AIMS Mathematics

http://www.aimspress.com/journal/Math

AIMS Mathematics, 9(6): 13861–13873.
DOI: 10.3934/math.2024674
Received: 17 January 2024
Revised: 27 March 2024
Accepted: 08 April 2024
Published: 15 April 2024

Research article

New results for fractional ordinary differential equations in fuzzy metric space

Li Chen*, **Suyun Wang, Yongjun Li and Jinying Wei**

School of Information Engineering, Lanzhou City University, Lanzhou, 730070, China

* **Correspondence:** Email: Lichen_79@163.com; Tel: +8613919958535.

Abstract: In this paper, we primarily focused on the existence and uniqueness of the initial value problem for fractional order fuzzy ordinary differential equations in a fuzzy metric space. First, definitions and relevant properties of the Gamma function and Beta function within a fuzzy metric space were provided. Second, by employing the principle of fuzzy compression mapping and Choquet integral of fuzzy numerical functions, we established the existence and uniqueness of solutions to initial value problems for fuzzy ordinary differential equations. Finally, several examples were presented to demonstrate the validity of our obtained results.

Keywords: fuzzy metric space; fractional fuzzy differential equation; power compression mapping principle; existence and uniqueness
Mathematics Subject Classification: 34A08, 34B15, 35J05

1. Introduction

The field of fractional calculus is a natural extension of traditional integral. The discussion of fractional calculus and the simplest fractional differential equations in the form of letters date back to as early as the 17th century, when integer order calculus was still under development [1]. With the advancement and practical implementation of scientific and mathematical knowledge, B. Roth organized the inaugural international conference on the theory and application of fractional calculus at New Haven University in 1974. This pivotal event not only propelled fractional calculus into a prominent research area worldwide, but also marked its formal establishment as a specialized discipline for scholarly exploration [2]. Fractional differential equations are very suitable for describing processes with heritability, such as semiconductor, physics, viscoelastic systems, electrolytic chemistry dispersion in multi-space and fractal media, etc., see [3–5]. At present, there have been many theoretical achievements on fractional differential equations, and many practical problems have been solved by these theoretical achievements. However, due to the lack of the fractional differential equation theory in fuzzy space, its application in the complex environment is limited.

There are a lot of results of fuzzy mathematics theory having been applied to the fields of artificial intelligence, knowledge engineering, game theory, economics, sociology, statistics, and so on [6–9]. Recently, because of the existence of uncertainties and disturbances in dynamic systems subject, fractional fuzzy differential equations have emerged as a significant topic, and the consideration and analysis of fractional fuzzy differential equations are essential in both research and practice [10–12]. In fuzzy metric space, based on H differentiability or gH differentiability, fractional calculus is considered in a number of papers: see [13–15]. specially, the fuzzy measure-based fuzzy numerical function integral is a widely used nonlinear function in dealing with nonlinear problems [16–18].

Nowadays, some scholars have tried to use the classical Schauder fixed point theorem and compression mapping method technique in functional analysis to study the existence and uniqueness of the solutions to the initial value problems of ordinary differential equations. In classical metric space, there have been relatively complete studies on such problems. However, there is still a lack of research on the existence and uniqueness of initial value problems of fractional-order ordinary differential equations based on fuzzy metric space. In particular, the initial value problem of the fractional ordinary differential equation based on the fuzzy interval number is less studied.

In this paper, we will use the fuzzy numerical function integral to study, the existence and uniqueness of the initial value problems of fuzzy differential equations in fuzzy metric space based on the principle of power compression in fuzzy metric space. The initial value problem of the following equation will be studied.

$$\begin{cases} {}^{c}D_t^{\lambda}\widetilde{u}(t) = \widetilde{f}(t, \widetilde{u}(t)), \quad 0 < \lambda < 1, t > 0, \\ \widetilde{u}(0) = \widetilde{u}_0. \end{cases} \tag{1.1}$$

In view of the knowledge that we now know about the existence and uniqueness of solutions to initial value problems of fuzzy ordinary differential equations, we combined with the classical Banach power compression mapping principle, which makes the existence interval of the solution is more larger than that of directly using the Banach compression mapping theorem (see Theorems 3.1 and 3.2 for the details). However, the Gamma function, Beta function and Stirling formula are used in this paper, which leads the calculations to being more complex.

The rest of this study is organized as follows. In Section 2, some symbols, notations, definitions, and preliminary facts used throughout this paper are reviewed. Section 3 gives the application of the power compression mapping principle in fractional ordinary differential equations in fuzzy space. To show the validity of the derived results, an appropriate example and applications are also discussed in Section 4. Finally, conclusions are made in Section 5.

2. Preliminaries

In this section, we present some basic concepts and lemmas needed in the paper.

Definition 2.1. *[16] Let X be a nonempty set, then* $\widetilde{A} = \{(u_{\widetilde{A}(x)}, x) | x \in X\}$ *is called fuzzy set on X. Here* $u_{\widetilde{A}}(x)$ *is the number specified on* $[0, 1]$ *and called the membership degree of a point x to a set* $\widetilde{A}$*, that is*

$$u_{\widetilde{A}}(x) : X \to [0, 1],$$

$$x \to u_{\widetilde{A}}(x).$$

We denote by $\mathcal{F}(X)$ the collection of all fuzzy subsets of X.

We identify a fuzzy set with its membership function. Other notations that can be used are the following: $u_{\widetilde{A}}(x) = \widetilde{A}(x)$.

Let us denote by $R_{\mathcal{F}}$ the space of fuzzy numbers.

For $r \in (0, 1]$, we denote

$$[u]_r = \{x \in R : u(x) \geqslant r\}$$

and

$$[u]_0 = \{x \in R : u(x) \geqslant 0\}.$$

Thus u_r is called the r-level set of the fuzzy number u. The 1-level set is called the core of the fuzzy number, while the 0-level set is called the support of the fuzzy number.

