return to Step 1.

Motivated and inspired by the work in the literature, the main purpose of this article was to design two modified inertial composite subgradient extragradient implicit rules for solving the SGEP with the VIP and CFPP constraints. The suggested algorithms consisted of the subgradient extragradient rule, inertial iteration approach, and hybrid deepest-descent technique. We proved that the proposed algorithms converge to a solution of the SGEP with the VIP and CFPP constraints, which also solved some HVI.

2. Preliminaries

Let C be a nonempty, convex, and closed subset of a real Hilbert space H. For all $v, w \in C$, an operator $T : C \to H$ is called

- asymptotically nonexpansive if $\exists \{\varpi_m\}_{m=1}^\infty \subset [0, +\infty)$ satisfying $\varpi_m \to 0$ ($m \to \infty$) and

$$\|T^m v - T^m w\| \leq \varpi_m \|v - w\| + \|v - w\|, \ \ \forall m \geq 1.$$

In particular, in the case of $\varpi_m = 0$, $\forall m \geq 1$, and T is known as being nonexpansive.

- α-Lipschitzian if $\exists\alpha > 0$ such that $\|Tv - Tw\| \leq \alpha\|v - w\|$;
- monotone if $\langle Tv - Tw, v - w\rangle \geq 0$;
- strongly monotone if there is $\rho > 0$ such that $\langle Tv - Tw, v - w\rangle \geq \rho\|v - w\|^2$;
- pseudomonotone if $\langle Tv, w - v\rangle \geq 0 \Rightarrow \langle Tw, w - v\rangle \geq 0$;
- σ inverse strongly monotone (σ-ism) if there is $\sigma > 0$ such that $\langle Tv - Tw, v - w\rangle \geq \sigma\|Tv - Tw\|^2$;
- sequentially weakly continuous if $\forall\{v_l\} \subset C$, $v_l \rightharpoonup v \Rightarrow Tv_l \rightharpoonup Tv$.

Recall that the metric (or nearest point) projection from H onto C is the mapping $P_C : H \to C$ which assigns to each point $x \in C$ the unique point $P_C(x) \in C$ satisfying the property

$$\|x - P_C(x)\| = \inf_{y\in C} \|x - y\|.$$

The following results are well-known ([12]):
(a) $\|P_C(y) - P_C(z)\|^2 \leq \langle y - z, P_C(y) - P_C(z)\rangle, \ \forall y, z \in H$;
(b) $z = P_C(y) \Leftrightarrow \langle x - z, y - z\rangle \leq 0, \ \forall y \in H, x \in C$;
(c) $\|y - z\|^2 \geq \|z - P_C(y)\|^2 + \|y - P_C(y)\|^2, \ \forall y \in H, z \in C$;
(d) $\|y - z\|^2 = \|y\|^2 - 2\langle y - z, z\rangle - \|z\|^2, \ \forall y, z \in H$;
(e) $\|ty + (1 - t)x\|^2 = t\|y\|^2 + (1 - t)\|x\|^2 - t(1 - t)\|y - x\|^2, \ \forall x, y \in H, t \in [0, 1]$.

Lemma 2.1. *[6] Suppose $B : H \to H$ is an η-ism. Then,*

$$\|(I - \alpha B)y - (I - \alpha B)z\|^2 \leq \|y - z\|^2 - \alpha(2\eta - \alpha)\|By - Bz\|^2, \forall y, z \in H, \forall\alpha \geq 0.$$

When $0 \leq \alpha \leq 2\eta$, we have that $I - \alpha B$ is nonexpansive.

Lemma 2.2. *[6] Let $B_1, B_2 : H \to H$ be ρ-ism and σ-ism, respectively. Suppose that the bifunctions $\Theta_1, \Theta_2 : C \times C \to \mathbf{R}$ satisfy the hypotheses (H1)–(H4). Then, $G := T^{\Theta_1}_{\alpha_1}(I - \alpha_1 B_1)T^{\Theta_2}_{\alpha_2}(I - \alpha_2 B_2)$ is nonexpansive when $0 < \alpha_1 \leq 2\rho$ and $0 < \alpha_2 \leq 2\sigma$.*

In particular, if $\Theta_1 = \Theta_2 = 0$, using Lemma 1.1, we deduce that $T^{\Theta_1}_{\alpha_1} = T^{\Theta_2}_{\alpha_2} = P_C$.

Corollary 2.1. *[5] Let $B_1 : H \to H$ be ρ-ism and $B_2 : H \to H$ σ-ism. Define an operator $G : H \to C$ by $G := P_C(I - \alpha_1 B_1)P_C(I - \alpha_2 B_2)$. Then G is nonexpansive when $0 < \alpha_1 \leq 2\rho$ and $0 < \alpha_2 \leq 2\sigma$.*

Lemma 2.3. *[9] If the operator $A : C \to H$ is continuous pseudomonotone, then $v \in VI(C, A)$ if and only if $\langle Aw, w - v\rangle \geq 0, \ \forall w \in C$.*

Lemma 2.4. *[30] Suppose $\{a_l\} \subset [0, \infty)$ s.t. $a_{l+1} \leq (1 - \omega_l)a_l + \omega_l\nu_l, \ \forall l \geq 1$, where $\{\omega_l\}$ and $\{\nu_l\}$ satisfy: (i) $\{\omega_l\} \subset [0, 1]$; (ii) $\sum_{l=1}^{\infty} \omega_l = \infty$, and (iii) $\limsup_{l\to\infty} \nu_l \leq 0$ or $\sum_{l=1}^{\infty} |\omega_l\nu_l| < \infty$. Then, we have $\lim_{l\to\infty} a_l = 0$.*

Lemma 2.5. *[18] Let X be a Banach space with a weakly continuous duality mapping. Let C be a nonempty closed convex subset of X and $T : C \to C$ an asymptotically nonexpansive mapping such that $\mathrm{Fix}(T) \neq \emptyset$. Then $I - T$ is demiclosed at zero.*

Lemma 2.6. *[19] Suppose that the real number sequence $\{\Gamma_m\}$ is not decreasing at infinity: $\exists\{\Gamma_{m_k}\} \subset \{\Gamma_m\}$ s.t. $\Gamma_{m_k} < \Gamma_{m_k+1}, \ \forall k \geq 1$. Let $\{\phi(m)\}_{m\geq m_0}$ be an integer sequence defined by*

$$\phi(m) = \max\{k \leq m : \Gamma_k < \Gamma_{k+1}\}.$$

Then,

(i) $\phi(m_0) \leq \phi(m_0 + 1) \leq \cdots$ *and* $\phi(m) \to \infty$ *as* $m \to \infty$*;*

(ii) *For all* $m \geq m_0$, $\Gamma_{\phi(m)} \leq \Gamma_{\phi(m)+1}$ *and* $\Gamma_m \leq \Gamma_{\phi(m)+1}$.

Lemma 2.7. *[30] Let* $\lambda \in (0, 1]$, $S : C \to C$ *be a nonexpansive operator, and* $F : C \to H$ *be a* κ*-Lipschitzian and* η*-strongly monotone operator. Set* $S^\lambda v := (I - \lambda\alpha F)Sv$, $\forall v \in C$. *If* $0 < \alpha < \frac{2\eta}{\kappa^2}$, *then* $\|S^\lambda v - S^\lambda w\| \leq (1 - \lambda\tau)\|v - w\|$, $\forall v, w \in C$, *where* $\tau = 1 - \sqrt{1 - \alpha(2\eta - \alpha\kappa^2)} \in (0, 1]$.

3. Algorithms and convergence theorems

Let the operator S_r be nonexpansive on H for all $r = 1, ..., N$ and $S : H \to H$ be a ϖ_n-asymptotically nonexpansive operator. Let $A : H \to H$ be an L-Lipschitz pseudomonotone operator satisfying $\|Ax\| \leq \liminf_{n\to\infty} \|Ax_n\|$ when $x_n \rightharpoonup x$. Let $\Theta_1, \Theta_2 : C \times C \to \mathbf{R}$ be two bifunctions fulfilling the hypotheses (H1)–(H4). Let $B_1 : H \to H$ be ρ-ism and $B_2 : H \to H$ be σ-ism. Let $f : H \to H$ be δ-contractive and $F : H \to H$ be κ-Lipschitz η-strongly monotone with $\delta < \tau := 1 - \sqrt{1 - \alpha(2\eta - \alpha\kappa^2)}$ for $\alpha \in (0, \frac{2\eta}{\kappa^2})$. Suppose that the sequences $\{\varepsilon_n\} \subset [0, 1]$, $\{\xi_n\} \subset (0, 1]$, and $\{\rho_n\}, \{\sigma_n\} \subset (0, 1)$ satisfy

(i) $\lim_{n\to\infty} \rho_n = 0$ and $\sum_{n=1}^{\infty} \rho_n = \infty$;

(ii) $\lim_{n\to\infty} \frac{\varpi_n}{\rho_n} = 0$ and $\sup_{n\geq1} \frac{\varepsilon_n}{\rho_n} < \infty$;

(iii) $0 < \liminf_{n\to\infty} \sigma_n \leq \limsup_{n\to\infty} \sigma_n < 1$;

(iv) $\limsup_{n\to\infty} \xi_n < 1$.

Let $\gamma > 0$, $\nu \in (0, 1)$, $\ell \in (0, 1)$, $\alpha_1 \in (0, 2\rho)$, and $\alpha_2 \in (0, 2\sigma)$ be five constants. Set $S_0 := S$ and $G := T^{\Theta_1}_{\alpha_1}(I - \alpha_1 B_1)T^{\Theta_2}_{\alpha_2}(I - \alpha_2 B_2)$. Suppose that $\Delta := \bigcap_{r=0}^{N} \mathrm{Fix}(S_r) \cap \mathrm{Fix}(G) \cap \mathrm{VI}(C, A) \neq \emptyset$.

Algorithm 3.1. *Let* $x_1, x_0 \in H$ *be arbitrary. Let* x_n *be known and compute* x_{n+1} *below:*

Step 1. *Set* $q_n = S^n x_n + \varepsilon_n(S^n x_n - S^n x_{n-1})$ *and calculate*

$$\begin{cases} p_n = \xi_n q_n + (1 - \xi_n)u_n, \\ v_n = T^{\Theta_2}_{\alpha_2}(p_n - \alpha_2 B_2 p_n), \\ u_n = T^{\Theta_1}_{\alpha_1}(v_n - \alpha_1 B_1 v_n). \end{cases}$$

Step 2. *Compute* $y_n = P_C(p_n - \zeta_n A p_n)$*, where* ζ_n *is the largest* $\zeta \in \{\gamma, \gamma\ell, \gamma\ell^2, ...\}$ *s.t.*

$$\zeta\|Ap_n - Ay_n\| \leq \nu\|p_n - y_n\|. \tag{3.1}$$

Step 3. *Compute* $t_n = \sigma_n x_n + (1 - \sigma_n)z_n$ *with* $z_n = P_{C_n}(p_n - \zeta_n A y_n)$ *and*

$$C_n := \{y \in H : \langle p_n - \zeta_n A p_n - y_n, y - y_n\rangle \leq 0\}.$$

Step 4. *Compute*

$$x_{n+1} = \rho_n f(x_n) + (I - \rho_n \alpha F)S_n t_n, \tag{3.2}$$

where S_n *is constructed as in Algorithm 1.1. Let* $n := n + 1$ *and return to Step 1.*

Lemma 3.1. *[21]* $\min\{\gamma, \nu\ell/L\} \leq \zeta_n \leq \gamma$.

Lemma 3.2. *Let $p \in \Delta$ and $q = T_{\alpha_2}^{\Theta_2}(p - \alpha_2 B_2 p)$. Then,*

$$\begin{aligned}\|z_n - p\|^2 \leq \|q_n - p\|^2 - \alpha_1(\alpha_1 - 2\rho)\|B_1 v_n - B_1 q\|^2 - (1 - \xi_n)[\|q_n - p_n\|^2 \\ - (1 - \nu)[\|y_n - z_n\|^2 + \|y_n - p_n\|^2] - \alpha_2(\alpha_2 - 2\sigma)\|B_2 p_n - B_2 p\|^2],\end{aligned}$$

where $v_n = T_{\alpha_2}^{\Theta_2}(p_n - \alpha_2 B_2 p_n)$.

