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Approximation of a generalized Lipschitz class function by Euler - Cesàro means of Fourier series

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Abstract

In this paper, I have taken product of two summability methods, Euler and Cesàro; and establish a new theorem on the degree of approximation of the function f belonging to $W(L^p, \xi(t))$ classes by Euler - Cesàro method.

Key words and phrases: Degree of approximation; (E,1) (C,1) Summability; Fourier series.

1. Definitions and Notations

A function $f(x) \in Lip\alpha$, if

$|f(x+t) - f(x)| = O(|t|^{\alpha})$ for $0 < \alpha \le 1$ and $f \in Lip(\alpha, p)$, if

$$\left(\int_0^{2\pi} |f(x+t) - f(x)|^p dx \right)^{\frac{1}{p}} = O(|t|^{\alpha}), \; 0 < \alpha \le 1, \; p \ge 1.$$

Given a positive increasing function $\xi(t)$, $p \ge 1$,

$f(x) \in Lip(\xi(t), p)$, if

$$\left(\int_0^{2\pi} |f(x+t) - f(x)|^p dx \right)^{\frac{1}{p}} = O(\xi(t)) \text{ and } f \in W(L^p, \xi(t)), \text{ if}$$

$$\left(\int_0^{2\pi} \left| (f(x+t) - f(x)) \sin^{\beta} x \right|^p dx \right)^{\frac{1}{p}} = O(\xi(t)), \; (\beta \ge 0). \quad [1]$$

It is noted that, $W(L^p, \xi(t)) \xrightarrow{\beta=0} Lip(\xi(t), p) \xrightarrow{\xi(t)=t^{\alpha}} Lip(\alpha, p) \xrightarrow{p\to\infty} Lip\alpha$

So, $Lip\,\alpha \subseteq Lip(\alpha, p) \subseteq Lip(\xi(t), p) \subseteq W(L^p, \xi(t))$ for $0 < \alpha \le 1$ and $p \ge 1$.

We define the norm $\|\ \|_p$ by $\|f\|_p = \left\{\int_0^{2\pi} |f(x)|^p dx\right\}^{\frac{1}{p}}$, $p \geq 1$.

The degree of approximation $E_n(f)$ of function f: R→R is given by

$$E_n(f) = \mathrm{Min}\|t_n - f\|_p.$$

Where t_n is trigonometric polynomial of degree n [2].

Let f be 2π periodic, integrable over $(-\pi,\pi)$ in the sense of Lebesgue and belonging to $W(L^p, \xi(t))$ class, then its "Fourier series" is given by

$$f(t) = \frac{1}{2}a_o + \sum_{n=1}^{\infty} (a_n \cos nt + b_n \sin nt) \quad . \tag{1}$$

Let $\sum_{n=0}^{\infty} u_n$ be the infinite series whose nth partial sum is given by $S_n = \sum_{i=0}^{n} u_i$.

The Cesåro means (C, 1) of sequence $\{S_n\}$ is $\sigma_n = \frac{1}{n+1}\sum_{k=0}^{n} S_k$.

If $\sigma_n \to S$, as $n \to \infty$ then sequence $\{S_n\}$ or the infinite series $\sum_{n=0}^{\infty} u_n$ is said to be summable by Cesåro means method (C,1) to S. It is denoted by

$$\sigma_n \to S(C,1), \text{ as } n \to \infty \text{ [3].}$$

The Euler means (E, 1) of sequence $\{S_n\}$ is $E_n^1 = \frac{1}{2^n}\sum_{k=0}^{n}\binom{n}{k}S_k$

If $E_n^1 \to S$ as, $n \to \infty$ then sequence $\{S_n\}$ or infinite series $\sum_{n=0}^{\infty} u_n$ is said to be summable by Euler means method (E, 1) to S. It is denoted by

$$E_n^1 \to S(E,1), \text{ as } n \to \infty \text{ [4].}$$

The E_n^1 transformation of $\{\sigma_n\}$ is denoted by $t_n^{E_1,C_1}$, which is (E, 1) (C, 1) transformation of $\{S_n\}$ and defined as $t_n^{E_1,C_1} = \frac{1}{2^n}\sum_{k=0}^{n}\binom{n}{k}\sigma_k = \frac{1}{2^n}\sum_{k=0}^{n}\binom{n}{k}\frac{1}{k+1}\sum_{r=o}^{k} S_r$.

