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Research article

The explicit formula and parity for some generalized Euler functions

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Abstract: Utilizing elementary methods and techniques, the explicit formula for the generalized Euler function $\varphi_e(n)(e = 8, 12)$ has been developed. Additionally, a sufficient and necessary condition for $\varphi_8(n)$ or $\varphi_{12}(n)$ to be odd has been obtained, respectively.

Keywords: generalized Euler function; explicit formula; Möbius function; parity
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1. Introduction

Let $\mathbb{Z}$ and $\mathbb{P}$ denote the set of integers and primes, respectively. In order to generalize Lehmer's congruence (see [4] or [7]) for modulo prime squares to be modulo integer squares, Cai et al. [1] defined the following generalized Euler function for a positive integer n related to a given positive integer e:

$$\varphi_e(n) = \sum_{i=1,\, \gcd(i,n)=1}^{[\frac{n}{e}]} 1,$$

where $[x]$ is the greatest integer not more than x, i.e., $\varphi_e(n)$ is the number of positive integers not greater than $[\frac{n}{e}]$ and prime to n. It is clear that $\varphi_1(n) = \varphi(n)$ is just the Euler function of n, $\varphi_2(n) = \frac{1}{2}\varphi(n)$, and

$$\varphi_e(n) = \sum_{d|n} \mu\Big(\frac{n}{d}\Big)\Big[\frac{d}{e}\Big], \tag{1.1}$$

where $\mu(n)$ is the Möbius function. There are some good results for the generalized Euler function and its applications, especially those concerning $\varphi_e(n)\,(e = 2, 3, 4, 6)$, which can be seen in [3].

In 2013, Cai et al [2] gave the explicit formula for $\varphi_3(n)$ and obtained a criterion regarding the parity for $\varphi_2(n)$ or $\varphi_3(n)$, respectively. In [8], the authors derived the explicit formulae for $\varphi_4(n)$ and $\varphi_6(n)$, and then they obtained some sufficient and necessary conditions for the case that $\varphi_e(n)$ or $\varphi_e(n+1)$ is odd or even, respectively.

Recently, Wang and Liao [9] gave the formula for $\varphi_5(n)$ in some special cases and then obtained some sufficient conditions for the case that $\varphi_5(n)$ is even. Liao and Luo [5] gave a computing formula for $\varphi_e(n)$ ($e = p, p^2, pq$), where p and q are distinct primes, and n satisfies some certain conditions. Liao [6] obtained the explicit formula for a special class of generalized Euler functions. However, the explicit formula for $\varphi_e(n)$ ($e \neq 3, 4, 6$) was not obtained in the general case.

In this paper, utilizing the methods and techniques given in [2, 5, 8], we study the explicit formula and the parity for $\varphi_e(n)$ ($e = 8, 12$), obtain the corresponding computing formula, and then give a sufficient and necessary condition for the case that $\varphi_e(n)$ ($e = 8, 12$) is odd or even, respectively.

For convenience, throughout the paper, we denote $\Omega(n)$ and $\omega(n)$ to be the number of prime factors and distinct prime factors of a positive integer n, respectively. And for k primes $p_1, \ldots, p_k$, set $\mathbb{P}_k = \{p_1, \ldots, p_k\}$,

$$R_{\mathbb{P}_k} = \{r_i \mid p_i \equiv r_i (\bmod 8), 0 \le r_i \le 7, p_i \in \mathbb{P}_k, 1 \le i \le k\},$$

and

$$R'_{\mathbb{P}_k} = \{r_i \mid p_i \equiv r_i (\bmod 12), 0 \le r_i \le 11, p_i \in \mathbb{P}_k, 1 \le i \le k\}.$$

We have organized this paper as follows. In Section 2, we obtain the obvious formulas for $[\frac{m}{8}]$ and $[\frac{m}{12}]$ based on Jacobi symbol, and some important lemmas are given. In Sections 3 and 4, according to (1.1), and by using the property of the Möbius function $\mu(n)$, we derive the expressions for $\varphi_e(8)$ and $\varphi_e(12)$. In Section 5, we give the parities of $\varphi_8(n)$ and $\varphi_{12}(n)$, respectively. In the last section, we summarize the main advantage of the proposed method, and propose a further problem to be studied.

2. Preliminaries

In this section, we first present Lemmas 2.1 and 2.2, which are necessary for the derivations of both $[\frac{m}{8}]$ and $[\frac{m}{12}]$.

Lemma 2.1. For any odd positive integer m, we have

$$\left[\frac{m}{8}\right] = \frac{1}{8}\Big(m - 4 + 2\Big(\frac{-2}{m}\Big) + \Big(\frac{-1}{m}\Big)\Big). \tag{2.1}$$

Furthermore, if $\gcd(m, 6) = 1$, then we have

$$\left[\frac{m}{12}\right] = \frac{1}{12}\Big(m - 6 + 3\Big(\frac{-1}{m}\Big) + 2\Big(\frac{-3}{m}\Big)\Big), \tag{2.2}$$

where $(\frac{a}{m})$ is the Jacobi symbol.

Proof. For any odd positive integer m, by properties of the Jacobi symbol, we have

$$\Big(\frac{-1}{m}\Big) = \begin{cases} 1, & m \equiv 1 (\bmod 4), \\ -1, & m \equiv 3 (\bmod 4), \end{cases} \quad \text{and} \quad \Big(\frac{-2}{m}\Big) = \begin{cases} 1, & m \equiv 1, 3 (\bmod 8), \\ -1, & m \equiv 5, 7 (\bmod 8). \end{cases}$$

Thus from $m \equiv 1 (\bmod 8)$, we can get that $\frac{1}{8}(m - 4 + 2(\frac{-2}{m}) + (\frac{-1}{m})) = \frac{1}{8}(m - 1)$ and $[\frac{m}{8}] = \frac{1}{8}(m - 1)$, namely, (2.1) is true. Similarly, if $m \equiv 3, 5, 7$ (mod 8), by direct computation, (2.1) holds.

Furthermore, if $\gcd(m, 6) = 1$, then by the properties for the Jacobi symbol and the quadratic reciprocity law, we have

$$\Big(\frac{-3}{m}\Big) = \Big(\frac{-1}{m}\Big)\Big(\frac{m}{3}\Big)(-1)^{\frac{1}{4}(3-1)(m-1)} = \begin{cases} 1, & m \equiv 1, 7 (\bmod 12), \\ -1, & m \equiv 5, 11 (\bmod 12). \end{cases}$$

Thus by $m \equiv 1 \ (\bmod 12)$, we have that $\frac{1}{12}(m - 6 + 3(\frac{-1}{m}) + 2(\frac{-3}{m})) = \frac{1}{12}(m-1) = [\frac{m}{12}]$, i.e., (2.2) is true. Similarly, if $m \equiv 5, 7, 11 \ (\bmod 12)$, one can get (2.2) also.

This completes the proof of Lemma 2.1.

Now, we give a property for the Möbius function, which unifies the cases of Lemma 1.5 in [2] and Lemmas 1.4 and 1.5 in [8].

Lemma 2.2. Let a be a nonzero integer, $p_1, \ldots, p_k$ be distinct odd primes, and $\alpha_1, \ldots, \alpha_k$ be positive integers. Suppose that $n = \prod_{i=1}^{k} p_i^{\alpha_i}$ and $\gcd(p_i, a) = 1 \ (1 \le i \le k)$; then,

$$\sum_{d|n} \mu\Big(\frac{n}{d}\Big)\Big(\frac{a}{d}\Big) = \prod_{i=1}^{k} \Big(\Big(\frac{a}{p_i}\Big)^{\alpha_i} - \Big(\frac{a}{p_i}\Big)^{\alpha_i - 1}\Big). \tag{2.3}$$

Proof. For a given integer x, set $f_x(m) = \sum_{d|m} \mu(\frac{m}{d})(\frac{x}{d})$.

First, if $m = p^{\alpha}$, where p is an odd prime and α is a positive integer, then, by the definition of the Möbius function, we have

$$f_a(m) = \mu(1)\Big(\frac{a}{p^{\alpha}}\Big) + \mu(p)\Big(\frac{a}{p^{\alpha-1}}\Big) = \Big(\frac{a}{p}\Big)^{\alpha} - \Big(\frac{a}{p}\Big)^{\alpha-1}.$$

Second, if $m = m_1 p^{\alpha}$, where α is a positive integer, p is an odd prime with $\gcd(m_1, p) = 1$, and m_1 is an odd positive integer, then we have

$$\begin{aligned} f_a(m) &= \sum_{d|m_1} \mu\Big(\frac{m_1}{d}\Big)\Big(\frac{a}{dp^{\alpha}}\Big) + \sum_{d|m_1} \mu(p)\mu\Big(\frac{m_1}{d}\Big)\Big(\frac{a}{dp^{\alpha-1}}\Big) \\ &= \Big(\frac{a}{p}\Big)^{\alpha} \sum_{d|m_1} \mu\Big(\frac{m_1}{d}\Big)\Big(\frac{a}{d}\Big) - \Big(\frac{a}{p}\Big)^{\alpha-1} \sum_{d|m_1} \mu\Big(\frac{m_1}{d}\Big)\Big(\frac{a}{d}\Big). \\ &= \Big(\Big(\frac{a}{p}\Big)^{\alpha} - \Big(\frac{a}{p}\Big)^{\alpha-1}\Big) f_a(m_1). \end{aligned}$$

This means that $f_a(m)$ is a multiplicative function. Now denote $p^{\alpha} \| n$ to be the case for both $p^{\alpha} \mid n$ and $p^{\alpha+1} \nmid n$; then, we can get

$$f_a(n) = \prod_{p^{\alpha} \| n} \Big(\Big(\frac{a}{p}\Big)^{\alpha} - \Big(\frac{a}{p}\Big)^{\alpha-1}\Big) = \prod_{i=1}^{k} \Big(\Big(\frac{a}{p_i}\Big)^{\alpha_i} - \Big(\frac{a}{p_i}\Big)^{\alpha_i-1}\Big).$$

This completes the proof of Lemma 2.2.

The following lemmas are necessary for proving our main results.

Lemma 2.3. *[2]* Let $p_1, \ldots, p_k$ be distinct primes and $\alpha, \alpha_1, \ldots, \alpha_k$ be non-negative integers. If $n = 3^{\alpha} \prod_{i=1}^{k} p_i^{\alpha_i} > 3$ and $\gcd(p_i, 3) = 1 \ (1 \le i \le k)$, then

$$\varphi_3(n) = \begin{cases} \frac{1}{3}\varphi(n) + \frac{1}{3}(-1)^{\Omega(n)} 2^{\omega(n)-\alpha-1}, & \text{if } \alpha = 0 \text{ or } 1,\ p_i \equiv 2 (\bmod 3), \\ \frac{1}{3}\varphi(n), & \text{otherwise.} \end{cases}$$

Lemma 2.4. *[8]* Let $p_1,\ldots,p_k$ be distinct odd primes and $\alpha,\alpha_1,\ldots,\alpha_k$ be non-negative integers. If $n = 2^{\alpha}\prod_{i=1}^{k} p_i^{\alpha_i} > 4$, then

$$\varphi_4(n) = \begin{cases} \frac{1}{4}\,\varphi(n) + \frac{1}{4}\,(-1)^{\Omega(n)} 2^{\omega(n)-\alpha}, & \text{if } \alpha = 0 \text{ or } 1,\ p_i \equiv 3 (\operatorname{mod} 4), \\ \frac{1}{4}\,\varphi(n), & \text{otherwise.} \end{cases}$$

Lemma 2.5. *[8]* Let $p_1,\ldots,p_k$ be distinct primes and $\alpha,\beta,\alpha_1,\ldots,\alpha_k$ be non-negative integers. If $n = 2^{\alpha}\,3^{\beta}\prod_{i=1}^{k} p_i^{\alpha_i} > 6$ and $\gcd(p_i, 6) = 1\ (1 \le i \le k)$, then

$$\varphi_6(n) = \begin{cases} \frac{1}{6}\,\varphi(n) + \frac{1}{6}\,(-1)^{\Omega(n)}\, 2^{\,\omega(n)+1-\beta}, & \text{if } \alpha = 0 \text{ and } \beta = 0 \text{ or } 1,\ p_i \equiv 5 (\operatorname{mod} 6), \\ \frac{1}{6}\,\varphi(n) + \frac{1}{6}\,(-1)^{\Omega(n)}\, 2^{\,\omega(n)-1-\beta}, & \text{if } \alpha = 1 \text{ and } \beta = 0 \text{ or } 1,\ p_i \equiv 5 (\operatorname{mod} 6), \\ \frac{1}{6}\,\varphi(n) - \frac{1}{6}\,(-1)^{\Omega(n)}\, 2^{\,\omega(n)-\beta}, & \text{if } \alpha \ge 2 \text{ and } \beta = 0 \text{ or } 1,\ p_i \equiv 5 (\operatorname{mod} 6), \\ \frac{1}{6}\,\varphi(n), & \text{otherwise.} \end{cases}$$

Lemma 2.6. *[6]* Let $p_1,\ldots,p_k$ be distinct primes and $\alpha_1,\ldots,\alpha_k$ be positive integers. If $n = \prod_{i=1}^{k} p_i^{\alpha_i}$ and $e = \prod_{i=1}^{k} p_i^{\beta_i}$ with $0 \le \beta_i \le \alpha_i - 1\ (1 \le i \le k)$, then

$$\varphi_e(n) = \frac{1}{e}\,\varphi(n). \tag{2.4}$$

3. The explicit formula for $\varphi_8(n)$

First, for a fixed positive integer α and $n = 2^{\alpha}$, by Lemma 2.6 we can obtain the following:

$$\varphi_8(2^{\alpha}) = \begin{cases} 0, & \text{if } \alpha = 1, 2, \\ 1, & \text{if } \alpha = 3, \\ 2^{\alpha-4}, & \text{if } \alpha \ge 4. \end{cases} \tag{3.1}$$

Next, we consider the case that $n = 2^{\alpha} n_1$, where $n_1 > 1$ is an odd integer. We have the following theorem.

