Teachers' Reflections on Nonstandard Students' Work*

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One of the tasks of teaching (Ball, Thames, & Phelps, 2008) concerns the work of interpreting student error and evaluating alternative algorithms used by students. Teachers' abilities to understand nonstandard student work affects their instructional decisions, the explanations they provide in the classroom, the way they guide their students, and how they conduct mathematical discussions. However, their knowledge or their perceptions of the knowledge may not correspond to the actual level of knowledge that will support flexibility and fluency in a mathematics classroom. In this paper, we focus on Norwegian and Portuguese teachers' reflections when trying to give sense to students' use of nonstandard subtraction algorithms and of the mathematics imbedded in such. By discussing teachers' mathematical knowledge associated with these situations and revealed in their reflections, we can perceive the difficulties teachers have in making sense of students' solutions that differ from those most commonly reached.

Introduction

Inspired by Shulman's (1987) ideas about subject matter knowledge and pedagogical content knowledge, Ball and colleagues at the University of Michigan have developed a conceptualization of math teachers' knowledge called Mathematical Knowledge for Teaching (MKT). They describe MKT as the "mathematical knowledge, skills, habits of mind, and insight" used to carry out the work of teaching mathematics (Ball, et al., 2008). In part of their work, they focus on the development of an instrument[i] intended to measure teachers' MKT (Ball et al., 2008; Hill, Sleep, Lewis, & Ball, 2007).

Scholars around the world have shown interest in the MKT conceptualization and in the MKT measurement instrument, and MKT measures have been adapted and used in several countries (e.g. Indonesia: Ng, 2012; Ireland: Delaney, Ball, Hill, Schilling, & Zopf, 2008; and Norway: Fauskanger, Jakobsen, Mosvold, & Bjuland, 2012). In Norway, a complete form[ii] with thirty items from the Learning Mathematics for Teaching (LMT) project (e.g. Learning Mathematics for Teaching, 2011) at the University of Michigan were translated and adapted for use among Norwegian teachers (Mosvold, Fauskanger, Jakobsen, & Melhus, 2009). The primary purpose of such a study was to see if the MKT measure could give valuable information about Norwegian teachers' knowledge, which might then contribute to the design/conceptualization of suitable professional development (PD)

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[i] This is composed by a set of items (questions) in which the teachers have to select the correct answer for each situation proposed from a set of given answers. For more information on these items see Hill, Schilling, and Ball (2004), and Learning Mathematics for Teaching, (2011).

[ii] Elementary form A, MSP_A04.

courses for teachers. Another approach was chosen in Portugal, where conceptualization features of the MKT model and the fundamental ideas in which the items were grounded have been used for discussing and reflecting on classroom practices and/or prospective teachers' knowledge. This was done by accessing and discussing teachers' knowledge, its critical features, and what promotes or limits its development—ultimately aimed to design PD programs that enable the development of teachers' MKT (e.g., Caseiro & Ribeiro, 2012; Ribeiro & Carrillo, 2011).

Besides measuring teachers' knowledge, we perceive the items (both from the LMT project or others grounded in the same ideas) as a privileged starting point for reflection and discussion on the MKT involved in approaching different topics and situations. As part of adapting MKT measures for use in Norway, seven focus group interviews were held with fifteen teachers who had participated in the pilot study using the MKT measures (Wilson, 1998). In these interviews, teachers discussed and reflected on the items they had answered in terms of format, content and relevance. One portion of the work done in Portugal concerned early year future teachers' MKT, focusing on how they acquire it, how it evolves, and what factors (in terms of learning opportunities) influence such evolution. Fifty-three trainee teachers answered a questionnaire within the framework of the LMT items, but with open questions. Both of these approaches provide valuable information about teaching, MKT, and insights on the knowledge involved in improving teaching. Because such knowledge can be taught (Hill & Ball, 2004), this can help mathematics educators to improve their training programs (both in terms of conceptualization and implementation), and ultimately to become more effective and more conscientious of their MKT.