Lemma 2.1. *[19] If $u \in R_{\mathcal{F}}$ is a fuzzy number and u_r is its level-sets, then:*

(1) u_r is a closed interval $u_r = [\underline{u}_r, \overline{u}_r]$, for any $r \in [0, 1]$;
(2) functions $\underline{u}_r, \overline{u}_r : [0, 1] \to R$;
(3) $\underline{u}_r = \underline{u} \in R$ is a bounded, nondecreasing, left-continuous function in (0, 1] and right-continuous at 0;
(4) $\overline{u}_r = \overline{u} \in R$ is a bounded, nonincreasing, left-continuous function in (0, 1] and right-continuous at 0;
(5) $\underline{u} \leqslant \overline{u}$.

According to the Lemma 2.1, we can denote zero fuzzy number by $[\widetilde{0}]_r = \widetilde{0}_r = [0, 0]$, for any $r \in [0, 1]$.

Definition 2.2. *[16] Let $u, v \in R_{\mathcal{F}}$, $r \in [0, 1]$*

(1) if $\underline{u}_r \leqslant \underline{v}_r$ and $\overline{u}_r \leqslant \overline{v}_r$, then $u \leqslant v$;
(2) if $u_r + v_r = w_r$, then $u \oplus v = w$.

The Hukuhara difference (H-difference $\ominus_H$) is defined by $u \ominus_H v = w \Longleftrightarrow u = v + w$, being + the standard fuzzy addition. If $u \ominus_H v$ exists, its r-cuts are $[u \ominus_H v]_r = [\underline{u}_r - \underline{v}_r, \overline{u}_r - \overline{v}_r]$. It is easy to verify that $u \ominus_H u = 0$ for any fuzzy numbers u, but as we have earlier discussed $u - u \neq 0$.

Definition 2.3. *[20] Let $u, v \in R_{\mathcal{F}}$. The generalized Hukuhara differential is defined as follows*

$$u \ominus_{gH} v = w \Leftrightarrow \begin{cases} (1)\ \ u = v \oplus w, & or \\ (2)\ v = u \ominus w. \end{cases}$$

Proposition 2.1. *For any $u, v \in R_{\mathcal{F}}$ we have*

$$[u \ominus_{gH} v]_r = [min\{\underline{u}_r - \underline{v}_r, \overline{u}_r - \overline{v}_r\}, max\{\underline{u}_r - \underline{v}_r, \overline{u}_r - \overline{v}_r\}].$$

Definition 2.4. *[21] Let $D_\infty : R_{\mathcal{F}} \times R_{\mathcal{F}} \longrightarrow R_+ \bigcup\{0\}$,*

$$D(u, v) = \sup_{0 \leqslant r \leqslant 1} max\{|\underline{u}_r - \underline{v}_r|, |\overline{u}_r - \overline{v}_r|\} = \sup_{0 \leqslant r \leqslant 1} d_H([u]_r, [v]_r),$$

where d_H is the classical Hausdorff Pompeiu distance between real intervals, then D_∞ is called the Hausdorff distance between fuzzy numbers.

Lemma 2.2. *[21]* $(\mathrm{R}_{\mathcal{F}}, D_\infty)$ *is a complete metric space.*

Let us denote $(X, N, *)$ as a fuzzy metric space and $\|u\|_{\mathcal{F}} = D_\infty(u, 0)$ as the norm of a fuzzy number. $\|u\|_{\mathcal{F}}$ has the following properties:

(1) $\|u\|_{\mathcal{F}} = 0 \Longleftrightarrow u = 0$;
(2) $\|\lambda \cdot u\|_{\mathcal{F}} = |\lambda|\|u\|_{\mathcal{F}}, \forall \lambda \in \mathrm{R}, u \in \mathrm{R}_{\mathcal{F}}$;
(3) $\|u + v\|_{\mathcal{F}} \leqslant \|u\|_{\mathcal{F}} + \|v\|_{\mathcal{F}}, \forall u, v \in \mathrm{R}_{\mathcal{F}}$;
(4) $|\|u\|_{\mathcal{F}} - \|v\|_{\mathcal{F}}| \leqslant D_\infty(u, v), \forall u, v \in \mathrm{R}_{\mathcal{F}}$;
(5) $D(a \cdot u, b \cdot u) = |b - a| \cdot \|u\|_{\mathcal{F}}, \forall u \in \mathrm{R}_{\mathcal{F}}$;
(6) $D_\infty(u, v) = \|u \ominus_{gH} v\|_{\mathcal{F}}, \forall u, v \in \mathrm{R}_{\mathcal{F}}$.

Lemma 2.3. *Let* $(X, N, *)$ *be a complete fuzzy metric space and* $T : X \to X$ *a fuzzy compression mapping, then* T *has a unique fixed point.*

Lemma 2.4. *Let* $(X, N, *)$ *be a complete fuzzy metric space and* $T : X \to X$*. If* T^n *is a fuzzy compression mapping (* $T : X \to X$ *is called fuzzy power compression mapping), then* T *has a unique fixed point.*

Definition 2.5. *[22] Let* $f : (a, b) \to \mathrm{R}_{\mathcal{F}}, x_0 \in (a, b)$*, then the fuzzy gH-derivative of a function* f *at* x_0 *is defined*

$$\mathcal{D}_{gH} f(x_0) = \lim_{h \to 0} \frac{1}{h} \cdot [f(x_0 + h) \ominus_{gH} f(x_0)].$$

If $\mathcal{D}_{gH} f(x_0) \in \mathrm{R}_{\mathcal{F}}$ *exists, then* f *is called gH−differentiable at* x_0*.*

Definition 2.6. *[22] Let* $f \in C^1_{gH}((a, b), \mathrm{R}_{\mathcal{F}})$ *with* $[f(x)]_r = [\underline{f}_r, \overline{f}_r]$ *for all* $x \in (a, b)$*,* $r \in [0, 1]$*. We call*

