Fermat's principle and the formal equivalence of local light-ray rotation and refraction at the interface between homogeneous media with a complex refractive index ratio

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We derive a formal description of local light-ray rotation in terms of complex refractive indices. We show that Fermat's principle holds, and we derive an extended Snell's law. The change in the angle of a light ray with respect to the normal of a refractive index interface is described by the modulus of the refractive index ratio; the rotation around the interface normal is described by the argument of the refractive index ratio. © 2009 Optical Society of America

OCIS codes: 080.2720, 160.1245, 240.3990.

We recently started to investigate ray-optical analogs of metamaterials. Like metamaterials [1,2], these so-called metamaterials for rays (METATOYs) [3] are capable of performing positive and negative refraction [4,5]. In addition, METATOYs can perform local light-ray rotation around the interface normal [6].

Here we describe local light-ray rotation around the interface normal in terms of Fermat's principle. Fermat's principle can be treated as a basic theorem of geometrical optics, but "is itself only understandable in terms of a wave theory" [7]. On the other hand, local light-ray rotation around the interface normal has no wave-optical analog in the sense that it is not always possible to construct a wave in which the phase-front normal—the geometrical-optics light-ray direction—has been rotated as required [3]. What, then, happens if we attempt to describe local light-ray rotation around the interface normal in terms of Fermat's principle? Here we do just that. We find a "natural" formulation of Fermat's principle in which ray rotation is described by an interface between homogeneous media with a complex refractive index ratio. This leads to an extended form of Snell's law that uses complex refractive indices. It is important to note that the meaning of the imaginary part of the complex refractive index we introduce here is different from that commonly used in optics [8]; in the former it is associated with ray rotation, in the latter with attenuation.

Fermat's principle [9] states that a light ray traveling between two points takes a path with a stationary optical path length; that is, for small variations in the path taken the optical path length stays the same. The whole path can be calculated using the calculus of variations, but simplified calculations can be performed using ordinary calculus.

Figure 1(a) shows a planar interface between two media with different refractive indices, n_1 and n_2. When light travels between two fixed points A and B on either side of the interface, via a point P on the interface, but not otherwise fixed, it covers an optical path length

$$\Delta = n_1 d_1 + n_2 d_2 \tag{1}$$

$$= n_1 \sqrt{x_1^2 + y_1^2 + z_1^2} + n_2 \sqrt{x_2^2 + y_2^2 + z_2^2}, \tag{2}$$

where (x_1, y_1, z_1) and (x_2, y_2, z_2) are the components of the vector AP (that is, the vector from A to P) and PB, respectively, in a Cartesian coordinate system whose (x,y) plane coincides with the plane of the interface. We can use the equations

$$X = x_1 + x_2, \quad Y = y_1 + y_2, \tag{3}$$

which describe the constant separations between the fixed points A and B in the x and the y directions to
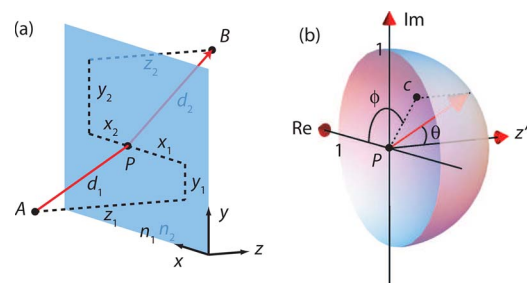

Fig. 1. (Color online) Geometry of refraction at a planar interface between two media with different refractive indices, n_1 and n_2. (a) Light ray travels from a point A in front of the interface to a point P on the interface and then to a point B behind the interface. The geometrical distance between A and P is d_1 and that between P and B is d_2. (b) Light-ray direction can be represented by two angles, θ and ϕ, which, respectively, represent the angle with respect to the interface normal (the z axis) and the angle of the projection into the interface plane with respect to the x axis. Alternatively, the light-ray direction can be described by the projection of the normalized direction vector into the interface plane. With a complex plane in the interface plane as shown, this projection can then be described by a single complex number, c.