Proof. According to Lemma 2.2, there exists the unique point $p_n \in H$ satisfying $p_n = \xi_n q_n + (1-\xi_n)Gp_n$. Since $p \in C_n$, we have

$$\begin{aligned}\|z_n - p\|^2 &\leq \langle p_n - \zeta_n A y_n - p, z_n - p\rangle \\ &= \frac{1}{2}(\|p_n - p\|^2 - \|z_n - p_n\|^2 + \|z_n - p\|^2) - \zeta_n\langle Ay_n, z_n - p\rangle,\end{aligned}$$

which implies that

$$\|z_n - p\|^2 \leq \|p_n - p\|^2 - \|z_n - p_n\|^2 - 2\zeta_n\langle Ay_n, z_n - p\rangle.$$

Noting that $z_n = P_{C_n}(p_n - \zeta_n A y_n)$, we have $\langle p_n - \zeta_n A p_n - y_n, z_n - y_n\rangle \leq 0$. Owing to the pseudomonotonicity of A, by (3.1), we get

$$\begin{aligned}\|z_n - p\|^2 &\leq \|p_n - p\|^2 - \|z_n - p_n\|^2 - 2\zeta_n\langle Ay_n, y_n - p + z_n - y_n\rangle \\ &\leq \|p_n - p\|^2 - \|z_n - p_n\|^2 - 2\zeta_n\langle Ay_n, z_n - y_n\rangle \\ &= \|p_n - p\|^2 + 2\langle p_n - \|y_n - p_n\|^2 - \zeta_n A y_n - y_n, z_n - y_n\rangle - \|z_n - y_n\|^2 \\ &= \|p_n - p\|^2 - \|z_n - y_n\|^2 + 2\langle p_n - \zeta_n A p_n - y_n, z_n - y_n\rangle - \|y_n - p_n\|^2 \\ &\quad + 2\zeta_n\langle Ap_n - Ay_n, z_n - y_n\rangle \\ &\leq \|p_n - p\|^2 + 2\nu\|p_n - y_n\|\|z_n - y_n\| - \|z_n - y_n\|^2 - \|y_n - p_n\|^2 \\ &\leq \|p_n - p\|^2 - \|y_n - p_n\|^2 + \nu(\|p_n - y_n\|^2 + \|z_n - y_n\|^2) - \|z_n - y_n\|^2 \\ &= \|p_n - p\|^2 - (1 - \nu)[\|y_n - p_n\|^2 + \|y_n - z_n\|^2].\end{aligned} \tag{3.3}$$

Observe that $u_n = T_{\alpha_1}^{\Theta_1}(v_n - \alpha_1 B_1 v_n)$, $v_n = T_{\alpha_2}^{\Theta_2}(p_n - \alpha_2 B_2 p_n)$, and $q = T_{\alpha_2}^{\Theta_2}(p - \alpha_2 B_2 p)$. Then $u_n = Gp_n$. Applying Lemma 2.1 to get

$$\|u_n - p\|^2 \leq \|v_n - q\|^2 + \alpha_1(\alpha_1 - 2\rho)\|B_1 v_n - B_1 q\|^2$$

and

$$\|v_n - q\|^2 \leq \|p_n - p\|^2 + \alpha_2(\alpha_2 - 2\sigma)\|B_2 p_n - B_2 p\|^2.$$

Then,

$$\|u_n - p\|^2 \leq \|p_n - p\|^2 + \alpha_1(\alpha_1 - 2\rho)\|B_1 v_n - B_1 q\|^2 + \alpha_2(\alpha_2 - 2\sigma)\|B_2 p_n - B_2 p\|^2.$$

Besides, thanks to $p_n = \xi_n q_n + (1 - \xi_n)u_n$, we get $\|p_n - p\|^2 \leq \xi_n\langle q_n - p, p_n - p\rangle + (1 - \xi_n)\|p_n - p\|^2$, which results in $\|p_n - p\|^2 \leq \langle q_n - p, p_n - p\rangle = \frac{1}{2}[\|q_n - p\|^2 + \|p_n - p\|^2 - \|q_n - p_n\|^2]$. So,

$$\|p_n - p\|^2 \leq \|q_n - p\|^2 - \|q_n - p_n\|^2. \tag{3.4}$$

Then,

$$\begin{aligned}
||p_n - p||^2 &\leq (1-\xi_n)||u_n - p||^2 + \xi_n||q_n - p||^2 \\
&\leq \xi_n||q_n - p||^2 + (1-\xi_n)[||p_n - p||^2 + \alpha_1(\alpha_1 - 2\rho)||B_1 v_n - B_1 q||^2 \\
&\quad + \alpha_2(\alpha_2 - 2\sigma)||B_2 p_n - B_2 p||^2] \\
&\leq \xi_n||q_n - p||^2 + (1-\xi_n)[||q_n - p||^2 - ||q_n - p_n||^2 + \alpha_2(\alpha_2 - 2\sigma) \\
&\quad \times ||B_2 p_n - B_2 p||^2 + \alpha_1(\alpha_1 - 2\rho)||B_1 v_n - B_1 q||^2] \\
&= ||q_n - p||^2 - (1-\xi_n)[||q_n - p_n||^2 - \alpha_1(\alpha_1 - 2\rho)||B_1 v_n - B_1 q||^2 \\
&\quad - \alpha_2(\alpha_2 - 2\sigma)||B_2 p_n - B_2 p||^2],
\end{aligned}$$

which, together with (3.3), yields

$$\begin{aligned}
||z_n - p||^2 &\leq ||p_n - p||^2 - (1-\nu)[||y_n - p_n||^2 + ||y_n - z_n||^2] \\
&\leq ||q_n - p||^2 - (1-\xi_n)[||q_n - p_n||^2 - \alpha_1(\alpha_1 - 2\rho)||B_1 v_n - B_1 q||^2 \\
&\quad - \alpha_2(\alpha_2 - 2\sigma)||B_2 p_n - B_2 p||^2] - (1-\nu)[||y_n - p_n||^2 + ||y_n - z_n||^2].
\end{aligned}$$

This ensures that the conclusion holds. □

Lemma 3.3. *Assume that*

(i) the sequences $\{p_n\}, \{q_n\}, \{y_n\}$, *and* $\{z_n\}$ *are bounded;*
(ii) $\lim_{n\to\infty}(x_{n+1} - x_n) = \lim_{n\to\infty}(q_n - z_n) = \lim_{n\to\infty}(x_n - y_n) = \lim_{n\to\infty}(S^{n+1}x_n - S^n x_n) = 0.$

Then $\omega_w(x_n) \subset \Delta$, *where* $\omega_w(x_n) = \{z \in H,$ and there is some $\{x_{n_i}\} \subset \{x_n\}$ such that $x_{n_i} \rightharpoonup z\}$.

Proof. Take an arbitrary fixed $z \in \omega_w(\{x_n\})$. Then, there is some $\{n_i\} \subset \{n\}$ such that $x_{n_i} \rightharpoonup z$ and $y_{n_i} \rightharpoonup z \in H$. Next, we show $z \in \Delta$. Using Lemma 3.2, we deduce

$$\begin{aligned}
&(1-\xi_n)[||q_n - p_n||^2 - \alpha_1(\alpha_1 - 2\rho)||B_1 v_n - B_1 q||^2 - \alpha_2(\alpha_2 - 2\sigma)||B_2 p_n - B_2 p||^2] \\
&\quad + (1-\nu)[||y_n - p_n||^2 + ||y_n - z_n||^2] \\
&\leq ||q_n - p||^2 - ||z_n - p||^2 \leq ||q_n - z_n||(||q_n - p|| + ||z_n - p||).
\end{aligned}$$

Because $q_n - z_n \to 0,\ \nu \in (0,1),\ \alpha_1 \in (0, 2\rho),\ \alpha_2 \in (0, 2\sigma)$, and $0 < \liminf_{n\to\infty}(1-\xi_n)$, we deduce that

$$\lim_{n\to\infty} ||B_2 p_n - B_2 p|| = \lim_{n\to\infty} ||B_1 v_n - B_1 q|| = 0, \tag{3.5}$$

and

$$\lim_{n\to\infty} ||y_n - z_n|| = \lim_{n\to\infty} ||q_n - p_n|| = \lim_{n\to\infty} ||y_n - p_n|| = 0.$$

Hence,

$$||x_n - q_n|| \leq ||x_n - y_n|| + ||y_n - z_n|| + ||z_n - q_n|| \to 0 \quad (n \to \infty),$$

$$||x_n - p_n|| \leq ||x_n - q_n|| + ||q_n - p_n|| \to 0 \quad (n \to \infty),$$

$$||p_n - z_n|| \leq ||p_n - y_n|| + ||y_n - z_n|| \to 0 \quad (n \to \infty),$$

and

$$||x_n - z_n|| \leq ||x_n - q_n|| + ||q_n - p_n|| + ||p_n - z_n|| \to 0 \quad (n \to \infty).$$

Note that
$$\|q_n - S^n x_n\| = \varepsilon_n \|S^n x_n - S^n x_{n-1}\| \le (1+\varpi_n)\|x_n - x_{n-1}\| \to 0.$$
Therefore,
$$\|x_n - S^n x_n\| \le \|x_n - q_n\| + \|q_n - S^n x_n\| \to 0 \quad (n\to\infty).$$
Note that
$$\begin{aligned}\|x_n - Sx_n\| &\le \|x_n - S^n x_n\| + \|Sx_n - S^{n+1}x_n\| + \|S^{n+1}x_n - S^n x_n\| \\ &\le \|x_n - S^n x_n\| + \|S^n x_n - S^{n+1}x_n\| + (1+\varpi_1)\|S^n x_n - x_n\| \\ &= (2+\varpi_1)\|x_n - S^n x_n\| + \|S^n x_n - S^{n+1}x_n\| \to 0.\end{aligned} \tag{3.6}$$
Observe that
$$\begin{aligned}\|u_n - p\|^2 &\le \langle v_n - q, u_n - p\rangle + \alpha_1\langle B_1 q - B_1 v_n, u_n - p\rangle \\ &\le \frac{1}{2}[\|v_n - q\|^2 - \|v_n - u_n + p - q\|^2 + \|u_n - p\|^2] \\ &\quad + \alpha_1\|B_1 q - B_1 v_n\|\|u_n - p\|,\end{aligned}$$
which arrives at
$$\|u_n - p\|^2 \le \|v_n - q\|^2 - \|v_n - u_n + p - q\|^2 + 2\alpha_1\|B_1 q - B_1 v_n\|\|u_n - p\|.$$
Similarly, we get
$$\|v_n - q\|^2 \le \|p_n - p\|^2 - \|p_n - v_n + q - p\|^2 + 2\alpha_2\|B_2 p - B_2 p_n\|\|v_n - q\|.$$
Combining the last two inequalities, we deduce that
$$\begin{aligned}\|u_n - p\|^2 &\le \|p_n - p\|^2 - \|p_n - v_n + q - p\|^2 - \|v_n - u_n + p - q\|^2 \\ &\quad + 2\alpha_1\|B_1 q - B_1 v_n\|\|u_n - p\| + 2\alpha_2\|B_2 p - B_2 p_n\|\|v_n - q\|.\end{aligned}$$
Hence,
$$\begin{aligned}\|z_n - p\|^2 &\le \|p_n - p\|^2 \le \xi_n\|q_n - p\|^2 + (1-\xi_n)\|u_n - p\|^2 \\ &\le \xi_n\|q_n - p\|^2 + (1-\xi_n)[\|q_n - p\|^2 - \|p_n - v_n + q - p\|^2 - \|v_n - u_n + p - q\|^2 \\ &\quad + 2\alpha_1\|B_1 q - B_1 v_n\|\|u_n - p\| + 2\alpha_2\|B_2 p - B_2 p_n\|\|v_n - q\|] \\ &\le \|q_n - p\|^2 - (1-\xi_n)[\|p_n - v_n + q - p\|^2 + \|v_n - u_n + p - q\|^2] + 2\alpha_2\|B_2 p - B_2 p_n\| \\ &\quad \times \|v_n - q\| + 2\alpha_1\|B_1 q - B_1 v_n\|\|u_n - p\|.\end{aligned}$$
This immediately implies that
$$\begin{aligned}&(1-\xi_n)[\|p_n - v_n + q - p\|^2 + \|v_n - u_n + p - q\|^2] \\ &\le \|q_n - p\|^2 - \|z_n - p\|^2 + 2\alpha_2\|B_2 p - B_2 p_n\|\|v_n - q\| + 2\alpha_1\|B_1 q - B_1 v_n\|\|u_n - p\| \\ &\le \|q_n - z_n\|(\|q_n - p\| + \|z_n - p\|) + 2\alpha_2\|B_2 p - B_2 p_n\|\|v_n - q\| + 2\alpha_1\|B_1 q - B_1 v_n\|\|u_n - p\|.\end{aligned}$$
Since $q_n - z_n \to 0$, and $0 < \liminf_{n\to\infty}(1-\xi_n)$, from (3.5) and the boundedness of $\{u_n\}, \{v_n\}, \{q_n\}$, and $\{z_n\}$ we get that
$$\lim_{n\to\infty}\|p_n - v_n + q - p\| = \lim_{n\to\infty}\|v_n - u_n + p - q\| = 0,$$

which hence yields

$$\|p_n - Gp_n\| = \|p_n - u_n\| \le \|p_n - v_n + q - p\| + \|v_n - u_n + p - q\| \to 0 \quad (n \to \infty).$$