Theorem 3.1. Suppose that α is a non-negative integer, $p_1,\ldots,p_k$ are distinct odd primes, and $n = 2^{\alpha}\prod_{i=1}^{k} p_i^{\alpha_i} > 8$. Then we have the following:

$$\varphi_8(n) = \begin{cases} \frac{1}{8}\,\varphi(n) + \frac{1}{4}\,(-1)^{\Omega(n)}\, 2^{\,\omega(n)-\alpha}, & \\ \qquad \text{if } \alpha = 0, 1, \text{ and } R_{\mathbb{P}_k} = \{5, 7\}, \{5\}; & \\ \frac{1}{8}\,\varphi(n) + \frac{1}{8}\,(-1)^{\Omega(n)-[\frac{\alpha+1}{2}]}\, 2^{\,\omega(n)-\frac{1}{2}(1-(-1)^{\alpha})}, & \\ \qquad \text{if } \alpha = 0, 1, 2, \text{ and } R_{\mathbb{P}_k} = \{3, 7\}, \{3\}; & \\ \frac{1}{8}\,\varphi(n) + \frac{1}{8}\,(-1)^{\Omega(n)-[\frac{\alpha}{2}]}\, 2^{\,\omega(n)-\frac{1}{2}(1-(-1)^{\alpha})} & \\ \qquad + \frac{1-[\frac{\alpha+1}{2}]}{4}\,(-1)^{\Omega(n)}\, 2^{\omega(n)}, & \\ \qquad \text{if } \alpha = 0, 1, 2, \text{ and } R_{\mathbb{P}_k} = \{7\}; & \\ \frac{1}{8}\,\varphi(n), & \text{otherwise.} \end{cases} \tag{3.2}$$

Proof. For $n = 2^{\alpha}\prod_{i=1}^{k} p_i^{\alpha_i} > 8$, set $n_1 = \prod_{i=1}^{k} p_i^{\alpha_i}$; then, $\gcd(n_1, 2) = 1$. There are 4 cases as follows.

Case 1. $\alpha = 0$, i.e., $n = n_1 > 8$. By (1.1), (2.1) and Lemmas 2.1 and 2.2, we have

$$\begin{aligned}
\varphi_8(n) &= \sum_{d\,|\,n_1} \mu\Big(\frac{n_1}{d}\Big)\Big[\frac{d}{8}\Big] = \frac{1}{8}\sum_{d\,|\,n_1} \mu\Big(\frac{n_1}{d}\Big)\Big(d - 4 + 2\Big(\frac{-2}{d}\Big) + \Big(\frac{-1}{d}\Big)\Big) \\
&= \frac{1}{8}\sum_{d\,|\,n_1} \mu\Big(\frac{n_1}{d}\Big)d - \frac{1}{2}\sum_{d\,|\,n_1} \mu\Big(\frac{n_1}{d}\Big) + \frac{1}{4}\sum_{d\,|\,n_1} \mu\Big(\frac{n_1}{d}\Big)\Big(\frac{-2}{d}\Big) \\
&\quad + \frac{1}{8}\sum_{d\,|\,n_1} \mu\Big(\frac{n_1}{d}\Big)\Big(\frac{-1}{d}\Big) \\
&= \frac{1}{8}\,\varphi(n_1) + \frac{1}{4}\prod_{i=1}^{k}\Big(\Big(\frac{-2}{p_i}\Big)^{\alpha_i} - \Big(\frac{-2}{p_i}\Big)^{\alpha_i - 1}\Big) \\
&\quad + \frac{1}{8}\prod_{i=1}^{k}\Big(\Big(\frac{-1}{p_i}\Big)^{\alpha_i} - \Big(\frac{-1}{p_i}\Big)^{\alpha_i - 1}\Big). \qquad (3.3)
\end{aligned}$$

If $1 \in R_{\mathbb{P}_k}$, i.e., there exists an $i\,(1 \le i \le k)$ such that $p_i \equiv 1 \pmod 8$, then $(\frac{-2}{p_i}) = (\frac{-1}{p_i}) = 1$. Now by (3.3) we have

$$\varphi_8(n) = \frac{1}{8}\,\varphi(n_1) = \frac{1}{8}\,\varphi(n). \qquad (3.4)$$

If $\{3, 5\} \subseteq R_{\mathbb{P}_k}$, i.e., there exist $i \neq j$ such that $p_i \equiv 3 \pmod 8$ and $p_j \equiv 5 \pmod 8$, which means that $(\frac{-2}{p_i}) = (\frac{-1}{p_j}) = 1$, then, by (3.3) we also have

$$\varphi_8(n) = \frac{1}{8}\,\varphi(n_1) = \frac{1}{8}\,\varphi(n).$$

If $R_{\mathbb{P}_k} = \{5, 7\}$ or $\{5\}$, i.e., for any $p \in \mathbb{P}_k$, we have that $p \equiv 5, 7 \pmod 8$ or $p \equiv 5 \pmod 8$, respectively. This means that there exists a prime p such that $(\frac{-2}{p}) = -1$ and $(\frac{-1}{p}) = 1$. Thus by (3.3) we can obtain

$$\varphi_8(n) = \frac{1}{8}\,\varphi(n_1) + \frac{1}{4}\prod_{i=1}^{k}\Big(2 \cdot (-1)^{\alpha_i}\Big) = \frac{1}{8}\,\varphi(n) + \frac{1}{4}\,(-1)^{\Omega(n)}\,2^{\,\omega(n)}. \qquad (3.5)$$

If $R_{\mathbb{P}_k} = \{3, 7\}$ or $\{3\}$, i.e., for any $p \in \mathbb{P}_k$, $p \equiv 3, 7 \pmod 8$ or $p \equiv 3 \pmod 8$, respectively. This implies that for any $p \in \mathbb{P}_k$, $(\frac{-1}{p}) = -1$, and there exists a prime $p' \in \mathbb{P}_k$ such that $p' \equiv 3 \pmod 8$; then, $(\frac{-2}{p'}) = 1$. Thus by (3.3) we have

$$\varphi_8(n) = \frac{1}{8}\,\varphi(n_1) + \frac{1}{8}\prod_{i=1}^{k}\Big(2 \cdot (-1)^{\alpha_i}\Big) = \frac{1}{8}\,\varphi(n) + \frac{1}{8}\,(-1)^{\Omega(n)}\,2^{\,\omega(n)}. \qquad (3.6)$$

If $R_{\mathbb{P}_k} = \{7\}$, i.e., for any $p \in \mathbb{P}_k$, $p \equiv 7 \pmod 8$, then $(\frac{-2}{p}) = (\frac{-1}{p}) = -1$. Thus by (3.3) we have

$$\varphi_8(n) = \frac{1}{8}\,\varphi(n_1) + \frac{3}{8}\prod_{i=1}^{k}\Big(2 \cdot (-1)^{\alpha_i}\Big) = \frac{1}{8}\,\varphi(n) + \frac{3}{8}\,(-1)^{\Omega(n)}\,2^{\,\omega(n)}. \qquad (3.7)$$

Now from (3.5)–(3.7) we know that Theorem 3.1 is true.

Case 2. $\alpha = 1$, i.e., $n = 2n_1 > 8$. Then from the definition we have

$$\begin{aligned} \varphi_8(n) &= \sum_{d|n_1} \mu\Big(\frac{2n_1}{d}\Big)\Big[\frac{d}{8}\Big] + \sum_{d|n_1} \mu\Big(\frac{2n_1}{2d}\Big)\Big[\frac{2d}{8}\Big] \\ &= -\varphi_8(n_1) + \varphi_4(n_1). \end{aligned}$$

Now by Lemma 2.4 and the proof for Case 1, we can get the following:

$$\varphi_8(n) = \begin{cases} \frac{1}{8}\,\varphi(n) + \frac{1}{4}\,(-1)^{\Omega(n)}\,2^{\,\omega(n)-1}, & \text{if } R_{\mathbb{P}_k} = \{5,7\}, \{5\}; \\ \frac{1}{8}\,\varphi(n) + \frac{1}{8}\,(-1)^{\Omega(n)-1}\,2^{\,\omega(n)-1}, & \text{if } R_{\mathbb{P}_k} = \{3,7\}, \{3\}; \\ \frac{1}{8}\,\varphi(n) + \frac{1}{8}\,(-1)^{\Omega(n)}\,2^{\,\omega(n)-1}, & \text{if } R_{\mathbb{P}_k} = \{7\}; \\ \frac{1}{8}\,\varphi(n), & \text{otherwise.} \end{cases} \tag{3.8}$$

This means that Theorem 3.1 is true in this case.

Case 3. $\alpha = 2$, i.e., $n = 4n_1 > 8$. Then from the definition we have

$$\begin{aligned} \varphi_8(n) &= \sum_{d\,|\,n_1} \mu\Big(\frac{4n_1}{d}\Big)\Big[\frac{d}{8}\Big] + \sum_{d\,|\,n_1} \mu\Big(\frac{4n_1}{2d}\Big)\Big[\frac{2d}{8}\Big] + \sum_{d\,|\,n_1} \mu\Big(\frac{4n_1}{4d}\Big)\Big[\frac{4d}{8}\Big] \\ &= \sum_{d\,|\,n_1} \mu\Big(\frac{2n_1}{d}\Big)\Big[\frac{d}{4}\Big] + \sum_{d\,|\,n_1} \mu\Big(\frac{n_1}{d}\Big)\Big[\frac{d}{2}\Big] \\ &= \varphi_2(n_1) - \varphi_4(n_1) = \frac{1}{2}\varphi(n_1) - \varphi_4(n_1). \end{aligned}$$

Now from Lemma 2.4 and the proof for Case 1, we can also get the following:

$$\varphi_8(n) = \begin{cases} \frac{1}{8}\,\varphi(n), & \text{if } R_{\mathbb{P}_k} = \{5,7\}, \{5\}, \\ \frac{1}{8}\,\varphi(n) + \frac{1}{8}\,(-1)^{\Omega(n)-1}\,2^{\,\omega(n)}, & \text{if } R_{\mathbb{P}_k} = \{3,7\}, \{3\}; \\ \frac{1}{8}\,\varphi(n) + \frac{1}{8}\,(-1)^{\Omega(n)-1}\,2^{\,\omega(n)}, & \text{if } R_{\mathbb{P}_k} = \{7\}; \\ \frac{1}{8}\,\varphi(n), & \text{otherwise.} \end{cases} \tag{3.9}$$

This means that Theorem 3.1 holds in this case.

Case 4. $\alpha \geq 3$. Note that $\mu(2^\gamma) = 0$ for any positive integer $\gamma \geq 2$; thus, by (1.1) and Lemma 2.4 we have

$$\varphi_8(n) = \sum_{d\,|\,n_1} \mu\Big(\frac{2n_1}{d}\Big)\Big[\frac{2^{\alpha-1}d}{8}\Big] + \sum_{d\,|\,n_1} \mu\Big(\frac{n_1}{d}\Big)\Big[\frac{2^{\alpha}d}{8}\Big]. \tag{3.10}$$

If $\alpha = 3$, then

$$\varphi_8(n) = -\sum_{d\,|\,n_1} \mu\Big(\frac{n_1}{d}\Big)\Big[\frac{d}{2}\Big] + \sum_{d\,|\,n_1} \mu\Big(\frac{n_1}{d}\Big)d = -\frac{1}{2}\varphi(n_1) + \varphi(n_1) = \frac{1}{2}\varphi(n_1) = \frac{1}{8}\,\varphi(n)$$

If $\alpha \geq 4$, then $\varphi_8(n) = -2^{\alpha-4}\varphi(n_1) + 2^{\alpha-3}\varphi(n_1) = 2^{\alpha-4}\varphi(n_1) = \frac{1}{8}\,\varphi(n)$, which means that Theorem 3.1 also holds.