In this paper we focus on discussions and reflections stemming from situations in which trainees and qualified teachers were asked to make sense of nonstandard students' work (after they have solved the situation themselves), and in particular in connection with students answering problems by using nonstandard algorithms. We do not give a full report about these two separate projects, but rather report from observations made independently, which together have developed a momentum and consensus of their own. By discussing and reflecting on what we are learning about teachers' MKT, we hope to contribute to the discussion about how to improve and develop training focused on teachers' MKT, both with trainees in teacher education and with qualified teachers in professional development programs. With this goal, we address the following question: What can be learned from teachers' reflections on situations in which they struggle to understand nonstandard students' work in order to conceptualize ways of improving teachers' MKT?

Theoretical Framework

Teachers play a key role in students' learning at all educational levels (Rowan, Correnti, & Miller, 2002). Several studies have documented that teachers have a greater impact than any other factor (e.g. class size, school size, and school system) when it comes to student achievement (Nye, Konstantopoulos, & Hedges, 2004). Despite these findings, we still know little about what constitutes effective teaching, which specific tasks of teaching teachers find hard or easy, and what can be done to improve teaching and learning. What we do know is that researchers have found that teachers' mathematical knowledge and experience, broadly construed, are not consistently associated with greater student learning (Begle, 1972, 1979; Ball, Lubienski, & Mewborn, 2001; National Mathematics Advisory Panel, 2008). We see, however, the mathematical knowledge associated with achievement gains for students is specifically related to the work of teaching and the mathematical tasks that constitute that work (for a review see Hill, Rowan and Ball, 2005). It is this evidence that led Ball and Bass (2003) to develop a theory of mathematical knowledge for teaching, where the "for teaching," in this context, means a practice-based characterization. Ball, Thames, and Phelps (2008, p. 399) define MKT to be mathematical knowledge "entailed by teaching"—in other words, mathematical knowledge implicated by the demands of teaching and central to performing the recurrent tasks of teaching mathematics to students. From our view, the insistence that claims about what teachers ought to know be warranted in practice brings a direct challenge to intuitive notions about what would be "good for" teachers.

Mathematical Knowledge for Teaching

Grounded in Shulman's (1987) categories of subject matter knowledge (SMK) and pedagogical content knowledge (PCK), Ball et al. (2008) introduced a model to allow a better understanding of the MKT conceptualization, still under development, with some refinements and new subgroups regarding teachers' knowledge (Figure 1). Such conceptualization emerged from analyzing classroom teaching from a mathematical perspective. They have suggested dividing SMK and PCK into three distinct sub-domains, and have indicated that MKT is a multidimensional construct.

In this paper we only discuss the sub-domains of SMK, as our approach focuses on teachers' reflections about their MKT when giving sense to nonstandard students' answers. In this sense, we focus on the mathematical knowledge involved in sustainable practice related to knowing how to perform an operation (through an algorithm), and the knowledge that would allow teachers to go beyond knowing how to perform, and thus give sense of other answers and/or explain their possible reasoning.