(1) f *is (i)-gH differentiable at* $x_0 \in (a, b)$ *if* $[\mathcal{D}_{gH} f(x_0)]_r = [(\underline{f}_r)'(t_0), (\overline{f}_r)'(x_0)]$ *(denoted by* $\mathcal{D}^i_{gH} f(x_0)$*);*
(2) f *is (ii)-gH differentiable at* $x_0 \in (a, b)$ *if* $[\mathcal{D}_{gH} f(t_0)]_r = [(\overline{f}_r)'(x_0), (\underline{f}_r)'(x_0)]$ *(denoted by* $\mathcal{D}^{ii}_{gH} f(x_0)$*).*

Definition 2.7. *[6, 22] Let* $\widetilde{f} : [a, b] \to \mathrm{R}_{\mathcal{F}}$*,* $[\widetilde{f}(x)]_r = [\underline{f}_r, \overline{f}_r]$ *for all* $x \in [a, b], r \in [0, 1]$*, where* $\underline{f}_r$*,* $\overline{f}_r$ *are measurable and Lebesgue integrable on* $[a, b]$*, then the Choquet integral of* $\widetilde{f}$ *(is denoted by* $\int_{(a,b)} f(x)dx$*) can be defined as*

$$\Big[\int_{(a,b)} f(x)dx \Big]_\alpha = \Big(\int_a^b \underline{f}_\alpha(x)dx, \int_a^b \overline{f}_\alpha(t)dt \Big),$$

for all $r \in [0, 1]$.

Definition 2.8. *[23] Let* $0 < \lambda < 1$*. The fuzzy fractional differential equation (1.1) is equivalent to one of the following integral equations:*

$$\widetilde{u}(t) = \widetilde{u}_0 \oplus \frac{1}{\Gamma(\lambda)} \int_0^t \frac{\widetilde{f}(s, \widetilde{u}(s))}{(t - s)^{1-\lambda}} ds, \ t \in [a, b],$$

if $\widetilde{u}$ *is (i)-gH differentiable, and*

$$\widetilde{u}(t) = \widetilde{u}_0 \ominus \frac{-1}{\Gamma(\lambda)} \int_0^t \frac{\widetilde{f}(s, \widetilde{u}(s))}{(t - s)^{1-\lambda}} ds, \ t \in [a, b],$$

if u *is (ii)-gH differentiable, provided that the Hukuhara difference exists.*

Definition 2.9. *The solution of the initial value problem* (1.1) *is a fuzzy numerical function u that satisfies (1.1). We say that $\widetilde{u}$ is* (i)*-solution if the solution of the initial value problem* (1.1) *is Caputo* (i)*-gH differentiable; we say that $\widetilde{u}$ is* (ii)*-solution if the solution of the initial value problem* (1.1) *is Caputo* (ii)*-gH differentiable.*

3. Application of power compression mapping principle in fractional ordinary differential equations in fuzzy space

In this section, by employing the principle of fuzzy compression mapping and Choquet integral of fuzzy numerical functions, we will establish the existence and uniqueness of solutions to initial value problems for fuzzy ordinary differential equations.

Let's first provide the definitions and relevant properties of the gamma function and beta function within a fuzzy metric space.

Definition 3.1. *Gamma function* $\Gamma(\cdot)$ *is defined:*

$$\Gamma(s) = \Big[\int_{[0,\infty)} \widetilde{x}^{s-1} e^{-\widetilde{x}} d\widetilde{x}\Big]_\alpha = \Big[\int_{[0,\infty)} \underline{x}_\alpha^{s-1} e^{-\underline{x}} d\underline{x}, \int_{[0,\infty)} \overline{x}_\alpha^{s-1} e^{-\overline{x}} d\overline{x}\Big],$$

where $s > 0$.

Definition 3.2. *Beta function* $B(\cdot,\cdot)$ *is defined:*

$$B(p,q) = \Big[\int_{[0,1]} \widetilde{x}^{p-1}(1 \ominus \widetilde{x})^{q-1} d\widetilde{x}\Big]_\alpha = \Big[\int_{[0,1]} \underline{x}_\alpha^{p-1}(1-\underline{x}_\alpha)^{q-1} d\underline{x}, \int_{[0,1]} \overline{x}_\alpha^{p-1}(1-\overline{x}_\alpha)^{q-1} d\overline{x}\Big],$$

where $p, q > 0$.

Proposition 3.1. *The relationship of Gamma function* $\Gamma(\cdot)$ *and Beta function* $B(\cdot,\cdot)$ *is as follows:*

$$B(p,q) = \frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)}, \qquad \alpha\Gamma(\alpha) = \Gamma(\alpha+1).$$

Next, we prove the existence and uniqueness of the initial value problem of Eq (1.1) in the fuzzy metric space by employing the principle of fuzzy compression mapping and Choquet integral of fuzzy numerical functions.

By employing the principle of fuzzy compression mapping and Choquet integral of fuzzy numerical functions, if u is (i)-gH differentiable, that is,

$$\widetilde{u}(t) = \widetilde{u}_0 \oplus \frac{1}{\Gamma(\lambda)} \int_0^t \frac{\widetilde{f}(s,\widetilde{u}(s))}{(t-s)^{1-\lambda}} ds,\ t \in [a,b],$$

we can come to the following results: Theorems 3.1 and 3.2.