From the above, we have completed the proof of Theorem 3.1.

4. The explicit formula for $\varphi_{12}(n)$

In this section, we give the explicit formula for $\varphi_{12}(n)$. Obviously, $\varphi_{12}(n) = 0$ when $n < 12$, and $\varphi_{12}(n) = 1$ when $n = 12$ or 24 ; then, we consider $n > 12$ and $n \neq 24$.

Theorem 4.1. Let α and β be non-negative integers. If $n = 2^{\alpha} 3^{\beta} > 12$ and $n \neq 24$, then the following holds:

$$\varphi_{12}(n) = \begin{cases} \frac{1}{2}(3^{\beta-2} - (-1)^{\alpha+\beta}), & \text{if } \alpha = 0, \ or\, \alpha \geq 1, \beta \geq 2; \\ 2^{\alpha-2} \cdot 3^{\beta-2}, & \text{if } \alpha \geq 2, \beta \geq 2; \\ \frac{1}{3}(2^{\alpha+\beta-3} + (-1)^{\alpha+\beta}), & \text{if } \alpha \geq 4, \beta = 0, 1. \end{cases} \tag{4.1}$$

Proof. (1) For the case that $\alpha = 0$, i.e., $n = 3^{\beta} > 12$, and $\beta \geq 3$, then we have

$$\begin{aligned} \varphi_{12}(3^{\beta}) &= \sum_{d \mid 3^{\beta}} \mu\Big(\frac{3^{\beta}}{d}\Big)\Big[\frac{d}{12}\Big] = \Big[\frac{3^{\beta}}{12}\Big] - \Big[\frac{3^{\beta-1}}{12}\Big] = \Big[\frac{3^{\beta-1}}{4}\Big] - \Big[\frac{3^{\beta-2}}{4}\Big] \\ &= \frac{1}{4}(3^{\beta-1} - 2 + (-1)^{\beta-1}) - \frac{1}{4}(3^{\beta-2} - 2 + (-1)^{\beta-2}) \\ &= \frac{1}{2}(3^{\beta-2} - (-1)^{\beta}). \end{aligned}$$

(2) For the case that $\alpha = 1$, i.e., $n = 2 \cdot 3^{\beta} > 12$, and $\beta \geq 2$, by Lemma 2.5,

$$\begin{aligned} \varphi_{12}(2 \cdot 3^{\beta}) &= \sum_{d \mid 3^{\beta}} \mu\Big(\frac{2 \cdot 3^{\beta}}{d}\Big)\Big[\frac{d}{12}\Big] + \sum_{d \mid 3^{\beta}} \mu\Big(\frac{2 \cdot 3^{\beta}}{2d}\Big)\Big[\frac{2d}{12}\Big] \\ &= -\varphi_{12}(3^{\beta}) + \varphi_{6}(3^{\beta}) = -\frac{1}{12}\varphi(3^{\beta}) + \frac{1}{2}(-1)^{\beta} + \frac{1}{6}\varphi(3^{\beta}) \\ &= \frac{1}{2}(3^{\beta-2} - (-1)^{\beta+1}). \end{aligned}$$

(3) For the case that $\alpha = 2$, i.e., $n = 4 \cdot 3^{\beta} > 12$, and so $\beta \geq 2$, then we have

$$\begin{aligned} \varphi_{12}(4 \cdot 3^{\beta}) &= \sum_{d \mid 4 \cdot 3^{\beta}} \mu\Big(\frac{4 \cdot 3^{\beta}}{d}\Big)\Big[\frac{d}{12}\Big] \\ &= \mu(1)\Big[\frac{4 \cdot 3^{\beta}}{12}\Big] + \mu(2)\Big[\frac{2 \cdot 3^{\beta}}{12}\Big] + \mu(3)\Big[\frac{4 \cdot 3^{\beta-1}}{12}\Big] + \mu(6)\Big[\frac{2 \cdot 3^{\beta-1}}{12}\Big] \\ &= 3^{\beta-1} - \Big[\frac{3^{\beta-1}}{2}\Big] - 3^{\beta-2} + \Big[\frac{3^{\beta-2}}{2}\Big] \\ &= 3^{\beta-2}. \end{aligned}$$

(4) For the case that $\alpha = 3$, i.e., $n = 8 \cdot 3^{\beta} > 12$ and $n \neq 24$, and $\beta \geq 2$, then

$$\begin{aligned} \varphi_{12}(8 \cdot 3^{\beta}) &= \sum_{d \mid 8 \cdot 3^{\beta}} \mu\Big(\frac{8 \cdot 3^{\beta}}{d}\Big)\Big[\frac{d}{12}\Big] \\ &= \mu(1)\Big[\frac{8 \cdot 3^{\beta}}{12}\Big] + \mu(2)\Big[\frac{4 \cdot 3^{\beta}}{12}\Big] + \mu(3)\Big[\frac{8 \cdot 3^{\beta-1}}{12}\Big] + \mu(6)\Big[\frac{4 \cdot 3^{\beta-1}}{12}\Big] \end{aligned}$$

$$
\begin{aligned}
&= 2\cdot 3^{\beta-1} - 3^{\beta-1} - 2\cdot 3^{\beta-2} + 3^{\beta-2}\\
&= 2\cdot 3^{\beta-2}.
\end{aligned}
$$

(5) For the case that $\alpha \geq 4$, i.e., $n = 2^{\alpha}\cdot 3^{\beta} > 12$, and so $\beta \geq 0$, if $\beta = 0$, i.e., $n = 2^{\alpha}(\alpha \geq 4)$, then we have

$$
\begin{aligned}
\varphi_{12}(2^{\alpha}) &= \sum_{d\,|\,2^{\alpha}} \mu\Big(\frac{2^{\alpha}}{d}\Big)\Big[\frac{d}{12}\Big] = \Big[\frac{2^{\alpha-2}}{3}\Big] - \Big[\frac{2^{\alpha-3}}{3}\Big]\\
&= \frac{1}{3}\Big(2^{\alpha-2} - \frac{1}{2}(3-(-1)^{\alpha-2})\Big) - \frac{1}{3}\Big(2^{\alpha-3} - \frac{1}{2}(3-(-1)^{\alpha-3})\Big)\\
&= \frac{1}{3}\Big(2^{\alpha-3} + (-1)^{\alpha}\Big)
\end{aligned}
$$

If $\beta = 1$, i.e., $n = 3\cdot 2^{\alpha}$, then we have

$$
\begin{aligned}
\varphi_{12}(3\cdot 2^{\alpha}) &= \sum_{d\,|\,3\cdot 2^{\alpha}} \mu\Big(\frac{3\cdot 2^{\alpha}}{d}\Big)\Big[\frac{d}{12}\Big] = 2^{\alpha-2} - 2^{\alpha-3} - \Big[\frac{2^{\alpha-2}}{3}\Big] + \Big[\frac{2^{\alpha-3}}{3}\Big]\\
&= \frac{1}{3}\Big(2^{\alpha-2} + (-1)^{\alpha+1}\Big).
\end{aligned}
$$

If $\beta \geq 2$, we have

$$
\begin{aligned}
\varphi_{12}(2^{\alpha}\cdot 3^{\beta}) &= \sum_{d\,|\,2^{\alpha}\cdot 3^{\beta}} \mu\Big(\frac{2^{\alpha}\cdot 3^{\beta}}{d}\Big)\Big[\frac{d}{12}\Big]\\
&= \mu(1)\Big[\frac{2^{\alpha}\cdot 3^{\beta}}{12}\Big] + \mu(2)\Big[\frac{2^{\alpha-1}\cdot 3^{\beta}}{12}\Big] + \mu(3)\Big[\frac{2^{\alpha}\cdot 3^{\beta-1}}{12}\Big] + \mu(6)\Big[\frac{2^{\alpha-1}\cdot 3^{\beta-1}}{12}\Big]\\
&= 2^{\alpha-2}\cdot 3^{\beta-1} - \Big[\frac{2^{\alpha-2}\cdot 3^{\beta-1}}{2}\Big] - 2^{\alpha-2}\cdot 3^{\beta-2} + \Big[\frac{2^{\alpha-2}\cdot 3^{\beta-2}}{2}\Big]\\
&= 2^{\alpha-2}\cdot 3^{\beta-2}.
\end{aligned}
$$

This completes the proof of Theorem 4.1.

Now consider the case that $n = 2^{\alpha}\, 3^{\beta} n_1$, where $n_1 > 1$ and $\gcd(n_1, 6) = 1$. We have the following theorem.

Theorem 4.2. Let α and β be non-negative integers, k, α_i $(1 \leq i \leq k)$ be positive integers, and $p_1, \ldots, p_k$ be distinct primes. Suppose that $\gcd(p_i, 6) = 1$ $(1 \leq i \leq k)$ and $n = 2^{\alpha}\, 3^{\beta} \prod_{i=1}^{k} p_i^{\alpha_i} > 12$; then, we have the following:

$$\varphi_{12}(n)=\begin{cases}\frac{1}{12}\,\varphi(n)+\frac{1}{4}\,(-1)^{\Omega(n)}\cdot 2^{\,\omega(n)-\alpha}, & \\ \qquad\qquad \text{if } \alpha=0,1,\beta=0, \text{ and } R'_{\mathbb{P}_k}=\{7,11\},\{7\}; & \\ \frac{1}{12}\,\varphi(n)+\frac{1}{4}\,(-1)^{\Omega(n)+1}\cdot 2^{\,\omega(n)-\alpha}, & \\ \qquad\qquad \text{if } \alpha=0,1,\beta\geq 2, \text{ and } R'_{\mathbb{P}_k}=\{7,11\},\{7\},\{11\}; & \\ \frac{1}{12}\,\varphi(n)+\frac{1}{6}\,(-1)^{\Omega(n)+[\frac{\alpha+1}{2}]}\cdot 2^{\,\omega(n)-[\frac{\alpha+1}{2}]-\beta}, & \\ \qquad\qquad \text{if } \alpha=0,1,2,\beta=0,1, \text{ and } R'_{\mathbb{P}_k}=\{5,11\},\{5\}, & \\ \qquad\qquad \text{or } \alpha=0,1,\beta=1, \text{ and } R'_{\mathbb{P}_k}=\{11\}; & \\ \qquad\qquad \text{or } \alpha=2,\beta=0,1, \text{ and } R'_{\mathbb{P}_k}=\{11\}; & \\ \frac{1}{12}\,\varphi(n)+\frac{1}{6}\,(-1)^{\Omega(n)}\cdot 2^{\,\omega(n)-\beta}, & \\ \qquad\qquad \text{if } \alpha\geq 3,\beta=0,1, \text{ and } R'_{\mathbb{P}_k}=\{5,11\},\{5\},\{11\}; & \\ \frac{1}{12}\,\varphi(n)+\frac{5}{12}\,(-1)^{\Omega(n)}\cdot 2^{\,\omega(n)}, & \\ \qquad\qquad \text{if } \alpha=0,\beta=0, \text{ and } R'_{\mathbb{P}_k}=\{11\}; & \\ \frac{1}{12}\,\varphi(n)+\frac{1}{12}\,(-1)^{\Omega(n)}\cdot 2^{\,\omega(n)-1}, & \\ \qquad\qquad \text{if } \alpha=1,\beta=0, \text{ and } R'_{\mathbb{P}_k}=\{11\}; & \\ \frac{1}{12}\,\varphi(n), \quad \text{otherwise.} & \end{cases} \tag{4.2}$$

Proof. Set $n_1=\prod_{i=1}^{k}p_i^{\alpha_i}$; then, $\gcd(n_1,6)=1$ and $n=2^{\alpha}\,3^{\beta}\,n_1$.
Case 1. $\alpha=0$.
(A) If $\beta=0$, then $n_1>1$. Thus by (1.1), (2.2) and Lemmas 2.1 and 2.2, we have

$$\begin{aligned}\varphi_{12}(n) &= \varphi_{12}(n_1)=\sum_{d\,|\,n_1}\mu\Big(\frac{n_1}{d}\Big)\Big[\frac{d}{12}\Big]\\ &= \frac{1}{12}\sum_{d\,|\,n_1}\mu\Big(\frac{n_1}{d}\Big)\Big(d-6+3\Big(\frac{-1}{d}\Big)+2\Big(\frac{-3}{d}\Big)\Big)\\ &= \frac{1}{12}\sum_{d\,|\,n_1}\mu\Big(\frac{n_1}{d}\Big)d-\frac{1}{2}\sum_{d\,|\,n_1}\mu\Big(\frac{n_1}{d}\Big)+\frac{1}{4}\sum_{d\,|\,n_1}\mu\Big(\frac{n_1}{d}\Big)\Big(\frac{-1}{d}\Big)\\ &\quad+\frac{1}{6}\sum_{d\,|\,n_1}\mu\Big(\frac{n_1}{d}\Big)\Big(\frac{-3}{d}\Big)\\ &= \frac{1}{12}\,\varphi(n_1)+\frac{1}{4}\prod_{i=1}^{k}\Big(\Big(\frac{-1}{p_i}\Big)^{\alpha_i}-\Big(\frac{-1}{p_i}\Big)^{\alpha_i-1}\Big)\\ &\quad+\frac{1}{6}\prod_{i=1}^{k}\Big(\Big(\frac{-3}{p_i}\Big)^{\alpha_i}-\Big(\frac{-3}{p_i}\Big)^{\alpha_i-1}\Big).\end{aligned} \tag{4.3}$$