This immediately implies that

$$\|x_n - Gx_n\| \le \|x_n - p_n\| + \|p_n - Gp_n\| + \|Gp_n - Gx_n\| \le 2\|x_n - p_n\| + \|p_n - Gp_n\| \to 0 \quad (n \to \infty). \quad (3.7)$$

Next, we prove $\lim_{n\to\infty} \|x_n - S_n x_n\| = 0$. In fact, we have

$$\|t_n - x_n\| \le \|x_n - z_n\| \to 0,$$

and

$$\begin{aligned}\|S_n t_n - x_n\| &= \|x_{n+1} - x_n - \rho_n(f(x_n) - \alpha F S_n t_n)\| \\ &\le \|x_{n+1} - x_n\| + \rho_n(\|f(x_n)\| + \|\alpha F S_n t_n\|) \to 0.\end{aligned}$$

Hence,

$$\|x_n - S_n x_n\| \le \|x_n - S_n t_n\| + \|S_n x_n - S_n t_n\| \le \|x_n - S_n t_n\| + \|x_n - t_n\| \to 0.$$

Now, we show $x_n - S_r x_n \to 0, \forall r \in \{1, ..., N\}$. For $1 \le l \le N$, it holds that

$$\begin{aligned}\|x_n - S_{n+l} x_n\| &\le \|x_n - x_{n+l}\| + \|S_{n+l} x_{n+l} - S_{n+l} x_n\| + \|x_{n+l} - S_{n+l} x_{n+l}\| \\ &\le 2\|x_n - x_{n+l}\| + \|x_{n+l} - S_{n+l} x_{n+l}\|.\end{aligned}$$

This, together with assumptions, implies that $x_n - S_{n+l} x_n \to 0, 1 \le l \le N$. So,

$$\lim_{n\to\infty} \|x_n - S_r x_n\| = 0, 1 \le r \le N. \quad (3.8)$$

Now, we show $z \in \mathrm{VI}(C, A)$. If $Az = 0$, then $z \in \mathrm{VI}(C, A)$. Next, we suppose that $Az \ne 0$. By the condition, we conclude that $0 < \|Az\| \le \liminf_{i\to\infty} \|Ay_{n_i}\|$ because $y_{n_i} \rightharpoonup z$. Observe that $y_n = P_C(p_n - \zeta_n A p_n)$. It follows that $\langle p_n - \zeta_n A p_n - y_n, y - y_n\rangle \le 0, \ \forall y \in C$. Therefore,

$$\frac{1}{\zeta_n}\langle p_n - y_n, y - y_n\rangle + \langle Ap_n, y_n - p_n\rangle \le \langle Ap_n, y - p_n\rangle, \quad \forall y \in C. \quad (3.9)$$

Since $\{Ap_n\}$ and $\{y_n\}$ are all bounded, from (3.9) and Lemma 3.1, we obtain $\liminf_{i\to\infty}\langle Ap_{n_i}, y - p_{n_i}\rangle \ge 0, \ \forall y \in C$. Meanwhile, $\langle Ay_n, y - y_n\rangle = \langle Ay_n - Ap_n, y - p_n\rangle + \langle Ap_n, y - p_n\rangle + \langle Ay_n, p_n - y_n\rangle$. Using $p_n - y_n \to 0$ and the uniform continuity of A, we get $Ap_n - Ay_n \to 0$, which hence attains $\liminf_{i\to\infty}\langle Ay_{n_i}, y - y_{n_i}\rangle \ge 0, \ \forall y \in C$.

To attain $z \in \mathrm{VI}(C, A)$, let $\{\lambda_i\} \subset (0, 1)$ be a sequence such that $\lambda_i \downarrow 0 (i \to \infty)$. For every $i \ge 1$, let k_i be the smallest positive integer satisfying

$$\langle Ay_{n_j}, y - y_{n_j}\rangle + \lambda_i \ge 0, \quad \forall j \ge k_i. \quad (3.10)$$

Put $\upsilon_{k_i} = \frac{Ay_{k_i}}{\|Ay_{k_i}\|^2}$, and hence get $\langle Ay_{k_i}, \upsilon_{k_i}\rangle = 1, \ \forall i \ge 1$. So, from (3.10), one gets $\langle Ay_{k_i}, y + \lambda_i \upsilon_{k_i} - y_{k_i}\rangle \ge 0$, $\forall i \ge 1$. Since A is pseudomonotone, we have $\langle A(y + \lambda_i \upsilon_{k_i}), y + \lambda_i \upsilon_{k_i} - y_{k_i}\rangle \ge 0, \ \forall i \ge 1$. So,

$$\langle Ay, y - y_{k_i}\rangle \ge \langle Ay - A(y + \lambda_i \upsilon_{k_i}), y + \lambda_i \upsilon_{k_i} - y_{k_i}\rangle - \lambda_i\langle Ay, \upsilon_{k_i}\rangle, \quad \forall i \ge 1. \quad (3.11)$$

Let us show that $\lim_{i\to\infty}\lambda_i\upsilon_{k_i}=0$. In fact, note that $\{y_{k_i}\}\subset\{y_{n_i}\}$ and $\lambda_i\downarrow 0$ as $i\to\infty$. So it follows that $0\le\limsup_{i\to\infty}\|\lambda_i\upsilon_{k_i}\|=\limsup_{i\to\infty}\frac{\lambda_i}{\|Ay_{k_i}\|}\le\frac{\limsup_{i\to\infty}\lambda_i}{\liminf_{i\to\infty}\|Ay_{n_i}\|}=0$. Therefore, one gets $\lambda_i\upsilon_{k_i}\to 0$ as $i\to\infty$. Thus, letting $i\to\infty$ in (3.11), we deduce that $\langle Ay,y-z\rangle=\liminf_{i\to\infty}\langle Ay,y-y_{k_i}\rangle\ge 0,\ \forall y\in C$. We apply Lemma 2.3 to conclude $z\in \mathrm{VI}(C,A)$.

Finally, we prove $z\in\Delta$. Note that $x_{n_i}\rightharpoonup z$ and $x_{n_i}-S_rx_{n_i}\to 0,\ \forall r\in\{1,...,N\}$ (due to (3.8)). Since $I-S_r(1\le r\le N)$ is demiclosed by Lemma 2.5, we attain $z\in\bigcap_{r=1}^N\mathrm{Fix}(S_r)$. By (3.6) and (3.7), we have $x_{n_i}-Sx_{n_i}\to 0$ and $x_{n_i}-Gx_{n_i}\to 0$, respectively. Similarly, $I-S$ and $I-G$ are all demiclosed at zero, and we have $z\in\mathrm{Fix}(S)\cap\mathrm{Fix}(G)$. Therefore, $z\in\bigcap_{r=0}^N\mathrm{Fix}(S_r)\cap\mathrm{Fix}(G)\cap\mathrm{VI}(C,A)=\Delta$. □

Theorem 3.1. *We have the following equivalent relation:*

$$x_n\to u^\dagger\in\Delta \quad\Leftrightarrow\quad \begin{cases} S^{n+1}x_n-S^nx_n\to 0,\\ x_{n+1}-x_n\to 0,\end{cases}$$

where $u^\dagger\in\Delta$ solves the HVI: $\langle(\alpha F-f)u^\dagger,p-u^\dagger\rangle\ge 0,\ \ \forall p\in\Delta$.

Proof. According to the condition, we assume that $\varpi_n\le\frac{\rho_n(\tau-\delta)}{2}$ and $\{\sigma_n\}\subset[a,b]\subset(0,1)$ for all $n\ge 1$. $\forall x,y\in H$, by Lemma 2.7, we obtain

$$\|P_\Delta(I-\alpha F+f)(x)-P_\Delta(I-\alpha F+f)(y)\|\le[1-(\tau-\delta)]\|x-y\|,$$

which implies that $P_\Delta(I-\alpha F+f)$ is contractive. Set $u^\dagger=P_\Delta(I-\alpha F+f)(u^\dagger)$. Therefore, there is the unique solution $u^\dagger\in\Delta=\bigcap_{r=0}^N\mathrm{Fix}(S_r)\cap\mathrm{Fix}(G)\cap\mathrm{VI}(C,A)$ of the HVI:

$$\langle(\alpha F-f)u^\dagger,p-u^\dagger\rangle\ge 0,\quad\forall p\in\Delta. \tag{3.12}$$

If $x_n\to u^\dagger\in\Delta$, then we know that $u^\dagger=Su^\dagger$ and

$$\begin{aligned}\|S^nx_n-S^{n+1}x_n\|&\le\|S^nx_n-u^\dagger\|+\|u^\dagger-S^{n+1}x_n\|\\&\le(1+\varpi_n)\|x_n-u^\dagger\|+(1+\varpi_{n+1})\|u^\dagger-x_n\|\\&=(2+\varpi_n+\varpi_{n+1})\|x_n-u^\dagger\|\to 0.\end{aligned}$$

Note that

$$\|x_{n+1}-x_n\|\le\|u^\dagger-x_{n+1}\|+\|x_n-u^\dagger\|\to 0.$$

Now, we prove the sufficiency.