Theorem 3.1. *Let the nonlinear fuzzy number value function $\widetilde{f}(t,\widetilde{u}(t))$ be continuous over the rectangular region $S = \{(t, \|\widetilde{u}\|) | 0 \leqslant t \leqslant a, |\widetilde{u} \ominus \widetilde{u}_0| \leqslant b\}$, and satisfy the Lispchitz condition about u, Then the initial value problem of fuzzy fractional differential equation* (1.1) *has a unique* (i)*-solution on interval $I = [0,h]$, where*

$$h = \min\Big\{a, (\frac{b\Gamma(\lambda+1)}{M})^{\frac{1}{\lambda}}\Big\}, \quad M = \max_{(t,\|\widetilde{u}\|)\in S} |\widetilde{f}(t,\widetilde{u}(t)) \ominus \widetilde{0}|. \tag{3.1}$$

Proof. If u is (i)-gH differentiable, we can define the operator $A : C(I, \mathrm{R}_{\mathcal{F}}) \to C(I, \mathrm{R}_{\mathcal{F}})$ as follows:

$$A\widetilde{u}(t) = \widetilde{u}_0 \oplus \frac{1}{\Gamma(\lambda)} \int_0^t (t-s)^{\lambda-1} \widetilde{f}(s, \widetilde{u}(s)) ds. \tag{3.2}$$

It is shown by Definition 2.8 that the solution of the initial value problem of fractional differential equation (1.1) is equivalent to the fixed point of operator A defined by (3.2). The fuzzy power compression mapping fixed point theorem is used to find the fixed point of operator A. If we define

$$D = \{\widetilde{u} \in C(I, R_F) : |\widetilde{u}(t) \ominus \widetilde{u}_0| \leqslant b, t \in I\},$$

then D is a bounded fuzzy convex closed set in $C(I, \mathcal{T})$.

First of all, we can prove $A : D \to D$. In fact, for $\forall u \in D$ and $t \in I$, according to Proposition 3.1 and (3.2), it can be obtained by calculation that

$$\begin{aligned}|A\underline{u}(t) - \underline{u}_0| &= \frac{1}{\Gamma(\lambda)} \int_0^t (t-s)^{\lambda-1} f(s, \underline{u}(s)) ds \\ &\leqslant \frac{1}{\Gamma(\lambda)} \int_0^t (t-s)^{\lambda-1} |f(s, \underline{u}(s)) \ominus 0| ds \\ &\leqslant \frac{M}{\Gamma(\lambda)} \frac{t^\lambda}{\lambda} = \frac{Mt^\lambda}{\Gamma(\lambda+1)} \leqslant b.\end{aligned}$$

We can obtain in the same way $|A\overline{u}(t) - \overline{u}_0| \leqslant b$, so we get $|Au(t) \ominus u_0| \leqslant b$ by Definition 2.2, furthermore, $Au(D) \in D$, that is $A : D \to D$.

Additionally, we can prove that $A : D \to D$ is a power compression mapping. In fact, for $\forall u_1, u_1 \in D$, according to Proposition 3.1, (3.2), and Lispchitz continuity of nonlinear fuzzy number value function $f(t, u(t))$ about the second argument, it can be obtained by calculation that

$$\begin{aligned}|A\underline{u}_2(t) - A\underline{u}_1(t)| &\leqslant \frac{1}{\Gamma(\lambda)} \int_0^t |(t-s)^{\lambda-1}(f(s, \underline{u}_2(s)) - f(s, \underline{u}_1(s)))| ds \\ &\leqslant \frac{L}{\Gamma(\lambda)} \frac{t^\lambda}{\lambda} \|\underline{u}_2 - \underline{u}_1\| = \frac{Lt^\lambda}{\Gamma(\lambda+1)} \|\underline{u}_2 - \underline{u}_1\|.\end{aligned}$$

We can obtain in the same way:

$$|A\overline{u}_2(t) - A\overline{u}_1(t)| \leqslant \frac{Lt^\lambda}{\Gamma(\lambda+1)} \|\overline{u}_2 - \overline{u}_1\|.$$

By Definition 2.2, we get

$$|A\widetilde{u}_2(t) \ominus A\widetilde{u}_1(t)| \leqslant \frac{Lt^\lambda}{\Gamma(\lambda+1)} D_C(u_2, u_1). \tag{3.3}$$

We can obtain, according to Proposition 3.1, (3.2), (3.3), and Lispchitz continuity of nonlinear fuzzy number value function $f(t, u(t))$, with respect to the second argument,

$$
\begin{aligned}
&|A^2\underline{u}_2(t) - A^2\underline{u}_1(t)| \\
=&|\frac{L}{\Gamma(\lambda)}\int_0^t (t-s)^{\lambda-1}(f(s, A\underline{u}_2(s)) - f(s, A\underline{u}_1(s)))ds| \\
\leqslant&\frac{L}{\Gamma(\lambda)}\int_0^t (t-s)^{\lambda-1}|(A\underline{u}_2(s) - A\underline{u}_1(s)|ds \\
\leqslant&\frac{L}{\Gamma(\lambda)}\int_0^t (t-s)^{\lambda-1}\frac{Ls^{\lambda}}{\Gamma(\lambda+1)}\|\underline{u}_2 - \underline{u}_1\|ds \\
=&\frac{L^2}{\Gamma(\lambda)\Gamma(\lambda+1)}\int_0^t (t-s)^{\lambda-1}s^{\lambda}ds\|\underline{u}_2 - \underline{u}_1\| \\
=&\frac{L^2}{\Gamma(\lambda)\Gamma(\lambda+1)}\int_0^1 (t-st)^{\lambda-1}s^{\lambda}t^{\lambda}tds\|\underline{u}_2 - \underline{u}_1\| \\
=&\frac{L^2t^{2\lambda}}{\Gamma(\lambda)\Gamma(\lambda+1)}\int_0^1 (1-s)^{\lambda-1}s^{\lambda}ds\|\underline{u}_2 - \underline{u}_1\| \\
=&\frac{L^2t^{2\lambda}}{\Gamma(\lambda)\Gamma(\lambda+1)}B(\lambda+1, \lambda)\|\underline{u}_2 - \underline{u}_1\| \\
=&\frac{L^2t^{2\lambda}}{\Gamma(\lambda)\Gamma(\lambda+1)}\frac{\Gamma(\lambda)\Gamma(\lambda+1)}{\Gamma(2\lambda+1)}\|\underline{u}_2 - \underline{u}_1\| \\
=&\frac{L^2t^{2\lambda}}{\Gamma(2\lambda+1)}\|\underline{u}_2 - \underline{u}_1\|.
\end{aligned} \tag{3.4}
$$