eliminate x_2 and y_2 from the expression for the path difference [Eq. (2)]. This gives

$$\Delta = n_1\sqrt{x_1^2 + y_1^2 + z_1^2} + n_2\sqrt{(X - x_1)^2 + (Y - y_1)^2 + z_2^2}.$$

(4)

We now find the values x_1 and y_1 for which the function Δ is stationary. According to Fermat's principle, these values then correspond to the point P through which the light ray would actually travel. These values x_1 and y_1 have to satisfy the equations

$$\frac{\partial \Delta}{\partial x_1} = 0, \quad \frac{\partial \Delta}{\partial y_1} = 0.$$

(5)

Substitution of the expression for Δ into these equations gives

$$n_1\frac{x_1}{r_1} - n_2\frac{x_2}{r_2} = 0, \quad n_1\frac{y_1}{r_1} - n_2\frac{y_2}{r_2} = 0,$$

(6)

where $r_j = (x_j^2 + y_j^2 + z_j^2)^{1/2}$ (with $j = 1,2$). The terms can be translated into spherical coordinates ϕ (the azimuthal angle) and θ [the angle with the z axis, see Fig. 1(b)] using the equations

$$\frac{x_j}{r_j} = \sin\theta_j \cos\phi_j, \quad \frac{y_j}{r_j} = \sin\theta_j \sin\phi_j.$$

(7)

In spherical coordinates, Eqs. (6) are therefore the real and the imaginary parts, respectively, of the equation

$$n_1 \sin\theta_1 \exp(i\phi_1) = n_2 \sin\theta_2 \exp(i\phi_2).$$

(8)

This equation is the basis of the remainder of this Letter. Like in Snell's law, it is not the individual refractive indices that matter but their ratio. As Eq. (8) is a complex equation, it is natural to allow the refractive indices (and their ratio) to be complex numbers. Equation (8) is then an extension of Snell's law; for real refractive index ratios, it describes ordinary refraction according to Snell's law, expressing (unlike the Snell's law formula) the fact that the refracted ray lies in the same plane as the incident ray and the interface normal, and it leads to local light-ray rotation as a natural extension of refraction with complex refractive index ratios. We discuss these properties below.

For the following discussion it is useful to visualize the extended Snell's law as follows. We interpret the plane of the refractive index interface as a complex plane, centered at the point P where the light ray intersects the plane, and with the real axis in the x direction and the imaginary axis in the y direction; z' is the normal to the interface at P. Figure 1(b) shows this coordinate system.

We now consider a unit vector in the direction of the incident light ray, starting at the origin. We define the complex number c as the orthographic projection of this unit vector into the complex plane. If we assume that light travels in the positive z' direc-

tion, then this projection uniquely defines the ray direction. For a unit vector with spherical-coordinate angles θ and ϕ, c is

$$c = \sin\theta \exp(i\phi).$$

(9)

The extended Snell's law, Eq. (8), can then be written in the form

$$n_1 c_1 = n_2 c_2,$$

(10)

where c_1 and c_2 are the complex numbers corresponding to the incident and the refracted ray directions, respectively. In other words, the product of refractive index and the complex number representing the ray direction remains constant. We can emphasize the dependence on the refractive index ratio by writing Eq. (10) in the form

$$\frac{n_1}{n_2} = \frac{c_2}{c_1}.$$

(11)

According to Eq. (11), the ratio c_2/c_1 is the same as the refractive index ratio n_1/n_2. Therefore, if the refractive index ratio is real then so is the ratio of the direction projections.

This has a simple interpretation. The argument of c_1—the spherical-coordinates angle ϕ_1—defines the plane of incidence. Specifically, it describes the angle between the plane of incidence and the x axis. This, together with the fact that the plane of incidence also contains the z axis, completely determines the plane of incidence. A real ratio c_2/c_1 means that c_2 lies on the same line through the origin as c_1 [Figs. 2(a) and 2(b)], and therefore the refracted ray is also in the plane of incidence.