If $t_n^{E_1,C_1} \to S$, as $n \to \infty$ then sequence $\{S_n\}$ or infinite series $\sum_{n=0}^{\infty} u_n$ is said to be summable by (E, 1) (C, 1) means method to S. It is denoted by

$$t_n^{E_1,C_1} \to S(E,1)\,(C,1), \quad \text{as } n \to \infty.$$

We use following notations.

$$\phi(t) = f(x+t) + f(x-t) - 2f(x) \tag{2}$$

$$N_n^{E_1,C_1} = \frac{1}{2^{n+1}\pi}\sum_{k=0}^{n}\binom{n}{k}\frac{\sin^2 (k+1)\frac{t}{2}}{(k+1)\sin^2\frac{t}{2}} \tag{3}$$

2. Main Theorem

In present paper, the degree of approximation of a function $f \in W\left(L^p, \xi(t)\right)$ class by (E,1) (C,1) means of a Fourier series has been determined in the following form:

Theorem: If f: R $\rightarrow$ R is 2π periodic, Lebesgue integrable function in $(-\pi, \pi)$ and is $W\left(L^p, \xi(t)\right)$, then the degree of approximation of function f by (E,1)(C,1) means of Fourier series (1) satisfies,

$$\left\| t_n^{E_1, C_1} - f \right\|_p = O\left((n+1)^{\beta + \frac{1}{p}} \xi\left(\tfrac{1}{n+1}\right)\right), \text{ for } n = 1,2,3,4,..........$$

Provided $\xi(t)$ satisfy the following conditions;

$$\left\{\frac{\xi(t)}{t}\right\} \text{ is monotonic decreasing} \tag{4}$$

$$\left\{ \int_0^{\frac{1}{n+1}} \left(\frac{t|\phi(t)|}{\xi(t)} \right)^p \sin^{\beta p} t \, dt \right\}^{\frac{1}{p}} = O\left(\frac{1}{n+1} \right), \tag{5}$$

$$\left\{ \int_{\frac{1}{n+1}}^{\pi} \left(\frac{t^{-\delta}|\phi(t)|}{\xi(t)} \right)^p dt \right\}^{\frac{1}{p}} = O\left((n+1)^{\delta} \right) \tag{6}$$

where δ is an arbitrary number such that $q(1-\delta)-1 > 0$, condition (5) and (6) hold uniformly in x.

3. Lemmas

We need the following Lemmas for the proof of our theorem.

Lemma 1: Let $$N_n^{E_1, C_1}(t) = \frac{1}{2^{n+1}\pi} \sum_{k=0}^{n} \binom{n}{k} \frac{\sin^2 (k+1)\frac{t}{2}}{(k+1)\sin^2 \frac{t}{2}}, \text{ then}$$

$$N_n^{E_1, C_1}(t) = O(n+1), \text{ for } 0 < t < \frac{1}{n+1}.$$

Proof: $$N_n^{E_1, C_1}(t) = \frac{1}{2^{n+1}\pi} \sum_{k=0}^{n} \binom{n}{k} \frac{\sin^2 (k+1)\frac{t}{2}}{(k+1)\sin^2 \frac{t}{2}}$$

$$\leq \frac{1}{2^{n+1}\pi} \sum_{k=0}^{n} \binom{n}{k} \frac{(k+1)^2 \sin^2 \frac{t}{2}}{(k+1)\sin^2 \frac{t}{2}}$$

$$= \frac{1}{2^{n+1}\pi} \sum_{k=0}^{n} \binom{n}{k} (k+1)$$

$$= \frac{1}{2^{n+1}\pi}\left[\sum_{k=0}^{n} k\binom{n}{k} + \sum_{k=0}^{n}\binom{n}{k}\right]$$

$$= \frac{1}{2^{n+1}\pi}\left[n\,2^{n-1} + 2^{n}\right]$$

$$= \left(\frac{n+2}{4\pi}\right)$$

$$\le \frac{(n+1)}{2\pi}$$

$$= O(n+1) \tag{7}$$

Lemma 2: Let $N_n^{E_1, C_1}$ be given as Lemma I, then

$$N_n^{E_1, C_1}(t) = O\left(\frac{1}{(n+1)t^2}\right), \quad \text{for} \quad \frac{1}{n+1} < t < \pi$$

