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Research article

Normalized ground states to the nonlinear Choquard equations with local perturbations

Xudong Shang*

School of Mathematics, Nanjing Normal University Taizhou College, Taizhou 225300, China

* **Correspondence:** Email: shangxudong@nnutc.edu.cn.

Abstract: In this paper, we considered the existence of ground state solutions to the following Choquard equation

$$\begin{cases} -\Delta u = \lambda u + (I_\alpha * F(u))f(u) + \mu|u|^{q-2}u \quad \text{in } \mathbb{R}^N, \\ \displaystyle\int_{\mathbb{R}^N} |u|^2 dx = a > 0, \end{cases}$$

where $N \geq 3$, I_α is the Riesz potential of order $\alpha \in (0, N)$, $2 < q \leq 2 + \frac{4}{N}$, $\mu > 0$ and $\lambda \in \mathbb{R}$ is a Lagrange multiplier. Under general assumptions on $F \in C^1(\mathbb{R}, \mathbb{R})$, for a L^2-subcritical and L^2-critical of perturbation $\mu|u|^{q-2}u$, we established several existence or nonexistence results about the normalized ground state solutions.

Keywords: Choquard equation; normalized solution; local perturbation

1. Introduction

In this paper, we are looking for solutions to the following Choquard equation

$$\begin{cases} -\Delta u = \lambda u + (I_\alpha * F(u))f(u) + \mu|u|^{q-2}u \quad \text{in } \mathbb{R}^N, \\ \displaystyle\int_{\mathbb{R}^N} |u|^2 dx = a > 0, \end{cases} \tag{1.1}$$

where $N \geq 3$, $\alpha \in (0, N)$, $F(s) := \int_0^s f(t)dt$, $\mu > 0$, $2 < q \leq \overline{q} := 2 + \frac{4}{N}$, $a > 0$ is a given mass, and $\lambda \in \mathbb{R}$ appears as an unknown Lagrange multiplier. $I_\alpha : \mathbb{R}^N \setminus \{0\} \mapsto \mathbb{R}$ is the Riesz potential defined by

$$I_\alpha(x) := \frac{A_\alpha}{|x|^{N-\alpha}} \quad \text{with } A_\alpha = \frac{\Gamma(\frac{N-\alpha}{2})}{2^\alpha \pi^{\frac{N}{2}} \Gamma(\frac{\alpha}{2})}.$$

The Choquard equation

$$-\Delta u + u = (I_\alpha * F(u))f(u) \quad \text{in } \mathbb{R}^N, \tag{1.2}$$

has been studied extensively for its profound physical backgrounds. In particular, when $N = 3$, $\alpha = 2$, and $F(s) = \frac{s^2}{2}$, Eq (1.2) turns into

$$-\Delta u + u = (I_2 * |u|^2)u \quad \text{in } \mathbb{R}^3, \tag{1.3}$$

which was introduced by Pekar [1] to describe the quantum theory of a polaron at rest and then used by Choquard [2] to study steady states of the one one-component plasma approximation to the Hartree-Fock theory. Also, Eq (1.3) reemerged as a model of self-gravitating matter [3,4], and in that context it is known as the Schrödinger-Newton equation.

The pioneering mathematical research dates back to Lieb [2], in which the author proved the existence and uniqueness for Eq (1.3) by variational methods. Later, Lions [5] obtained the existence of normalized solutions. In the homogeneous nonlinearity case of Eq (1.2) with $F(s) = \frac{1}{p}|s|^p$, Moroz and Van Schaftingen in [6] established the existence of ground states to Eq (1.2) with an optimal range $\frac{N+\alpha}{N} < p < \frac{N+\alpha}{N-2}$. Moreover, Ghimenti and Van Schaftingen in [7] obtained solutions which are odd with respect to a hyperplane of $\mathbb{R}^N$. The existence of saddle type nodal solution for the Choquard equation was proven in the study [8]. For a more general nonlinearity, Moroz and Van Schaftingen [9] proved Eq (1.2) has a ground solution when the nonlinearity satisfies Berestycki-Lions type condition and $N \geq 3$, see also [10] for the case $N = 2$. For the Choquard equation with a local nonlinear perturbation

$$-\Delta u + u = (I_\alpha * |u|^p)|u|^{p-2}u + |u|^{q-2}u, \quad \text{in } \mathbb{R}^N. \tag{1.4}$$

When $N = 3$, $0 < \alpha < 1$, $p = 2$, and $4 \leq q < 6$, Chen and Gao in [11] obtained the existence of solutions of Eq (1.4). For $q = 2^*$ in Eq (1.4), Seok in [12] constructed a family of nontrivial solutions. In [13], the authors studied Eq (1.4) with a general local nonlinearity $f(x,u)$ subcritical type instead of $|u|^{q-2}u$.

From a physical point of view, it is interesting to find solutions with prescribed L^2- norm, since there is a conservation of mass. Solutions of this type are often referred to as normalized solutions. In recent years, normalized solutions to nonlinear elliptic problems have attracted much attention from researchers. In [14], Jeanjean using a mountain pass structure for a stretched functional to consider the equation

$$-\Delta u = \lambda u + f(u), \quad \text{in } \mathbb{R}^N. \tag{1.5}$$

He proved the existence of at least one normalized solution of Eq (1.5) in a purely L^2- supercritical case. Later, Bartsch and Valeriola [15] obtained the existence of infinitely many normalized solutions by a new linking geometry for the stretched functional. For more general nonlinearity $f(s)$ has L^2- subcritical growth, Shibata [16] obtained the existence and nonexistence of normalized ground state solution of Eq (1.5) via minimizing method, Jeanjean and Lu in [17] showed the existence of nonradial normalized solutions for any $N \geq 4$. For more results on normalized solutions for Schrödinger equations by variational methods, we would like to refer [18–23]. Also, for the evolution equations with the singular potentials whose the steady state equations are the nonlinear elliptic equations, see [24].

Concerning the normalized solutions of Choquard equations, Li and Ye in [25] considered the existence of normalized solutions to the following equation

$$-\Delta u = \lambda u + (I_\alpha * F(u))f(u), \quad x \in \mathbb{R}^N, \tag{1.6}$$

under a set of assumptions on f, where f takes the special case $f(s) = C_1|s|^{r-2}s + C_2|s|^{p-2}s$ requires that $\frac{N+2+\alpha}{N} < r \le p < \frac{N+\alpha}{N-2}$. Later, Yuan et al. [26] generalized the results in [25] to more general $f \in C(\mathbb{R}, \mathbb{R})$. In [27], Bartsch et al. obtained the existence of infinitely many normalized solutions of (1.6). For $F(u) = \frac{|u|^p}{p}$, Ye in [28] studied the qualitative properties including existence and nonexistence of minimizers of the functional associated to the Eq (1.6). Yao et al. in [29] considered normalized solutions for the following problems

$$-\Delta u + \lambda u = \gamma(I_\alpha * |u|^p)|u|^{p-2}u + \mu|u|^{q-2}u, \text{ in } \mathbb{R}^N. \tag{1.7}$$

Under different assumptions on γ, μ, p, and q, they proved several existence, multiplicity, and nonexistence results. For more related topics, we refer the reader to [30–33].

Motivated by the above papers, it is natural to ask if the nonlinearity $f(u)$ in Eq (1.1) satisfies general growth assumptions, and if the normalized ground states still exist. In the present paper, we attempt to study this kind of problem. In order to prove the existence of normalized ground state solutions to Eq (1.1), assuming that nonlinearity $f \in C(\mathbb{R}, \mathbb{R})$ satisfies the growth assumptions:

(f_1) $\lim_{s\to 0} \frac{f(s)}{|s|^{\frac{\alpha}{N}}} = 0$,

(f_2) $\lim_{|s|\to +\infty} \frac{f(s)}{|s|^{\frac{2+\alpha}{N}}} = 0$,

(f_3) f is odd and f does not change sign on $(0, +\infty)$,

(f_4) $\lim_{s\to 0} \frac{F(s)}{|s|^{\frac{N+2+\alpha}{N}}} = +\infty$,

(f_5) $\limsup_{s\to 0} \frac{F(s)}{|s|^{\frac{N+2+\alpha}{N}}} < +\infty$.

To find solutions of Eq (1.1), we define functional $I_\mu(u) : H^1(\mathbb{R}^N) \to \mathbb{R}$ by

$$I_\mu(u) = \frac{1}{2}\int_{\mathbb{R}^N} |\nabla u|^2 dx - \frac{1}{2}\int_{\mathbb{R}^N} (I_\alpha * F(u))F(u)dx - \frac{\mu}{q}\int_{\mathbb{R}^N} |u|^q dx.$$

For $a > 0$, set

$$S(a) = \{u \in H^1(\mathbb{R}^N) : |u|_2^2 = a\}.$$

Since $f \in C(\mathbb{R}, \mathbb{R})$ satisfies (f_1) and (f_2), using the Hardy-Littlewood-Sobolev inequality, we see that $I_\mu \in C^1(H^1(\mathbb{R}^N), \mathbb{R})$. Normalized solutions of Eq (1.1) can be obtained by looking for critical points of I_μ on the constraint $S(a)$. It is standard that each critical point $u \in S(a)$ of $I_\mu|_{S(a)}$, corresponds a Lagrangian multiplier $\lambda \in \mathbb{R}$ such that (u, λ) solves Eq (1.1). We will be interested in ground state solutions, and following [34], we say that $u \in S(a)$ is a normalized ground state solution to Eq (1.1) if $(I_\mu|_{S(a)})'(u) = 0$ and

$$I_\mu(u) = \inf\{I_\mu(v) : v \in S(a), (I_\mu|_{S(a)})'(v) = 0\}.$$

In particular, if $u \in S(a)$ is a minimizer of the minimization problem

$$m(a) = \inf_{u\in S(a)} I_\mu(u),$$

then u is a critical point of $I_\mu|_{S(a)}$ as well as a normalized ground state to Eq (1.1).