Suppose that it holds for $n = k - 1$, and we get

$$
|A^{k-1}\underline{u}_2(t) - A^{k-1}\underline{u}_1(t)| \leqslant \frac{(Lt^{\lambda})^{k-1}}{\Gamma((k-1)\lambda+1)}\|\underline{u}_2 - \underline{u}_1\|. \tag{3.5}
$$

When $n = k$, according to Proposition 3.1, (3.2), (3.5), and Lispchitz continuity of nonlinear fuzzy number value function $f(t, u(t))$ about the second argument, we have

$$
\begin{aligned}
&|A^k\underline{u}_2(t) - A^k\underline{u}_1(t)| \\
=&|\frac{L}{\Gamma(\lambda)}\int_0^t (t-s)^{\lambda-1}(f(s, A^{k-1}\underline{u}_2(s)) - f(s, A^{k-1}\underline{u}_1(s)))ds| \\
\leqslant&\frac{L}{\Gamma(\lambda)}\int_0^t (t-s)^{\lambda-1}|(A^{k-1}\underline{u}_2(s) - A^{k-1}\underline{u}_1(s)|ds \\
\leqslant&\frac{L}{\Gamma(\lambda)}\int_0^t (t-s)^{\lambda-1}\frac{(Ls^{\lambda})^{k-1}}{\Gamma((k-1)\lambda+1)}\|\underline{u}_2 - \underline{u}_1\|ds \\
=&\frac{L^k}{\Gamma(\lambda)\Gamma((k-1)\lambda+1)}\int_0^t (t-s)^{\lambda-1}s^{(k-1)\lambda}ds\|\underline{u}_2 - \underline{u}_1\| \\
=&\frac{L^kt^{k\lambda}}{\Gamma(\lambda)\Gamma((k-1)\lambda+1)}B((k-1)\lambda+1, \lambda)\|\underline{u}_2 - \underline{u}_1\| \\
=&\frac{L^kt^{k\lambda}}{\Gamma(\lambda)\Gamma((k-1)\lambda+1)}\frac{\Gamma((k-1)\lambda+1)\Gamma(\lambda)}{\Gamma(k\lambda+1)}\|\underline{u}_2 - \underline{u}_1\| \\
=&\frac{L^kt^{k\lambda}}{\Gamma(k\lambda+1)}\|\underline{u}_2 - \underline{u}_1\|.
\end{aligned} \tag{3.6}
$$

By mathematical induction, for every positive integer n and $t \in I$, we have

$$|A^n \underline{u}_2(t) - A^n \underline{u}_1(t)| \leqslant \frac{(Lh^\lambda)^n}{\Gamma(n\lambda + 1)} \|\underline{u}_2 - \underline{u}_1\|.$$

We can obtain in the same way:

$$|A^n \overline{u}_2(t) - A^n \overline{u}_1(t)| \leqslant \frac{(Lh^\lambda)^n}{\Gamma(n\lambda + 1)} \|\underline{u}_2 - \underline{u}_1\|.$$

By Definition 2.2, we get

$$|A^n u_2(t) \ominus A^n u_1(t)| \leqslant \frac{(Lh^\lambda)^n}{\Gamma(n\lambda + 1)} D_C(u_2, u_1),$$

which means that

$$D_C(A^n u_2, A^n u_1) \leqslant \frac{(Lh^\lambda)^n}{\Gamma(n\lambda + 1)} D_C(u_2, u_1). \tag{3.7}$$

We have, by the Stirling formula,

$$\Gamma(n\lambda + 1) = \sqrt{2\pi n\lambda}(\frac{n\lambda}{e})^{n\lambda} e^{\frac{\theta}{12n\lambda}}, 0 < \theta < 1,$$

then

$$\frac{(Lh^\lambda)^n}{\Gamma(n\lambda + 1)} = \frac{(Lh^\lambda)^n}{\sqrt{2\pi n\lambda}(\frac{n\lambda}{e})^{n\lambda} e^{\frac{\theta}{12n\lambda}}} \to 0 \ \ (n \to \infty).$$

So, there exists a sufficiently large integer n_0 such that

$$\frac{(Lh^\lambda)^{n_0}}{\Gamma(n_0\lambda + 1)} < 1. \tag{3.8}$$

By combining (3.7) and (3.8), it can be obtained

$$D_C(A^{n_0} u_2, A^{n_0} u_1) < D_C(\widetilde{u}_2, \widetilde{u}_1).$$

That is, A^{n_0} is a fuzzy compression operator, so A is a fuzzy power compression operator. Therefore, according to Lemma 2.4, operator A has a unique fixed point $\widetilde{u} \in D$. This fixed point is the unique (i)-solution of the initial value problem of fuzzy fractional differential equation (1.1) in the interval $I = [0, h]$. □

Theorem 3.2. *If all the assumptions of Theorem 3.1 are satisfied, then the initial value problem of fuzzy fractional differential equation* (1.1) *has a unique* (i)*-solution on* $I' = [0, h']$, *where*

$$h' < \min\left\{a, (\frac{b\Gamma(\lambda + 1)}{M})^{\frac{1}{\lambda}}, (\frac{\Gamma(\lambda + 1)}{L})^{\frac{1}{\lambda}}\right\}, \quad M = \max_{(t,\|\widetilde{u}\|)\in S} |\widetilde{f}(t, \widetilde{u}(t)) \ominus \widetilde{0}|.$$

Proof. If $\widetilde{u}$ is (i)-gH differentiable, we can define the operator $A : C(I', R_{\mathcal{F}}) \to C(I', R_{\mathcal{F}})$ as follows:

$$A\widetilde{u}(t) = \widetilde{u}_0 \oplus \frac{1}{\Gamma(\lambda)} \int_0^t (t-s)^{\lambda-1} \widetilde{f}(s, \widetilde{u}(s)) ds. \tag{3.9}$$