If $1\in R'_{\mathbb{P}_k}$ or $\{5,7\}\subseteq R'_{\mathbb{P}_k}$, then there exists $p_i\equiv 1\,(\mathrm{mod}\,12)$, or there exist p_j and p_l such that $p_j\equiv 5(\mathrm{mod}\,12)$ and $p_l\equiv 7\,(\mathrm{mod}\,12)$; then, $(\frac{-1}{p_i})=(\frac{-3}{p_i})=1$ or $(\frac{-1}{p_j})=(\frac{-3}{p_l})=1$, respectively. Thus by (4.3) we can get

$$\varphi_{12}(n)=\frac{1}{12}\,\varphi(n_1)=\frac{1}{12}\,\varphi(n). \tag{4.4}$$

If $R'_{\mathbb{P}_k} = \{7, 11\}$ or $\{7\}$, i.e., for any $p \in \mathbb{P}_k$, we have that $p \equiv 7, 11 \pmod{12}$ or $p \equiv 7 \pmod{12}$, respectively. This means that $(\frac{-1}{p}) = -1$ and there exists a prime $p' \equiv 7 \pmod{12}$, i.e., $(\frac{-3}{p'}) = 1$, in either of the two cases. Thus by (4.3) we can obtain

$$\varphi_{12}(n) = \frac{1}{12}\varphi(n_1) + \frac{1}{4}\prod_{i=1}^{k}\left(2\,(-1)^{\alpha_i}\right) = \frac{1}{12}\varphi(n) + \frac{1}{4}(-1)^{\Omega(n)}\,2^{\omega(n)}. \tag{4.5}$$

If $R'_{\mathbb{P}_k} = \{5, 11\}$ or $\{5\}$, i.e., for any $p \in \mathbb{P}_k$, $p \equiv 5, 11 \pmod{12}$ or $p \equiv 5 \pmod{12}$, respectively. Then $(\frac{-3}{p}) = -1$, and there exists a prime $p' \equiv 5 \pmod{12}$, i.e., $(\frac{-1}{p'}) = 1$ in either case. Thus by (4.3) we can get

$$\varphi_{12}(n) = \frac{1}{12}\varphi(n_1) + \frac{1}{6}\prod_{i=1}^{k}\left(2\,(-1)^{\alpha_i}\right) = \frac{1}{12}\varphi(n) + \frac{1}{6}(-1)^{\Omega(n)}\,2^{\omega(n)}. \tag{4.6}$$

If $R'_{\mathbb{P}_k} = \{11\}$, i.e., for any $p \in \mathbb{P}_k$, $p \equiv 11 \pmod{12}$; then, $(\frac{-1}{p}) = (\frac{-3}{p}) = -1$. Thus by (4.3) we have

$$\varphi_{12}(n) = \frac{1}{12}\varphi(n_1) + \frac{5}{12}\prod_{i=1}^{k}\left(2\,(-1)^{\alpha_i}\right) = \frac{1}{12}\varphi(n) + \frac{5}{12}(-1)^{\Omega(n)}\,2^{\,\omega(n)}. \tag{4.7}$$

(B) If $\beta \geq 1$, then by (1.1) we have

$$\begin{aligned}
\varphi_{12}(n) &= \varphi_{12}(3^\beta n_1) = \sum_{d\,|\,n_1}\mu\Big(\frac{3^\beta n_1}{d}\Big)\Big[\frac{d}{12}\Big] + \sum_{d\,|\,3^{\beta-1}n_1}\mu\Big(\frac{3^\beta n_1}{3d}\Big)\Big[\frac{3d}{12}\Big] \\
&= \mu(3^\beta)\,\varphi_{12}(n_1) + \varphi_4(3^{\beta-1}n_1).
\end{aligned}$$

Now from $\beta = 1$, Lemma 2.4 and Case 1, we can get the following:

$$\begin{aligned}
\varphi_{12}(n) &= -\varphi_{12}(n_1) + \varphi_4(n_1) \\
&= \begin{cases}
\frac{1}{12}\varphi(n), & \text{if } R'_{\mathbb{P}_k} = \{7, 11\}, \{7\}, \\
\frac{1}{12}\varphi(n) + \frac{1}{6}(-1)^{\Omega(n)}\,2^{\,\omega(n)-1}, & \text{if } R'_{\mathbb{P}_k} = \{5, 11\}, \{5\}, \\
\frac{1}{12}\varphi(n) + \frac{1}{6}(-1)^{\Omega(n)}\,2^{\,\omega(n)-1}, & \text{if } R'_{\mathbb{P}_k} = \{11\}, \\
\frac{1}{12}\varphi(n), & \text{otherwise.}
\end{cases}
\end{aligned} \tag{4.8}$$

For the case that $\beta \geq 2$, note that $\mu(3^\gamma) = 0$ with $\gamma \geq 2$; thus, by Lemma 2.4 we have the following:

$$\begin{aligned}
\varphi_{12}(n) &= \varphi_4(3^{\beta-1}n_1) \\
&= \begin{cases}
\frac{1}{12}\varphi(n) + \frac{1}{4}(-1)^{\Omega(n)+1}\,2^{\,\omega(n)}, & \text{if } R'_{\mathbb{P}_k} = \{7, 11\}, \{7\}, \\
\frac{1}{12}\varphi(n), & \text{if } R'_{\mathbb{P}_k} = \{5, 11\}, \{5\}, \\
\frac{1}{12}\varphi(n) + \frac{1}{4}(-1)^{\Omega(n)+1}\,2^{\,\omega(n)}, & \text{if } R'_{\mathbb{P}_k} = \{11\}, \\
\frac{1}{12}\varphi(n), & \text{otherwise.}
\end{cases}
\end{aligned} \tag{4.9}$$

From the above (4.3)–(4.9), Theorem 4.2 is proved in this case.

Case 2. $\alpha = 1$.

(A) If $\beta = 0$, i.e., $n = 2n_1$, then by (1.1), Case 1 and Lemma 2.4, we have

$$\begin{aligned}
\varphi_{12}(n) &= \sum_{d\,|\,n_1}\mu\Big(\frac{2n_1}{d}\Big)\Big[\frac{d}{12}\Big] + \sum_{d\,|\,n_1}\mu\Big(\frac{2n_1}{2d}\Big)\Big[\frac{2d}{12}\Big] \\
&= -\varphi_{12}(n_1) + \varphi_6(n_1) \\
&= \begin{cases} \frac{1}{12}\,\varphi(n) + \frac{1}{4}\,(-1)^{\Omega(n)}\,2^{\,\omega(n)-1}, & \text{if } R'_{\mathbb{P}_k} = \{7,11\},\{7\}, \\ \frac{1}{12}\,\varphi(n) + \frac{1}{12}\,(-1)^{\Omega(n)+1}\,2^{\,\omega(n)}, & \text{if } R'_{\mathbb{P}_k} = \{5,11\},\{5\}, \\ \frac{1}{12}\,\varphi(n) + \frac{1}{12}\,(-1)^{\Omega(n)}\,2^{\,\omega(n)-1}, & \text{if } R'_{\mathbb{P}_k} = \{11\}, \\ \frac{1}{12}\,\varphi(n), & \text{otherwise.} \end{cases}
\end{aligned} \tag{4.10}$$

(B) If $\beta = 1$, i.e., $n = 6n_1$, then from (1.1) we can get

$$\begin{aligned}
\varphi_{12}(n) &= \sum_{d\,|\,n_1}\mu\Big(\frac{6n_1}{d}\Big)\Big[\frac{d}{12}\Big] + \sum_{d\,|\,n_1}\mu\Big(\frac{6n_1}{2d}\Big)\Big[\frac{2d}{12}\Big] + \sum_{d\,|\,n_1}\mu\Big(\frac{6n_1}{3d}\Big)\Big[\frac{3d}{12}\Big] \\
&\quad + \sum_{d\,|\,n_1}\mu\Big(\frac{6n_1}{6d}\Big)\Big[\frac{6d}{12}\Big] \\
&= \varphi_{12}(n_1) - \varphi_6(n_1) - \varphi_4(n_1) + \varphi_2(n_1).
\end{aligned}$$

Now by Lemmas 2.4 and 2.5 and Case 1, we have the following:

$$\varphi_{12}(n) = \begin{cases} \frac{1}{12}\,\varphi(n), & \text{if } R'_{\mathbb{P}_k} = \{7,11\},\{7\}, \\ \frac{1}{12}\,\varphi(n) + \frac{1}{12}\,(-1)^{\Omega(n)+1}\,2^{\,\omega(n)-1}, & \text{if } R'_{\mathbb{P}_k} = \{5,11\},\{5\}, \\ \frac{1}{12}\,\varphi(n) + \frac{1}{12}\,(-1)^{\Omega(n)+1}\,2^{\,\omega(n)-1}, & \text{if } R'_{\mathbb{P}_k} = \{11\}, \\ \frac{1}{12}\,\varphi(n), & \text{otherwise.} \end{cases} \tag{4.11}$$

(C) If $\beta \geq 2$, then by (1.1) one can easily see that

$$\begin{aligned}
\varphi_{12}(n) &= \sum_{d\,|\,n_1}\mu\Big(\frac{2\cdot 3^{\beta}n_1}{d}\Big)\Big[\frac{d}{12}\Big] + \sum_{d\,|\,n_1}\mu\Big(\frac{2\cdot 3^{\beta}n_1}{2d}\Big)\Big[\frac{2d}{12}\Big] \\
&\quad + \sum_{d\,|\,3^{\beta-1}n_1}\mu\Big(\frac{2\cdot 3^{\beta}n_1}{3d}\Big)\Big[\frac{3d}{12}\Big] + \sum_{d\,|\,3^{\beta-1}n_1}\mu\Big(\frac{2\cdot 3^{\beta}n_1}{6d}\Big)\Big[\frac{6d}{12}\Big] \\
&= -\varphi_4(3^{\beta-1}n_1) + \varphi_2(3^{\beta-1}n_1).
\end{aligned}$$

Now by Lemma 2.4 we can get the following:

$$\varphi_{12}(n) = \begin{cases} \frac{1}{12}\,\varphi(n) + \frac{1}{4}\,(-1)^{\Omega(n)+1}\,2^{\,\omega(n)-1}, & \text{if } R'_{\mathbb{P}_k} = \{7,11\},\{7\}, \\ \frac{1}{12}\,\varphi(n), & \text{if } R'_{\mathbb{P}_k} = \{5,11\},\{5\}, \\ \frac{1}{12}\,\varphi(n) + \frac{1}{4}\,(-1)^{\Omega(n)+1}\,2^{\,\omega(n)-1}, & \text{if } R'_{\mathbb{P}_k} = \{11\}, \\ \frac{1}{12}\,\varphi(n), & \text{otherwise.} \end{cases} \tag{4.12}$$

From the above (4.10) and (4.12), Theorem 4.2 is true in this case.

Case 3. $\alpha = 2$.