Our main results dealing with problem (1.1) are the following:

Theorem 1.1. *Assume that* (f_1)*–*(f_3) *hold. Let* $\mu > 0$ *and* $2 < q < \overline{q}$*. Then, for any* $a > 0$*, there exists a global minimizer* $\widetilde{u}$ *with respect to* $m(a)$*, which solves Eq (1.1) for some* $\widetilde{\lambda} < 0$*. Moreover,* $\widetilde{u}$ *is a ground state solution of Eq (1.1) which has constant sign, is radially symmetric with respect to some point in* $\mathbb{R}^N$*, and is decreasing.*

Theorem 1.2. *Assume that* (f_1)*–*(f_3) *hold and* $q = \overline{q}$*. For any* $\mu > 0$ *small enough, there exist a* $a_0 = a_0(\mu) > 0$ *and* $a^* \in [0, a_0)$ *such that:*
(i) for any $a \in (0, a^*)$*, there is no global minimizer with respect to* $m(a)$*.*
(ii) for any $a \in (a^*, a_0)$*, there exists a global minimizer u with respect to* $m(a)$*, which solves Eq (1.1) for some* $\lambda < 0$*. Moreover, u is a ground state solution of Eq (1.1) which has constant sign, is radially symmetric with respect to some point in* $\mathbb{R}^N$*, and is decreasing.*
(iii) if (f_4) *holds, then* $a^* = 0$*, and if* (f_5) *holds, then* $a^* > 0$*.*

Remark 1. The value $a_0 = a_0(\mu) > 0$ is explicit and is given in Lemma 4.1. In particular, $a_0 > 0$ can be taken arbitrary large by taking $\mu > 0$ small enough.

The following result positively answers the existence of global minimizer with respect to $m(a^*)$ for the sharp threshold $a^* > 0$.

Theorem 1.3. *Assume that* (f_1)*–*(f_3) *and*
(f_6) $\limsup_{s\to 0} \frac{F(s)}{|s|^{\frac{N+2+\alpha}{N}}} = 0$ *holds.*
Let $q = \overline{q}$*. For* $a = a^*$*, there exists a global minimizer v with respect to* $m(a^*) = 0$*, which solves Eq (1.1) for some* $\lambda < 0$*. In particular, v is a ground state solution of Eq (1.1) which has constant sign and is radially symmetric with respect to some point in* $\mathbb{R}^N$*.*

To the best of our knowledge, the main results in this paper are new. This is a complement of the results for Choquard equations about the existence of normalized solutions. Our main theorems can be viewed as an extension of some results in [25, 29] to more general cases. In our setting, we only consider the situation when the constraint functional $I_\mu|_{S(a)}$ is bounded from below and is coercive. As we will see, the existence of normalized states of Eq (1.1) are strongly effected by further assumptions on the exponent q. We are first devoted to prove the existence of ground state solutions of Eq (1.1) with $2 < q < 2 + \frac{4}{N}$ by application of the concentration-compactness principle [35]. In this case, $m(a) < 0$ for any $a > 0$ and the strict subadditivity inequality

$$m(a + b) < m(a) + m(b) \quad \text{for all } a, b > 0 \tag{1.8}$$

holds, which permits us to exclude the dichotomy of the minimizing sequence. However, in the case of $q = \overline{q} := 2 + \frac{4}{N}$, compare to [29], for a general f, the strict subadditivity inequality (1.8) does not hold, and $m(a) < 0$ for all $a > 0$ may not be satisfied. This prevents us from using the concentration-compactness principle in a standard way. In the proof, we adopt some ideas in [16, 19] to recover the compactness of minimizing sequence with respect to $m(a)$.

The paper is organized as follows. In Section 2, we introduce the variational framework and give some preliminary results. In Section 3, we discuss the case of $2 < q < \overline{q}$ and prove Theorem 1.1. In Section 4, we deal with the case of $q = \overline{q}$ and prove Theorems 1.2 and 1.3.

In this paper, we will use the following notations:
- $H^1(\mathbb{R}^N)$ is the usual Sobolev space endowed with the norm $\|u\|^2 = \int_{\mathbb{R}^N}(|\nabla u|^2 + |u|^2)dx$.

• $L^s(\mathbb{R}^N)$, for $1 \leq s < \infty$, denotes the Lebesgue space with the norm $|u|_s^s = \int_{\mathbb{R}^N} |u|^s dx$.
• For any $r > 0$ and $x \in \mathbb{R}^N$, $B_r(x)$ denotes the ball of radius of r centered at the x.
• The letters $C, C_0, C', C'', C_1, C_2 \cdots$ are positive (possibly different) constants.
• $o_n(1)$ denotes the vanishing quantities as $n \to \infty$.

2. Preliminaries

In this section, we give some results which will be useful in forthcoming sections. First, let us recall the following Hardy-Littlewood-Sobolev inequality which will be frequently used throughout the paper.

Lemma 2.1. *(See [[36], Theorem4.3]). Suppose $\alpha \in (0, N)$, and $s, r > 1$ with $\frac{1}{r} + \frac{1}{s} = 1 + \frac{\alpha}{N}$. Let $f \in L^s(\mathbb{R}^N)$ and $g \in L^r(\mathbb{R}^N)$. Then, there exists a constant $C(N, \alpha, s, r) > 0$ such that*

$$\left| \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} f(x)|x-y|^{\alpha-N} g(y) dx dy \right| \leq C(N, \alpha, s, r)|f|_s|g|_r.$$

In particular, if $r = s = \frac{2N}{N+\alpha}$, then

$$C(N, \alpha, s, r) = C_\alpha := \pi^{\frac{N-\alpha}{2}} \Gamma(\frac{\alpha}{2}) \Gamma^{-1}(\frac{N+\alpha}{2})[\Gamma(\frac{N}{2})\Gamma^{-1}(N)]^{-\frac{\alpha}{N}}.$$

Let $N \geq 1$ and $\frac{N+\alpha}{N} < p < 2_\alpha^* := \frac{N+\alpha}{N-2}$. Then, we introduce the following Gagliardo-Nirenberg inequality of Hartree type ([28])

$$\int_{\mathbb{R}^N} (I_\alpha * |u|^p)|u|^p dx \leq \overline{C}_{N,p} |\nabla u|_2^{Np-N-\alpha} |u|_2^{N+\alpha-p(N-2)}. \tag{2.1}$$

We also recall the following Gagliardo-Nirenberg inequality. For $p \in (2, 2^*)$ and $u \in H^1(\mathbb{R}^N)$,

$$|u|_p \leq C_{N,p} |\nabla u|_2^{\gamma_p} |u|_2^{1-\gamma_p}, \text{ where } \gamma_p = \frac{N(p-2)}{2p}. \tag{2.2}$$

Lemma 2.2. *Assume that (f_1)-(f_2) hold. Let $\{u_n\} \subset H^1(\mathbb{R}^N)$ be a bounded sequence. If either* $\lim_{n\to\infty} |u_n|_2 = 0$ *or* $\lim_{n\to\infty} |u_n|_{\frac{2(N+\alpha+2)}{N+\alpha}} = 0$ *holds, then*

$$\lim_{n\to\infty} \int_{\mathbb{R}^N} (I_\alpha * F(u_n))F(u_n) dx = 0.$$

Proof. By (f_1) and (f_2), for any $\varepsilon > 0$, there exists $C_\varepsilon > 0$ such that

$$|F(s)| \leq \varepsilon |s|^{\frac{N+\alpha}{N}} + C_\varepsilon |s|^{\frac{N+2+\alpha}{N}}.$$

Then, using Lemma 2.1, for $u \in H^1(\mathbb{R}^N)$ we obtain

$$\left| \int_{\mathbb{R}^N} (I_\alpha * F(u))F(u) dx \right| \leq \varepsilon^2 \int_{\mathbb{R}^N} (I_\alpha * |u|^{\frac{N+\alpha}{N}})|u|^{\frac{N+\alpha}{N}} dx$$

$$
\begin{aligned}
&+2\varepsilon C_\varepsilon \int_{\mathbb{R}^N} (I_\alpha * |u|^{\frac{N+\alpha}{N}})|u|^{\frac{N+\alpha+2}{N}} dx \\
&+C_\varepsilon^2 \int_{\mathbb{R}^N} (I_\alpha * |u|^{\frac{N+2+\alpha}{N}})|u|^{\frac{N+\alpha+2}{N}} dx \\
\leq\ & C_1\varepsilon^2 |u|_2^{\frac{2(N+2)}{N}} + C_2 C_\varepsilon^2 |u|_{\frac{2(N+2+\alpha)}{N+\alpha}}^{\frac{2(N+2+\alpha)}{N}}.
\end{aligned}
\tag{2.3}
$$

By Eq (2.2), we have

$$
|u|_{\frac{2(N+2+\alpha)}{N+\alpha}}^{\frac{2(N+2+\alpha)}{N}} \leq C_3 |\nabla u|_2^2 |u|_2^{\frac{2(2+\alpha)}{N}}. \tag{2.4}
$$

If $\lim_{n\to\infty} |u_n|_2 = 0$, by Eqs (2.3), (2.4), and the boundedness of $\{u_n\}$, we get

$$
\lim_{n\to\infty} \int_{\mathbb{R}^N} (I_\alpha * F(u_n))F(u_n) dx = 0.
$$

If $\lim_{n\to\infty} |u_n|_{\frac{2(N+\alpha+2)}{N+\alpha}} = 0$, by Eq (2.3), we have

$$
\limsup_{n\to\infty} |\int_{\mathbb{R}^N} (I_\alpha * F(u))F(u) dx| \leq C_1 \varepsilon^2 |u_n|_2^{\frac{2(N+2)}{N}}.
$$

Since $\{u_n\}$ is bounded in $H^1(\mathbb{R}^N)$ and $\varepsilon > 0$ is arbitrary, the conclusion holds. The proof is completed.

In our subsequent arguments, we will use the following nonlocal version of the Brezis-Lieb lemma.