Step 1. Let $p\in\Delta$. Then $Gp=p$, $p=P_C(p-\zeta_nAp)$, $S_np=p$, $\forall n\ge 0$, and the inequalities (3.3) and (3.4) hold, i.e.,

$$\|z_n-p\|^2\le\|p_n-p\|^2-(1-\nu)\|y_n-p_n\|^2-(1-\nu)\|y_n-z_n\|^2, \tag{3.13}$$

and

$$\|p_n-p\|^2\le\|q_n-p\|^2-\|q_n-p_n\|^2. \tag{3.14}$$

Combining (3.13) and (3.14) guarantees that

$$\|z_n-p\|\le\|p_n-p\|\le\|q_n-p\|. \tag{3.15}$$

Observe that

$$\begin{aligned}\|q_n - p\| &\le \|S^n x_n - p\| + \varepsilon_n\|S^n x_n - S^n x_{n-1}\| \\ &\le (1+\varpi_n)[\|x_n - p\| + \varepsilon_n\|x_n - x_{n-1}\|] \\ &= (1+\varpi_n)[\|x_n - p\| + \rho_n \cdot \frac{\varepsilon_n}{\rho_n}\|x_n - x_{n-1}\|].\end{aligned} \tag{3.16}$$

Since $\sup_{n\ge1} \frac{\varepsilon_n}{\rho_n}\|x_n - x_{n-1}\| < \infty$, there is a constant $M_1 > 0$ satisfying

$$\frac{\varepsilon_n}{\rho_n}\|x_n - x_{n-1}\| \le M_1. \tag{3.17}$$

Combining (3.15)–(3.17), we get

$$\|z_n - p\| \le \|p_n - p\| \le \|q_n - p\| \le (1+\varpi_n)[\|x_n - p\| + \rho_n M_1], \quad \forall n \ge 1. \tag{3.18}$$

Also, it is readily known that

$$\|t_n - p\| \le \sigma_n\|x_n - p\| + (1-\sigma_n)\|z_n - p\| \le (1+\varpi_n)[\|x_n - p\| + \rho_n M_1]. \tag{3.19}$$

Thus, using (3.19) and $\varpi_n \le \frac{\rho_n(\tau-\delta)}{2}$ $\forall n \ge 1$, from Lemma 2.7, we receive

$$\begin{aligned}\|x_{n+1} - p\| &= \|\rho_n f(x_n) + (I - \rho_n\alpha F)S_n t_n - p\| \\ &= \|\rho_n(f(x_n) - f(p)) + (I - \rho_n\alpha F)S_n t_n - (I - \rho_n\alpha F)p + \rho_n(f - \alpha F)p\| \\ &\le \rho_n\delta\|x_n - p\| + (1-\rho_n\tau)\|t_n - p\| + \rho_n\|(f-\alpha F)p\| \\ &\le \rho_n\delta\|x_n - p\| + (1-\rho_n\tau)(1+\varpi_n)[\|x_n - p\| + \rho_n M_1] + \rho_n\|(f-\alpha F)p\| \\ &\le \rho_n\delta\|x_n - p\| + [(1-\rho_n\tau) + \varpi_n]\|x_n - p\| + \rho_n(1+\varpi_n)M_1 + \rho_n\|(f-\alpha F)p\| \\ &\le \rho_n\delta\|x_n - p\| + [(1-\rho_n\tau) + \frac{\rho_n(\tau-\delta)}{2}]\|x_n - p\| + 2\rho_n M_1 + \rho_n\|(f-\alpha F)p\| \\ &= [1 - \frac{\rho_n(\tau-\delta)}{2}]\|x_n - p\| + \frac{\rho_n(\tau-\delta)}{2} \cdot \frac{2(2M_1 + \|(f-\alpha F)p\|)}{\tau-\delta}.\end{aligned}$$

Hence,

$$\|x_n - p\| \le \max\{\|x_1 - p\|, \frac{2(2M_1 + \|(f-\alpha F)p\|)}{\tau-\delta}\}, \quad \forall n \ge 1.$$

We deduce that the sequences $\{x_n\}, \{p_n\}, \{q_n\}, \{y_n\}, \{z_n\}, \{t_n\}, \{f(x_n)\}, \{S_n t_n\}$, and $\{S^n x_n\}$ are bounded.

Step 2. Observe that $t_n - p = \sigma_n(x_n - p) + (1-\sigma_n)(z_n - p)$ and

$$x_{n+1} - p = \rho_n(f(x_n) - f(p)) + (I - \rho_n\alpha F)S_n t_n - (I - \rho_n\alpha F)p + \rho_n(f - \alpha F)p.$$

Utilizing Lemma 2.7, we attain

$$\begin{aligned}\|x_{n+1} - p\|^2 &\le \|\rho_n(f(x_n) - f(p)) + (I - \rho_n\alpha F)S_n t_n - (I - \rho_n\alpha F)p\|^2 \\ &\quad + 2\rho_n\langle (f-\alpha F)p, x_{n+1} - p\rangle \\ &\le [\rho_n\delta\|x_n - p\| + (1-\rho_n\tau)\|t_n - p\|]^2 + 2\rho_n\langle (f-\alpha F)p, x_{n+1} - p\rangle \\ &\le \rho_n\delta\|x_n - p\|^2 + (1-\rho_n\tau)\|t_n - p\|^2 + 2\rho_n\langle (f-\alpha F)p, x_{n+1} - p\rangle \\ &= \rho_n\delta\|x_n - p\|^2 + (1-\rho_n\tau)[\sigma_n\|x_n - p\|^2 + (1-\sigma_n)\|z_n - p\|^2 \\ &\quad - \sigma_n(1-\sigma_n)\|x_n - z_n\|^2] + 2\rho_n\langle (f-\alpha F)p, x_{n+1} - p\rangle \\ &\le \rho_n\delta\|x_n - p\|^2 + (1-\rho_n\tau)[\sigma_n\|x_n - p\|^2 + (1-\sigma_n)\|z_n - p\|^2] \\ &\quad - (1-\rho_n\tau)\sigma_n(1-\sigma_n)\|x_n - z_n\|^2 + \rho_n M_2,\end{aligned} \tag{3.20}$$

where $M_2 > 0$ is a constant such that $\sup_{n\geq 1} 2\|(f-\alpha F)p\|\|x_n - p\| \leq M_2$. By (3.20) and Lemma 3.2, we have

$$\begin{aligned}
\|x_{n+1}-p\|^2 \leq\ & \rho_n\delta\|x_n-p\|^2 + (1-\rho_n\tau)[\sigma_n\|x_n-p\|^2 + (1-\sigma_n)\|z_n-p\|^2] \\
& - (1-\rho_n\tau)\sigma_n(1-\sigma_n)\|x_n-z_n\|^2 + \rho_n M_2 \\
\leq\ & \rho_n\delta\|x_n-p\|^2 + (1-\rho_n\tau)\{\sigma_n\|x_n-p\|^2 + (1-\sigma_n)[\|q_n-p\|^2 \\
& - (1-\xi_n)\|q_n-p_n\|^2 - (1-\nu)(\|y_n-z_n\|^2 + \|y_n-p_n\|^2)]\} \\
& - (1-\rho_n\tau)\sigma_n(1-\sigma_n)\|x_n-z_n\|^2 + \rho_n M_2.
\end{aligned} \tag{3.21}$$

Taking (3.18) into account, we obtain

$$\begin{aligned}
\|q_n-p\|^2 &\leq (1+\varpi_n)^2(\|x_n-p\| + \rho_n M_1)^2 \\
&= (\|x_n-p\| + \rho_n M_1)^2 + \varpi_n(2+\varpi_n))(\|x_n-p\| + \rho_n M_1)^2 \\
&= \|x_n-p\|^2 + \rho_n\{M_1(2\|x_n-p\| + \rho_n M_1) + \frac{\varpi_n(2+\varpi_n)}{\rho_n}(\|x_n-p\| + \rho_n M_1)^2\} \\
&\leq \|x_n-p\|^2 + \rho_n M_3,
\end{aligned} \tag{3.22}$$

where $M_3 > 0$ is a constant such that $\sup_{n\geq 1}\{(2\|x_n-p\| + \rho_n M_1)M_1 + \frac{\varpi_n(2+\varpi_n)}{\rho_n}(\|x_n-p\| + \rho_n M_1)^2\} \leq M_3$. Based on (3.21) and (3.22), we get

$$\begin{aligned}
\|x_{n+1}-p\|^2 \leq\ & \rho_n\delta\|x_n-p\|^2 + (1-\rho_n\tau)\{\sigma_n\|x_n-p\|^2 + (1-\sigma_n)[\|x_n-p\|^2 + \rho_n M_3 \\
& - (1-\xi_n)\|q_n-p_n\|^2 - (1-\nu)(\|y_n-z_n\|^2 + \|y_n-p_n\|^2)]\} \\
& - (1-\rho_n\tau)\sigma_n(1-\sigma_n)\|x_n-z_n\|^2 + \rho_n M_2 \\
\leq\ & [1-\rho_n(\tau-\delta)]\|x_n-p\|^2 - (1-\rho_n\tau)(1-\sigma_n)[(1-\xi_n)\|q_n-p_n\|^2 + \rho_n M_3 \\
& + (1-\nu)(\|y_n-z_n\|^2 + \|y_n-p_n\|^2)] - (1-\rho_n\tau)\sigma_n(1-\sigma_n)\|x_n-z_n\|^2 + \rho_n M_2 \\
\leq\ & \|x_n-p\|^2 - (1-\rho_n\tau)(1-\sigma_n)[(1-\xi_n)\|q_n-p_n\|^2 + (1-\nu)(\|y_n-z_n\|^2 \\
& + \|y_n-p_n\|^2)] - (1-\rho_n\tau)\sigma_n(1-\sigma_n)\|x_n-z_n\|^2 + \rho_n M_4,
\end{aligned}$$

where $M_4 := M_3 + M_2$. This immediately implies that

$$\begin{aligned}
& (1-\rho_n\tau)(1-\sigma_n)[(1-\xi_n)\|q_n-p_n\|^2 + (1-\nu)(\|y_n-z_n\|^2 + \|y_n-p_n\|^2)] \\
& \quad + (1-\rho_n\tau)\sigma_n(1-\sigma_n)\|x_n-z_n\|^2 \leq \|x_n-p\|^2 - \|x_{n+1}-p\|^2 + \rho_n M_4.
\end{aligned} \tag{3.23}$$

Step 3. Note that

$$\begin{aligned}
\|q_n-p\|^2 &\leq (1+\varpi_n)^2(\|x_n-p\| + \varepsilon_n\|x_n-x_{n-1}\|)^2 \\
&= (1+\varpi_n)^2\|x_n-p\|^2 + (1+\varpi_n)^2\varepsilon_n\|x_n-x_{n-1}\|(2\|x_n-p\| + \varepsilon_n\|x_n-x_{n-1}\|) \\
&= (1+\varpi_n(2+\varpi_n))\|x_n-p\|^2 + (1+\varpi_n)^2\varepsilon_n\|x_n-x_{n-1}\|(2\|x_n-p\| \\
&\quad + \varepsilon_n\|x_n-x_{n-1}\|).
\end{aligned} \tag{3.24}$$

Combining (3.18), (3.20), and (3.24), we receive

$$\begin{aligned}
\|x_{n+1}-p\|^2 &\le \rho_n\delta\|x_n-p\|^2+(1-\rho_n\tau)[\sigma_n\|x_n-p\|^2+(1-\sigma_n)\|z_n-p\|^2]\\
&\quad+2\rho_n\langle(f-\alpha F)p,x_{n+1}-p\rangle\\
&\le [1-\rho_n(\tau-\delta)](1+\varpi_n)^2(\|x_n-p\|+\varepsilon_n\|x_n-x_{n-1}\|)^2\\
&\quad+2\rho_n\langle(f-\alpha F)p,x_{n+1}-p\rangle\\
&\le [1-\rho_n(\tau-\delta)]\{(1+\varpi_n(2+\varpi_n))\|x_n-p\|^2+(1+\varpi_n)^2\varepsilon_n\|x_n-x_{n-1}\|\\
&\quad\times(2\|x_n-p\|+\varepsilon_n\|x_n-x_{n-1}\|)\}+2\rho_n\langle(f-\alpha F)p,x_{n+1}-p\rangle\\
&\le [1-\rho_n(\tau-\delta)]\|x_n-p\|^2+\varepsilon_n\|x_n-x_{n-1}\|(1+\varpi_n)^2(2\|x_n-p\|\\
&\quad+\varepsilon_n\|x_n-x_{n-1}\|)+\varpi_n(2+\varpi_n)\|x_n-p\|^2+2\rho_n\langle(f-\alpha F)p,x_{n+1}-p\rangle\\
&\le [1-\rho_n(\tau-\delta)]\|x_n-p\|^2+(\varepsilon_n\|x_n-x_{n-1}\|3(1+\varpi_n)^2+\varpi_n(2+\varpi_n))M\\
&\quad+2\rho_n\langle(f-\alpha F)p,x_{n+1}-p\rangle\\
&= [1-\rho_n(\tau-\delta)]\|x_n-p\|^2+\rho_n(\tau-\delta)[\frac{2\langle(f-\alpha F)p,x_{n+1}-p\rangle}{\tau-\delta}\\
&\quad+\frac{M}{\tau-\delta}(\frac{\varepsilon_n}{\rho_n}\|x_n-x_{n-1}\|\times 3(1+\varpi_n)^2+\frac{\varpi_n(2+\varpi_n)}{\rho_n})],
\end{aligned}\tag{3.25}$$

where $M>0$ is a constant such that $\sup_{n\ge1}\{\|x_n-p\|,\varepsilon_n\|x_n-x_{n-1}\|,\|x_n-p\|^2\}\le M$.
Step 4. Taking $p=u^\dagger$, by (3.25), we have

$$\begin{aligned}
\|x_{n+1}-u^\dagger\|^2 \le [1-\rho_n(\tau-\delta)]\|x_n-u^\dagger\|^2+\rho_n(\tau-\delta)[\frac{2\langle(f-\alpha F)u^\dagger,x_{n+1}-u^\dagger\rangle}{\tau-\delta}\\
+\frac{M}{\tau-\delta}(\frac{\varepsilon_n}{\rho_n}\|x_n-x_{n-1}\|3(1+\varpi_n)^2+\frac{\varpi_n(2+\varpi_n)}{\rho_n})].
\end{aligned}\tag{3.26}$$