It is shown by Definition 2.8 that the solution of the initial value problem of fuzzy fractional differential equation (1.1) is equivalent to the fixed point of operator A defined by (3.9). In the following, we find the fixed point of operator A by use of the compression mapping fixed point theorem. If we define $D' = \{u \in C(I', R_{\mathcal{F}}) : |\widetilde{u}(t) \ominus \widetilde{u}_0| \leqslant b, t \in I'\}$, then D' is a bounded fuzzy convex closed set in $C(I', R_{\mathcal{F}})$. To begin, we know from the proof of Theorem 3.1 that $A : D' \to D'$. Now we prove that $A : D' \to D'$ is a fuzzy compression operator. For every $u_1, u_2 \in D'$, we get

$$\begin{aligned}
&|A\underline{u}_2(t) - A\underline{u}_1(t)| \\
=&|\frac{L}{\Gamma(\lambda)} \int_0^t (t-s)^{\lambda-1}(f(s, \underline{u}_2(s)) - f(s, \underline{u}_1(s)))ds| \\
\leqslant&\frac{L}{\Gamma(\lambda)} \int_0^t (t-s)^{\lambda-1}|(f(s, \underline{u}_2(s)) - f(s, \underline{u}_1(s)))|ds \\
\leqslant&\frac{L}{\Gamma(\lambda)} \frac{t^{\lambda}}{\lambda}\|\underline{u}_2 - \underline{u}_1\| \\
<&\frac{L(h')\lambda}{\Gamma(\lambda+1)}\|\underline{u}_2 - \underline{u}_1\| < \|\underline{u}_2 - \underline{u}_1\|.
\end{aligned} \tag{3.10}$$

We can obtain in the same way:

$$|A\overline{u}_2(t) - A\overline{u}_1(t)| < \|\overline{u}_2 - \overline{u}_1\|.$$

Therefore, by Definition 2.2, we get

$$D_C(A\widetilde{u}_2, A\widetilde{u}_1) < D_C(\widetilde{u}_2, \widetilde{u}_1).$$

That is, $A : D' \to D'$ is a fuzzy compression operator. Therefore, according to Lemma 2.3, operator A has a unique fixed point $u \in D'$, and u is the unique (i)-solution of the initial value problem of fuzzy fractional differential equation (1.1) in the interval $I' = [0, h']$. □

If u is (ii)-gH differentiable, that is,

$$\widetilde{u}(t) = \widetilde{u}_0 \ominus \frac{-1}{\Gamma(\lambda)} \int_0^t \frac{\widetilde{f}(s, \widetilde{u}(s))}{(t-s)^{1-\lambda}} ds, \ t \in [a, b],$$

we can come to the following conclusion: Theorems 3.3 and 3.4.

Theorem 3.3. *Suppose $C([0, a], R_{\mathcal{F}}) \neq \emptyset$, and for any $\widetilde{u} \in C([0, a], R_{\mathcal{F}})$, $\widetilde{u}_0 \ominus \frac{-1}{\Gamma(\lambda)} \int_0^t \frac{\widetilde{f}(s,\widetilde{u}(s))}{(t-s)^{1-\lambda}} ds$ exists for all $t \in [0, a]$. If all the conditions in Theorem 3.1 are satisfied, then the initial value problem (1.1) has a unique (ii)-solution.*

Proof. If u is (ii)-gH differentiable, we can define the operator $A : C(I, R_{\mathcal{F}}) \to C(I, R_{\mathcal{F}})$ as follows:

$$A\widetilde{u}(t) = \widetilde{u}_0 \ominus \frac{-1}{\Gamma(\lambda)} \int_0^t (t-s)^{\lambda-1} \widetilde{f}(s, \widetilde{u}(s)) ds,$$

and the remaining proof is similar to Theorem 3.1. □

Theorem 3.4. *Suppose $C([0,a],R_{\mathcal{F}}) \neq \emptyset$, and for any $u \in C([0,a],R_{\mathcal{F}})$, $\widetilde{u}_0 \ominus \frac{-1}{\Gamma(\lambda)} \int_0^t \frac{\widetilde{f}(s,\widetilde{u}(s))}{(t-s)^{1-\lambda}} ds$ exists for all $t \in [0,a]$. If all the conditions in Theorem 3.2 are satisfied, then the initial value problem (1.1) has a unique (ii)-solution.*

Proof. If u is (ii)-gH differentiable, we can define the operator $A : C(I, R_{\mathcal{F}}) \to C(I, R_{\mathcal{F}})$ as follows:

$$A\widetilde{u}(t) = \widetilde{u}_0 \ominus \frac{-1}{\Gamma(\lambda)} \int_0^t (t-s)^{\lambda-1} \widetilde{f}(s, \widetilde{u}(s)) ds,$$

and the remaining proof is similar to Theorem 3.2. □

4. Examples

The existence and uniqueness of the solution to the initial value problem of the fractional-order differential equation is an important mathematical concept. Studying the fractional-order differential equation initial value problem can help us understand the actual situation better and solve the problem accurately. However for this kind of problem, the theory of fuzzy metric space is still lacking, and the lack of theory greatly limits the practical application of fractional-order ordinary differential equations. In the face of some complex environments, the usual practice is to combine the solution of the system to minimize the complexity of the initial value problem to reduce the complexity of the solution. However, this operation will lead to the loss of a lot of information, so the accuracy and effectiveness of the actual problem cannot be guaranteed. Therefore, we study the existence and uniqueness of initial value problems of fractional ordinary differential equations in fuzzy metric space. By the following example, to show the validity of the derived results, an appropriate example and applications are discussed in this section.