Lemma 2.3. *(See [[37], Lemma 2.2]). Assume $\alpha \in (0, N)$ and there exists a constant $C > 0$ such that*

$$
|f(s)| \leq C(|s|^{\frac{\alpha}{N}} + |s|^{\frac{2+\alpha}{N-2}}), \quad s \in \mathbb{R}.
$$

Let $\{u_n\} \subset H^1(\mathbb{R}^N)$ be such that $u_n \rightharpoonup u$ weakly in $H^1(\mathbb{R}^N)$ and almost everywhere in $\mathbb{R}^N$ as $n \to \infty$. Then,

$$
\begin{aligned}
\int_{\mathbb{R}^N} (I_\alpha * F(u_n))F(u_n) dx &= \int_{\mathbb{R}^N} (I_\alpha * F(u_n - u))F(u_n - u) dx \\
&\quad + \int_{\mathbb{R}^N} (I_\alpha * F(u))F(u) dx + o_n(1).
\end{aligned}
$$

Using a similar argument as the proof of ([9] Theorem 3), we have the following Pohožaev of Eq (1.1).

Lemma 2.4. *Assume that $N \geq 3$ and $\alpha \in (0, N)$. If $f \in C(\mathbb{R}, \mathbb{R})$ satisfies (f_1) and (f_2), and if $(u, \lambda) \in H^1(\mathbb{R}^N) \times \mathbb{R}$ solves problem (1.1), then*

$$
P(u) = \frac{N-2}{2} |\nabla u|_2^2 - \frac{N}{2} \lambda |u|_2^2 - \frac{N+\alpha}{2} \int_{\mathbb{R}^N} (I_\alpha * F(u))F(u) dx - \frac{N\mu}{q} |u|_q^q = 0.
$$

For each $u \in S(a)$ and $t > 0$, we define the scaling function

$$u^t(x) := t^{\frac{N}{2}} u(tx).$$

It is clear that $u^t \in S(a)$ and

$$I_\mu(u^t) = \frac{t^2}{2}|\nabla u|_2^2 - \frac{1}{2t^{N+\alpha}} \int_{\mathbb{R}^N} (I_\alpha * F(t^{\frac{N}{2}} u))F(t^{\frac{N}{2}} u)dx - \frac{\mu}{q} t^{\frac{N(q-2)}{2}} |u|_q^q.$$

3. L^2-subcritical perturbation

In this section, we deal with the case $2 < q < \overline{q}$ and prove Theorem 1.1.

Lemma 3.1. *Assume that (f_1) and (f_2) hold. For any $2 < q < \overline{q}$ and $a, \mu > 0$, we have*

$$-\infty < m(a) = \inf_{u \in S(a)} I_\mu(u) < 0.$$

Proof. By (f_1) and (f_2), for any $\varepsilon > 0$, there exists $C_\varepsilon > 0$ such that

$$|F(s)| \leq C_\varepsilon |s|^{\frac{N+\alpha}{N}} + \varepsilon |s|^{\frac{N+2+\alpha}{N}}.$$

Using Lemma 2.1 and Eq (2.2), we obtain, for $u \in H^1(\mathbb{R}^N)$

$$|\int_{\mathbb{R}^N} (I_\alpha * F(u))F(u)dx| \leq C_3 C_\varepsilon^2 |u|_2^{\frac{2(N+2)}{N}} + C_4 \varepsilon^2 |u|_2^{\frac{2(\alpha+2)}{N}} |\nabla u|_2^2. \tag{3.1}$$

Then, for any $u \in S(a)$, by Eqs (2.2) and (3.1), we get

$$I_\mu(u) \geq \frac{1}{2}|\nabla u|_2^2 - C_3 C_\varepsilon^2 a^{\frac{N+2}{N}} - C_4 \varepsilon^2 a^{\frac{\alpha+2}{N}} |\nabla u|_2^2 - \frac{\mu C_{N,q}^q}{q} a^{\frac{q(1-\gamma_q)}{2}} |\nabla u|_2^{q\gamma_q}. \tag{3.2}$$

Choosing $\varepsilon = (4C_4 a^{\frac{2+\alpha}{N}})^{-\frac{1}{2}}$, it follows from Eq (3.2) that

$$I_\mu(u) \geq \frac{1}{4}|\nabla u|_2^2 - C a^{\frac{N+2}{N}} - \mu C_{N,q}^q q^{-1} a^{\frac{q(1-\gamma_q)}{2}} |\nabla u|_2^{q\gamma_q}$$

for every $u \in S(a)$. Since $2 < q < \overline{q}$, we see that $0 < q\gamma_q < 2$, and hence I_μ is coercive on $S(a)$, which provides that $m(a) > -\infty$.

On the other hand, for $u \in S(a)$

$$\begin{aligned} I_\mu(u^t) &\leq \frac{t^2}{2}|\nabla u|_2^2 - \frac{\mu}{q} t^{\frac{N(q-2)}{2}} |u|_q^q \\ &= t^{\frac{N(q-2)}{2}} \left(\frac{1}{2} t^{2-\frac{N(q-2)}{2}} |\nabla u|_2^2 - \frac{\mu}{q}|u|_q^q \right). \end{aligned}$$

Noticing that $2 < q < \overline{q}$, we have $2 - \frac{N(q-2)}{2} > 0$, and hence $I_\mu(u^t) < 0$ for every $u \in S(a)$ with $t > 0$ small enough. Therefore, we have that $m(a) < 0$ for any $a > 0$.

Since $m(a) < 0$ for any $a > 0$, we can give the following strict sub-additivity.

Lemma 3.2. *Let $a_1, a_2 > 0$ be such that $a_1 + a_2 = a$. Then,*

$$m(a) < m(a_1) + m(a_2).$$

Proof. For $u \in S(a)$ and $\theta > 1$, we set $\overline{u}(x) = u(\theta^{-\frac{1}{N}}x)$. Then, $\overline{u}(x) \in S(\theta a)$. Let $\{u_n\} \subset S(a)$ be a minimizing sequence for $m(a)$. Since $\theta > 1$, we have

$$\begin{aligned} m(\theta a) &\leq I_\mu(\overline{u}_n) = \frac{\theta^{1-\frac{2}{N}}}{2}|\nabla u_n|_2^2 - \frac{\theta^{1+\frac{\alpha}{N}}}{2}\int_{\mathbb{R}^N}(I_\alpha * F(u_n))F(u_n)dx - \frac{\mu\theta}{q}|u_n|_q^q \\ &< \theta I_\mu(u_n) = \theta m(a) + o_n(1). \end{aligned}$$

As a consequence,

$$m(\theta a) \leq \theta m(a),$$

with equality if and only if

$$\lim_{n\to\infty}(I_\mu(\overline{u}_n) - \theta I_\mu(u_n)) = 0. \tag{3.3}$$

But this is can not occur. Otherwise, by Eq (3.3), we find

$$\lim_{n\to\infty}\left(\frac{\theta^{-\frac{2}{N}} - 1}{2}|\nabla u_n|_2^2 + \frac{1 - \theta^{\frac{\alpha}{N}}}{2}\int_{\mathbb{R}^N}(I_\alpha * F(u_n))F(u_n)dx\right) = 0.$$

By Lemma 3.1, I_μ is coercive on $S(a)$ and we have $\{u_n\}$ is bounded in $H^1(\mathbb{R}^N)$. It follows from $\theta > 1$ that

$$\lim_{n\to\infty}|\nabla u_n|_2^2 = 0 = \lim_{n\to\infty}\int_{\mathbb{R}^N}(I_\alpha * F(u_n))F(u_n)dx.$$

Combining Eq (2.2) we get $\lim_{n\to\infty}|u_n|_q^q = 0$. Then, by Lemma 3.1, we obtain

$$0 > m(a) = \lim_{n\to\infty} I_\mu(u_n) = 0,$$

a contradiction. Thus, we have the strict inequality

$$m(\theta a) < \theta m(a) \qquad \text{for any } \theta > 1. \tag{3.4}$$

Next, we show that $m(a) < m(a_1) + m(a_2)$. We may assume that $a_1 \geq a_2$, by Eq (3.4) we have

$$m(a) = m(\frac{a}{a_1}\cdot a_1) < \frac{a}{a_1}m(a_1) = m(a_1) + \frac{a_2}{a_1}m(a_1) \leq m(a_1) + m(a_2).$$

The proof is completed.

Lemma 3.3. *Assume that (f_1) and (f_2) hold and $a, \mu > 0$. Let $\{u_n\} \subset H^1(\mathbb{R}^N)$ be a sequence such that*

$$\lim_{n\to\infty} I_\mu(u_n) = m(a), \qquad \lim_{n\to\infty} |u_n|_2^2 = a. \tag{3.5}$$

Then, taking a subsequence if necessary, there exist $\widetilde{u} \in S(a)$ and a family $\{y_n\} \subset \mathbb{R}^N$ such that $u_n(\cdot + y_n) \to \widetilde{u}$ in $H^1(\mathbb{R}^N)$ as $n \to \infty$. Specifically, $\widetilde{u}$ is a global minimizer.

Proof. Since $\{u_n\} \subset H^1(\mathbb{R}^N)$ satisfies Eq (3.5), it is easy to see that $\{u_n\}$ is bounded in $H^1(\mathbb{R}^N)$. From the concentration-compactness lemma [35], there exists a subsequence of $\{u_n\}$ (denoted in the same way) satisfying one of the three following possibilities:

vanishing: for all $R > 0$

$$\lim_{n\to\infty} \sup_{y\in\mathbb{R}^N} \int_{B_R(y)} |u_n|^2 dx = 0;$$

dichotomy: there exists a constant $b \in (0, a)$, sequences $\{u_n^{(1)}\}$, $\{u_n^{(2)}\}$ bounded in $H^1(\mathbb{R}^N)$ such that as $n \to \infty$

$$\begin{cases} \left|u_n - (u_n^{(1)} + u_n^{(2)})\right|_p \to 0, \quad \text{for } 2 \le p < 2^*, \\ |u_n^{(1)}|_2^2 \to b, \ |u_n^{(2)}|_2^2 \to a - b, \ \operatorname{dist}(\operatorname{supp} u_n^{(1)}, \operatorname{supp} u_n^{(2)}) \to +\infty, \\ \liminf\limits_{n\to\infty} \displaystyle\int_{\mathbb{R}^N} (|\nabla u_n|^2 - |\nabla u_n^{(1)}|^2 - |\nabla u_n^{(2)}|^2) dx \ge 0; \end{cases} \tag{3.6}$$

compactness: there exists a sequence $\{y_n\} \subset \mathbb{R}^N$ with the following property: for any $\varepsilon > 0$, there exists $R > 0$ such that

$$\int_{B_R(y_n)} |u_n|^2 dx \ge a - \varepsilon. \tag{3.7}$$

Claim 1. Vanishing does not occur.