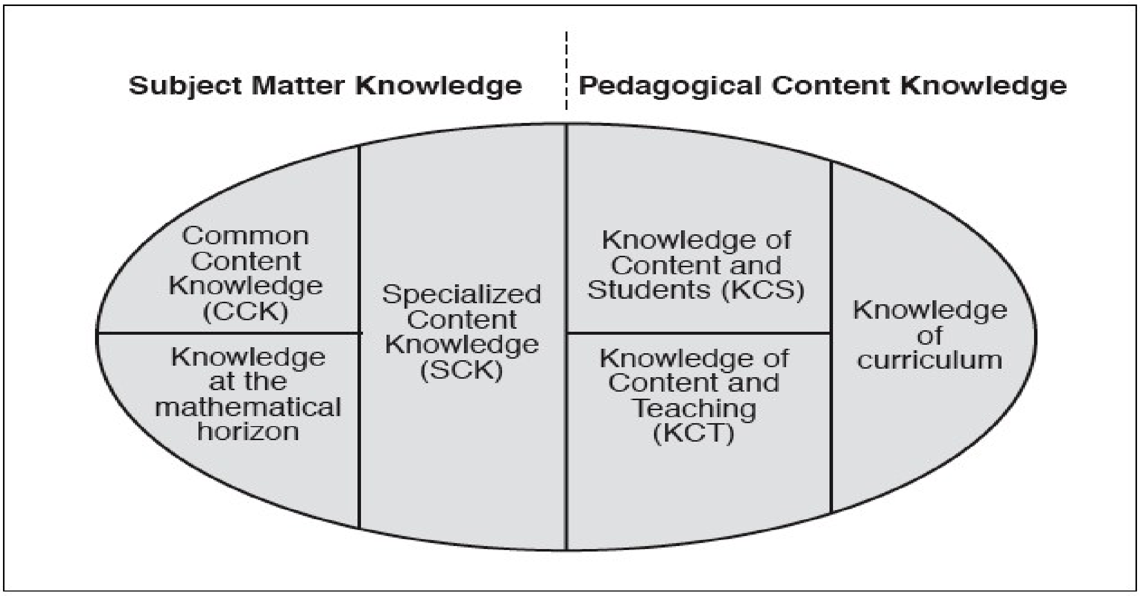

Figure 1. Domains of Mathematical Knowledge for Teaching (Ball et al., 2008, p 403)

Within SMK is included common content knowledge (CCK), which corresponds to the sort of general mathematical knowledge that is used by people in their life and work: "schoolchild" math; specialized content knowledge (SCK), which concerns the knowledge that allows the teacher to know how to make the subject understandable to others, and which is a crucial part of the knowledge needed by teachers to do the mathematical work of teaching; and a provisional category, horizon content knowledge (HCK) that describes how “mathematical topics are related over the span of mathematics included in the curriculum” (Ball et al., 2008, p. 403). In other words, content knowledge needs to be complemented by an understanding of how to make the given content accessible to students, and this includes knowing where and why students might encounter difficulties (Ribeiro & Carrillo, 2011).

SCK is a different type of mathematical knowledge than CCK. While CCK is about being able to solve mathematical problems correctly, SCK is the mathematical knowledge unique for teaching, which complements CCK. It concerns the knowledge that allows the teacher to engage in particular teaching tasks, including how to accurately represent mathematical ideas, provide mathematical explanations for common rules and procedure, and examine and understand unusual solution methods to problems (Ball et al., 2005).

This can perhaps best be illustrated by a subtraction computation problem: 51-17. Most of us will use some kind of algorithm to produce the answer. But which algorithm is common can vary from country to country, and sometimes even within the same country.

Figure 2. Two distinct possible algorithms to perform a subtraction computation

In Figure 2, a) is the algorithm most commonly used in Norway, while b) shows the typical algorithm used in Portugal. Teachers must be able to perform this calculation, whether by using an algorithm or not, and to know when the result is (in)correct. This is generally common knowledge among other professions outside teaching and mathematical fields (in this case, the algorithm involves a very basic mathematical knowledge, typically at the level of primary school pupils). However, being able to interpret students’ use of nonstandard algorithms involves a different type of mathematical knowledge that is not part of CCK. The answer is correct, as is the thinking used in solving the problem in both ways, and the procedures can also be generalized, but they involve two different approaches to working with numbers. At first sight, the reasoning might not be obvious for a reader that is not familiar with the algorithm. A teacher should be able to figure out such nonstandard work and determine whether the thinking is mathematically correct for the problem, and whether the approach used would work in general (Ball et al., 2008). This is an example that teachers need knowledge (here related to subtraction) that is not part of common content knowledge and not necessarily needed in other professions. Understanding nonstandard student work is one example of a task of teaching and is considered as part of teachers SCK. Besides this specificity, teachers need to know when results are not correct, and to identify the errors, as mentioned previously, which are seen as CCK. Besides being able to give sense to nonstandard students’ solutions, teachers must be in possession of knowledge that allows them to understand the mathematical motives that lead to any errors and to not only identify them, but be able to correct them and promote in students a true understanding of the mathematics embedded in the errors and the correct answer (SCK).