Set $\Gamma_n=\|x_n-u^\dagger\|^2$.
Case 1. There is an integer $n_0\ge1$ such that $\{\Gamma_n\}$ is nonincreasing. In this case, $\lim_{n\to\infty}\Gamma_n=\hbar<+\infty$. From (3.23), we have

$$\begin{aligned}
&(1-\rho_n\tau)(1-b)[(1-\xi_n)\|q_n-p_n\|^2+(1-\nu)(\|y_n-z_n\|^2+\|y_n-p_n\|^2)]\\
&\quad+(1-\rho_n\tau)a(1-b)\|x_n-z_n\|^2\\
&\le(1-\rho_n\tau)(1-\sigma_n)[(1-\xi_n)\|q_n-p_n\|^2+(1-\nu)(\|y_n-z_n\|^2+\|y_n-p_n\|^2)]\\
&\quad+(1-\rho_n\tau)\sigma_n(1-\sigma_n)\|x_n-z_n\|^2\\
&\le\|x_n-u^\dagger\|^2-\|x_{n+1}-u^\dagger\|^2+\rho_nM_4=\Gamma_n-\Gamma_{n+1}+\rho_nM_4.
\end{aligned}$$

Noticing $0<\liminf_{n\to\infty}(1-\xi_n)$, $\rho_n\to0$ and $\Gamma_n-\Gamma_{n+1}\to0$, for $\nu\in(0,1)$ one has $\lim_{n\to\infty}\|q_n-p_n\|=\lim_{n\to\infty}\|y_n-z_n\|=0$, and $\lim_{n\to\infty}\|y_n-p_n\|=\lim_{n\to\infty}\|x_n-z_n\|=0$. Thus, we get

$$\|x_n-y_n\|\le\|x_n-z_n\|+\|y_n-z_n\|\to0,\tag{3.27}$$

and

$$\|q_n-z_n\|\le\|q_n-p_n\|+\|p_n-y_n\|+\|y_n-z_n\|\to0\to0\quad(n\to\infty).\tag{3.28}$$

Since $\{x_n\}$ is bounded, there is a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ satisfying $x_{n_k} \rightharpoonup \widetilde{x}$ and

$$\limsup_{n\to\infty}\langle (f-\alpha F)u^\dagger, x_n - u^\dagger\rangle = \lim_{k\to\infty}\langle (f-\alpha F)u^\dagger, x_{n_k} - u^\dagger\rangle. \tag{3.29}$$

In the light of (3.29), one gets

$$\limsup_{n\to\infty}\langle (f-\alpha F)u^\dagger, x_n - u^\dagger\rangle = \lim_{k\to\infty}\langle (f-\alpha F)u^\dagger, x_{n_k} - u^\dagger\rangle = \langle (f-\alpha F)u^\dagger, \widetilde{x} - u^\dagger\rangle. \tag{3.30}$$

Since $x_{n+1} - x_n \to 0,\ y_n - x_n \to 0,\ z_n - q_n \to 0$ and $S^{n+1}x_n - S^n x_n \to 0$, applying Lemma 3.3, we conclude that $\widetilde{x} \in \omega_w(\{x_n\}) \subset \Delta$. Combining (3.12) and (3.30), we get

$$\limsup_{n\to\infty}\langle (f-\alpha F)u^\dagger, x_n - u^\dagger\rangle = \langle (f-\alpha F)u^\dagger, \widetilde{x} - u^\dagger\rangle \le 0. \tag{3.31}$$

Since $x_n - x_{n+1} \to 0$, we have

$$\begin{aligned}
&\limsup_{n\to\infty}\langle (f-\alpha F)u^\dagger, x_{n+1} - u^\dagger\rangle \\
&= \limsup_{n\to\infty}[\langle (f-\alpha F)u^\dagger, x_{n+1} - x_n\rangle + \langle (f-\alpha F)u^\dagger, x_n - u^\dagger\rangle] \\
&\le \limsup_{n\to\infty}[\|(f-\alpha F)u^\dagger\|\|x_{n+1} - x_n\| + \langle (f-\alpha F)u^\dagger, x_n - u^\dagger\rangle] \le 0.
\end{aligned}$$

Note that

$$\limsup_{n\to\infty}[\frac{2\langle (f-\alpha F)u^\dagger, x_{n+1} - u^\dagger\rangle}{\tau-\delta} + \frac{M}{\tau-\delta}(\frac{\varepsilon_n}{\rho_n}\|x_n - x_{n-1}\|3(1+\varpi_n)^2 + \frac{\varpi_n(2+\varpi_n)}{\rho_n})] \le 0.$$

According to (3.26) and Lemma 2.4, we deduce that $\lim_{n\to\infty}\|x_n - u^\dagger\|^2 = 0$.
Case 2. Suppose that $\exists\{\Gamma_{n_k}\} \subset \{\Gamma_n\}$ s.t. $\Gamma_{n_k} < \Gamma_{n_k+1},\ \forall k \in \mathbf{N}$. Let $\phi : \mathbf{N} \to \mathbf{N}$ be a mapping defined by

$$\phi(n) := \max\{k \le n : \Gamma_k < \Gamma_{k+1}\}.$$

Based on Lemma 2.6, we have

$$\Gamma_{\phi(n)} \le \Gamma_{\phi(n)+1} \quad \text{and} \quad \Gamma_n \le \Gamma_{\phi(n)+1}.$$

Putting $p = u^\dagger$, from (3.23), we have

$$\begin{aligned}
&(1-\rho_{\phi(n)}\tau)(1-b)[(1-\xi_{\phi(n)})\|q_{\phi(n)} - p_{\phi(n)}\|^2 + (1-\nu)(\|y_{\phi(n)} - z_{\phi(n)}\|^2 \\
&\quad + \|y_{\phi(n)} - p_{\phi(n)}\|^2)] + (1-\rho_{\phi(n)}\tau)a(1-b)\|x_{\phi(n)} - z_{\phi(n)}\|^2 \\
&\le (1-\rho_{\phi(n)}\tau)(1-\sigma_{\phi(n)})[(1-\xi_{\phi(n)})\|q_{\phi(n)} - p_{\phi(n)}\|^2 + (1-\nu)(\|y_{\phi(n)} - z_{\phi(n)}\|^2 \\
&\quad + \|y_{\phi(n)} - p_{\phi(n)}\|^2)] + (1-\rho_{\phi(n)}\tau)\sigma_{\phi(n)}(1-\sigma_{\phi(n)})\|x_{\phi(n)} - z_{\phi(n)}\|^2 \\
&\le \|x_{\phi(n)} - u^\dagger\|^2 - \|x_{\phi(n)+1} - u^\dagger\|^2 + \rho_{\phi(n)}M_4 = \Gamma_{\phi(n)} - \Gamma_{\phi(n)+1} + \rho_{\phi(n)}M_4,
\end{aligned} \tag{3.32}$$

which immediately yields $\lim_{n\to\infty}\|q_{\phi(n)} - p_{\phi(n)}\| = \lim_{n\to\infty}\|y_{\phi(n)} - z_{\phi(n)}\| = 0$ and $\lim_{n\to\infty}\|y_{\phi(n)} - p_{\phi(n)}\| = \lim_{n\to\infty}\|x_{\phi(n)} - z_{\phi(n)}\| = 0$. Therefore,

$$\lim_{n\to\infty}\|x_{\phi(n)} - y_{\phi(n)}\| = \lim_{n\to\infty}\|q_{\phi(n)} - z_{\phi(n)}\| = 0, \tag{3.33}$$

and

$$\limsup_{n\to\infty}\langle(f-\alpha F)u^{\dagger}, x_{\phi(n)+1}-u^{\dagger}\rangle \le 0. \tag{3.34}$$

At the same time, by (3.26), we known that

$$\begin{aligned}\rho_{\phi(n)}(\tau-\delta)\Gamma_{\phi(n)} &\le \Gamma_{\phi(n)}-\Gamma_{\phi(n)+1}+\rho_{\phi(n)}(\tau-\delta)[\frac{2\langle(f-\alpha F)u^{\dagger}, x_{\phi(n)+1}-u^{\dagger}\rangle}{\tau-\delta}\\ &\quad+\frac{M}{\tau-\delta}(\frac{\varepsilon_{\phi(n)}}{\rho_{\phi(n)}}\|x_{\phi(n)}-x_{\phi(n)-1}\|3(1+\varpi_{\phi(n)})^2+\frac{\varpi_{\phi(n)}(2+\varpi_{\phi(n)})}{\rho_{\phi(n)}})]\\ &\le \rho_{\phi(n)}(\tau-\delta)[\frac{2\langle(f-\alpha F)u^{\dagger}, x_{\phi(n)+1}-u^{\dagger}\rangle}{\tau-\delta}+\frac{M}{\tau-\delta}(\frac{\varepsilon_{\phi(n)}}{\rho_{\phi(n)}}\|x_{\phi(n)}-x_{\phi(n)-1}\|\\ &\quad\times 3(1+\varpi_{\phi(n)})^2+\frac{\varpi_{\phi(n)}(2+\varpi_{\phi(n)})}{\rho_{\phi(n)}})],\end{aligned}$$

which hence arrives at

$$\begin{aligned}\limsup_{n\to\infty}\Gamma_{\phi(n)} &\le \limsup_{n\to\infty}[\tfrac{2\langle(f-\alpha F)u^{\dagger}, x_{\phi(n)+1}-u^{\dagger}\rangle}{\tau-\delta}\\ &\quad+\tfrac{M}{\tau-\delta}(\tfrac{\varepsilon_{\phi(n)}}{\rho_{\phi(n)}}\|x_{\phi(n)}-x_{\phi(n)-1}\|3(1+\varpi_{\phi(n)})^2+\tfrac{\varpi_{\phi(n)}(2+\varpi_{\phi(n)})}{\rho_{\phi(n)}})]\le 0.\end{aligned}$$

Thus, $\lim_{n\to\infty}\|x_{\phi(n)}-u^{\dagger}\|^2=0$. Also, note that

$$\begin{aligned}\|x_{\phi(n)+1}-u^{\dagger}\|^2-\|x_{\phi(n)}-u^{\dagger}\|^2 &= 2\langle x_{\phi(n)+1}-x_{\phi(n)}, x_{\phi(n)}-u^{\dagger}\rangle+\|x_{\phi(n)+1}-x_{\phi(n)}\|^2\\ &\le 2\|x_{\phi(n)+1}-x_{\phi(n)}\|\|x_{\phi(n)}-u^{\dagger}\|+\|x_{\phi(n)+1}-x_{\phi(n)}\|^2.\end{aligned} \tag{3.35}$$

Since $\Gamma_n \le \Gamma_{\phi(n)+1}$, we have

$$\begin{aligned}\|x_n-u^{\dagger}\|^2 &\le \|x_{\phi(n)+1}-u^{\dagger}\|^2\\ &\le \|x_{\phi(n)}-u^{\dagger}\|^2+2\|x_{\phi(n)+1}-x_{\phi(n)}\|\|x_{\phi(n)}-u^{\dagger}\|+\|x_{\phi(n)+1}-x_{\phi(n)}\|^2 \to 0 \quad (n\to\infty).\end{aligned}$$

So, $x_n \to u^{\dagger}$. □

According to Theorem 3.1, we have the following corollary.