Example 4.1. *Consider the initial problem of fractional fuzzy differential equations as follows:*

$$\begin{cases} {}^cD_t^{\frac{1}{2}} \widetilde{u}(t) = \frac{1}{1 \oplus |u(t)|} \odot \cos t, t > 0, \\ \widetilde{u}(0) = \widetilde{u}_0 \in \mathbb{R}_{\mathscr{F}}. \end{cases} \tag{4.1}$$

The conditions corresponding to Theorem 3.1 yield the following data information: $\lambda = \frac{1}{2}, \widetilde{f}(t, \widetilde{u}(t)) = \frac{1}{1 \oplus |u(t)|} \odot \cos t, a = 1$, and $b = 1$. Set $u_0 = (1,2,3) \in \mathbb{R}_{\mathscr{F}}$, then $\alpha = 0$ level set $(u_0)_0 = [1,3]$ of symmetric triangular fuzzy number u_0. It can be obtained that the construction of fuzzy-valued function $f(t, \widetilde{u}(t))$ is continuous over the rectangular region $S = \{(t, \|\widetilde{u}\|) | 0 \leqslant t \leqslant 1, |\widetilde{u} \ominus \widetilde{u}_0| \leqslant 1\}$. Let's verify that the function $f(t, \widetilde{u}(t))$ satisfies the Lispchitz condition about u on the rectangular region $S = \{(t, \|\widetilde{u}\|) | 0 \leqslant t \leqslant 1, |\widetilde{u} \ominus \widetilde{u}_0| \leqslant 1\}$ as follows. By Definition 2.7, we can obtain

$$\begin{aligned} & \|f(t, \underline{u}_2(t)) - f(t, \underline{u}_1(t))\| \\ =& \| \frac{1}{1 + |\underline{u}_2(t)|} \cos t - \frac{1}{1 + |\underline{u}_1(t)|} \cos t \| \\ =& \| \frac{1}{1 + |\underline{u}_2(t)|} - \frac{1}{1 + |\underline{u}_1(t)|} \| \cdot |\cos t| \\ =& \| \frac{|\underline{u}_2(t)| - |\underline{u}_1(t)|}{(1 + |\underline{u}_1(t)|) \cdot (1 + |\underline{u}_2(t)|)} \| \cdot |\cos t| \\ \leqslant & \|\underline{u}_2 - \underline{u}_1\| \cdot |\cos t|. \end{aligned} \tag{4.2}$$

We can get in the same way

$$\|f(t,\overline{u}_2(t)) - f(t,\overline{u}_1(t))\| \leqslant \|\overline{u}_2 - \overline{u}_1\| \cdot |\cos t|. \tag{4.3}$$

Thus, the initial value problem of above fuzzy fractional differential equation (4.1) has a unique (i)-solution on the fuzzy interval $I = [0,h]$, where

$$h = \min\left\{1, (\frac{\Gamma(\frac{1}{2}+1)}{M})^2\right\} = 0.785, \quad M = \max_{(t,u)\in S} |\widetilde{f}(t,\widetilde{u}(t)) \ominus \widetilde{0}| = 2. \tag{4.4}$$

Example 4.2. *Consider the initial problem of fractional fuzzy differential equations as follows:*

$$\begin{cases} {}^cD_t^\lambda \widetilde{u}(t) = 2\sin\mu(A) \oplus e^{tA}, 0 < \lambda < 1, t > 0, \\ \widetilde{u}(0) = \widetilde{0} \in \mathbb{R}_{\mathscr{F}}, \end{cases} \tag{4.5}$$

where $\widetilde{f}(t,\widetilde{u}(t)) = 2\sin\mu(A) \oplus e^{tA}$ and $A = (1,2,3) \in \mathbb{R}_{\mathscr{F}}$ is a symmetric triangular fuzzy number. Clearly, it can be obtained that the construction of fuzzy-valued function $\widetilde{f}(t,\widetilde{u}(t))$ is continuous over the rectangular region $S = \{(t,\|\widetilde{u}\|)|0 \leqslant t \leqslant 1, |\widetilde{u} \ominus \widetilde{u}_0| \leqslant 1\}$, Let's verify that the function $f(t,\widetilde{u}(t))$ satisfies the Lispchitz condition about u with $L = 2$ on the rectangular region $S = \{(t,\|\widetilde{u}\|)|0 \leqslant t \leqslant 1, |\widetilde{u} \ominus \widetilde{u}_0| \leqslant 1\}$ as follows. By definition 2.7, we can obtain

$$\begin{aligned} &\|f(t,\underline{u}_2(t)) - f(t,\underline{u}_1(t))\| \\ =&\|(2\sin\underline{u}_2(A) + e^{tA}) - (2sin\underline{u}_1(A) + e^{tA})\| \\ \leqslant& 2\|\underline{u}_2 - \underline{u}_1\|. \end{aligned} \tag{4.6}$$

We can get in the same way,

$$\|f(t,\overline{u}_2(t)) - f(t,\overline{u}_1(t))\| \leqslant 2\|\overline{u}_2 - \overline{u}_1\|. \tag{4.7}$$

Thus Eq (4.5) has a unique solution according to the Theorem 3.1.

As can be seen from the above example, we do not need to idealize the actual information. We can directly draw the desired conclusion, reducing the error value of the research on the actual problem.

5. Conclusions

In this work, the initial value problem to a class of fractional fuzzy differential equations is studied. We obtain new existence and uniqueness of solutions for initial value problems of fractional ordinary differential equations in fuzzy metric space by means of the fuzzy power compression mapping principle and the related properties of the Gamma function under the assumption that the nonlinear functions satisfy the Lispchitz condition. It is proved that the existence interval of the solution is larger than that of directly using the Banach compression mapping theorem. However, due to the lack of research on this kind of problem in fuzzy metric space, the calculation involved is relatively difficult, and few practical application examples can be found. In the future, we will continue to promote this work, strive to combine theoretical research with practice, and make breakthroughs in numerical simulation.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

Acknowledgments

The authors would like to thank the referees and editors for providing very helpful comments and suggestions. This work is supported by Young Doctor Fund project of Gansu Province(No. 2022QB-173) and Scientific Research Projects of Colleges and Universities in Gansu Province (No. 2023B-149).