(A) If $\beta = 0$, i.e., $n = 4n_1$, then from Lemmas 2.3 and 2.5, we can obtain

$$\begin{aligned}
\varphi_{12}(n) &= \sum_{d\,|\,4n_1} \mu\Big(\frac{4n_1}{d}\Big)\Big[\frac{d}{12}\Big] \\
&= \sum_{d\,|\,n_1} \mu\Big(\frac{4n_1}{d}\Big)\Big[\frac{d}{12}\Big] + \sum_{d\,|\,n_1} \mu\Big(\frac{4n_1}{2d}\Big)\Big[\frac{2d}{12}\Big] + \sum_{d\,|\,n_1} \mu\Big(\frac{4n_1}{4d}\Big)\Big[\frac{4d}{12}\Big] \\
&= -\varphi_6(n_1) + \varphi_3(n_1) \\
&= \begin{cases} \frac{1}{12}\,\varphi(n), & \text{if } R'_{\mathbb{P}_k} = \{7, 11\}, \{7\}, \\ \frac{1}{12}\,\varphi(n) + \frac{1}{12}\,(-1)^{\Omega(n)+1}\,2^{\,\omega(n)}, & \text{if } R'_{\mathbb{P}_k} = \{5, 11\}, \{5\}, \\ \frac{1}{12}\,\varphi(n) + \frac{1}{12}\,(-1)^{\Omega(n)+1}\,2^{\,\omega(n)}, & \text{if } R'_{\mathbb{P}_k} = \{11\}, \\ \frac{1}{12}\,\varphi(n), & \text{otherwise.} \end{cases}
\end{aligned} \tag{4.13}$$

(B) If $\beta = 1$, i.e., $n = 12\,n_1$, then by the definition we have

$$\begin{aligned}
\varphi_{12}(n) &= \sum_{d|12n_1} \mu\Big(\frac{12n_1}{d}\Big)\Big[\frac{d}{12}\Big] \\
&= \sum_{d\,|\,n_1} \mu\Big(\frac{12n_1}{d}\Big)\Big[\frac{d}{12}\Big] + \sum_{d\,|\,n_1} \mu\Big(\frac{12n_1}{2d}\Big)\Big[\frac{2d}{12}\Big] + \sum_{d\,|\,n_1} \mu\Big(\frac{12n_1}{4d}\Big)\Big[\frac{4d}{12}\Big] \\
&\quad + \sum_{d\,|\,n_1} \mu\Big(\frac{12n_1}{3d}\Big)\Big[\frac{3d}{12}\Big] + \sum_{d\,|\,n_1} \mu\Big(\frac{12n_1}{6d}\Big)\Big[\frac{6d}{12}\Big] + \sum_{d\,|\,n_1} \mu\Big(\frac{12n_1}{12d}\Big)\Big[\frac{12d}{12}\Big] \\
&= \varphi_6(n_1) - \varphi_3(n_1) - \varphi_2(n_1) + \varphi(n_1).
\end{aligned}$$

Now by Lemmas 2.3 and 2.4 and Case 1, we can get the following:

$$\varphi_{12}(n) = \begin{cases} \frac{1}{12}\,\varphi(n), & \text{if } R'_{\mathbb{P}_k} = \{7, 11\}, \{7\}, \\ \frac{1}{12}\,\varphi(n) + \frac{1}{12}\,(-1)^{\Omega(n)+1}\,2^{\,\omega(n)-1}, & \text{if } R'_{\mathbb{P}_k} = \{5, 11\}, \{5\}, \\ \frac{1}{12}\,\varphi(n) + \frac{1}{12}\,(-1)^{\Omega(n)+1}\,2^{\,\omega(n)-1}, & \text{if } R'_{\mathbb{P}_k} = \{11\}, \\ \frac{1}{12}\,\varphi(n), & \text{otherwise.} \end{cases} \tag{4.14}$$

(C) If $\beta \geq 2$, then from $n = 4 \cdot 3^{\beta}\, n_1$ and the definition, we know that

$$\begin{aligned}
\varphi_{12}(n) &= \sum_{d\,|\,4\cdot 3^{\beta} n_1} \mu\Big(\frac{4 \cdot 3^{\beta} n_1}{d}\Big)\Big[\frac{d}{12}\Big] = \sum_{d\,|\,2\cdot 3^{\beta-1} n_1} \mu\Big(\frac{4 \cdot 3^{\beta} n_1}{6d}\Big)\Big[\frac{6d}{12}\Big] \\
&= \sum_{d\,|\,3^{\beta-1} n_1} \mu\Big(\frac{2 \cdot 3^{\beta} n_1}{2d}\Big)\Big[\frac{2d}{2}\Big] + \sum_{d\,|\,3^{\beta-1} n_1} \mu\Big(\frac{2 \cdot 3^{\beta} n_1}{d}\Big)\Big[\frac{d}{2}\Big] \\
&= \varphi(3^{\beta-1} n_1) - \varphi_2(3^{\beta-1} n_1) = \frac{1}{2}\varphi(3^{\beta-1} n_1) \\
&= \frac{1}{12}\varphi(4 \cdot 3^{\beta} n_1) = \frac{1}{12}\varphi(n).
\end{aligned} \tag{4.15}$$

From the above (4.13)–(4.15), Theorem 4.2 is proved in this case.

Case 4. $\alpha \geq 3$.

(A) If $\beta = 0$, i.e., $n = 2^{\alpha} n_1$, then by Lemma 2.5 we have

$$\begin{aligned}\varphi_{12}(n) &= \sum_{d\,|\,n_1} \mu\Big(\frac{2^{\alpha} n_1}{d}\Big)\Big[\frac{d}{12}\Big] + \sum_{d\,|\,2^{\alpha-1} n_1} \mu\Big(\frac{2^{\alpha} n_1}{2d}\Big)\Big[\frac{2d}{12}\Big] \\ &= \varphi_6(2^{\alpha-1} n_1) \\ &= \begin{cases} \frac{1}{12}\,\varphi(n), & \text{if } R'_{\mathbb{P}_k} = \{7,11\}, \{7\}, \\ \frac{1}{12}\,\varphi(n) + \frac{1}{6}\,(-1)^{\Omega(n)}\, 2^{\,\omega(n)}, & \text{if } R'_{\mathbb{P}_k} = \{5,11\}, \{5\}, \\ \frac{1}{12}\,\varphi(n) + \frac{1}{6}\,(-1)^{\Omega(n)}\, 2^{\,\omega(n)}, & \text{if } R'_{\mathbb{P}_k} = \{11\}, \\ \frac{1}{12}\,\varphi(n), & \text{otherwise.} \end{cases}\end{aligned} \tag{4.16}$$

(B) If $\beta = 1$, i.e., $n = 3 \cdot 2^{\alpha} n_1$, then by the definition we have

$$\begin{aligned}\varphi_{12}(n) &= \sum_{d\,|\,n_1} \mu\Big(\frac{3 \cdot 2^{\alpha} n_1}{d}\Big)\Big[\frac{d}{12}\Big] + \sum_{d\,|\,2^{\alpha-1} n_1} \mu\Big(\frac{3 \cdot 2^{\alpha} n_1}{2d}\Big)\Big[\frac{2d}{12}\Big] \\ &\quad + \sum_{d\,|\,n_1} \mu\Big(\frac{3 \cdot 2^{\alpha} n_1}{3d}\Big)\Big[\frac{3d_1}{12}\Big] + \sum_{d\,|\,2^{\alpha-1} n_1} \mu\Big(\frac{3 \cdot 2^{\alpha} n_1}{6d}\Big)\Big[\frac{6d}{12}\Big] \\ &= -\varphi_6(2^{\alpha-1} n_1) + \varphi_2(2^{\alpha-1} n_1) \\ &= \begin{cases} \frac{1}{12}\,\varphi(n), & \text{if } R'_{\mathbb{P}_k} = \{7,11\}, \{7\}, \\ \frac{1}{12}\,\varphi(n) + \frac{1}{12}\,(-1)^{\Omega(n)}\, 2^{\,\omega(n)}, & \text{if } R'_{\mathbb{P}_k} = \{5,11\}, \{5\}, \\ \frac{1}{12}\,\varphi(n) + \frac{1}{12}\,(-1)^{\Omega(n)}\, 2^{\,\omega(n)}, & \text{if } R'_{\mathbb{P}_k} = \{11\}, \\ \frac{1}{12}\,\varphi(n), & \text{otherwise.} \end{cases}\end{aligned} \tag{4.17}$$

(C) If $\beta \geq 2$, then by Lemma 2.6 we can get

$$\varphi_{12}(n) = \frac{1}{12}\varphi(2^{\alpha} \cdot 3^{\beta} n_1) = \frac{1}{12}\varphi(n). \tag{4.18}$$

Now from (4.16)–(4.18), Theorem 4.2 is proved in this case.

From the above, we complete the proof for Theorem 4.2.

5. The parity of the generalized Euler functions $\varphi_8(n)$ and $\varphi_{12}(n)$

Based on Theorems 3.1, 4.1 and 4.2, this section gives the parity of $\varphi_8(n)$ and $\varphi_{12}(n)$, respectively.

Theorem 5.1. If n is a positive integer, then $\varphi_8(n)$ is odd if and only if $n = 8, 16$ or n is given by Table 1.

Table 1. the conditions of $\varphi_8(n)$ is odd.

n	Conditions
p^α	$p \equiv 9, 15 \pmod{16}$; $p \equiv 3, 5 \pmod{16}$, $2 \mid \alpha$; $p \equiv 11, 13 \pmod{16}$, $2 \nmid \alpha$;
$2p^\alpha$	$p \equiv 7, 9 \pmod{16}$; $p \equiv 3, 13 \pmod{16}$, $2 \mid \alpha$; $p \equiv 5, 11 \pmod{16}$, $2 \nmid \alpha$;
$4p^\alpha$	$p \equiv 3, 5 \pmod 8$;
$8p^\alpha$	$p \equiv 3, 7 \pmod 8$;
$p_1^{\alpha_1} p_2^{\alpha_2}$	$p_1 \equiv p_2 \equiv 3 \pmod 8$; $p_1 \equiv p_2 \equiv 5 \pmod 8$; $p_1 \equiv 3 \pmod 8$, $p_2 \equiv 5 \pmod 8$;
$2p_1^{\alpha_1} p_2^{\alpha_2}$	$p_1 \equiv p_2 \equiv 3 \pmod 8$; $p_1 \equiv p_2 \equiv 5 \pmod 8$; $p_1 \equiv 3 \pmod 8$, $p_2 \equiv 5 \pmod 8$.

In the above table, p, p_1, p_2 are odd primes with $p_1 \neq p_2$, and $\alpha, \alpha_1, \alpha_2$ are positive integers.

Proof. For $n = 2^\alpha$, by (3.1) we know that $\varphi_8(n)$ is odd if and only if $n = 8, 16$.

Now suppose that $n = 2^\alpha \prod_{i=1}^{k} p_i^{\alpha_i}$, where $\alpha \geq 0$, $\alpha_1, \ldots, \alpha_k$ are positive integers, and $p_1, \ldots, p_k$ are distinct odd primes. Set $n_1 = \prod_{i=1}^{k} p_i^{\alpha_i}$; then, $n_1 > 1$ is odd. By Theorem 3.1, we have the following four cases.

Case 1. $R_{\mathbb{P}_k} = \{5, 7\}$ or $\{5\}$.

(A) If $\alpha = 0$, i.e., $n = n_1$ is odd, then, by (3.2) we have that $\varphi_8(n) = \frac{1}{8}\varphi(n) + \frac{1}{4}(-1)^{\Omega(n)} 2^{\omega(n)}$. Note that there exists a prime factor p of n such that $p \equiv 5 \pmod 8$; thus, we must have that $\omega(n) \leq 2$ if $\varphi_8(n)$ is odd. For $\omega(n) = 2$, i.e., $n = p_1^{\alpha_1} p_2^{\alpha_2}$, by (3.2) we have

$$\varphi_8(n) = \frac{1}{8}\, p_1^{\alpha_1 - 1}(p_1 - 1)\, p_2^{\alpha_2 - 1}(p_2 - 1) + (-1)^{\alpha_1 + \alpha_2}.$$

Therefore $\varphi_8(n)$ is odd if and only if $p_1 \equiv p_2 \equiv 5 \pmod 8$, which is true. Now for $\omega(n) = 1$, i.e., $n = p_1^{\alpha_1}$ with $p_1 \equiv 5 \pmod 8$, similarly, by (3.2) we have

$$\varphi_8(n) = \frac{1}{8}\, p_1^{\alpha_1 - 1}(p_1 - 1) + \frac{1}{2}(-1)^{\alpha_1} = \frac{1}{8}\,(p_1^{\alpha_1 - 1}(p_1 - 1) + 4 \cdot (-1)^{\alpha_1}).$$

From $p_1 \equiv 5 \pmod 8$, we have that $p_1 \equiv 5, 13 \pmod{16}$. If $p_1 \equiv 5 \pmod{16}$, then

$$p_1^{\alpha_1 - 1}(p_1 - 1) + 4(-1)^{\alpha_1} \equiv 4 \cdot 5^{\alpha_1 - 1} + 4(-1)^{\alpha_1} \pmod{16}.$$

Thus, $\varphi_8(n)$ is odd if and only if $2 \mid \alpha_1$. If $p_1 \equiv 13 \pmod{16}$, then

$$p_1^{\alpha_1 - 1}(p_1 - 1) + 4(-1)^{\alpha_1} \equiv 12 \cdot (-3)^{\alpha_1 - 1} + 4(-1)^{\alpha_1} \pmod{16}.$$

Thus, $\varphi_8(n)$ is odd if and only if α_1 is odd.

(B) If $\alpha = 1$, i.e., $\omega(n) \geq 2$, by (3.2) we have that $\varphi_8(n) = \frac{1}{8}\varphi(n) + \frac{1}{4}(-1)^{\Omega(n)} 2^{\omega(n) - 1}$. Then we must have that $\omega(n) \leq 3$ if $\varphi_8(n)$ is odd. For $\omega(n) = 3$, namely, $n = 2p_1^{\alpha_1} p_2^{\alpha_2}$, using the same method as (A), $\varphi_8(n)$ is odd if and only if $p_1 \equiv p_2 \equiv 5 \pmod 8$. Now for $\omega(n) = 2$, i.e., $n = 2p_1^{\alpha_1}$ with $p_1 \equiv 5 \pmod 8$, similar to (A), $\varphi_8(n)$ is odd if and only if $p_1 \equiv 5 \pmod{16}$ and α_1 is odd, or if $p_1 \equiv 13 \pmod{16}$ and $2 \mid \alpha_1$.