Proof:

$$\left|N_n^{E_1, C_1}(t)\right| \le \frac{1}{2^{n+1}\pi}\sum_{k=0}^{n}\binom{n}{k}\frac{\left|\sin^2 (k+1)\frac{t}{2}\right|}{(k+1)\left|\sin^2 \frac{t}{2}\right|}$$

$$= \frac{1}{2^{n+1}\pi}\sum_{k=0}^{n}\binom{n}{k}\frac{\left|1-\cos (k+1)t\right|}{2(k+1)\left|\sin^2 \frac{t}{2}\right|}$$

$$\le \frac{1}{2^{n+1}\pi}\sum_{k=0}^{n}\binom{n}{k}\frac{1}{(k+1)\left|\sin^2 \frac{t}{2}\right|}$$

$$= \frac{\pi}{2^{n+1}t^2}\sum_{k=0}^{n}\binom{n}{k}\frac{1}{(k+1)}$$

$$= \frac{\pi}{2^{n+1}t^2}\left(\frac{2^{n+1}-1}{n+1}\right)$$

$$= \frac{\pi}{(n+1)t^2}\left(1-\frac{1}{2^{n+1}}\right)$$

$$\le \frac{\pi}{(n+1)t^2}$$

$$= O\left(\frac{1}{(n+1)t^2}\right). \tag{8}$$

4. Proof of the Theorem

Following Titchmarsh [5], n^{th} partial sum of Fourier series (1) at t = x is given by

$$S_n(x)-f(x)=\frac{1}{2\pi}\int_0^{\pi}\phi(t)\frac{\sin\left(n+\frac{t}{2}\right)t}{\sin\frac{t}{2}}\,dt.$$

(C,1) transform of S_n i.e. σ_n is given by

$$\frac{1}{n+1}\sum_{k=0}^{n}\left(S_k(x)-f(x)\right)=\frac{1}{2(n+1)\pi}\int_0^{\pi}\frac{\phi(t)}{\sin\frac{t}{2}}\sum_{k=0}^{n}\sin\left(k+\frac{t}{2}\right)t\,dt$$

$$\sigma_n(x)-f(x)=\frac{1}{2(n+1)\pi}\int_0^{\pi}\phi(t)\frac{\sin^2(n+1)\frac{t}{2}}{\sin^2\frac{t}{2}}\,dt.$$

Similarly, (E,1) transform of σn i.e. $t_n^{E_1,C_1}$ is

$$\frac{1}{2^n}\sum_{k=0}^{n}\binom{n}{k}\left(\sigma_k(x)-f(x)\right)=\frac{1}{2^{n+1}\pi}\int_0^{\pi}\phi(t)\sum_{k=0}^{n}\binom{n}{k}\frac{\sin^2(k+1)\frac{t}{2}}{(k+1)\sin^2\frac{t}{2}}\,dt$$

$$t_n^{E_1,C_1}(x)-f(x)=\int_0^{\pi}\phi(t)\frac{1}{2^{n+1}\pi}\sum_{k=0}^{n}\binom{n}{k}\frac{\sin^2(k+1)\frac{t}{2}}{(k+1)\sin^2\frac{t}{2}}\,dt$$

$$=\int_0^{\pi}\phi(t)\,N_n^{E_1,C_1}(t)\,dt$$

$$=\int_0^{\frac{1}{n+1}}\phi(t)N_n^{E_1,C_1}(t)\,dt+\int_{\frac{1}{n+1}}^{\pi}\phi(t)N_n^{E_1,C_1}(t)\,dt$$