Corollary 3.1. *Suppose that $S: C\to C$ is a nonexpansive mapping. For two fixed points $x_1, x_0 \in H$, let the sequence $\{x_n\}$ be defined by*

$$\begin{cases} q_n = Sx_n+\varepsilon_n(Sx_n-Sx_{n-1}),\\ p_n = \xi_n q_n+(1-\xi_n)u_n,\\ v_n = T^{\Theta_2}_{\alpha_2}(p_n-\alpha_2 B_2 p_n),\\ u_n = T^{\Theta_1}_{\alpha_1}(v_n-\alpha_1 B_1 v_n),\\ y_n = P_C(p_n-\zeta_n A p_n),\\ z_n = P_{C_n}(p_n-\zeta_n A y_n),\\ t_n = \sigma_n x_n+(1-\sigma_n)z_n,\\ x_{n+1} = \rho_n f(x_n)+(I-\rho_n\alpha F)S_n t_n, \quad \forall n\ge 1,\end{cases} \tag{3.36}$$

where C_n and ζ_n have the same form as in Algorithm 3.1. Then, $x_n\to u^{\dagger}\in\Delta \Leftrightarrow x_{n+1}-x_n\to 0$, where $u^{\dagger}\in\Delta$ is the unique solution of the HVI: $\langle(\alpha F-f)u^{\dagger}, p-u^{\dagger}\rangle\ge 0,\ \forall p\in\Delta$.

Next, we put forth another modification of the inertial composite subgradient extragradient implicit rule with line-search process.

Algorithm 3.2. *Let $x_1, x_0 \in H$ be two fixed points. Let x_n be given. Compute x_{n+1} via the following iterative steps:*
Step 1. *Set $q_n = S^n x_n + \varepsilon_n(S^n x_n - S^n x_{n-1})$ and calculate*

$$\begin{cases} p_n = \xi_n q_n + (1-\xi_n)u_n, \\ v_n = T_{\alpha_2}^{\Theta_2}(p_n - \alpha_2 B_2 p_n), \\ u_n = T_{\alpha_1}^{\Theta_1}(v_n - \alpha_1 B_1 v_n). \end{cases}$$

Step 2. *Compute $y_n = P_C(p_n - \zeta_n A p_n)$, with ζ_n being chosen to be the largest $\zeta \in \{\gamma, \gamma\ell, \gamma\ell^2, ...\}$ s.t.*

$$\zeta\|Ap_n - Ay_n\| \le \nu\|p_n - y_n\|.$$

Step 3. *Compute $t_n = \sigma_n z_n + (1-\sigma_n)S_n t_n$ with $z_n = P_{C_n}(p_n - \zeta_n A y_n)$ and*

$$C_n := \{y \in H : \langle p_n - \zeta_n A p_n - y_n, y - y_n\rangle \le 0\}.$$

Step 4. *Compute*

$$x_{n+1} = \rho_n f(x_n) + (I - \rho_n \alpha F)S_n t_n,$$

where S_n is constructed as in Algorithm 1.1. Set $n := n+1$ and go to Step 1.

Theorem 3.2. *Let the sequence $\{x_n\}$ be generated by Algorithm 3.2. Then*

$$x_n \to u^\dagger \in \Delta \quad \Leftrightarrow \quad \begin{cases} S^{n+1}x_n - S^n x_n \to 0, \\ x_{n+1} - x_n \to 0, \end{cases}$$

where $u^\dagger \in \Delta$ is the unique solution of the HVI: $\langle(\alpha F - f)u^\dagger, p - u^\dagger\rangle \ge 0, \ \ \forall p \in \Delta$.

Proof. The necessity is obvious. Next, we prove the sufficiency.

Note that

$$\|t_n - p\| \le \sigma_n\|z_n - p\| + (1-\sigma_n)\|S_n t_n - p\| \le \sigma_n\|z_n - p\| + (1-\sigma_n)\|t_n - p\|.$$

This, together with (3.18), ensures that

$$\|t_n - p\| \le \|z_n - p\| \le \|p_n - p\| \le \|q_n - p\| \le (1+\varpi_n)[\|x_n - p\| + \rho_n M_1], \quad \forall n \ge 1. \tag{3.37}$$

By (3.37) and Lemma 2.7, we have

$$\begin{aligned} \|x_{n+1} - p\| &= \|\rho_n(f(x_n) - f(p)) + (I - \rho_n\alpha F)S_n t_n - (I - \rho_n\alpha F)p + \rho_n(f - \alpha F)p\| \\ &\le \rho_n\delta\|x_n - p\| + (1-\rho_n\tau)\|t_n - p\| + \rho_n\|(f-\alpha F)p\| \\ &\le \rho_n\delta\|x_n - p\| + (1-\rho_n\tau)(1+\varpi_n)[\|x_n - p\| + \rho_n M_1] + \rho_n\|(f-\alpha F)p\| \\ &\le [1 - \tfrac{\rho_n(\tau-\delta)}{2}]\|x_n - p\| + \tfrac{\rho_n(\tau-\delta)}{2} \cdot \tfrac{2(2M_1 + \|(f-\alpha F)p\|)}{\tau-\delta}. \end{aligned}$$

It follows that $\|x_n - p\| \le \max\{\|x_1 - p\|, \frac{2(2M_1+\|(f-\alpha F)p\|)}{\tau-\delta}\} \ \forall n \ge 1$. Therefore, $\{x_n\}$, $\{p_n\}$, $\{q_n\}$, $\{y_n\}$, $\{z_n\}$, $\{t_n\}$, $\{f(x_n)\}$, $\{S_n t_n\}$, and $\{S^n x_n\}$ are bounded.

According to Lemma 2.7, we get

$$\begin{aligned}
\|x_{n+1}-p\|^2 &\le \|\rho_n(f(x_n)-f(p))+(I-\rho_n\alpha F)S_nt_n-(I-\rho_n\alpha F)p\|^2 \\
&\quad +2\rho_n\langle(f-\alpha F)p,x_{n+1}-p\rangle \\
&\le \rho_n\delta\|x_n-p\|^2+(1-\rho_n\tau)\|t_n-p\|^2+2\rho_n\langle(f-\alpha F)p,x_{n+1}-p\rangle \\
&= \rho_n\delta\|x_n-p\|^2+(1-\rho_n\tau)[\sigma_n\|z_n-p\|^2+(1-\sigma_n)\|S_nt_n-p\|^2 \\
&\quad -\sigma_n(1-\sigma_n)\|z_n-S_nt_n\|^2]+2\rho_n\langle(f-\alpha F)p,x_{n+1}-p\rangle \\
&\le \rho_n\delta\|x_n-p\|^2+(1-\rho_n\tau)[\sigma_n\|z_n-p\|^2+(1-\sigma_n)\|t_n-p\|^2] \\
&\quad -(1-\rho_n\tau)\sigma_n(1-\sigma_n)\|z_n-S_nt_n\|^2+\rho_nM_2,
\end{aligned} \tag{3.38}$$

where $M_2>0$ is a constant such that $\sup_{n\ge1}2\|(f-\alpha F)p\|\|x_n-p\|\le M_2$. Using Lemma 3.2, from (3.37) and (3.38), we have

$$\begin{aligned}
\|x_{n+1}-p\|^2 &\le \rho_n\delta\|x_n-p\|^2+(1-\rho_n\tau)\|z_n-p\|^2 \\
&\quad -(1-\rho_n\tau)\sigma_n(1-\sigma_n)\|z_n-S_nt_n\|^2+\rho_nM_2 \\
&\le \rho_n\delta\|x_n-p\|^2+(1-\rho_n\tau)\{\|q_n-p\|^2-(1-\xi_n)\|q_n-p_n\|^2-(1-\nu) \\
&\quad \times(\|y_n-z_n\|^2+\|y_n-p_n\|^2)\}-(1-\rho_n\tau)\sigma_n(1-\sigma_n)\|z_n-S_nt_n\|^2+\rho_nM_2.
\end{aligned} \tag{3.39}$$

Also, using the same inferences as those of (3.22) of Theorem 3.1, we have

$$\|q_n-p\|^2\le\|x_n-p\|^2+\rho_nM_3, \tag{3.40}$$

where $\sup_{n\ge1}\{M_1(2\|x_n-p\|+\rho_nM_1)+\frac{\varpi_n(2+\varpi_n)}{\rho_n}(\|x_n-p\|+\rho_nM_1)^2\}\le M_3$ for some constant M_3. By (3.39) and (3.40), we attain

$$\begin{aligned}
\|x_{n+1}-p\|^2 &\le \rho_n\delta\|x_n-p\|^2+(1-\rho_n\tau)\{\|x_n-p\|^2+\rho_nM_3-(1-\xi_n)\|q_n-p_n\|^2 \\
&\quad -(1-\nu)(\|y_n-z_n\|^2+\|y_n-p_n\|^2)\}-(1-\rho_n\tau)\sigma_n(1-\sigma_n)\|z_n-S_nt_n\|^2+\rho_nM_2 \\
&\le [1-\rho_n(\tau-\delta)]\|x_n-p\|^2-(1-\rho_n\tau)[(1-\xi_n)\|q_n-p_n\|^2+(1-\nu)(\|y_n-z_n\|^2 \\
&\quad +\|y_n-p_n\|^2)]-(1-\rho_n\tau)\sigma_n(1-\sigma_n)\|z_n-S_nt_n\|^2+\rho_nM_3+\rho_nM_2 \\
&\le \|x_n-p\|^2-(1-\rho_n\tau)[(1-\xi_n)\|q_n-p_n\|^2+(1-\nu)(\|y_n-z_n\|^2 \\
&\quad +\|y_n-p_n\|^2)]-(1-\rho_n\tau)\sigma_n(1-\sigma_n)\|z_n-S_nt_n\|^2+\rho_nM_4,
\end{aligned}$$

where $M_4:=M_3+M_2$. Hence, we attain the assertion.

By the same argument as those of (3.24), we have

$$\|q_n-p\|^2\le(1+\varpi_n(2+\varpi_n))\|x_n-p\|^2+(1+\varpi_n)^2\varepsilon_n\|x_n-x_{n-1}\|(2\|x_n-p\|+\varepsilon_n\|x_n-x_{n-1}\|). \tag{3.41}$$

By (3.37), (3.38), and (3.41), we obtain

$$
\begin{aligned}
\|x_{n+1}-p\|^2 &\le \rho_n\delta\|x_n-p\|^2+(1-\rho_n\tau)[(1-\sigma_n)\|t_n-p\|^2+\sigma_n\|z_n-p\|^2]\\
&\quad +2\rho_n\langle (f-\alpha F)p, x_{n+1}-p\rangle\\
&\le [1-\rho_n(\tau-\delta)](1+\varpi_n)^2(\|x_n-p\|+\varepsilon_n\|x_n-x_{n-1}\|)^2\\
&\quad +2\rho_n\langle (f-\alpha F)p, x_{n+1}-p\rangle\\
&\le [1-\rho_n(\tau-\delta)]\|x_n-p\|^2+\varepsilon_n\|x_n-x_{n-1}\|(1+\varpi_n)^2(2\|x_n-p\| \\
&\quad +\varepsilon_n\|x_n-x_{n-1}\|)+\varpi_n(2+\varpi_n)\|x_n-p\|^2+2\rho_n\langle (f-\alpha F)p, x_{n+1}-p\rangle\\
&\le [1-\rho_n(\tau-\delta)]\|x_n-p\|^2+\rho_n(\tau-\delta)[\frac{2\langle (f-\alpha F)p, x_{n+1}-p\rangle}{\tau-\delta}\\
&\quad +\frac{M}{\tau-\delta}(\frac{\varepsilon_n}{\rho_n}\|x_n-x_{n-1}\|\times 3(1+\varpi_n)^2+\frac{\varpi_n(2+\varpi_n)}{\rho_n})],
\end{aligned}
\tag{3.42}
$$

where $\sup_{n\ge1}\{\|x_n-p\|, \varepsilon_n\|x_n-x_{n-1}\|, \|x_n-p\|^2\}\le M$ for some constant M.

Setting $p=u^\dagger$, by (3.42), we have

$$
\begin{aligned}
\|x_{n+1}-u^\dagger\|^2 &\le [1-\rho_n(\tau-\delta)]\|x_n-u^\dagger\|^2+\rho_n(\tau-\delta)[\tfrac{2\langle (f-\alpha F)u^\dagger, x_{n+1}-u^\dagger\rangle}{\tau-\delta}\\
&\quad +\tfrac{M}{\tau-\delta}(\tfrac{\varepsilon_n}{\rho_n}\|x_n-x_{n-1}\|3(1+\varpi_n)^2+\tfrac{\varpi_n(2+\varpi_n)}{\rho_n})].
\end{aligned}
$$

Set $\Gamma_n=\|x_n-u^\dagger\|^2$.