(C) If $\alpha = 2$, i.e., $\omega(n) \geq 2$, then by (3.2), we have that $\varphi_8(n) = \frac{1}{8}\varphi(n) = \frac{1}{4} \prod_{i=1}^{k} p_i^{\alpha_i - 1}(p_i - 1)$. Thus from the assumption that $p_i \equiv 5, 7 \pmod 8$ or $p_i \equiv 5 \pmod 8$, we know that $\omega(n) = 2$ if $\varphi_8(n)$ is odd. In this case, $n = 4p_1^{\alpha_1}$ with $p_1 \equiv 5 \pmod 8$; then, $p_1^{\alpha_1 - 1}(p_1 - 1) \equiv 4 \pmod 8$, namely, $\varphi_8(n)$ is odd.

(D) If $\alpha \geq 3$, then by (3.2), $\varphi_8(n) = \frac{1}{8}\varphi(n) = 2^{\alpha-4}\prod_{i=1}^{k} p_i^{\alpha_i-1}(p_i-1)$. Thus we must have that $\alpha = 3$ and $k = 1$ if $\varphi_8(n)$ is odd, namely, $n = 8p_1^{\alpha_1}$ with $p_1 \equiv 5 \pmod 8$. In this case, $\varphi_8(n) = \frac{1}{8}\varphi(n) = \frac{1}{2}p_1^{\alpha_1-1}(p_1-1)$ is always even.

Case 2. $R_{\mathbb{P}_k} = \{3,7\}$ or $\{3\}$.

(A) If $\alpha = 0$, by (3.2) we have that $\varphi_8(n) = \frac{1}{8}\varphi(n) + \frac{1}{8}(-1)^{\Omega(n)}2^{\omega(n)}$. Thus we must have that $\omega(n) \leq 3$ if $\varphi_8(n)$ is odd. For the case that $\omega(n) = 3$, i.e, $n = p_1^{\alpha_1}p_2^{\alpha_2}p_3^{\alpha_3}$, where $p_i \equiv 3 \pmod 4$ $(i = 1,2,3)$, it is easy to see that $\varphi_8(n)$ is always even in this case. Therefore we must have that $\omega(n) = 1,2$. Consider that $\omega(n) = 2$, i.e., $n = p_1^{\alpha_1}p_2^{\alpha_2}$. Note that $R_{\mathbb{P}_k} = \{3,7\}$ or $\{3\}$; then, by (3.2), $\varphi_8(n) = \frac{1}{8}(p_1^{\alpha_1-1}(p_1-1)\,p_2^{\alpha_2-1}(p_2-1)+4\cdot(-1)^{\alpha_1+\alpha_2})$ is odd if and only if $p_1 \equiv p_2 \equiv 3 \pmod 8$. Now, for $\omega(n) = 1$, i.e., $n = p_1^{\alpha_1}$ with $p_1 \equiv 3 \pmod 8$, then by (3.2) we have that $\varphi_8(n) = \frac{1}{8}(p_1^{\alpha_1-1}(p_1-1) + 2(-1)^{\alpha_1})$. Thus, $\varphi_8(n)$ is odd if and only if $p_1 \equiv 3 \pmod{16}$ and $2 \mid \alpha_1$, or if $p_1 \equiv 11 \pmod{16}$ and α_1 is odd.

(B) If $\alpha = 1$, i.e., $\omega \geq 2$, by (3.2) we have that $\varphi_8(n) = \frac{1}{8}\varphi(n) + \frac{1}{8}(-1)^{\Omega(n)-1}2^{\omega(n)-1}$. Thus we must have that $\omega(n) \leq 3$ if $\varphi_8(n)$ is odd. Using the same method as (A) in case 1, we can get that $\varphi_8(n)$ is odd if and only if $n = 2p_1^{\alpha_1}p_2^{\alpha_2}$ with $p_1 \equiv p_2 \equiv 3 \pmod 8$, or if $n = 2p_1^{\alpha_1}$ with $p_1 \equiv 3 \pmod{16}$ and $2 \mid \alpha_1$, or if $p_1 \equiv 11 \pmod{16}$ and α_1 is odd.

(C) If $\alpha = 2$, i.e., $\omega(n) \geq 2$, by (3.2) we have that $\varphi_8(n) = \frac{1}{8}\varphi(n) + \frac{1}{8}(-1)^{\Omega(n)-1}2^{\omega(n)}$. Therefore we must have that $\omega(n) \leq 3$ if $\varphi_8(n)$ is odd. For the case that $\omega(n) = 3$, i.e., $n = 4p_1^{\alpha_1}p_2^{\alpha_2}$, we know that

$$\varphi_8(n) = \frac{1}{4}p_1^{\alpha_1-1}(p_1-1)p_2^{\alpha_2-1}(p_2-1) + (-1)^{\alpha_1+\alpha_2+1},$$

which is always even. Now for the case that $\omega(n) = 2$, i.e., $n = 4p_1^{\alpha_1}$ with $p_1 \equiv 3 \pmod 8$, by (3.2) we have that $\varphi_8(n) = \frac{1}{4}(p_1^{\alpha_1-1}(p_1-1) + 2(-1)^{\alpha_1+1})$. Since

$$p_1^{\alpha_1-1}(p_1-1) + 2(-1)^{\alpha_1+1} \equiv 2\cdot 3^{\alpha_1-1} + 2(-1)^{\alpha_1+1} \equiv 4 \pmod 8,$$

it follows that $\varphi_8(n)$ is odd.

(D) If $\alpha \geq 3$, by (3.2) we have that $\varphi_8(n) = \frac{1}{8}\varphi(n) = 2^{\alpha-4}\prod_{i=1}^{k} p_i^{\alpha_i-1}(p_i-1)$. From $R_{\mathbb{P}_k} = \{3,7\}$ or $\{3\}$, we must have that $\alpha = 3$ and $k = 1$ if $\varphi_8(n)$ is odd, namely, $n = 8p_1^{\alpha_1}$ with $p_1 \equiv 3 \pmod 8$. Obviously, $\varphi_8(n) = \frac{1}{2}p_1^{\alpha_1-1}(p_1-1)$ is odd in this case.

Case 3. $R_{\mathbb{P}_k} = \{7\}$.

(A) If $\alpha = 0$, by (3.2), $\varphi_8(n) = \frac{1}{8}\varphi(n) + \frac{3}{8}(-1)^{\Omega(n)}2^{\omega(n)}$. Then we must have that $\omega(n) \leq 2$ if $\varphi_8(n)$ is odd. For $\omega(n) = 2$, i.e., $n = p_1^{\alpha_1}p_2^{\alpha_2}$, it follows that

$$\varphi_8(n) = \frac{1}{2}\Big(p_1^{\alpha_1-1}p_2^{\alpha_2-1}\cdot\frac{p_1-1}{2}\cdot\frac{p_2-1}{2} + 3\cdot(-1)^{\alpha_1+\alpha_2}\Big).$$

Since $p_1 \equiv p_2 \equiv 7 \pmod 8$, we have that $\frac{p_1-1}{2}\cdot\frac{p_2-1}{2} \equiv 1 \pmod 4$ and

$$p_1^{\alpha_1-1}p_2^{\alpha_2-1}\cdot\frac{p_1-1}{2}\cdot\frac{p_2-1}{2} + 3\cdot(-1)^{\alpha_1+\alpha_2} \equiv (-1)^{\alpha_1+\alpha_2-2} + 3\cdot(-1)^{\alpha_1+\alpha_2} \equiv 0 \pmod 4,$$

which means that $\varphi_8(n)$ is even. Now for $\omega(n) = 1$, i.e., $n = p_1^{\alpha_1}$, by (3.2) we have

$$\varphi_8(n) = \frac{1}{4}\Big(p_1^{\alpha_1-1}\cdot\frac{p_1-1}{2} + 3\cdot(-1)^{\alpha_1}\Big).$$

Now from $p_1 \equiv 7 \pmod 8$, we have that $p_1 \equiv 7, 15 \pmod{16}$. If $p_1 \equiv 7 \pmod{16}$, then

$$p_1^{\alpha_1-1} \cdot \frac{p_1-1}{2} + 3 \cdot (-1)^{\alpha_1} \equiv 3 \cdot (-1)^{\alpha_1-1} + 3 \cdot (-1)^{\alpha_1} \equiv 0 \pmod 8,$$

namely, $\varphi_8(n)$ is even. Thus, $p_1 \equiv 15 \pmod{16}$, then

$$p_1^{\alpha_1-1} \cdot \frac{p_1-1}{2} + 3 \cdot (-1)^{\alpha_1} \equiv 7 \cdot (-1)^{\alpha_1-1} + 3 \cdot (-1)^{\alpha_1} \equiv 4 \pmod 8,$$

namely, $\varphi_8(n)$ is odd.

(B) If $\alpha = 1$, by (3.2), $\varphi_8(n) = \frac{1}{8}\varphi(n) + \frac{1}{8}(-1)^{\Omega(n)} 2^{\,\omega(n)-1}$. Using a similar proof as that for (A) in case 1, $\varphi_8(n)$ is odd if and only if $n = 2p_1^{\alpha_1}$ and $p_1 \equiv 7 \pmod{16}$.

(C) If $\alpha = 2$, i.e., $\omega(n) \geq 2$, by (3.2), $\varphi_8(n) = \frac{1}{8}\varphi(n) + \frac{1}{8}(-1)^{\Omega(n)-1} 2^{\,\omega(n)}$. Then we must have that $\omega(n) \leq 3$ if $\varphi_8(n)$ is odd. For $\omega(n) = 3$, i.e., $n = 4p_1^{\alpha_1}p_2^{\alpha_2}$ with $p_1 \equiv p_2 \equiv 7 \pmod 8$, we know that

$$\varphi_8(n) = p_1^{\alpha_1-1}p_2^{\alpha_2-1} \cdot \frac{p_1-1}{2} \cdot \frac{p_2-1}{2} + (-1)^{\alpha_1+\alpha_2-1}$$

is always even. Now for $\omega(n) = 2$, i.e., $n = 4p_1^{\alpha_1}$ with $p_1 \equiv 7 \pmod 8$, we can verify that

$$\varphi_8(n) = \frac{1}{2}\Big(p_1^{\alpha_1-1} \cdot \frac{p_1-1}{2} + (-1)^{\alpha_1-1}\Big)$$

is also even.

(D) If $\alpha \geq 3$, by (3.2), $\varphi_8(n) = \frac{1}{8}\varphi(n) = 2^{\alpha-4}\prod_{i=1}^{k} p_i^{\alpha_i-1}(p_i-1)$. Hence, by $R_{\mathbb{P}_k} = \{7\}$ we know that $\varphi_8(n)$ is odd if and only if $\alpha = 3$ and $k = 1$, i.e., $n = 8p_1^{\alpha_1}$ with $p_1 \equiv 7 \pmod 8$.

Case 4. $\{3, 5\} \subseteq R_{\mathbb{P}_k}$ or $1 \in R_{\mathbb{P}_k}$.

(A) If $\{3, 5\} \subseteq R_{\mathbb{P}_k}$, i.e., $k \geq 2$, then by (3.2) we have that $\varphi_8(n) = \frac{1}{8}\varphi(n) = \frac{1}{8}\varphi(2^\alpha)\prod_{i=1}^{k} p_i^{\alpha_i-1}(p_i-1)$. Thus we must have that $k = 2$ and $\alpha \leq 1$ if $\varphi_8(n)$ is odd, namely, $n = p_1^{\alpha_1}p_2^{\alpha_2}$ or $2p_1^{\alpha_1}p_2^{\alpha_2}$, where $p_1 \equiv 3 \pmod 8$ and $p_2 \equiv 5 \pmod 8$. Obviously, $\varphi_8(n) = \frac{1}{8}p_1^{\alpha_1-1}(p_1-1)p_2^{\alpha_2-1}(p_2-1)$ is always odd in this case.

(B) If $1 \in R_{\mathbb{P}_k}$, by (3.2), $\varphi_8(n) = \frac{1}{8}\varphi(n) = \frac{1}{8}\varphi(2^\alpha)\prod_{i=1}^{k} p_i^{\alpha_i-1}(p_i-1)$. Thus, we must have that $\alpha \leq 1$ and $k = 1$ if $\varphi_8(n)$ is odd. Namely, $n = p_1^{\alpha_1}, 2p_1^{\alpha_1}$ with $p_1 \equiv 1 \pmod 8$; then, $\varphi_8(n) = \frac{1}{8}p^{\alpha_1-1}(p_1-1)$. Obviously, $\varphi_8(n)$ is odd if and only if $p_1 \equiv 9 \pmod{16}$.

From the above, we have completed the proof of Theorem 5.1.