$$=I_1+I_2\text{, say.}\tag{9}$$

For I_1, applying Holder inequality and fact that $\phi(t)\in W\left(L^p,\xi(t)\right)$, we have

$$|I_1|\le\left\{\int_0^{\frac{1}{n+1}}\left(\frac{t|\phi(t)|}{\xi(t)}\sin^\beta t\right)^p dt\right\}^{\frac{1}{p}}\left\{\int_0^{\frac{1}{n+1}}\left(\frac{\xi(t)}{t\sin^\beta t}N_n^{E_1,C_1}(t)\right)^q dt\right\}^{\frac{1}{q}}$$

$$=O\left(\tfrac{1}{n+1}\right)\left\{\int_0^{\frac{1}{n+1}}\left(\frac{\xi(t)(n+1)}{t^{\beta+1}}\right)^q dt\right\}^{\frac{1}{q}}$$

$$=O\left(\xi\left(\tfrac{1}{n+1}\right)\right)\left\{\int_{\varepsilon}^{\frac{1}{n+1}}t^{-q(\beta+1)}\,dt\right\}^{\frac{1}{q}}\text{, by the Mean Value Theorem, where }\quad 0<\varepsilon<\frac{1}{n+1}.$$

$$=O\left(\xi\left(\tfrac{1}{n+1}\right)\right)\left\{\left(\frac{t^{-q(\beta+1)+1}}{-q(\beta+1)+1}\right)_{\varepsilon}^{\frac{1}{n+1}}\right\}^{\frac{1}{q}}$$

$$=O\left(\xi\left(\tfrac{1}{n+1}\right)\right)\left\{(n+1)^{(\beta+1)q-1}\right\}^{\frac{1}{q}}$$

$$= O\left((n+1)^{\beta+1-\frac{1}{q}}\,\xi(\tfrac{1}{n+1})\right)$$

$$= O\left((n+1)^{\beta+\frac{1}{p}}\,\xi(\tfrac{1}{n+1})\right). \tag{10}$$

For I_2, applying Holder's inequality and taking δ as an arbitrary number such that $q(1-\delta)-1>0$, we have

$$|I_2| \le \left\{\int_{\frac{1}{n+1}}^{\pi}\left(\frac{t^{-\delta}|\phi(t)|}{\xi(t)}\sin^{\beta} t\right)^p dt\right\}^{\frac{1}{p}} \left\{\int_{\frac{1}{n+1}}^{\pi}\left(\frac{\xi(t)\,N_n^{E_1,C_1}(t)}{t^{-\delta}\sin^{\beta} t}\right)^q dt\right\}^{\frac{1}{q}}$$

$$= \left\{\int_{\frac{1}{n+1}}^{\pi}\left(\frac{t^{-\delta}|\phi(t)|}{\xi(t)}\right)^p dt\right\}^{\frac{1}{p}} \left\{\int_{\frac{1}{n+1}}^{\pi}\left(\frac{\xi(t)}{t^{-\delta+\beta+2}(n+1)}\right)^q dt\right\}^{\frac{1}{q}}$$

$$= O\left((n+1)^{\delta-1}\right)\left\{\int_{n+1}^{\frac{1}{\pi}}\left(\frac{\xi(\frac{1}{y})}{y^{\delta-\beta-2}}\right)^q\left(-\frac{dy}{y^2}\right)\right\}^{\frac{1}{q}}$$

$$= O\left((n+1)^{\delta}\;\xi\left(\frac{1}{n+1}\right)\right)\; O\left[\left\{\int_{\frac{1}{\pi}}^{n+1} y^{\,q(1+\beta-\delta)-2}\,dy\right\}^{\frac{1}{q}}\right] \text{ Using condition (4)}$$

$$= O\left((n+1)^{\delta}\,\xi(\tfrac{1}{n+1})\right)\, O\left[\left\{\left(\frac{y^{q(1+\beta-\delta)-1}}{q(1+\beta-\delta)-1}\right)_{\frac{1}{\pi}}^{n+1}\right\}^{\frac{1}{q}}\right]$$