Otherwise, by Lemma I.1 of [38], we get $u_n \to 0$ strongly in $L^p(\mathbb{R}^N)$ for $2 < p < 2^*$. Since $2 < \frac{2(N+2+\alpha)}{N+\alpha} < 2^*$, it follows from Lemma 2.2 that

$$\lim_{n\to\infty} \int_{\mathbb{R}^N} (I_\alpha * F(u_n)) F(u_n) dx = 0.$$

Then, by Lemma 3.1 and Eq (3.5) we have

$$0 > m(a) = \lim_{n\to\infty} I_\mu(u_n) = \lim_{n\to\infty} \frac{1}{2} |\nabla u_n|_2^2 \ge 0.$$

This contradiction proves Claim 1.

Claim 2. Dichotomy does not occur.

Otherwise, if dichotomy occurs, there exist $b \in (0, a)$ and sequences $\{u_n^{(1)}\}$, $\{u_n^{(2)}\}$ satisfying Eq (3.6). Furthermore, we may assume

$$u_n = u_n^{(1)} + u_n^{(2)} + v_n, \ \ u_n^{(1)} u_n^{(2)} = u_n^{(1)} v_n = u_n^{(2)} v_n = 0 \ \text{ almost everywhere in } \mathbb{R}^N.$$

Then, we have

$$|u_n|_q^q = |u_n^{(1)}|_q^q + |u_n^{(2)}|_q^q + |v_n|_q^q, \tag{3.8}$$

and

$$\begin{aligned}\int_{\mathbb{R}^N}(I_\alpha * F(u_n))F(u_n)dx &= \int_{\mathbb{R}^N}(I_\alpha * F(u_n^{(1)}))F(u_n^{(1)})dx + \int_{\mathbb{R}^N}(I_\alpha * F(u_n^{(2)}))F(u_n^{(2)})dx \\ &\quad +2\int_{\mathbb{R}^N}(I_\alpha * F(u_n^{(1)}))F(u_n^{(2)})dx + 2\int_{\mathbb{R}^N}(I_\alpha * F(u_n))F(v_n)dx \\ &\quad + \int_{\mathbb{R}^N}(I_\alpha * F(v_n))F(v_n)dx. \end{aligned} \tag{3.9}$$

By (f_1), (f_2), and Lemma 2.1, we obtain

$$\left|\int_{\mathbb{R}^N}(I_\alpha * F(u_n))F(v_n)dx\right| \le C\left(|u_n|_2^2 + |u_n|_{\frac{2(N+2+\alpha)}{N+\alpha}}^{\frac{2(N+2+\alpha)}{N+\alpha}}\right)^{\frac{N+\alpha}{2N}}\left(|v_n|_2^2 + |v_n|_{\frac{2(N+2+\alpha)}{N+\alpha}}^{\frac{2(N+2+\alpha)}{N+\alpha}}\right)^{\frac{N+\alpha}{2N}}.$$

By the definition of v_n and Eq (3.6), we obtain that $|v_n|_p \to 0$ for $2 \le p < 2^*$. It follows from the above inequality and the boundedness of $\{u_n\}$ that

$$\lim_{n\to\infty}\int_{\mathbb{R}^N}(I_\alpha * F(u_n))F(v_n)dx = 0. \tag{3.10}$$

Using Eq (3.6) and Lemma 2.2, we get

$$\lim_{n\to\infty}\int_{\mathbb{R}^N}(I_\alpha * F(v_n))F(v_n)dx = 0. \tag{3.11}$$

Moreover, by the Young's inequality ([36] Theorem 4.2) and $dist(supp\ u_n^{(1)}, supp\ u_n^{(2)}) \to +\infty$, we infer that

$$\lim_{n\to\infty}\int_{\mathbb{R}^N}(I_\alpha * F(u_n^{(1)}))F(u_n^{(2)})dx = 0. \tag{3.12}$$

Thus, Eqs (3.6) and (3.8)–(3.12) imply that

$$m(a) = \lim_{n\to\infty} I_\mu(u_n) \ge \limsup_{n\to\infty}(I_\mu(u_n^{(1)}) + I_\mu(u_n^{(2)})) \ge m(b) + m(a-b),$$

which contradicts to Lemma 3.2, and proves Claim 2.

Hence, the compactness holds, namely, there exists a subsequence $\{y_n\} \subset \mathbb{R}^N$ such that $\widetilde{u}_n = u_n(x+y_n) \to \widetilde{u}$ in $L^2(\mathbb{R}^N)$, $\widetilde{u}_n \rightharpoonup \widetilde{u}$ in $H^1(\mathbb{R}^N)$ and $\widetilde{u} \in S(a)$. It follows from interpolation inequality and Sobolev inequality that

$$\lim_{n\to\infty}|\widetilde{u}_n - \widetilde{u}|_p = 0, \quad \text{for } 2 < p < 2^*.$$

Then, Lemma 2.2 implies

$$\lim_{n\to\infty}\int_{\mathbb{R}^N}(I_\alpha * F(\widetilde{u}_n-\widetilde{u}))F(\widetilde{u}_n-\widetilde{u})dx=0.$$

Hence,

$$m(a)\le I_\mu(\widetilde{u})\le \liminf_{n\to\infty} I_\mu(\widetilde{u}_n)=\liminf_{n\to\infty} I_\mu(u_n)=m(a).$$

Thus, we have $I_\mu(\widetilde{u})=m(a)$ and $\|\widetilde{u}_n\|\to\|\widetilde{u}\|$ as $n\to\infty$. Moreover, $u_n(x+y_n)\to\widetilde{u}$ in $H^1(\mathbb{R}^N)$.

Proof. [Proof of Theorem 1.1] By Lemmas 3.1 and 3.3, there exists a global minimizer $\widetilde{u}$ for I_μ on $S(a)$ with $m(a)=I_\mu(\widetilde{u})<0$. Furthermore, $\widetilde{u}$ is a ground state solution of Eq (1.1) for some $\widetilde{\lambda}\in\mathbb{R}$. Then, by Lemma 2.4, we have

$$P(\widetilde{u})=\frac{N-2}{2}|\nabla\widetilde{u}|_2^2-\frac{N}{2}\widetilde{\lambda}|\widetilde{u}|_2^2-\frac{N+\alpha}{2}\int_{\mathbb{R}^N}(I_\alpha * F(\widetilde{u}))F(\widetilde{u})dx-\frac{N\mu}{q}|\widetilde{u}|_q^q=0.$$

Then,

$$\lambda a=2m(a)-\frac{2}{N}|\nabla\widetilde{u}|_2^2-\frac{\alpha}{N}\int_{\mathbb{R}^N}(I_\alpha * F(\widetilde{u}))F(\widetilde{u})dx. \tag{3.13}$$

Since $\int_{\mathbb{R}^N}(I_\alpha * F(\widetilde{u}))F(\widetilde{u})dx>0$, by Eq (3.13) we have $\widetilde{\lambda}<0$.

By (f_3), without loss of generality, we may assume that $f\ge 0$ on $(0,+\infty)$. Since f is odd, then F is even and thus for every $u\in H^1(\mathbb{R}^N)$, $I_\mu(|u|)=I_\mu(u)$. From this one easily obtain that the function $|\widetilde{u}|$ is also a ground state solution of Eq (1.1). By regularity properties of [9], $|\widetilde{u}|$ is continuous, we can apply the strong maximum principle get $|\widetilde{u}|>0$ on $\mathbb{R}^N$ and thus $\widetilde{u}$ has constant sign.

Finally, we prove the symmetry of $\widetilde{u}$. Assume that $H\subset\mathbb{R}^N$ is a closed half-space and that σ_H denotes the reflection with respect to ∂H. The polarization $\widetilde{u}^H(x):\mathbb{R}^N\to\mathbb{R}$ of $\widetilde{u}$ is defined for $x\in\mathbb{R}^N$ by

$$\widetilde{u}^H(x):=\begin{cases}\max\{\widetilde{u}(x),\widetilde{u}(\sigma_H(x))\} & \text{if } x\in H,\\ \min\{\widetilde{u}(x),\widetilde{u}(\sigma_H(x))\} & \text{if } x\notin H.\end{cases}$$

By the properties of polarization ([9] Lemma 5.4), we observe that

$$\int_{\mathbb{R}^N}|\nabla\widetilde{u}^H|^2dx=\int_{\mathbb{R}^N}|\nabla\widetilde{u}|^2dx,$$

and then

$$\int_{\mathbb{R}^N}|\widetilde{u}^H|^s dx=\int_{\mathbb{R}^N}|\widetilde{u}|^s dx,\quad \text{for any } s\in[0,+\infty).$$

Moreover, since F is nondecreasing on $(0, +\infty)$, we have $F(\widetilde{u}^H) = F(\widetilde{u})^H$. Therefore, by Lemma 5.5 of [9], we get

$$I_\mu(\widetilde{u}^H) \leq I_\mu(\widetilde{u}) \tag{3.14}$$

with equality if and only if

$$\text{either } F(\widetilde{u})^H = F(\widetilde{u}) \text{ or } F(\widetilde{u})^H = F(\widetilde{u}(\sigma_H)) \text{ in } \mathbb{R}^N. \tag{3.15}$$

On the other hand, since $\widetilde{u} \in S(a)$ and $|\widetilde{u}|_2^2 = |\widetilde{u}^H|_2^2$, we have $I_\mu(\widetilde{u}^H) \geq m(a)$. It follows from Eq (3.14) that $I_\mu(\widetilde{u}^H) = I_\mu(\widetilde{u})$, and thus Eq (3.15) holds. If $F(\widetilde{u})^H = F(\widetilde{u})$, for every $x \in H$,

$$\int_{\widetilde{u}(\sigma_H(x))}^{\widetilde{u}(x)} f(t)dt = F(\widetilde{u}(x)) - F(\widetilde{u}(\sigma_H(x))) = F(\widetilde{u}^H(x)) - F(\widetilde{u}(\sigma_H(x))) \geq 0.$$

Since F is nondecreasing on $(0, +\infty)$, we get $\widetilde{u}(\sigma_H(x)) \leq \widetilde{u}(x)$. In particular, $f(\widetilde{u}^H) = f(\widetilde{u})$ on $\mathbb{R}^N$, hence $\widetilde{u}^H = \widetilde{u}$. If $F(\widetilde{u}(\sigma_H)) = F(\widetilde{u})^H$, we similarly get $\widetilde{u}^H = \widetilde{u}(\sigma_H)$. Since the hyperplane H is arbitrary, in either case we conclude that the function $\widetilde{u}$ is radially symmetric with respect to some point $x_0 \in \mathbb{R}^N$, and is radially decreasing.