Theorem 5.2. If n is a positive integer, then $\varphi_{12}(n)$ is odd if and only if $n = 2^\alpha$ $(\alpha \geq 4)$, $3 \cdot 2^\alpha$ $(\alpha \geq 2)$, $2 \cdot 3^\beta$ $(\beta \geq 2)$, $4 \cdot 3^\beta$ $(\beta \geq 2)$, or if it satisfies the conditions given in Table 2.

Table 2. the conditions of $\varphi_{12}(n)$ is odd.

n	Conditions
p^α	$p \equiv 13, 17, 19, 23 \pmod{24}$;
$2p^\alpha$	$p \equiv 7, 11, 13, 17 \pmod{24}$;
$3p^\alpha$	$p \equiv 5, 7 \pmod{12}$;
$4p^\alpha$	$p \equiv 5, 7 \pmod{12}$;
$6p^\alpha$	$p \equiv 5, 7 \pmod{12}$;
$12p^\alpha$	$p \equiv 5, 11 \pmod{12}$.

Here, $p > 3$ is an odd prime and $\alpha \geq 1$.

Proof. Obviously, by the definition of $\varphi_{12}(n)$ we can get that $\varphi_{12}(n) = 0$ for $n < 12$ and $\varphi_{12}(n) = 1$ for $n = 12, 24$; then, we consider that $n > 12$ and $n \neq 24$. First, we consider the case that $n = 2^\alpha \cdot 3^\beta$.

If $\alpha = 0$, we have that $\beta \geq 3$; then, by (4.1), $\varphi_{12}(n) = \frac{1}{2}(3^{\beta-2} - (-1)^\beta)$ is even.

If $\alpha = 1$, we have that $\beta \geq 2$; then, by (4.1), $\varphi_{12}(n) = \frac{1}{2}(3^{\beta-2} - (-1)^{\beta+1})$ is odd.

If $\alpha = 2$, we have that $\beta \geq 2$; then, by (4.1), $\varphi_{12}(n) = 3^{\beta-2}$ is odd.

If $\alpha = 3$, we have that $\beta \geq 2$; then, by (4.1), $\varphi_{12}(8 \cdot 3^\beta) = 2 \cdot 3^{\beta-2}$ is even.

If $\alpha \geq 4$, then that $\beta \geq 0$. For $\beta = 0$, by (4.1), $\varphi_{12}(n) = \frac{1}{3}(2^{\alpha-3} + (-1)^\alpha)$ is odd. For $\beta = 1$, by (4.1), $\varphi_{12}(n) = \frac{1}{3}(2^{\alpha-2} + (-1)^{\alpha+1})$ is odd. For $\beta \geq 2$, by (4.1), $\varphi_{12}(2^\alpha \cdot 3^\beta) = 2^{\alpha-2} \cdot 3^{\beta-2}$, which is always even.

Next, we consider the case that $n = 2^\alpha 3^\beta n_1$, where $\alpha \geq 0, \beta \geq 0, n_1 > 1$ and $\gcd(n_1, 6) = 1$. For convenience, we set $n = 2^\alpha 3^\beta \prod_{i=1}^{k} p_i^{\alpha_i}$, where $\alpha_i \geq 1$, p_i is an odd prime and $p_i > 3\,(1 \leq i \leq k)$. By Theorem 4.2 we have the following four cases.

Case 1. $R'_{\mathbb{P}_k} = \{7, 11\}$ or $\{7\}$.

(A) $\alpha = 0$. If $\beta = 0$, i.e., $\omega(n) \geq 1$, from (4.2) we have that $\varphi_{12}(n) = \frac{1}{12}\varphi(n) + \frac{1}{4}(-1)^{\Omega(n)} 2^{\omega(n)}$. Thus, from the assumption that $R'_{\mathbb{P}_k} = \{7, 11\}$ or $\{7\}$, we must have that $\omega(n) \leq 2$ if $\varphi_{12}(n)$ is odd. For $\omega(n) = 2$, i.e., $n = p_1^{\alpha_1} p_2^{\alpha_2}$, note that $R'_{\mathbb{P}_k} = \{7, 11\}$ or $\{7\}$; then,

$$\varphi_{12}(n) = \frac{1}{3}\, p_1^{\alpha_1-1}\, p_2^{\alpha_2-1} \cdot \frac{p_1 - 1}{2} \cdot \frac{p_2 - 1}{2} + (-1)^{\alpha_1+\alpha_2}$$

is always even in this case. Thus, $\omega(n) = 1$, i.e., $n = p_1^{\alpha_1}$; then, by (4.5),

$$\varphi_{12}(n) = \frac{1}{12}(p_1^{\alpha_1-1}(p_1 - 1) + 6 \cdot (-1)^{\alpha_1}).$$

Note that $p_1 \equiv 7 \pmod{12}$, i.e., $p_1 \equiv 7, 19 \pmod{24}$. If $p_1 \equiv 7 \pmod{24}$, then $p_1^{\alpha_1-1}(p-1) + 6 \cdot (-1)^{\alpha_1} \equiv 0 \pmod{24}$, which means that $\varphi_{12}(n)$ is even. Thus, $p_1 \equiv 19 \pmod{24}$; then, $p_1^{\alpha_1-1}(p_1 - 1) + 6 \cdot (-1)^{\alpha_1} \equiv 12 \pmod{24}$, namely, $\varphi_{12}(n)$ is odd.

If $\beta = 1$, i.e., $\omega(n) \geq 2$, by (4.2), $\varphi_{12}(n) = \frac{1}{12}\varphi(n) = \frac{1}{6}\prod_{i=1}^{k} p_i^{\alpha_i-1}(p_i - 1)$. Similarly, we must have that $\omega(n) = 2$ if $\varphi_{12}(n)$ is odd. In this case, $n = 3p_1^\alpha$ with $p_1 \equiv 7 \pmod{12}$; then, $\varphi_{12}(n) = \frac{1}{6}p_1^{\alpha-1}(p_1 - 1)$, easy to see that $\varphi_{12}(n)$ is always even. If $\beta \geq 2$, i.e., $\omega(n) \geq 2$, by (4.2), $\varphi_{12}(n) = \frac{1}{12}\varphi(n) + \frac{1}{4}(-1)^{\Omega(n)+1} 2^{\omega(n)}$. Similarly, we must have that $\omega(n) = 2$, i.e., $n = 3^\beta p_1^{\alpha_1}$ if $\varphi_{12}(n)$ is odd. Since $p_1 \equiv 7 \pmod{12}$, it follows that $\varphi_{12}(n) = 3^{\beta-2} p_1^{\alpha_1-1} \cdot \frac{p_1-1}{2} + (-1)^{\beta+\alpha_1+1}$ is always even.

(B) $\alpha = 1$. If $\beta = 0$, i.e., $\omega(n) \geq 2$, by (4.10), $\varphi_{12}(n) = \frac{1}{12}\varphi(n) + \frac{1}{4}(-1)^{\Omega(n)} 2^{\omega(n)-1}$. Similarly, we must have that $\omega(n) \leq 3$ if $\varphi_{12}(n)$ is odd. For $\omega(n) = 3$, i.e., $n = 2p_1^{\alpha_1} p_2^{\alpha_2}$, note that $R'_{\mathbb{P}_k} = \{7, 11\}$ or $\{7\}$; then, it is easy to see that $\varphi_{12}(n)$ is always even. Thus, $\omega(n) = 2$, i.e., $n = 2p_1^{\alpha_1}$; note that $p_1 \equiv 7 \pmod{12}$, namely, $p_1 \equiv 7, 19 \pmod{24}$. In this case, $\varphi_{12}(n)$ is odd if and only if $p_1 \equiv 7 \pmod{24}$.

If $\beta = 1$, i.e., $\omega(n) \geq 3$, by (4.2), $\varphi_{12}(n) = \frac{1}{12}\varphi(n) = \frac{1}{6}\prod_{i=1}^{k} p_i^{\alpha_i-1}(p_i - 1)$. Similarly, from $R'_{\mathbb{P}_k} = \{7, 11\}$ or $\{7\}$, we can get that $\varphi_{12}(n)$ is odd if and only if $\omega(n) = 3$, i.e., $n = 6p_1^\alpha$ with $p_1 \equiv 7 \pmod{12}$.

If $\beta \geq 2$, i.e., $\omega(n) \geq 3$, by(4.2), $\varphi_{12}(n) = \frac{1}{12}\varphi(n) + \frac{1}{4}(-1)^{\Omega(n)+1} 2^{\omega(n)-1}$. Then we must have that $\omega(n) = 3$ if $\varphi_{12}(n)$ is odd, namely, $n = 2 \cdot 3^\beta p_1^{\alpha_1}$ with $p_1 \equiv 7 \pmod{12}$. Obviously, $\varphi_{12}(n) = 3^{\beta-2} p^{\alpha-1} \cdot \frac{p-1}{2} + (-1)^{2+\beta+\alpha}$ is always even in this case.

(C) $\alpha = 2$. If $\beta = 0$, i.e., $\omega(n) \geq 2$, by (4.2), $\varphi_{12}(n) = \frac{1}{12}\varphi(n) = \frac{1}{6}\prod_{i=1}^{k} p_i^{\alpha_i-1}(p_i - 1)$. Thus, we must have that $\omega(n) = 2$ if $\varphi_{12}(n)$ is odd. In this case, $n = 4p_1^{\alpha_1}$ with $p_1 \equiv 7 \pmod{12}$; then, $p_1^{\alpha_1}(p_1 - 1) \equiv 6 \pmod{12}$, which means that $\varphi_{12}(n)$ is always odd in this case.

If $\beta \geq 1$, i.e., $\omega(n) \geq 3$, by (4.2), $\varphi_{12}(n) = \frac{1}{12}\varphi(n) = 3^{\beta-2}\prod_{i=1}^{k} p_i^{\alpha_i-1}(p_i-1)$. Note that $R'_{\mathbb{P}_k} = \{7, 11\}$ or $\{7\}$; then, $\varphi_{12}(n)$ is always even.

(D) $\alpha \geq 3$. By (4.2) and $R'_{\mathbb{P}_k} = \{7, 11\}$ or $\{7\}$, $\varphi_{12}(n) = \frac{1}{12}\varphi(n) = \frac{1}{3}\cdot 2^{\alpha-3}\varphi(3^{\beta}n_1)$ is always even in this case.

Case 2. $R'_{\mathbb{P}_k} = \{5, 11\}$ or $\{5\}$.

(A) $\alpha = 0$. If $\beta = 0$, i.e., $\omega(n) \geq 1$, from (4.6), we can get that $\varphi_{12}(n) = \frac{1}{12}\varphi(n) + \frac{1}{6}(-1)^{\Omega(n)}2^{\omega(n)}$. Thus, we must have that $\omega(n) = 1$ if $\varphi_{12}(n)$ is odd, namely, $n = p_1^{\alpha_1}$ with $p_1 \equiv 5 \pmod{12}$. Hence

$$\varphi_{12}(p_1^{\alpha_1}) = \frac{1}{12}p_1^{\alpha_1-1}(p_1-1) + \frac{1}{3}(-1)^{\alpha_1} = \frac{1}{3}\Big(p_1^{\alpha_1-1}\cdot\frac{p_1-1}{4} + (-1)^{\alpha_1}\Big).$$

Note that $p_1 \equiv 5 \pmod{12}$, i.e., $p_1 \equiv 5, 17 \pmod{24}$. If $p_1 \equiv 5 \pmod{24}$, then $p_1^{\alpha-1}\cdot\frac{p_1-1}{4} + (-1)^{\alpha_1} \equiv 0 \pmod 6$, which means that $\varphi_{12}(p_1^{\alpha})$ is always even. Thus, $p_1 \equiv 17 \pmod{24}$, in this case $p_1^{\alpha_1-1}\cdot\frac{p_1-1}{4} + (-1)^{\alpha_1} \equiv 3 \pmod 6$, namely, $\varphi_{12}(p^{\alpha_1})$ is odd.

If $\beta = 1$, i.e., $\omega(n) \geq 2$, from (4.8) we have that $\varphi_{12}(n) = \frac{1}{12}\varphi(n) + \frac{1}{6}(-1)^{\Omega(n)}2^{\omega(n)-1}$. Thus, we must have that $\omega(n) = 2$ if $\varphi_{12}(n)$ is odd, namely, $n = 3p_1^{\alpha_1}$ with $p_1 \equiv 5 \pmod{12}$; in this case

$$\varphi_{12}(3p_1^{\alpha_1}) = \frac{1}{3}\Big(2\,p_1^{\alpha_1-1}\cdot\frac{p_1-1}{4} + (-1)^{\alpha_1}\Big)$$

is always odd.