$$= O\left((n+1)^{\delta}\,\xi(\tfrac{1}{n+1})\right)\, O\left[\left\{\frac{1}{q(1+\beta-\delta)-1}\left((n+1)^{q(1+\beta-\delta)-1}-(\tfrac{1}{\pi})^{q(1+\beta-\delta)-1}\right)\right\}^{\frac{1}{q}}\right]$$

$$= O\left((n+1)^{\delta}\,\xi(\tfrac{1}{n+1})\right)\; O\left((n+1)^{1+\beta-\delta-\frac{1}{q}}\right)$$

$$= O\left((n+1)^{\beta+1-\frac{1}{q}}\,\xi(\tfrac{1}{n+1})\right)$$

$$= O\left((n+1)^{\beta+\frac{1}{p}}\,\xi(\tfrac{1}{n+1})\right). \tag{11}$$

By (9), (10) and (11), we have

$$\left|t_n^{E_1,C_1}-f\right| = \left((n+1)^{\beta+\frac{1}{p}}\,\xi(\tfrac{1}{n+1})\right)$$

or $$\left\|t_n^{E_1,C_1}-f\right\|_p = O\left\{\int_0^{2\pi}\left((n+1)^{\beta+\frac{1}{p}}\;\xi(\tfrac{1}{n+1})\right)^p dx\right\}^{\frac{1}{p}}$$

$$= O\left((n+1)^{\beta+\frac{1}{p}}\xi(\tfrac{1}{n+1})\right)\left\{\int_0^{2\pi} dx\right\}^{\frac{1}{p}}$$

$$= O\left((n+1)^{\beta+\frac{1}{p}}\xi(\tfrac{1}{n+1})\right). \tag{12}$$

This completes the proof of theorem.

5. Corollaries

Corollary 1: If $\beta = o$ and $\xi(t) = t^{\alpha}$, $0 < \alpha \le 1$ then the degree of approximation of a 2π periodic function f belonging to class $Lip(\alpha, p)$ is given by

$$\left\| t_n^{E_1,C_1} - f \right\|_p = O\left(\frac{1}{(n+1)^{\alpha-\frac{1}{p}}}\right)$$

Proof:

$$\left\| t_n^{E_1,C_1} - f \right\|_p = O\left((n+1)^{\beta+\frac{1}{p}}\xi(\tfrac{1}{n+1})\right)$$

$$= O\left((n+1)^{\frac{1}{p}}\xi(\tfrac{1}{n+1})\right)$$

$$= O\left((n+1)^{\frac{1}{p}}\tfrac{1}{(n+1)^{\alpha}}\right)$$

$$= O\left(\frac{1}{(n+1)^{\alpha-\frac{1}{p}}}\right). \tag{13}$$

Corollary 2: If $p \to \infty$ in corollary 1 then the degree of approximation of a 2π periodic function f belonging to class $Lip\,\alpha$ $(0 < \alpha < 1)$ is given by

$$\left\| t_n^{E_1,C_1} - f \right\|_{\infty} = O\left(\frac{1}{(n+1)^{\alpha}}\right), \text{ for } 0 < \alpha < 1. \tag{14}$$

Remarks: An independent proof of corollary can be developed along the same line as the theorem.
Example: Consider the infinite series,

$$1 - 4\sum_{n=1}^{\infty}(-3)^{n-1}. \tag{15}$$

The (E,1) (C,1) means of the sequence $\{S_n\}$ is given by

$$t_n^{E_1,C_1} = \frac{1}{2^n}\sum_{k=0}^{n}\binom{n}{k}\sigma_k$$

$$= \frac{1}{2(n+1)}\left(1-(-1)^{n+1}\right). \tag{16}$$

The infinite series (15) is neither (C,1) nor (E,1) summable. But from (16), it is summable by (E,1) (C,1) method. Therefore product summability (E,1) (C,1) is more powerful than the individual methods (C,1) and (E,1). Consequently, (E,1) (C,1) means gives the better approximation than individual methods (C,1) and (E,1).

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