4. L^2-critical perturbation

In this section, we consider the case $q = \overline{q}$ and prove Theorems 1.2 and 1.3. Set $\overline{a}_N := \frac{\overline{q}}{2C_{N,\overline{q}}^{\overline{q}}} > 0$.

Lemma 4.1. *Assume that (f_1) and (f_2) hold. For any $\mu > 0$, there exists $a_0(\mu) = (\overline{a}_N \mu^{-1})^{\frac{N}{2}}$, such that for any $a \in (0, a_0)$, we have*

$$-\infty < m(a) \leq 0.$$

Proof. For every $u \in S(a)$, by Eqs (2.2) and (3.1), we get

$$I_\mu(u) \geq (\frac{1}{2} - \frac{\mu C_{N,\overline{q}}^{\overline{q}}}{\overline{q}} a^{\frac{2}{N}})|\nabla u|_2^2 - C_3 C_\varepsilon^2 a^{\frac{N+2}{N}} - C_4 \varepsilon^2 a^{\frac{\alpha+2}{N}} |\nabla u|_2^2.$$

Since $a < a_0$, we have $\frac{1}{2} - \frac{\mu C_{N,\overline{q}}^{\overline{q}}}{\overline{q}} a^{\frac{2}{N}} > 0$. We choose $\varepsilon > 0$ small such that $C_4 \varepsilon^2 a^{\frac{\alpha+2}{N}} = \frac{1}{2}(\frac{1}{2} - \frac{\mu C_{N,\overline{q}}^{\overline{q}}}{\overline{q}} a^{\frac{2}{N}})$, it follows that

$$I_\mu(u) \geq \frac{1}{2}\left(\frac{1}{2} - \frac{\mu C_{N,\overline{q}}^{\overline{q}}}{\overline{q}} a^{\frac{2}{N}}\right)|\nabla u|_2^2 - C a^{\frac{N+2}{N}}.$$

This implies $m(a) > -\infty$.

In addition, for $u \in S(a)$ we have

$$m(a) \leq I_\mu(u^t) = t^2 \left(\frac{1}{2}|\nabla u|_2^2 - \frac{\mu}{\overline{q}}|u|_{\overline{q}}^{\overline{q}}\right) - \frac{1}{2} \int_{\mathbb{R}^N} (I_\alpha * F(u^t)) F(u^t) dx. \tag{4.1}$$

Noticing that $a < a_0$, by Eq (2.2), we have $\frac{1}{2}|\nabla u|_2^2 - \frac{\mu}{\overline{q}}|u|_{\overline{q}}^{\overline{q}} > 0$, and

$$\lim_{t\to 0} t^2 \left(\frac{1}{2}|\nabla u|_2^2 - \frac{\mu}{\overline{q}}|u|_{\overline{q}}^{\overline{q}}\right) = 0.$$

Moreover, $|u^t|_{\frac{2(N+2+\alpha)}{N+\alpha}}^{\frac{2(N+2+\alpha)}{N+\alpha}} = t^{\frac{2N}{N+\alpha}}|u|_{\frac{2(N+2+\alpha)}{N+\alpha}}^{\frac{2(N+2+\alpha)}{N+\alpha}} \to 0$ as $t \to 0$. By Lemma 2.2, we have

$$\lim_{t\to 0} \int_{\mathbb{R}^N} (I_\alpha * F(u^t))F(u^t)dx = 0.$$

Then, by Eq (4.1) we infer that

$$m(a) \leq \lim_{t\to 0} I_\mu(u^t) = 0.$$

Thus, $m(a) \leq 0$. The proof is completed.

Next, we give some properties of $m(a)$.

Lemma 4.2. *It holds that*
(i) Let $a \in (0, a_0)$. Then, we have for all $b \in (0, a)$

$$m(a) \leq m(b) + m(a - b),$$

and if $m(b)$ or $m(a - b)$ is reached, then the inequality is strict.
(ii) Taking $\mu > 0$ small enough, there exists $b_0 > 0$ such that $0 < b_0 < a_0(\mu)$. Then, for any $a \in (b_0, a_0)$, we have $m(a) < 0$.
(iii) $a \in (0, a_0) \mapsto m(a)$ is continuous.

Proof. (i) Fix $b \in (0, a)$, we first show that

$$m(\theta b) \leq \theta m(b), \quad \text{for any } \theta \in (1, \frac{a}{b}], \tag{4.2}$$

and that if $m(b)$ is reached, the inequality is strict. By the definition of $m(b)$, for any $\varepsilon > 0$ sufficiently small, there exists a $u \in S(b)$ such that

$$I_\mu(u) \leq m(b) + \varepsilon. \tag{4.3}$$

Now set $\overline{u}(x) = u(\theta^{-\frac{1}{N}}x)$. Note that $\overline{u}(x) \in S(\theta b)$. It follows from Eq (4.3) that

$$m(\theta b) \leq I_\mu(\overline{u}) < \theta I_\mu(u) \leq \theta m(b) + \theta\varepsilon. \tag{4.4}$$

Since $\varepsilon > 0$ is arbitrary, we have that $m(\theta b) \leq \theta m(b)$. If $m(b)$ is reached, we can let $\varepsilon = 0$ in Eq (4.3), and thus Eq (4.4) implies $m(\theta b) < \theta m(b)$.

Then, by Eq (4.2) we have

$$m(a) \;=\; \frac{a-b}{a}m(a) + \frac{b}{a}m(a)$$

$$
\begin{aligned}
&= \frac{a-b}{a} m\left(\frac{a}{a-b}\cdot(a-b)\right) + \frac{b}{a} m\left(\frac{a}{b}\cdot b\right)\\
&\le m(a-b) + m(b),
\end{aligned}
$$

with a strict inequality if $m(b)$ is reached.

(ii) By (f_3), there exists $v \in H^1(\mathbb{R}^N)$ such that

$$\int_{\mathbb{R}^N} (I_\alpha * F(v))F(v)dx > 0.$$

For any $b > 0$, set $v_b(x) = v(b^{-\frac{1}{N}}|v|_2^{\frac{2}{N}} x)$. Obviously $v_b \in S(b)$. Then, we have

$$
\begin{aligned}
I_\mu(v_b) &= \frac{1}{2}\int_{\mathbb{R}^N} |\nabla v_b|^2 dx - \frac{1}{2}\int_{\mathbb{R}^N} (I_\alpha * F(v_b))F(v_b)dx - \frac{\mu}{\overline{q}}\int_{\mathbb{R}^N} |v_b|^{\overline{q}} dx\\
&= \frac{b^{\frac{N-2}{N}}}{2|v|_2^{\frac{2(N-2)}{N}}}|\nabla v|_2^2 - \frac{b^{\frac{N+\alpha}{N}}}{2|v|_2^{\frac{2(N+\alpha)}{N}}}\int_{\mathbb{R}^N} (I_\alpha * F(v))F(v)dx\\
&\quad -\frac{\mu b}{\overline{q}|v|_2^2}\int_{\mathbb{R}^N} |v|^{\overline{q}} dx\\
&:= g(b).
\end{aligned}
$$

Since $g(b) \to -\infty$ as $b \to +\infty$ and by choosing $b_0 > 0$ large such that $I_\mu(v_{b_0}) < 0$, it follows that

$$m(b_0) \le I_\mu(v_{b_0}) < 0.$$

Now, taking $\mu > 0$ small enough such that $b_0 < a_0(\mu)$. For any $a \in (b_0, a_0)$, by Lemma 4.1 and (i), we obtain

$$m(a) \le m(a - b_0) + m(b_0) \le m(b_0) < 0.$$

(iii) Let $a \in (0, a_0)$ be arbitrary and $\{a_n\} \subset (0, a_0)$ be such that $a_n \to a$ as $n \to \infty$. By the definition of $m(a_n)$, for every n there exists $u_n \in S(a_n)$ such that

$$I_\mu(u_n) \le m(a_n) + \frac{1}{n}. \tag{4.5}$$

Since $m(a_n) \le 0$, by the proof of Lemma 4.1, the sequence $\{u_n\}$ is bounded in $H^1(\mathbb{R}^N)$. Set $v_n = \sqrt{\frac{a}{a_n}} u_n$. It is clear that $v_n \in S(a)$. Then, we can write

$$m(a) \le I_\mu(v_n) = I_\mu(u_n) + (I_\mu(v_n) - I_\mu(u_n)), \tag{4.6}$$

where

$$
\begin{aligned}
I_\mu(v_n) - I_\mu(u_n) &= \frac{1}{2}\left(\frac{a}{a_n} - 1\right)|\nabla u_n|_2^2 - \frac{\mu}{\overline{q}}\left[\left(\frac{a}{a_n}\right)^{\frac{\overline{q}}{2}} - 1\right]|u_n|_{\overline{q}}^{\overline{q}}\\
&\quad -\frac{1}{2}\int_{\mathbb{R}^N} \left[(I_\alpha * F(v_n))F(v_n) - (I_\alpha * F(u_n))F(u_n)\right] dx.
\end{aligned} \tag{4.7}
$$