Conflict of interest

The authors declare no conflict of interest in this paper.

References

1. K. S. Miller, B. Ross, *An introduction to the fractional calculus and fractional differential equations*, New York: Wiley, 1993.
2. A. A. Kilbas, H. M. Srivastava, J. J. Trujillo, *Theory and applications of fractional differential equations*, Amsterdam, Boston: Elsevier, 2006.
3. N. A. Albasheir, A. Alsinai, A. U. K. Niazi, R. Shafqat, Romana, M. Alhagyan, et al., A theoretical investigation of Caputo variable order fractional differential equations: Existence, uniqueness, and stability analysis, *Comput. Appl. Math.*, **42** (2023), 367. https://doi.org/10.1007/S40314-023-02520-6
4. V. Lakshmikantham, A. S. Vatsala, Basic theory of fractional differential equations, *Nonlinear Anal.*, **69** (2008), 2677–2682. https://doi.org/10.1016/j.na.2007.08.042
5. M. Al-Refai, D. Baleanu, Comparison principles of fractional differential equations with non-local derivative and their applications, *AIMS Mathematics*, **6** (2021), 1443–1451. https://doi.org/10.3934/math.2021088
6. L. Chen, G. Duan, S. Y. Wang, J. F. Ma, A Choquet integral based fuzzy logic approach to solve uncertain multi-criteria decision making problem, *Expert Syst. Appl.*, **149** (2020), 113303. https://doi.org/10.1016/j.eswa.2020.113303
7. M. H. Alshayeji, S. C. Sindhu, S. Abed, Viral genome prediction from raw human DNA sequence samples by combining natural language processing and machine learning techniques, *Expert Syst. Appl.*, **218** (2023), 119641. https://doi.org/10.1016/j.eswa.2023.119641
8. K. Tamilselvan, V. Visalakshi, P. Balaji, Applications of picture fuzzy filters: Performance evaluation of an employee using clustering algorithm, *AIMS Mathematics*, **8** (2023), 21069–21088. https://doi.org/10.3934/math.20231073
9. S. Salahshour, A. Ahmadian, B. A. Pansera, M. Ferrara, Uncertain inverse problem for fractional dynamical systems using perturbed collage theorem, *Commun. Nonlinear Sci. Numer. Simul.*, **94** (2021), 105553. https://doi.org/10.1016/j.cnsns.2020.105553

10. M. Z. Ahmad, M. K. Hasan, B. D. Baets, Analytical and numerical solutions of fuzzy differential equations, *Inf. Sci.*, **236** (2013), 156–167. https://doi.org/10.1016/j.ins.2013.02.026
11. M. Mazandarani, N. Pariz, A. V. Kamyad, Granular differentiability of fuzzy-number-valued functions, *IEEE Trans. Fuzzy Syst.*, **26** (2018), 310–323. https://doi.org/10.1109/TFUZZ.2017.2659731
12. Y. B. Shao, Q. Mou, Z. T. Gong, On retarded fuzzy functional differential equations and nonabsolute fuzzy integrals, *Fuzzy Sets Syst.*, **375** (2019), 121–140. https://doi.org/10.1016/j.fss.2019.02.005
13. T. Allahviranloo, A. Armand, Z. Gouyandeh, Fuzzy fractional differential equations under generalized fuzzy Caputo derivative, *J. Intell. Fuzzy Syst.*, **26** (2014), 1481–1490.
14. N. V. Hoa, H. Vu, T. M. Duc, Fuzzy fractional differential equations under Caputo-Katugampola fractional derivative approach, *Fuzzy Sets Syst.*, **375** (2019), 70–99. https://doi.org/10.1016/j.fss.2018.08.001
15. M. Mazandarani, A. V. Kamyad, Modified fractional Euler method for solving fuzzy fractional intial value problem, *Commun. Nonlinear Sci. Numer. Simul.*, **18** (2013), 12–21. https://doi.org/10.1016/j.cnsns.2012.06.008
16. C. X. Wu, Z. T. Gong, On Henstock integrals of interval-valued and fuzzy-number-valued functions, *Fuzzy Sets Syst.*, **115** (2000), 377–391. https://doi.org/10.1016/S0165-0114(98)00277-2
17. D. Denneberg, *Non-additive measure and integral*, Dordrecht: Springer, 1994. https://doi.org/10.1007/978-94-017-2434-0
18. Z. T. Gong, L. Chen, G. Duan, Choquet integral of fuzzy-number-valued functions: The differentiability of the primitive with respect to fuzzy measures and choquet integral equations, *Abstr. Appl. Anal.*, **2014** (2014), 953893. https://doi.org/10.1155/2014/953893
19. C. V. Negoiţă , D. A. Ralescu, *Application of fuzzy sets to system analysis*, Basel: Springer, 1975. https://doi.org/10.1007/978-3-0348-5921-9
20. L. Stefanini, A generalization of Hukuhara difference and division for interval and fuzzy arithmetic, *Fuzzy Sets Syst.*, **161** (2010), 1564–1584. https://doi.org/10.1016/j.fss.2009.06.009
21. P. Diamond, P. Kloeden, Metric topology of fuzzy numbers and fuzzy analysis, In: *Fundamentals of fuzzy sets*, Boston: Springer, 2000. https://doi.org/10.1007/978-1-4615-4429-6_12
22. B. Bede, L. Stefanini, Generalized differentiability of fuzzy-valued functions, *Fuzzy Sets Syst.*, **230** (2013), 119–141. https://doi.org/10.1016/j.fss.2012.10.003
23. S. Salahshour, T. Allahviranloo, S. Abbasbandy, D. Baleanu, Existence and uniqueness results for fractional differential equations with uncertainty, *Adv. Differ. Equ.*, **2012** (2012), 112. https://doi.org/10.1186/1687-1847-2012-112

AIMS Press