Case 1. Assume $\{\Gamma_n\}$ is nonincreasing when $n\ge n_0$. Then, $\lim_{n\to\infty}\Gamma_n=\hbar<+\infty$. Choosing $p=u^\dagger$, from (3.38), we have

$$
\begin{aligned}
&(1-\rho_n\tau)[(1-\xi_n)\|q_n-p_n\|^2+(1-\nu)(\|y_n-z_n\|^2+\|y_n-p_n\|^2)]\\
&\quad +(1-\rho_n\tau)a(1-b)\|z_n-S_nt_n\|^2\\
&\le (1-\rho_n\tau)[(1-\xi_n)\|q_n-p_n\|^2+(1-\nu)(\|y_n-z_n\|^2+\|y_n-p_n\|^2)]\\
&\quad +(1-\rho_n\tau)\sigma_n(1-\sigma_n)\|z_n-S_nt_n\|^2\\
&\le \|x_n-u^\dagger\|^2-\|x_{n+1}-u^\dagger\|^2+\rho_nM_4=\Gamma_n-\Gamma_{n+1}+\rho_nM_4.
\end{aligned}
$$

Since $\Gamma_n-\Gamma_{n+1}\to0$ for $\nu\in(0,1)$, $\lim_{n\to\infty}\|q_n-p_n\|=\lim_{n\to\infty}\|y_n-z_n\|=0$, and $\lim_{n\to\infty}\|y_n-p_n\|=\lim_{n\to\infty}\|z_n-S_nt_n\|=0$. Observe that

$$
\begin{aligned}
\|z_n-x_n\| &\le \|z_n-S_nt_n\|+\|S_nt_n-x_n\|\\
&= \|z_n-S_nt_n\|+\|x_{n+1}-x_n-\rho_n(f(x_n)-\alpha FS_nt_n)\|\\
&\le \|z_n-S_nt_n\|+\|x_{n+1}-x_n\|+\rho_n(\|f(x_n)\|+\|\alpha FS_nt_n\|)\to0 \quad (n\to\infty).
\end{aligned}
$$

By the similar arguments as those in Theorem 3.1, we deduce $\lim_{n\to\infty}\|x_n-u^\dagger\|^2=0$.

Case 2. Assume $\exists\{\Gamma_{n_k}\}\subset\{\Gamma_n\}$ s.t. $\Gamma_{n_k}<\Gamma_{n_k+1},\ \forall k\in\mathbf{N}$. Let $\phi:\mathbf{N}\to\mathbf{N}$ be a mapping defined by

$$\phi(n)=\max\{k\le n:\Gamma_k<\Gamma_{k+1}\}.$$

By Lemma 2.6, we have

$$\Gamma_n\le\Gamma_{\phi(n)+1}\quad\text{and}\quad\Gamma_{\phi(n)}\le\Gamma_{\phi(n)+1}.$$

Set $p=u^\dagger$. Then,

$$
\begin{aligned}
&(1-\rho_{\phi(n)}\tau)[(1-\xi_{\phi(n)})\|q_{\phi(n)}-p_{\phi(n)}\|^2+(1-\nu)(\|y_{\phi(n)}-z_{\phi(n)}\|^2\\
&\quad+\|y_{\phi(n)}-p_{\phi(n)}\|^2)]+(1-\rho_{\phi(n)}\tau)a(1-b)\|z_{\phi(n)}-S_{\phi(n)}t_{\phi(n)}\|^2\\
&\le(1-\rho_{\phi(n)}\tau)[(1-\xi_{\phi(n)})\|q_{\phi(n)}-p_{\phi(n)}\|^2+(1-\nu)(\|y_{\phi(n)}-z_{\phi(n)}\|^2\\
&\quad+\|y_{\phi(n)}-p_{\phi(n)}\|^2)]+(1-\rho_{\phi(n)}\tau)\sigma_{\phi(n)}(1-\sigma_{\phi(n)})\|z_{\phi(n)}-S_{\phi(n)}t_{\phi(n)}\|^2\\
&\le\|x_{\phi(n)}-u^\dagger\|^2-\|x_{\phi(n)+1}-u^\dagger\|^2+\rho_{\phi(n)}M_4=\Gamma_{\phi(n)}-\Gamma_{\phi(n)+1}+\rho_{\phi(n)}M_4,
\end{aligned}
$$

which immediately yields $\lim_{n\to\infty}\|q_{\phi(n)}-p_{\phi(n)}\|=\lim_{n\to\infty}\|y_{\phi(n)}-z_{\phi(n)}\|=0$ and $\lim_{n\to\infty}\|y_{\phi(n)}-p_{\phi(n)}\|=\lim_{n\to\infty}\|z_{\phi(n)}-S_{\phi(n)}t_{\phi(n)}\|=0$. Therefore, $\lim_{n\to\infty}\|z_{\phi(n)}-x_{\phi(n)}\|=0$. Finally, using the similar arguments to those in Theorem 3.1, we get the conclusion. □

Remark 3.1. Compared with the corresponding results in Cai, Shehu, and Iyiola [2], Ceng and Shang [4], and Thong and Hieu [28], our results improve and extend them in the following aspects:

(i) The problem of finding an element of $\mathrm{Fix}(S)\cap\mathrm{Fix}(G)$ (with $G=P_C(I-\mu_1B_1)P_C(I-\mu_2B_2)$) in [2] is extended to develop our problem of finding an element of $\bigcap_{r=0}^N\mathrm{Fix}(S_r)\cap\mathrm{Fix}(G)\cap\mathrm{VI}(C,A)$ where $G=T_{\mu_1}^{\Theta_1}(I-\mu_1B_1)T_{\mu_2}^{\Theta_2}(I-\mu_2B_2)$ and S is an asymptotically nonexpansive mapping. The modified viscosity implicit rule for finding an element of $\mathrm{Fix}(S)\cap\mathrm{Fix}(G)$ in [2] is extended to develop our modified inertial composite subgradient extragradient implicit rules with line-search process for finding an element of $\bigcap_{r=0}^N\mathrm{Fix}(S_r)\cap\mathrm{Fix}(G)\cap\mathrm{VI}(C,A)$, which is on the basis of the subgradient extragradient rule with line-search process, inertial iteration approach, viscosity approximation method, and hybrid deepest-descent technique.

(ii) The problem of finding an element of $\mathrm{Fix}(S)\cap\mathrm{VI}(C,A)$ with a quasi-nonexpansive mapping S in [4] is extended to develop our problem of finding an element of $\bigcap_{r=0}^N\mathrm{Fix}(S_r)\cap\mathrm{Fix}(G)\cap\mathrm{VI}(C,A)$ with an asymptotically nonexpansive mapping S. The inertial subgradient extragradient method with line-search process for finding an element of $\mathrm{Fix}(S)\cap\mathrm{VI}(C,A)$ in [28] is extended to develop our modified inertial composite subgradient extragradient implicit rules with line-search process for finding an element of $\bigcap_{r=0}^N\mathrm{Fix}(S_r)\cap\mathrm{Fix}(G)\cap\mathrm{VI}(C,A)$, which is on the basis of the subgradient extragradient rule with line-search process, inertial iteration approach, viscosity approximation method, and hybrid deepest-descent technique.

(iii) The problem of finding an element of $\Omega=\bigcap_{r=0}^N\mathrm{Fix}(S_r)\cap\mathrm{VI}(C,A)$ is extended to develop our problem of finding an element of $\Omega=\bigcap_{r=0}^N\mathrm{Fix}(S_r)\cap\mathrm{Fix}(G)\cap\mathrm{VI}(C,A)$ with $G=T_{\mu_1}^{\Theta_1}(I-\mu_1B_1)T_{\mu_2}^{\Theta_2}(I-\mu_2B_2)$. The hybrid inertial subgradient extragradient method with line-search process in [4] is extended to develop our modified inertial composite subgradient extragradient implicit rules with line-search process.

4. Examples

In this section, we give an example to show the feasibility of our algorithms. Put $\Theta_1=\Theta_2=0$, $\alpha=2$, $\alpha_1=\alpha_2=\frac{1}{3}$, $\gamma=1$, $\nu=\ell=\frac{1}{2}$, $\sigma_n=\xi_n=\frac{2}{3}$, and $\varepsilon_n=\rho_n=\frac{1}{3(n+1)}$, for all $n\ge0$. Now, we construct an example of $\Delta=\bigcap_{r=0}^N\mathrm{Fix}(S_r)\cap\mathrm{Fix}(G)\cap\mathrm{VI}(C,A)\ne\emptyset$ with $S_0:=S$ and $G=T_{\alpha_1}^{\Theta_1}(I-\alpha_1B_1)T_{\alpha_2}^{\Theta_2}(I-\alpha_2B_2)=P_C(I-\alpha_1B_1)P_C(I-\alpha_2B_2)$, where $A:H\to H$ is pseudomonotone and a Lipschitz continuous mapping, $B_1,B_2:H\to H$ are two inverse-strongly monotone mappings, $S:H\to H$ is asymptotical nonexpansive, and each $S_r:H\to H$ is nonexpansive for $r=1,...,N$.

Let $H=\mathbf{R}$ and use $\langle a,b\rangle=ab$ and $\|\cdot\|=|\cdot|$ to denote its inner product and induced norm, respectively. Set $C=[-2,4]$ and the starting point x_1 is arbitrarily chosen in C. Let $f(x)=F(x)=\frac{1}{2}x$,

$\forall x \in H$ with

$$\delta = \frac{1}{2} < \tau = 1 - \sqrt{1-\alpha(2\eta - \alpha\kappa^2)} = 1 - \sqrt{1 - 2(2\cdot\frac{1}{2} - 2(\frac{1}{2})^2)} = 1.$$

Let $B_1x = B_2x := Bx = x - \frac{1}{2}\sin x,\ \forall x \in C$. Let the operators $A, S, S_r : H \to H$ be defined by

$$Ax := \frac{1}{1+|\sin x|} - \frac{1}{1+|x|},\ Sx := \frac{3}{4}\sin x,\ S_rx := S_1x = \sin x\ (r = 1, \cdots, N), \forall x \in H.$$

We have the following assertions:
(i) A is 2-Lipschitz continuous, in fact, for each $x, y \in H$, we have

$$\begin{aligned}|Ax - Ay| &\le \left|\frac{|y|-|x|}{(1+\||y|)(1+|x|)}\right| + \left|\frac{|\sin y| - |\sin x|}{(1+|\sin y|)(1+|\sin x|)}\right| \\ &\le \frac{|x-y|}{(1+|x|)(1+|y|)} + \frac{|\sin x - \sin y|}{(1+|\sin x|)(1+|\sin y|)} \\ &\le |x-y| + |\sin x - \sin y| \\ &\le 2|x-y|.\end{aligned}$$

(ii) A is pseudomonotone, in fact, for each $x, y \in H$, if

$$\langle Ax, y-x\rangle = (\frac{1}{1+|\sin x|} - \frac{1}{1+|x|})(y-x) \ge 0,$$

then

$$\langle Ay, y-x\rangle = (\frac{1}{1+|\sin y|} - \frac{1}{1+|y|})(y-x) \ge 0.$$

(iii) B is $\frac{2}{9}$-inverse-strongly monotone. In fact, since B is $\frac{1}{2}$-strongly monotone and $\frac{3}{2}$-Lipschitz continuous, we know that B is $\frac{2}{9}$-inverse-strongly monotone with $\rho = \sigma = \frac{2}{9}$.