By the boundedness of $\{u_n\}$, we have $\{|u_n|_{\bar{q}}^{\bar{q}}\}$ is bounded. Thus, we have

$$\lim_{n\to\infty}(\frac{a}{a_n}-1)|\nabla u_n|_2^2 = 0 = \lim_{n\to\infty}[(\frac{a}{a_n})^{\frac{\bar{q}}{2}}-1]|u_n|_{\bar{q}}^{\bar{q}}. \tag{4.8}$$

Moreover,

$$\begin{aligned}&\int_{\mathbb{R}^N}[(I_\alpha * F(v_n))F(v_n)-(I_\alpha * F(u_n))F(u_n)]dx\\ &=\int_{\mathbb{R}^N}(I_\alpha * F(v_n))(F(v_n)-F(u_n))dx\\ &+\int_{\mathbb{R}^N}(I_\alpha * (F(v_n)-F(u_n)))F(u_n)dx.\end{aligned} \tag{4.9}$$

Since $a_n \to a$ as $n\to\infty$, then $\{\sqrt{\frac{a}{a_n}}\}$ is bounded. It follows from (f_1) and (f_2) that

$$\begin{aligned}|F(v_n)-F(u_n)| &\le \int_0^1 |f(u_n+t(v_n-u_n))|\cdot|v_n-u_n|dt\\ &\le C(\sqrt{\frac{a}{a_n}}-1)\left(|u_n|^{\frac{N+\alpha}{N}}+|u_n|^{\frac{N+\alpha+2}{N}}\right).\end{aligned}$$

Thus, using Lemma 2.1 and the Sobolev imbedding inequality, we get

$$\begin{aligned}\left|\int_{\mathbb{R}^N}(I_\alpha * (F(v_n)-F(u_n)))F(u_n)dx\right| &\le C_\alpha|F(u_n)|_{\frac{2N}{N+\alpha}}|F(v_n)-F(u_n)|_{\frac{2N}{N+\alpha}}\\ &\le C(\sqrt{\frac{a}{a_n}}-1)\left(|u_n|_2^2+|u_n|_{\frac{2(N+\alpha+2)}{N+\alpha}}^{\frac{2(N+\alpha+2)}{N+\alpha}}\right)^{\frac{N+\alpha}{N}}.\end{aligned}$$

Thus,

$$\lim_{n\to\infty}\int_{\mathbb{R}^N}(I_\alpha * (F(v_n)-F(u_n)))F(u_n)dx = 0.$$

Similarly,

$$\lim_{n\to\infty}\int_{\mathbb{R}^N}(I_\alpha * F(v_n))(F(v_n)-F(u_n))dx = 0.$$

Then, in view of Eq (4.9), we have

$$\lim_{n\to\infty}\int_{\mathbb{R}^N}[(I_\alpha * F(v_n))F(v_n)-(I_\alpha * F(u_n))F(u_n)]dx = 0. \tag{4.10}$$

By Eqs (4.7), (4.8), and (4.10), we have

$$\lim_{n\to\infty}(I_\mu(v_n) - I_\mu(u_n)) = 0. \tag{4.11}$$

It follows from Eqs (4.5) and (4.6) that

$$m(a) \le \liminf_{n\to\infty} m(a_n).$$

On the other hand, for any $\varepsilon > 0$, there exists $u \in S(a)$ such that

$$I_\mu(u) \le m(a) + \varepsilon. \tag{4.12}$$

Set $\widetilde{u}_n(x) = \sqrt{\frac{a_n}{a}}u(x)$. Then, $\widetilde{u}_n \in S(a_n)$. Similar to Eq (4.11), we have

$$I_\mu(\widetilde{u}_n) = I_\mu(u) + o_n(1).$$

It follows from Eq (4.12) that

$$m(a_n) \le I_\mu(\widetilde{u}_n) = I_\mu(u) + (I_\mu(\widetilde{u}_n) - I_\mu(u)) \le m(a) + \varepsilon + o_n(1).$$

Since $\varepsilon > 0$ is arbitrary, we get

$$\limsup_{n\to\infty} m(a_n) \le m(a).$$

Thus, we infer that $m(a_n) \to m(a)$ as $n \to \infty$.

Now, for any fixed $\mu > 0$ small enough, we define

$$a^* = \inf\{a : 0 < a < a_0(\mu), m(a) < 0\}.$$

By Lemma 4.1, $a^* \in [0, a_0(\mu))$ is well defined and satisfying

$$m(a) = 0 \text{ if } 0 < a \le a^*, \quad m(a) < 0 \text{ if } a^* < a < a_0. \tag{4.13}$$

Lemma 4.3. *Assume that (f_1)–(f_3) hold. For any $a \in (0, a_0)$, if (f_4) holds, then $a^* = 0$; if (f_5) holds, then $a^* > 0$.*

Proof. (i) Let $a \in (0, a_0)$, we choose $u \in S(a) \cap C_0^\infty(\mathbb{R}^N)$ and set

$$L = \left(\frac{2|\nabla u|_2^2}{\int_{\mathbb{R}^N} (I_\alpha * |u|^{\frac{N+2+\alpha}{N}})|u|^{\frac{N+2+\alpha}{N}} dx} \right)^{\frac{1}{2}} > 0.$$

By the assumption (f_4), there exists $\delta > 0$ such that

$$F(s) \ge L|s|^{\frac{N+2+\alpha}{N}} \quad \text{for all } |s| < \delta.$$

Since $|u^t|_\infty \leq \delta$ for sufficiently small $t > 0$, we have

$$\begin{aligned} I_\mu(u^t) &\leq \frac{1}{2}|\nabla u^t|_2^2 - \frac{1}{2}\int_{\mathbb{R}^N}(I_\alpha * F(u^t))F(u^t)dx \\ &\leq \frac{1}{2}t^2|\nabla u|_2^2 - \frac{L^2}{2}t^2\int_{\mathbb{R}^N}(I_\alpha * |u|^{\frac{N+2+\alpha}{N}})|u|^{\frac{N+2+\alpha}{N}}dx \\ &= -\frac{1}{2}t^2|\nabla u|_2^2 < 0. \end{aligned}$$

Thus, $m(a) \leq I_\mu(u^t) < 0$ for any $a \in (0, a_0)$. By the definition of a^*, we get $a^* = 0$.

(ii) By (f_2) and (f_5), there exists $C' > 0$ such that

$$|F(s)| \leq C'|s|^{\frac{N+2+\alpha}{N}} \quad \text{for all } s \in \mathbb{R}. \tag{4.14}$$

For $u \in S(a)$, by Eqs (2.1) and (4.14), we get

$$\int_{\mathbb{R}^N}(I_\alpha * F(u))F(u)dx \leq C''a^{\frac{2+\alpha}{N}}|\nabla u|_2^2.$$

It follows from Eq (2.2) that

$$I_\mu(u) \geq \frac{1}{2}|\nabla u|_2^2 - \frac{C''}{2}a^{\frac{2+\alpha}{N}}|\nabla u|_2^2 - \frac{\mu}{\overline{q}}C_{N,\overline{q}}^{\overline{q}}a^{\frac{2}{N}}|\nabla u|_2^2.$$

Taking $a > 0$ small enough such that $\frac{C''}{2}a^{\frac{2+\alpha}{N}} + \frac{\mu}{\overline{q}}C_{N,\overline{q}}^{\overline{q}}a^{\frac{2}{N}} \leq \frac{1}{2}$, we get $I_\mu(u) \geq 0$. This implies $m(a) \geq 0$ for a small $a > 0$. From Lemma 4.1 that $m(a) = 0$ for a small $a > 0$. Hence, we have $a^* > 0$.

Lemma 4.4. *Assume that (f_1)–(f_3) hold. Let $\mu > 0$ and $a \in (0, a_0)$. Let $\{u_n\} \subset S(a)$ be a minimizing sequence for $m(a)$. Then, one of the following holds:*
(i).

$$\limsup_{n\to\infty}\sup_{z\in\mathbb{R}^N}\int_{B_1(z)}|u_n|^2dx = 0.$$

(ii). Taking a subsequence if necessary, there exist $u \in S(a)$ and a family $\{y_n\} \subset \mathbb{R}^N$ such that $u_n(\cdot + y_n) \to u$ in $H^1(\mathbb{R}^N)$ as $n \to \infty$. Specifically, u is a global minimizer.

Proof. Suppose that $\{u_n\} \subset S(a)$ is a minimizing sequence which does not satisfy (i). Then, we have

$$0 < \lim_{n\to\infty}\sup_{z\in\mathbb{R}^N}\int_{B_1(z)}|u_n|^2dx \leq a.$$

Taking a subsequence if necessary, we may assume there exists a family $\{y_n\} \subset \mathbb{R}^N$ such that

$$0 < \lim_{n\to\infty}\int_{B_1(y_n)}|u_n|^2dx \leq a.$$

Let us consider $u_n(\cdot + y_n)$. Since $\{u_n\} \subset S(a)$ is a minimizing sequence of $m(a)$, by the proof of Lemma 4.1, $\{u_n\}$ is bounded in $H^1(\mathbb{R}^N)$, Hence, $\{u_n(\cdot + y_n)\}$ is bounded in $H^1(\mathbb{R}^N)$. Then, there exists $u \in H^1(\mathbb{R}^N)$ such that $u_n(\cdot + y_n) \rightharpoonup u$ in $H^1(\mathbb{R}^N)$, $u_n(\cdot + y_n) \to u$ in $L^p_{loc}(\mathbb{R}^N)$ for $2 \leq p < 2^*$, and $u_n(\cdot + y_n) \to u$ almost everywhere in $\mathbb{R}^N$. It follows that

$$\int_{B_1(0)} |u_n(x + y_n)|^2 dx > 0.$$

Then, by $u_n(\cdot + y_n) \to u$ in $L^2_{loc}(\mathbb{R}^N)$, we obtain that $u \neq 0$ and $|u|_2^2 > 0$. Set $v_n(x) = u_n(x + y_n) - u$. Then, $v_n \rightharpoonup 0$ in $H^1(\mathbb{R}^N)$, and we have

$$|u_n(\cdot + y_n)|_2^2 = |u_n|_2^2 = |v_n + u|_2^2 = |v_n|_2^2 + |u|_2^2 + o_n(1), \tag{4.15}$$

$$|\nabla u_n(\cdot + y_n)|_2^2 = |\nabla u_n|_2^2 = |\nabla(v_n + u)|_2^2 = |\nabla v_n|_2^2 + |\nabla u|_2^2 + o_n(1), \tag{4.16}$$

$$|u_n(\cdot + y_n)|_{\overline{q}}^{\overline{q}} = |u_n|_{\overline{q}}^{\overline{q}} = |v_n + u|_{\overline{q}}^{\overline{q}} = |v_n|_{\overline{q}}^{\overline{q}} + |u|_{\overline{q}}^{\overline{q}} + o_n(1). \tag{4.17}$$