This specificity of teachers’ mathematical knowledge when compared with that of any other professional who uses mathematics from the perspective of knowing how to do, has a direct relationship with the kind of tasks teachers are expected to develop in teaching.

These tasks can be called “tasks of teaching” (Ball et al., 2008) and the work “involves an uncanny kind of unpacking of mathematics that is not needed—or even desirable—in settings other than teaching” (Ball etl al, 2008, p. 400). These mathematical tasks of teaching comprise, amongst others: presenting mathematical ideas; responding to students’ “why” questions; linking representations to underlying ideas and to other representations; choosing and developing useable definitions; asking productive mathematical questions.

One factor that led us to the selection of the conceptualization of MKT over other conceptualizations of teachers’ knowledge is a response to traditions trying to address documented weaknesses in teachers’ content knowledge by increasing requirements for more advanced mathematical study in teachers’ education and professional development. Our selection is also a response to the many studies showing that teachers’ advanced coursework in mathematics has no positive effect on their students’ learning (e.g., Eisenberg, 1977). With our approach, besides identifying teachers’ mathematical and critical skills, we aim to get a deeper understanding of their motives in order to allow discussion and to conceptualize ways of improving teachers’ training (both initial and continuous).

Contexts and Methods

Here we will report on some data gathered from the focus group interviews with Norwegian teachers. For the sake of validating and testing out the U.S. developed MKT measures (Learning Mathematics for Teaching, 2011), one hundred and forty-two teachers from seventeen university practice schools participated in a pilot study (Fauskanger, Jakobsen, Mosvold, & Bjuland, 2012). From this sample, seven schools were selected and seven semi-structured focus group interviews were conducted with fifteen teachers after the participating teachers had individually answered all the items. Of these, six teachers worked in primary and middle schools (six to twelve years old in Norway), and nine teachers worked at lower secondary schools (thirteen to fifteen years old). As part of the focus group interviews, the teachers reflected on and discussed items. The interviews were audio recorded and transcribed, using Törner, Rolka, Rösken and Sriraman’s (2010) perspective. For the purpose of this article, we analyzed the transcriptions according to how the teachers commented on each individual item on issues related to the research question. Because the LMT items have not yet been released, we can’t divulge the actual question presented to teachers, but it involved possible students productions aligned with the two examples presented previously in Figure 2.

Concerning the Portuguese contexts, some questionnaires were applied to prospective early year teachers. These questionnaires were developed from the perspective of the LMT items, but in this case the aim was not to evaluate future teachers’ knowledge. The goal was to provide open questions, allowing prospective teachers to express their knowledge but also doubts and fears, and to access also their most critical features in terms of MKT. These questionnaires were applied to eighty prospective teachers in the context of a mathematics course, where one of the topics concerned operations and algorithms. The study we present here takes an instrumental case-study approach (Stake, 2005), using data from the two contexts (though the comments from the Portuguese prospective students are mentioned mainly to illustrate the fact that the comments/reflections from Norwegian teachers were not due to the context they were immersed in, neither in terms of social or cultural context, nor in terms of the nature of the tasks/items they were supposed to comment upon). We do not focus on the cases themselves, but rather on the information we can obtain from them, which may allow us to deepen the level of understanding of the phenomena under analysis and elaborate on the theorization thereof. As previously mentioned, we will present transcription from focus group interviews in Norway, and use the data from the prospective teachers' questionnaires to contrast and/or reinforce the ideas given by the Norwegian teachers.