Moreover, it is easy to check that S is asymptotically nonexpansive with $\varpi_n = (\frac{3}{4})^n,\ \forall n \ge 1$, such that $\|S^{n+1}x_n - S^nx_n\| \to 0$ as $n \to \infty$. In fact, note that

$$\|S^nx - S^ny\| \le \frac{3}{4}\|S^{n-1}x - S^{n-1}y\| \le \cdots \le (\frac{3}{4})^n\|x-y\| \le (1+\varpi_n)\|x-y\|,$$

and

$$\|S^{n+1}x_n - S^nx_n\| \le (\frac{3}{4})^{n-1}\|S^2x_n - Sx_n\| = (\frac{3}{4})^{n-1}|\frac{3}{4}\sin(Sx_n) - \frac{3}{4}\sin x_n| \le 2(\frac{3}{4})^n \to 0.$$

It is obvious that $\mathrm{Fix}(S) = \{0\}$ and

$$\lim_{n\to\infty}\frac{\varpi_n}{\rho_n} = \lim_{n\to\infty}\frac{(3/4)^n}{1/3(n+1)} = 0.$$

Accordingly, $\Delta = \mathrm{Fix}(S) \cap \mathrm{Fix}(S_1) \cap \mathrm{Fix}(G) \cap \mathrm{VI}(C, A) = \{0\} \ne \emptyset$. In this case, noticing $G = P_C(I - \alpha_1B_1)P_C(I - \alpha_2B_2) = [P_C(I - \frac{1}{3}B)]^2$, we rewrite Algorithm 3.1 as follows:

$$\begin{cases} q_n = S^n x_n + \frac{1}{3(n+1)}(S^n x_n - S^n x_{n-1}), \\ p_n = \frac{2}{3}q_n + \frac{1}{3}u_n, \\ v_n = P_C(p_n - \frac{1}{3}Bp_n), \\ u_n = P_C(v_n - \frac{1}{3}Bv_n), \\ y_n = P_C(p_n - \zeta_n Ap_n), \\ z_n = P_{C_n}(p_n - \zeta_n Ay_n), \\ t_n = \frac{2}{3}x_n + \frac{1}{3}z_n, \\ x_{n+1} = \frac{1}{3(n+1)} \cdot \frac{1}{2}x_n + (1 - \frac{1}{3(n+1)})S_1 t_n, \ \forall n \geq 1, \end{cases}$$

where C_n and ζ_n are chosen as in Algorithm 3.1. Then, $x_n \to 0 \in \Delta$.

In particular, since $Sx := \frac{3}{4}\sin x$ is also nonexpansive, we consider the modified version of Algorithm 3.1, that is,

$$\begin{cases} q_n = Sx_n + \frac{1}{3(n+1)}(Sx_n - Sx_{n-1}), \\ p_n = \frac{2}{3}q_n + \frac{1}{3}u_n, \\ v_n = P_C(p_n - \frac{1}{3}Bp_n), \\ u_n = P_C(v_n - \frac{1}{3}Bv_n), \\ y_n = P_C(p_n - \zeta_n Ap_n), \\ z_n = P_{C_n}(p_n - \zeta_n Ay_n), \\ t_n = \frac{2}{3}x_n + \frac{1}{3}z_n, \\ x_{n+1} = \frac{1}{3(n+1)} \cdot \frac{1}{2}x_n + (1 - \frac{1}{3(n+1)})S_1 t_n, \ \forall n \geq 1, \end{cases}$$

where C_n and ζ_n are chosen as above. Then, $x_n \to 0 \in \Delta$.

5. Conclusions

In a real Hilbert space, we have put forward two modified inertial composite subgradient extragradient implicit rules with line-search process for settling a generalized equilibrium problems system with constraints of a pseudomonotone variational inequality problem and a common fixed-point problem of finite nonexpansive mappings and an asymptotically nonexpansive mapping, respectively. Under the lack of the sequential weak continuity and Lipschitz constant of the cost operator A, we have demonstrated the strong convergence of the proposed algorithms to an element of the studied problem. In addition, an illustrated example was provided to demonstrate the feasibility of our proposed algorithms.

In the end, it is worthy to mention that part of our future research is aimed at acquiring the strong convergence results for the modifications of our proposed rules with a Nesterov inertial extrapolation step and adaptive stepsizes.

Use of AI tools declaration

The authors declare that they have not used artificial intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare that there are no conflicts of interest.

References

1. E. Blum, W. Oettli, From optimization and variational inequalities to equilibrium problems, *Math. Student*, **63** (1994), 123–145.
2. G. Cai, Y. Shehu, O. Iyiola, Strong convergence results for variational inequalities and fixed point problems using modified viscosity implicit rules, *Numer. Algor.*, **77** (2018), 535–558. http://dx.doi.org/10.1007/s11075-017-0327-8
3. L. Ceng, A. Petrusel, X. Qin, J. Yao, Two inertial subgradient extragradient algorithms for variational inequalities with fixed-point constraints, *Optimization*, **70** (2021), 1337–1358. http://dx.doi.org/10.1080/02331934.2020.1858832
4. L. Ceng, M. Shang, Hybrid inertial subgradient extragradient methods for variational inequalities and fixed point problems involving asymptotically nonexpansive mappings, *Optimization*, **70** (2021), 715–740. http://dx.doi.org/10.1080/02331934.2019.1647203
5. L. Ceng, C. Wang, J. Yao, Strong convergence theorems by a relaxed extragradient method for a general system of variational inequalities, *Math. Meth. Oper. Res.*, **67** (2008), 375–390. http://dx.doi.org/10.1007/s00186-007-0207-4
6. L. Ceng, J. Yao, A relaxed extragradient-like method for a generalized mixed equilibrium problem, a general system of generalized equilibria and a fixed point problem, *Nonlinear Anal.-Theor.*, **72** (2010), 1922–1937. http://dx.doi.org/10.1016/j.na.2009.09.033
7. J. Chen, S. Liu, X. Chang, Extragradient method and golden ratio method for equilibrium problems on Hadamard manifolds, *Int. J. Comput. Math.*, **98** (2021), 1699–1712. http://dx.doi.org/10.1080/00207160.2020.1846728
8. P. Combettes, S. Hirstoaga, Equilibrium programming in Hilbert spaces, *J. Nonlinear Convex Anal.*, **6** (2005), 117–136.

9. R. Cottle, J. Yao, Pseudomonotone complementarity problems in Hilbert space, *J. Optim. Theory Appl.*, **75** (1992), 281–295. http://dx.doi.org/10.1007/BF00941468
10. L. Deng, R. Hu, Y. Fang, Projection extragradient algorithms for solving nonmonotone and non-Lipschitzian equilibrium problems in Hilbert spaces, *Numer. Algor.*, **86** (2021), 191–221. http://dx.doi.org/10.1007/s11075-020-00885-x
11. S. Denisov, V. Semenov, L. Chabak, Convergence of the modified extragradient method for variational inequalities with non-Lipschitz operators, *Cybern. Syst. Anal.*, **51** (2015), 757–765. http://dx.doi.org/10.1007/s10559-015-9768-z
12. K. Goebel, S. Reich, *Uniform convexity, hyperbolic geometry, and nonexpansive mappings*, New York: Marcel Dekker, 1983.
13. L. He, Y. Cui, L. Ceng, T. Zhao, D. Wang, H. Hu, Strong convergence for monotone bilevel equilibria with constraints of variational inequalities and fixed points using subgradient extragradient implicit rule, *J. Inequal. Appl.*, **2021** (2021), 146. http://dx.doi.org/10.1186/s13660-021-02683-y
14. L. Jolaoso, Y. Shehu, J. Yao, Inertial extragradient type method for mixed variational inequalities without monotonicity, *Math. Comput. Simulat.*, **192** (2022), 353–369. http://dx.doi.org/10.1016/j.matcom.2021.09.010
15. G. Korpelevich, The extragradient method for finding saddle points and other problems, *Matecon*, **12** (1976), 747–756.
16. R. Kraikaew, S. Saejung, Strong convergence of the Halpern subgradient extragradient method for solving variational inequalities in Hilbert spaces, *J. Optim. Theory Appl.*, **163** (2014), 399–412. http://dx.doi.org/10.1007/s10957-013-0494-2
17. X. Li, Z. Liu, Sensitivity analysis of optimal control problems described by differential hemivariational inequalities, *SIAM J. Comtrol Optim.*, **56** (2018), 3569–3597. http://dx.doi.org/10.1137/17M1162275
18. T. Lim, H. Xu, Fixed point theorems for asymptotically nonexpansive mappings, *Nonlinear Anal.-Theor.*, **22** (1994), 1345–1355. http://dx.doi.org/10.1016/0362-546X(94)90116-3
19. P. Maingé, Strong convergence of projected subgradient methods for nonsmooth and nonstrictly convex minimization, *Set-Valued Anal.*, **16** (2008), 899–912. http://dx.doi.org/10.1007/s11228-008-0102-z
20. A. Moudafi, M. Théra, Proximal and dynamical approaches to equilibrium problems, In: *Ill-posed variational problems and regularization techniques*, Berlin: Springer, 1999. http://dx.doi.org/10.1007/978-3-642-45780-7_12
21. X. Qin, A. Petrusel, B. Tan, J. Yao, Efficient extragradient methods for bilevel pseudomonotone variational inequalities with non-Lipschitz operators and their applications, *Fixed Point Theor.*, **25** (2024), 309–332. http://dx.doi.org/10.24193/fpt-ro.2024.1.19
22. Y. Shehu, Q. Dong, D. Jiang, Single projection method for pseudo-monotone variational inequality in Hilbert spaces, *Optimization*, **68** (2019), 385–409. http://dx.doi.org/10.1080/02331934.2018.1522636

23. Y. Shehu, O. Iyiola, Strong convergence result for monotone variational inequalities, *Numer. Algor.*, **76** (2017), 259–282. http://dx.doi.org/10.1007/s11075-016-0253-1

24. Y. Song, O. Bazighifan, Two regularization methods for the variational inequality problem over the set of solutions of the generalized mixed equilibrium problem, *Mathematics*, **10** (2022), 2981. http://dx.doi.org/10.3390/math10162981

25. Y. Song, Y. Pei, A new viscosity semi-implicit midpoint rule for strict pseudo-contractions and (α,β)-generalized hybrid mappings, *Optimization*, **70** (2021), 2635–2653. http://dx.doi.org/10.1080/02331934.2020.1789640

26. G. Stampacchia, Formes bilineaires coercivities sur les ensembles convexes, *R. Acad. Scz. Paris*, **258** (1964), 4413–4416.

27. B. Tan, S. Li, Modified inertial projection and contraction algorithms with non-monotonic step sizes for solving variational inequalities and their applications, *Optimization*, **73** (2024), 793–832. http://dx.doi.org/10.1080/02331934.2022.2123705

28. D. Thong, D. Hieu, Inertial subgradient extragradient algorithms with line-search process for solving variational inequality problems and fixed point problems, *Numer. Algor.*, **80** (2019), 1283–1307. http://dx.doi.org/10.1007/s11075-018-0527-x

29. P. Vuong, Y. Shehu, Convergence of an extragradient-type method for variational inequality with applications to optimal control problems, *Numer. Algor.*, **81** (2019), 269–291. http://dx.doi.org/10.1007/s11075-018-0547-6

30. H. Xu, T. Kim, Convergence of hybrid steepest-descent methods for variational inequalities, *J. Optim. Theory Appl.*, **119** (2003), 185–201. http://dx.doi.org/10.1023/B:JOTA.0000005048.79379.b6

31. J. Yang, H. Liu, Z. Liu, Modified subgradient extragradient algorithms for solving monotone variational inequalities, *Optimization*, **67** (2018), 2247–2258. http://dx.doi.org/10.1080/02331934.2018.1523404

32. Y. Yao, O. Iyiola, Y. Shehu, Subgradient extragradient method with double inertial steps for variational inequalities, *J. Sci. Comput.*, **90** (2022), 71. http://dx.doi.org/10.1007/s10915-021-01751-1

33. Y. Yu, T. Yin, Weak convergence of a self-adaptive Tseng-type algorithm for solving variational inclusion problems, *U.P.B. Sci. Bull., Series A*, **85** (2023), 51–58.

34. Y. Yu, T. Yin, Strong convergence theorems for a nonmonotone equilibrium problem and a quasi-variational inclusion problem, *J. Nonlinear Convex Anal.*, **25** (2024), 503–512.

35. Z. Jing, Z. Liu, E. Vilches, C. Wen, J. Yao, Optimal control of an evolution hemivariational inequality involving history-dependent operators, *Commun. Nonlinear Sci.*, **103** (2021), 105992. http://dx.doi.org/10.1016/j.cnsns.2021.105992

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