Moreover, by Lemma 2.3, we have

$$\begin{aligned}\int_{\mathbb{R}^N} (I_\alpha * F(u_n))F(u_n)dx &= \int_{\mathbb{R}^N} (I_\alpha * F(v_n))F(v_n)dx \\ &\quad + \int_{\mathbb{R}^N} (I_\alpha * F(u))F(u)dx + o_n(1).\end{aligned} \tag{4.18}$$

Since $I_\mu(u_n) = I_\mu(u_n(\cdot + y_n)) = I_\mu(v_n + u)$, by Eqs (4.16)–(4.18), we obtain

$$I_\mu(u_n) = I_\mu(v_n) + I_\mu(u) + o_n(1). \tag{4.19}$$

Claim. $\lim_{n\to\infty} |v_n|_2^2 = 0$.

Denote $b = |u|_2^2 > 0$. By Eq (4.15), we see that $b \leq a$, and if we prove that $b = a$, then the claim holds. We suppose by contradiction that $b < a$. Since $v_n \in S(|v_n|_2^2)$, then

$$I_\mu(v_n) \geq m(|v_n|_2^2).$$

It follows from $I_\mu(u_n) \to m(a)$ and Eq (4.19) that

$$m(a) = I_\mu(v_n) + I_\mu(u) + o_n(1) \geq m(|v_n|_2^2) + I_\mu(u) + o_n(1).$$

By Lemma 4.2 (iii) and Eq (4.15), we obtain

$$m(a) \geq m(a - b) + I_\mu(u). \tag{4.20}$$

Note that $b = |u|_2^2$, we have $u \in S(b)$ and $I_\mu(u) \geq m(b)$. If $I_\mu(u) > m(b)$, then Lemma 4.2 (i) and Eq (4.20) imply

$$m(a) > m(a - b) + m(b) \geq m(a),$$

which is impossible. Hence, $I_\mu(u) = m(b)$, that is u is a minimizer of $m(b)$ on $S(b)$. It follows from Lemma 4.2 (i) and Eq (4.20) that

$$m(a) \geq m(a-b) + I_\mu(u) = m(a-b) + m(b) > m(a-b+b) = m(a),$$

which is a contradiction. Thus, the claim holds and Eq (4.15) implies $|u|_2^2 = a$.

Finally, we show that $\lim_{n\to\infty} |\nabla v_n|_2^2 = 0$. Since $u \in S(a)$, we have $I_\mu(u) \geq m(a)$. It follows from Eq (4.19) that

$$m(a) = \lim_{n\to\infty} I_\mu(v_n) + I_\mu(u) \geq \lim_{n\to\infty} I_\mu(v_n) + m(a),$$

this implies

$$\lim_{n\to\infty} I_\mu(v_n) \leq 0. \tag{4.21}$$

On the other hand, by Lemma 2.2, Eq (2.2), and $|v_n|_2^2 \to 0$, we infer that

$$\lim_{n\to\infty} \int_{\mathbb{R}^N} (I_\alpha * F(v_n))F(v_n)dx = 0, \quad \lim_{n\to\infty} \int_{\mathbb{R}^N} |v_n|^{\bar{q}} dx = 0.$$

In view of Eq (4.21), we get

$$\lim_{n\to\infty} \frac{1}{2}|\nabla v_n|_2^2 = \lim_{n\to\infty} I_\mu(v_n) \leq 0.$$

Then, we have $\lim_{n\to\infty} |\nabla v_n|_2^2 = 0$. It follows from $|v_n|_2^2 \to 0$ that $v_n \to 0$ in $H^1(\mathbb{R}^N)$. Hence, $u_n(\cdot + y_n) \to u$ in $H^1(\mathbb{R}^N)$ as $n \to \infty$.

Proof. [Proof of Theorem 1.2] (i) For the case $0 < a < a^*$, we assume by contradiction that there exists a global minimizer with respect to $m(a)$. By Eq (4.13), we know $m(a) = 0$. By Lemma 4.2 (i) with the strict inequality, we infer that

$$0 = m(a^*) < m(a^* - a) + m(a) = m(a) = 0,$$

which is a contradiction.

(ii) For the case $a^* < a < a_0$. By Eq (4.13), we have $m(a) < 0$. Let $\{u_n\} \subset S(a)$ satisfying $\lim_{n\to\infty} I_\mu(u_n) = m(a)$. It is easy to see that $\{u_n\}$ is bounded in $H^1(\mathbb{R}^N)$. Then, one of the two alternatives in Lemma 4.4 occurs. Let assume that (i) of Lemma 4.4 take place. The Lemma 1.1 of [38] implies $u_n \to 0$ in $L^t(\mathbb{R}^N)$ for $2 < t < 2^*$. Moreover, since $2 < \frac{2(N+2+\alpha)}{N+2} < 2^*$, by Lemma 2.2 we have

$$\lim_{n\to\infty} \int_{\mathbb{R}^N} (I_\alpha * F(u_n))F(u_n)dx = 0 \quad \text{and} \quad \lim_{n\to\infty} \int_{\mathbb{R}^N} |u_n|^{\bar{q}} dx = 0.$$

From this, we infer that

$$m(a) = \lim_{n\to\infty} I_\mu(u_n) = \lim_{n\to\infty} \frac{1}{2}|\nabla u_n|_2^2 \geq 0,$$

which contradicts to $m(a) < 0$. Thus, Lemma 4.4 (ii) holds, namely, there exist $u \in S(a)$ and a family $\{y_n\} \subset \mathbb{R}^N$ such that $u_n(\cdot + y_n) \to u$ in $H^1(\mathbb{R}^N)$ as $n \to \infty$, and u is a global minimizer to $m(a)$. Moreover, u is a ground state solution to Eq (1.1) for some $\lambda \in \mathbb{R}$. By Lemma 2.4, Eq (3.13), and $m(a) < 0$, we have $\lambda < 0$. Similar as the proof of Theorem 1.1, u has constant sign and is radially symmetric about a point in $\mathbb{R}^N$.

Proof. [Proof of Theorem 1.4] By (f_6) and (iii) of Theorem 1.2, we have $a^* > 0$. Let $a_n = a^* + \frac{l}{n}$ for any $n \in \mathbb{N}$, where $0 < l < a_0 - a^*$. By Eq (4.13) and Theorem 1.2 (ii), for every n, there exists a global minimizer $u_n \in S(a_n)$ such that

$$I_\mu(u_n) = m(a_n) < 0. \tag{4.22}$$

Then, $\{u_n\}$ is bounded in $H^1(\mathbb{R}^N)$. Since $a_n \to a^*$, we have $\lim_{n\to\infty} |u_n|_2^2 = a^*$. Set $v_n = \frac{\sqrt{a^*}}{|u_n|_2} \cdot u_n$. Clearly, $v_n \in S(a^*)$ and $\frac{\sqrt{a^*}}{|u_n|_2} \to 1$ as $n \to \infty$. By the similar proof of Lemma 4.2 (iii), we obtain

$$\lim_{n\to\infty} I_\mu(v_n) = \lim_{n\to\infty} I_\mu(u_n) = \lim_{n\to\infty} m(a_n) = m(a^*) = 0.$$

Thus, $\{v_n\}$ is a minimizing sequence with respect to $m(a^*)$.

Next, we prove that $\{v_n\}$ is non-vanishing, that is

$$\delta := \lim_{n\to\infty} \sup_{z\in\mathbb{R}^N} \int_{B_1(z)} |v_n|^2 dx > 0.$$

Otherwise, $\delta = 0$. By the definition of v_n and $\frac{\sqrt{a^*}}{|u_n|_2} \to 1$, we get

$$\lim_{n\to\infty} \sup_{z\in\mathbb{R}^N} \int_{B(z,1)} |u_n|^2 dx = 0. \tag{4.23}$$

By (ii) of Theorem 1.2, u_n satisfies Eq (1.1) and we may assume that u_n is radially symmetric with respect to the origin and decreasing for any $n \in \mathbb{N}$. Using the elliptic regularity theory, we see that $\{u_n\}$ is bounded in $C^1(B(0,1))$. Thus, by Eq (4.23) we get

$$u_n(0) = |u_n|_\infty \to 0 \ \text{ as } \ n \to \infty.$$

It follows from (f_6) that for any $\varepsilon > 0$ and sufficiently large n,

$$\begin{aligned}\int_{\mathbb{R}^N} (I_\alpha * F(u_n))F(u_n)dx &\le C\varepsilon^2 a_n^{\frac{2+\alpha}{N}} |\nabla u_n|_2^2 \\ &\le C\varepsilon^2 (a^* + l)^{\frac{2+\alpha}{N}} |\nabla u_n|_2^2.\end{aligned}$$

Choosing $\varepsilon > 0$ small enough such that $C\varepsilon^2(a^* + l)^{\frac{2+\alpha}{N}} \le \frac{1}{2} - \frac{\mu}{q} C_{N,\bar{q}}^{\bar{q}} (a^* + l)^{\frac{2}{N}}$, then by Eq (2.2) we have

$$I_\mu(u_n) \ge \frac{1}{2}\left(\frac{1}{2} - \frac{\mu}{q} C_{N,\bar{q}}^{\bar{q}} (a^* + l)^{\frac{2}{N}}\right) |\nabla u_n|_2^2 \ge 0,$$

which contradicts Eq (4.22), and hence $\delta > 0$.