Reflections on Nonstandard Students' Answers

Of the thirty LMT items considered and discussed in the Norwegian FGIs, one item addressed the task of teaching of understanding and giving sense to nonstandard student work. This item (Item 10) presented a simple subtraction problem solved correctly by a student in an algorithmic way different from that taught by teachers and commonly used by students in Norway. The nonstandard solution method is similar to the work done in Figure 2b. The item gave the teachers several possible explanations for what the student had done. This item was mentioned in two of the FGIs, in School 2 and School 6. Below is the transcription from the FGIs in School 2 and 6, starting with the discussion between the interviewer (I) and teachers T2A and T2B at School 2:

104. T2A: I had difficulties with [answering] that one. What did you answer?

105. T2B: I actually skipped that [item].

106. T2A: Yes, that was a rather tricky way of doing it. Is it d. that is the correct [answer] or is it…?

107. I: I don't have the solution with me; was this [item] tricky? Was it because of the calculations, the way it is done, or is there something you don't recognize?

108. T2B: Yes, you have to interpret… It is kind of like we have to understand what they actually have done, isn't it?

109. T2A: It is extremely important to understand what they have done, but I've never experienced something like this.

110. I: No, you have never seen this error pattern before, or perhaps it is not a pattern, but….

111. T2A: No, exactly like this I have never seen, but I have seen many different ways of doing things, and [I have] found out why they have done like that. But exactly like this, I have never seen before.

112. T2B: But when you are in the classroom…and see the error pattern, then you can ask the kids about what they have done and why…

113. T2A: [murmuring] homework, so I am very strict with…

114. T2B: Yes, and then you get an explanation.

115. T2A: …and…careful about asking what have you done and can you do this in this way, or is it correct?

Figure 3. Extract from transcription of a FGI, School 2

91. T6B: There you have a wonderful example of [an item] where I would have asked [the students]: What have you done, could you show me what you have done?

92. I: Yes.

93. T6B: Instead of me using twenty minutes to try and figure out what on earth they have done

94. I: You would have asked…?

95. T6B: Yes, I missed that as an option.

96. I: Yes.

97. T6B: But I have to reach an answer by guess work.

98. I: This could have been what you got back from written homework…if that was the case…then it would have been difficult to ask, you would have to make a decision about the work… perhaps it would not be possible for you to ask students directly.

99. T6B: Yes, yes….

100. I: That would require that you try to actually understand the student's thinking, when they suddenly use another approach than what we have learned…

101. T6B: Most likely the answer they gave is correct, and then….it is not easy to know what they have been thinking.

Figure 4. Extract from transcription of a FGI School 6

In both of these discussions, the teachers revealed that they didn't have knowledge other than CCK on this topic (being able to solve the computation and see that the students found the correct answer). Teacher T2B admitted that he skipped that item (105), and T2A also had difficulties (104). Teacher T2B understood that the item is actually about understanding the student's solution (108), but argued that if this had occurred in a classroom situation, he could ask the student about what he had done and why (112) and felt confident that he would get an explanation (114) without needing to understand the student's reasoning and the mathematical thinking behind the solution.

T6B offered similar reflections and argued that, instead of spending time trying to understand the work, he would have asked the student (91, 93). Since he doesn't know the answer and did not show any evidence of possessing the knowledge to make sense of the solution, he concludes that he would have to guess (97). In School 2, however, teacher T2A argues that it is extremely important to understand what students do (109). He also reveals that he has experience of seeing problems solved in many different ways and being able to figure out and understand what students have done (111). Teacher T6B argued that the answer the student reached was probably correct, but because it is not easy to know what the student was thinking (101), he is not sure. He does not even realize that he could have checked the correctness of the answer using an algorithm he was more familiar with.

The written explanation and justification (written in the questionnaire) given by the prospective Portuguese teachers when making sense of Figure 2 a) was of the same nature as the ones given by Norwegian teachers when making sense of 2 b), which was nonstandard to them. The Portuguese teachers' explanations and justifications can be summarized with the following two answers:

(A): I would not know what the students may have been thinking. I can't get into his/her head.

(B): The result is correct, the student got the same result as me, so he must have done the same process [traditional algorithm] but then made some kind of confusion when registering the steps.