If $\beta \geq 2$, i.e., $\omega(n) \geq 2$, from (4.9) we can get that $\varphi_{12}(n) = \frac{1}{12}\varphi(n)$. We must have that $\omega(n) = 2$, i.e., $n = 3^{\beta}p^{\alpha}(\beta \geq 2)$, if $\varphi_{12}(n)$ is odd. From the assumption $p_1 \equiv 5 \pmod{12}$, $\varphi_{12}(3^{\beta}p_1^{\alpha_1}) = \frac{1}{12}\varphi(3^{\beta}p_1^{\alpha_1}) = 2\cdot 3^{\beta-1}p_1^{\alpha_1-1}\cdot\frac{p_1-1}{4}$ is always even.

(B) $\alpha = 1$, i.e., $\omega(n) \geq 2$. By (4.10)–(4.12), we must have $\omega(n) \leq 3$ if $\varphi_{12}(n)$ is odd. Namely, $n = 2p_1^{\alpha_1}, 2p_1^{\alpha_1}p_2^{\alpha_2}, 6p_1^{\alpha_1}$, or $2\cdot 3^{\beta}p_1^{\alpha_1}$ $(\beta \geq 2)$. Similar to the proof of (A) in case 1, $\varphi_{12}(n)$ is odd if and only if $n = 2p_1^{\alpha_1}$ with $p_1 \equiv 17 \pmod{24}$, or if $n = 6p_1^{\alpha_1}$ with $p_1 \equiv 5 \pmod{12}$.

(C) $\alpha = 2$. If $\beta = 0$, i.e., $\omega(n) \geq 2$, by (4.13), $\varphi_{12}(n) = \frac{1}{12}\varphi(n) + \frac{1}{12}(-1)^{\Omega(n)+1}2^{\omega(n)}$; then, we must have that $\omega(n) = 2$ if $\varphi_{12}(n)$ is odd. In this case, $n = 4p_1^{\alpha_1}$ with $p_1 \equiv 5 \pmod{12}$. Hence, $\varphi_{12}(n) = \frac{1}{6}p_1^{\alpha_1-1}(p_1-1) + \frac{1}{3}(-1)^{\alpha_1+3} = \frac{1}{3}\Big(p_1^{\alpha_1-1}\frac{p_1-1}{2} + (-1)^{\alpha_1+3}\Big)$ is always odd.

If $\beta = 1$, i.e., $\omega(n) \geq 3$, by (4.14), $\varphi_{12}(n) = \frac{1}{12}\varphi(n) + \frac{1}{12}(-1)^{\Omega(n)+1}2^{\omega(n)-1}$; we must have that $\omega(n) = 3$ if $\varphi_{12}(n)$ is odd. In this case, $n = 12\,p_1^{\alpha_1}$ with $p_1 \equiv 5 \pmod{12}$; then, $\varphi_{12}(n) = \frac{1}{3}p_1^{\alpha_1-1}(p_1-1) + \frac{1}{3}(-1)^{\alpha_1+4} = \frac{1}{3}(p_1^{\alpha_1-1}(p_1-1) + (-1)^{\alpha_1+4})$ is odd.

If $\beta \geq 2$, i.e., $\omega(n) \geq 3$, by (4.15), $\varphi_{12}(n) = \frac{1}{12}\varphi(n)$; we must have that $\omega(n) = 3$ if $\varphi_{12}(n)$ is odd. Namely, $n = 4\cdot 3^{\beta}p_1^{\alpha_1}$ with $p_1 \equiv 5 \pmod{12}$; then,

$$\varphi_{12}(n) = \frac{1}{12}\varphi(n) = 3^{\beta-2}p_1^{\alpha_1-1}(p_1-1)$$

is always even.

(D) $\alpha \geq 3$, i.e., $\omega(n) \geq 2$. If $\beta = 0$, then by (4.16) and $R'_{\mathbb{P}_k} = \{5, 11\}$ or $\{5\}$, we konw that $\varphi_{12}(n) = \frac{1}{12}\varphi(n) + \frac{1}{6}(-1)^{\Omega(n)}2^{\omega(n)}$ is always even in this case.

If $\beta = 1$, by (4.17) and $R'_{\mathbb{P}_k} = \{5, 11\}$ or $\{5\}$, we know that $\varphi_{12}(n) = \frac{1}{12}\varphi(n) + \frac{1}{12}(-1)^{\Omega(n)}2^{\omega(n)}$ is always even in this case.

If $\beta \geq 2$, by (4.18) and $R'_{\mathbb{P}_k} = \{5, 11\}$ or $\{5\}$, $\varphi_{12}(n) = \frac{1}{12}\varphi(n)$ is always even in this case.

Case 3. $R'_{\mathbb{P}_k} = \{11\}$.
(A) $\alpha = 0$, i.e., $\omega(n) \geq 1$. From (4.7)–(4.9), we must have that $\omega(n) \leq 2$ if $\varphi_{12}(n)$ is odd. Consider that $\omega(n) = 2$, i.e., $n = p_1^{\alpha_1} p_2^{\alpha_2}$, or $3^\beta p_1^{\alpha_1}$ $(\beta \geq 1)$ with $p_1 \equiv p_2 \equiv 11 \pmod{12}$. Thus, by (4.7)–(4.9), $\varphi_{12}(n)$ is always even. Hence, $\omega(n) = 1$, i.e., $n = p_1^{\alpha_1}$ with $p \equiv 11 \pmod{12}$; then, (4.7) we can get

$$\varphi_{12}(p_1^{\alpha_1}) = \frac{1}{12} p_1^{\alpha_1 - 1}(p_1 - 1) + \frac{5}{6}(-1)^{\alpha_1} = \frac{1}{6}\Big(p_1^{\alpha_1 - 1} \cdot \frac{p_1 - 1}{2} + 5(-1)^{\alpha_1}\Big).$$

Note that $p_1 \equiv 11 (\mathrm{mod}\, 12)$, i.e., $p_1 \equiv 11, 23 \pmod{24}$. If $p_1 \equiv 11 \pmod{24}$, then

$$p_1^{\alpha_1 - 1} \cdot \frac{p_1 - 1}{2} + 5(-1)^{\alpha_1} \equiv 5(-1)^{\alpha_1 - 1} + 5(-1)^{\alpha_1} \equiv 0 \pmod{12},$$

namely, $\varphi_{12}(n)$ is even. If $p_1 \equiv 23 \pmod{24}$, then

$$p_1^{\alpha_1 - 1} \cdot \frac{p_1 - 1}{2} + 5(-1)^{\alpha_1} \equiv 11(-1)^{\alpha_1 - 1} + 5(-1)^{\alpha_1} \equiv 6 \pmod{12},$$

namely, $\varphi_{12}(n)$ is odd.
(B) $\alpha = 1$, i.e., $\omega(n) \geq 2$. From (4.10)–(4.12), we must have that $\omega(n) \leq 3$ if $\varphi_{12}(n)$ is odd. Namely, $n = 2p_1^{\alpha_1}, 2p_1^{\alpha_1} p_2^{\alpha_2}, 6p_1^{\alpha_1}$, or $2 \cdot 3^\beta p_1^{\alpha_1}$ $(\beta \geq 2)$ with $p_1 \equiv p_2 \equiv 11 \pmod{12}$. Using the same method as for (A) in case 1, $\varphi_{12}(n)$ is odd if and only if $n = 2p_1^{\alpha_1}$ with $p_1 \equiv 11 \pmod{24}$.
(C) $\alpha = 2$, i.e., $\omega(n) \geq 2$. If $\beta = 0$, by (4.13), we must have that $\omega(n) = 2$ if $\varphi_{12}(n)$ is odd, namely, $n = 4p_1^{\alpha_1}$ with $p_1 \equiv 11 \pmod{12}$. Then by (4.13),

$$\varphi_{12}(4p_1^{\alpha_1}) = \frac{1}{3}\Big(p_1^{\alpha_1 - 1}\frac{p_1 - 1}{2} + (-1)^{\alpha_1 + 3}\Big)$$

is always even.

If $\beta \geq 1$, i.e., $\omega(n) \geq 3$, by (4.14)–(4.15), we must have that $\omega(n) = 3$ if $\varphi_{12}(n)$ is odd. Namely, $n = 4 \cdot 3^\beta p_1^{\alpha_1}$ $(\beta \geq 1)$ with $p_1 \equiv 11 \pmod{12}$. If $\beta \geq 2$, then by (4.15),

$$\varphi_{12}(4 \cdot 3^\beta p_1^{\alpha_1}) = \frac{1}{12}\varphi(4 \cdot 3^\beta p_1^{\alpha_1}) = 3^{\beta - 2}\, p_1^{\alpha_1 - 1}(p_1 - 1)$$

is always even. Thus, $\beta = 1$; by (4.14), $\varphi_{12}(12\, p_1^{\alpha_1}) = \frac{1}{3}(p_1^{\alpha_1 - 1}(p_1 - 1) + (-1)^{\alpha_1 + 3})$ is odd.
(D) $\alpha \geq 3$. If $\beta = 0$, i.e., $\omega(n) \geq 2$, then by (4.16) and $R'_{\mathbb{P}_k} = \{11\}$, we know that $\varphi_{12}(n) = \frac{1}{12}\varphi(n) + \frac{1}{6}(-1)^{\Omega(n)}\, 2^{\,\omega(n)}$ is always even in this case.

If $\beta = 1$, i.e., $\omega(n) \geq 3$, then by (4.17) and $R'_{\mathbb{P}_k} = \{11\}$, we know that $\varphi_{12}(n) = \frac{1}{12}\varphi(n) + \frac{1}{12}(-1)^{\Omega(n)}\, 2^{\,\omega(n)}$ is always even in this case.

If $\beta \geq 2$, i.e., $\omega(n) \geq 3$, then by (4.18) and $R'_{\mathbb{P}_k} = \{11\}$, we know that $\varphi_{12}(n) = \frac{1}{12}\varphi(n) = 2^{\alpha - 2} \cdot 3^{\beta - 1} \prod_{i=1}^{k} p_i^{\alpha_i - 1}(p_i - 1)$ is always even.
Case 4. $\{5, 7\} \subseteq R'_{\mathbb{P}_k}$ or $1 \in R'_{\mathbb{P}_k}$.
(A) If $\{5, 7\} \subseteq R'_{\mathbb{P}_k}$, by (4.2) we have that $\varphi_{12}(n) = \frac{1}{12}\varphi(n)$ is always even.
(B) If $1 \in R'_{\mathbb{P}_k}$, then by (4.2), $\varphi_{12}(n) = \frac{1}{12}\varphi(n)$; thus, we must have that $k = 1$, $\alpha \leq 1$, and $\beta = 0$ if $\varphi_{12}(n)$ is odd. Namely, $n = p_1^{\alpha_1}$ or $2p_1^{\alpha_1}$ with $p_1 \equiv 1 \pmod{12}$. In this case, $\varphi_{12}(n) = \frac{1}{12}\, p_1^{\alpha_1 - 1}(p_1 - 1)$ is odd if and only if $p_1 \equiv 13 \pmod{24}$.

From the above, we have completed the proof of Theorem 5.2.

6. Final remark

In [2, 8], Cai, et al. gave the explicit formulae for the generalized Euler functions denoted by $\varphi_e(n)$ for $e = 3, 4, 6$. The key point is that the derivation of $[\frac{n}{e}]$ can be obtained by utilizing the corresponding Jacobi symbol for $e = 3, 4, 6$. In the present paper, by applying Lemmas 2.1 and 2.2, the exact formulae for $\varphi_8(n)$ and $\varphi_{12}(n)$ have been given and the parity has been determined. Therefore, the obvious expression for $[\frac{n}{e}]$ depends on the Jacobi symbol, seems to be the key to finding the exact formulae for $\varphi_e(n)$.

We propose the following conjecture.

Conjecture 6.1. Let $e > 1$ be a given integer. For any integer $d > 2$ with $\gcd(d, e) = 1$, there exist $u \in \mathbb{Q}$, a_1, a_2, a_3, $b_i\, (1 \le j \le r) \in \mathbb{Z}$, and $q_j\, (1 \le j \le r) \in \mathbb{P}$, such that

$$\Big[\frac{d}{e}\Big] = u\Big(a_1 d + a_2 + a_3\Big(\frac{-1}{d}\Big) + \sum_{j=1}^{r} b_j\Big(\frac{\varepsilon_j\, q_j}{d}\Big)\Big) \ \ (2 \nmid d\,), \tag{6.1}$$

or

$$\Big[\frac{d}{e}\Big] = u\Big(a_1 d + a_2 + \sum_{j=1}^{r} b_j\Big(\frac{\varepsilon_j\, d}{q_j}\Big)\Big) \ \ (2 \mid d\,), \tag{6.2}$$

where $r \ge 1$ and $\varepsilon_j \in \{1, -1\}$.

It is easy to see that Conjecture 6.1 is true for $e = 2, 3, 4, 6, 8$ and 12. (see [2, 8] and (2.1), (2.2)). If the formulas for (6.1) and (6.2) in the above conjecture can be obtained, then, by (1.1), using the properties of Möbius functions, we can find the exact formulae for $\varphi_e(n)$.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

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