Then Lemma 4.4 (ii) holds, that is, up to a subsequence, there exist $v \in S(a^*)$ and a family $\{y_n\} \subset \mathbb{R}^N$ such that $v_n(\cdot + y_n) \to v$ in $H^1(\mathbb{R}^N)$ as $n \to \infty$. Then, v is a global minimizer to $m(a^*)$. Moreover, v is a ground state solution to (1.1) for some $\lambda \in \mathbb{R}$. By Lemma 2.4 and Eq (3.13), we have

$$\begin{aligned}\lambda a^* &= 2m(a^*) - \frac{2}{N}|\nabla v|_2^2 - \frac{\alpha}{N}\int_{\mathbb{R}^N} (I_\alpha * F(v))F(v)dx \\ &= -\frac{2}{N}|\nabla v|_2^2 - \frac{\alpha}{N}\int_{\mathbb{R}^N} (I_\alpha * F(v))F(v)dx < 0,\end{aligned}$$

which implies $\lambda < 0$. By the same arguments of Theorem 1.1, we obtain the symmetric of v.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare there is no conflicts of interest.

References

1. S. Pekar, *Untersuchung über die Elektronentheorie der Kristalle*, Akademie Verlag, Berlin, 1954. https://doi.org/10.1515/9783112649305
2. E. H. Lieb, Existence and Uniqueness of the Minimizing Solution of Choquard's Nonlinear Equation, in *Inequalities* (eds. M, Loss and M. B. Ruskai), Springer, 2002. https://doi.org/10.1007/978-3-642-55925-9-37
3. K. R. W. Jones, Newtonian Quantum Gravity, *Aust. J. Phys.*, **48** (1995), 1055–1082. https://doi.org/10.1071/ph951055
4. R. Penrose, On gravity's role in quantum state reduction, *Gen. Relativ. Gravitatation*, **28** (1996), 581–600. https://doi.org/10.1007/BF02105068
5. P. L. Lions, The Choquard equation and related questions, *Nonlinear Anal.*, **4** (1980), 1063–1072. https://doi.org/10.1016/0362-546X(80)90016-4
6. V. Moroz, J. V. Schaftingen. Groundstates of nonlinear Choquard equations: Existence, qualitative properties and decay asymptotics, *J. Funct. Anal.*, **265** (2013), 153–184. https://doi.org/10.1016/j.jfa.2013.04.007
7. M. Ghimenti, J. V. Schaftingen, Nodal solutions for the Choquard equation, *J. Funct. Anal.*, **271** (2016), 107–135. https://doi.org/10.1016/j.jfa.2016.04.019
8. J. Xia, Z. Q. Wang, Saddle solutions for the Choquard equation, *Calc. Var. Partial Differ. Equations*, **58** (2019), 1–30. https://doi.org/10.1007/s00526-019-1546-8
9. V. Moroz, J. V. Schaftingen. Existence of Groundstates for a class of nonlinear Choquard equations, *Trans. Am. Math. Soc.*, **367** (2015), 6557–6579. https://doi.org/10.1515/ans-2016-0038
10. L. Battaglia, J. J. V. Schaftingen, Existence of groundstates for a class of nonlinear Choquard equations in the plane, *Adv. Nonlinear Stud.*, **17** (2017), 581–594. https://doi.org/10.1515/ans-2016-0038
11. J. Chen, B. Guo, Blow up solutions for onr class of system of Pekar-Choquard type nonlinear Schrödinger equation, *Appl. Math. Comput.*, **186** (2007), 83–92. https://doi.org/10.1016/j.amc.2006.07.089

12. J. Seok, Nonlinear Choquard equations involving a critical local term, *Appl. Math. Lett.*, **63** (2017), 77–87. https://doi.org/10.1016/j.aml.2016.07.027

13. J. V. Schaftingen, J. Xia, Groundstates for a local nonlinear perturbation of the Choquard equations with lower critical exponent, *J. Math. Anal. Appl.*, **464** (2018), 1184–1202. https://doi.org/10.1016/j.jmaa.2018.04.047

14. L. Jeanjean, Existence of solutions with prescribed norm for semilinear equations, *Nonlinear Anal.*, **28** (1997), 1633–1659. https://doi.org/10.1016/S0362-546X(96)00021-1

15. T. Bartsch, S. D. Valeriola, Normalized solutions of nonlinear Schrödinger equations, *Arch. Math.*, **100** (2013), 75–83. https://doi.org/10.1007/s00013-012-0468-x

16. M. Shibata, Stable standing waves of nonlinear Schrödinger equations with a general nonlinear term, *Manuscr. Math.*, **143** (2014), 221–237. https://doi.org/10.1007/s00229-013-0627-9

17. L. Jeanjean, S. S Lu, Nonradial normalized solutions for nonlinear scalar field equations, *Nonlinearity*, **32** (2019), 4942–4966. https://doi.org/10.1088/1361-6544/ab435e

18. S. Deng, Q. Wu, Existence of normalized solutions for the Schrödinger equation, *Commun. Anal. Mech.*, **15** (2023), 575–585. https://doi.org/10.3934/cam.2023028

19. L. Jeanjean, J. Jendrej, T. T. Le, N. Visciglia, Orbital stability of ground states for a Sobolev critical Schrödinger equation, *J. Math. Pures Appl.*, **164** (2022), 158–179. https://doi.org/10.1016/j.matpur.2022.06.005

20. W. Lv, Ground states of a Kirchhoff equation with the potential on the lattice graphs, *Commun. Anal. Mech.*, **15** (2023), 792–810. https://doi.org/10.3934/cam.2023038

21. N. Soave, Normalized ground states for the NLS equation with combined nonlinearities, *J. Differ. Equations*, **269** (2020), 6941–6987. https://doi.org/10.1016/j.jde.2020.05.016

22. C. O. Alves, C. Ji, O. H. Miyagaki, Normalized solutions for a Schrödinger equation with critical growth in $\mathbb{R}^N$, *Calc. Var. Partial Differ. Equations*, **61** (2022), 18. https://doi.org/10.1007/s00526-021-02123-1

23. D. Bonheure, J. B. Casteras, T. Gou, L. Jeanjean, Normalized solutions to the mixed dispersion nonlinear Schrödinger equation in the mass critical and supercritical regime, *Trans. Am. Math. Soc.*, **372** (2019), 2167–2212. https://doi.org/10.1090/tran/7769

24. H. Xu, Existence and blow-up of solutions for finitely degenerate semilinear parabolic equations with singular potentials, *Commun. Anal. Mech.*, **15** (2023), 132–161. https://doi.org/10.3934/cam.2023008

25. G. Li, H. Ye, The existence of positive solutions with prescribed L^2-norm for nonlinear Choquard equations, *J. Math. Phys.*, **55** (2014), 121501. https://doi.org/10.1063/1.4902386

26. S. Yuan, S. Chen, X. Tang, Normalized solutions for Choquard equations with general nonlinearities, *Electron. Res. Arch.*, **28** (2020), 291–309. https://doi.org/10.3934/era.2020017

27. T. Bartsch, Y. Liu, Z. Liu, Normalized solutions for a class of nonlinear Choquard equations, *SN Partial Differ. Equations Appl.*, **1** (2020), 1–25. https://doi.org/10.1007/s42985-020-00036-w

28. H. Ye, Mass minimizers and concentration for nonlinear Choquard equations in $\mathbb{R}^N$, *Topol. Methods Nonlinear Anal.*, **48** (2016), 393–417. https://doi.org/10.12775/TMNA.2016.066

29. S. Yao, J. Sun, T-F Wu, Normalized solutions for the Schrödinger equations with combined hartree type and power nonlinearities, preprint, arXiv:2102.10268. https://doi.org/10.48550/arXiv.2102.10268

30. X. Li, Standing waves to upper critical Choquard equation with a local perturbation: Multiplicity, qualitative properties and stability, *Adv. Nonlinear Anal.*, **11** (2022), 1134–1164. https://doi.org/10.1515/anona-2022-0230

31. Y. Li, D. Zhao, Q. Wang, Concentration behavior of nonlinear Hartree-type equation with almost mass critical exponent, *Z. Angew. Math. Phys.*, **70** (2019), 1–17. https://doi.org/10.1007/s00033-019-1172-5

32. J. Wang, W. Yang, Normalized solutions and asymptotical behavior of minimizer for the coupled Hartree equations, *J. Differ. Equations*, **265** (2018), 501–544. https://doi.org/10.1016/j.jde.2018.03.003

33. S. Yao, H. Chen, V. D. Rădulescu, J. Sun, Normalized solutions for lower critical Choquard equations with critical sobolev perturbation, *SIAM J. Math. Anal.*, **54** (2022), 3696–3723. https://doi.org/10.1137/21M1463136

34. J. Bellazzini, L. Jeanjean, On diplar quantum gases in the unstable regime, *SIAM J. Math. Anal.*, **48** (2016), 2028–2058. https://doi.org/10.1137/15M1015959

35. P.L. Lions, The concentration-compactness principle in the calculus of variations: the locally compact case, Part I, *Ann. inst. H. Poincaré Anal. Non Linéaire*, **2** (1984), 109–145. https://doi.org/10.1016/S0294-1449(16)30428-0

36. E. H. Lieb, M. Loss, *Analysis*, 2^{nd} edition, American Mathematical Society, Providence, 2001.

37. D. Cassani, J. Zhang, Choquard-type equations with Hardy-Littlewood-Sobolev upper-critical growth, *Adv. Nonlinear Anal.*, **8** (2019), 1184–1212. https://doi.org/10.1515/anona-2018-0019

38. P.L. Lions, The concentration-compactness principle in the calculus of variations: the locally compact case, Part II, *Ann. inst. H. Poincaré Anal. Non Linéaire*, **4** (1984), 223–283. https://doi.org/10.1016/S0294-1449(16)30422-X