Mathematically, it means that $\phi_1 = \phi_2$. This means that the $\exp(i\phi_j)$ terms ($j = 1,2$) can be cancelled in Eq. (8), which reduces to Snell's law

$$n_1 \sin\theta_1 = n_2 \sin\theta_2.$$

(12)

Perhaps the most simple examples of complex refractive index ratios are those of the form

$$\frac{n_1}{n_2} = \exp(i\alpha)$$

(13)

(where α is a real number). According to Eq. (11), this implies that
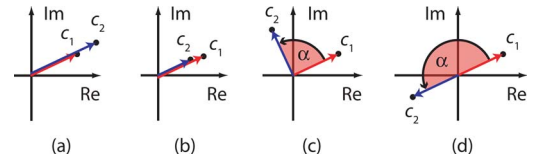

(a)   (b)   (c)   (d)

Fig. 2. (Color online) Plots of the complex numbers c_1 and c_2 representing various types of refraction. (a) and (b) Examples of standard refraction (n_1/n_2 real and positive). In (a) $n_1/n_2 > 1$, and in (b) $n_1/n_2 < 1$. (c) Complex refractive index ratio $n_1/n_2 = \exp(i\alpha)$ leads to local light-ray rotation through an angle α (here $\alpha = 90°$). (d) Light-ray rotation through 180° is equivalent to negative refraction with a refractive index ratio $n_1/n_2 = -1$.

$$\frac{c_2}{c_1} = \exp(i\alpha). \tag{14}$$

This means that c_2, which characterizes the projection of the refracted ray into the interface plane, is rotated with respect to c_1 through an angle α around the point P [Fig. 2(c)]. In three dimensions, it means that the direction of the refracted ray is that of the incident ray, rotated around the z' axis through an angle α. This is precisely the local light-ray rotation that can be achieved with METATOYs [3,6].

The case of general complex refractive index ratios n_1/n_2 can be approached by writing the left-hand side of Eq. (11) in terms of the modulus and argument of this ratio, namely,

$$\frac{n_1}{n_2} = \left| \frac{n_1}{n_2} \right| \exp\left[i \arg\left(\frac{n_1}{n_2} \right) \right] \tag{15}$$

and the right-hand side in the form

$$\frac{\sin\theta_2 \exp(i\phi_2)}{\sin\theta_1 \exp(i\phi_1)} = \frac{\sin\theta_2 \exp[i(\phi_1 + \alpha)]}{\sin\theta_1 \exp(i\phi_1)} = \frac{\sin\theta_2}{\sin\theta_1} \exp(i\alpha), \tag{16}$$

which expresses the direction of the refracted light ray's projection as that of the incident light ray, rotated through an angle α around P. A comparison of the moduli of Eqs. (15) and (16) reveals that the change of the angle between the ray and the z' axis is then given by the absolute value of the refractive-index ratio according to the equation

$$\left| \frac{n_1}{n_2} \right| = \frac{\sin\theta_2}{\sin\theta_1}, \tag{17}$$

which, for real and positive refractive index ratios, is the same as Snell's law [Eq. (12)]. A comparison of the arguments reveals that the rotation angle α is given by the argument of the refractive index ratio

$$\arg\left(\frac{n_1}{n_2} \right) = \alpha. \tag{18}$$

Now we discuss briefly the case of negative refraction [10] for which the refractive index ratio n_1/n_2 is real and negative. Negative refraction is fully described by Snell's law (and indeed the extended Snell's law) from which it then follows that the angle of the ray with the z' axis θ has to change sign. It can alternatively be described by a change of the angle θ without a sign change in combination with a ray rotation around the z' axis through 180°. Mathematically, the equivalence between negative refraction and ray rotation through 180° can be expressed as

$$c = \sin(-\theta)\exp(i\phi) = \sin\theta \exp[i(\phi + 180°)]. \tag{19}$$

The case $n_1/n_2 = -1$ is shown in Fig. 2(d). In fact, a Dove-prism-array structure that is ray-optically equivalent to a refractive index interface with $n_1/n_2 = -1$ [4] is a special case of a Dove-prism-array ray-rotator [6] for rotation angle $\alpha = 180°$.

Ray rotation is a concept that has no wave-optical analog. It is curious that it is possible to describe it—in such a natural manner—by using Fermat's principle, albeit with a complex refractive index ratio. More work may lead to a deeper understanding.

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