Besides the evidence of CCK only concerning the possible ways to obtain the result of the operation being present, the prospective teachers assume that there is only one possible way of getting the correct answer: the one they learned when they were students in primary school.

Final Comments and Reflections

These (prospective) teachers show evidence of gaps in MKT, specifically related to SCK, expressed in the inability to give sense to nonstandard solutions—in this particular case, a nonstandard algorithm (and even to ones which are not that far from the "standard" ones). In practice, they might "fill" such a gap by asking students to "explain what they had done." Such knowledge (or the lack thereof) may lead teachers to run blindly through a set of given procedures, and to not acknowledge the correctness of alternative algorithms used by students, which may then lead students to assume the teacher's knowledge (or the lack thereof) as their own.

Practice, and subsequent students' hypothetical opportunities for learning (Hiebert & Grouws, 2007) is grounded, necessarily, in the knowledge that teachers have (or assume they have) about each topic they are going to teach. Teachers' knowledge is fundamental in defining each teacher's core tasks of teaching, how they develop them, and the role of the different elements involved in the teaching/learning process. On the one hand, teachers' knowledge (or its lack) influences the nature of the tasks teachers prepare for students and the way they are carried out in the classroom (Charalambous, 2008), thus avoiding contingency moments (Rowland, Huckstep, & Thwaites, 2005). A lack of knowledge and flexibility will also limit the richness of the tasks given to students, and perhaps make teachers and students alike fear the "why" questions, as they may not feel confident about answering them in a mathematically correct and understandable way, nor about making sense of the answer and acknowledging what they do and why they do it.

By focusing on this specific task of teaching in two very different contexts, and on the mathematically critical features emerging from it, we get a better understanding of teachers' MKT that leads them to struggle in the understanding of nonstandard student solutions, which is part of SCK. This illustrates that teaching mathematics is not only about the level of knowing for ourselves (being able to do math, as a user), but also involves other aspects of teacher's knowledge (for example, SCK). In addition, we can draw attention to and get deeper insights on some of the whys and hows that may be at the basis of teachers' difficulties in understanding/giving sense to different ways of doing mathematics, and from those insights we are able to focus our efforts where they are most needed when preparing courses or professional development programs aimed at improving teachers' MKT and teachers' training in general.

In line with Kazemi and Franke (2004), we also assume that discussing and reflecting on students' answers (real or hypothetical) might contribute to promoting teachers' awareness of their own critical skills and lead to an improvement in teachers' MKT. First, we should work for a better and deeper understanding of the dimensions of teachers' MKT in all tasks. If we can find out how the different dimensions of MKT relate to and influence each other's, and how they influence teachers' reasoning and practice, it will give educators and future educators documentation about what matters in teaching. Then, hopefully, we can change the focus of teachers' training and teaching practice, ultimately improving students' learning opportunities and results. Much can be achieved if teachers become aware of the different dimensions of MKT in their own practice. By this we do not mean that the teaching of the MKT domains should be a focus of teaching in the courses, but we assume that the tasks prepared and implemented in teachers' training should focus on developing those domains, taking into consideration its specificities, as well as the specificities of the tasks of teaching in which such knowledge might be more easily seen. An awareness about these important aspects of teachers' knowledge and the effect it has on teaching practice can contribute to a widespread desire and plan to enrich and improve each teacher's own MKT. This can help in reducing procedural teaching, now far too common in many primary schools (Brocardo & Serrazina, 2008), and instead promote a teaching grounded in an effective understanding of the basic concepts and the mathematical justifications of various procedures and tasks.

This seems to be an area in which there is a need for further research, because one of the core tasks of teaching is to make sense of students' reasoning and solutions, even the nonstandard ones. This is thus an area in which further training can contribute to deepening teachers' understanding of the mathematical knowledge in order to teach it well (Ball, Lubienski, & Mewborn, 2001), and the research could contribute examples of mathematical critical situations to be further discussed in training.

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