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Research article

Stability analysis for a Rao-Nakra sandwich beam equation with time-varying weights and frictional dampings

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Abstract: In this paper, we studied the asymptotic behavior of solutions for a Rao-Nakra sandwich beam equation with time-varying weights and frictional damping terms acting complementarily in the domain. We studied the effect of the three damping on the asymptotic behavior of the energy function. Under nonrestrictive on the growth assumption on the frictional damping terms, we established exponential and general energy decay rates for this system by using the multiplier approach. The results generalized some earlier decay results on the Rao-Nakra sandwich beam equation.

Keywords: multilayer beam; multiplier method; stability
Mathematics Subject Classification: 35B40, 93D15, 93D20

1. Introduction

In this paper, we consider the following Rao-Nakra sandwich beam with time-varying weights and frictional dampings in $(x,t) \in (0,L) \times (0,\infty)$,

$$\begin{aligned} \rho_1 h_1 u_{tt} - E_1 h_1 u_{xx} - k(-u+v+\alpha w_x) + \alpha_1(t) f_1(u_t) &= 0, \\ \rho_3 h_3 v_{tt} - E_3 h_3 v_{xx} + k(-u+v+\alpha w_x) + \alpha_2(t) f_2(v_t) &= 0, \\ \rho h w_{tt} + EI w_{xxxx} - \alpha k(-u+v+\alpha w_x)_x + \alpha_3(t) f_3(w_t) &= 0, \end{aligned} \tag{1.1}$$

in which u and v denote the longitudinal displacement and shear angle of the bottom and top layers, and w represents the transverse displacement of the beam. The positive constants ρ_i, h_i, and E_i $(i = 1, 3)$ are physical parameters representing, respectively, density, thickness, and Young's modulus of the i-th layer for $i = 1, 2, 3$ and $\rho h = \rho_1 h_1 + \rho_2 h_2 + \rho_3 h_3$. $EI = E_1 I_1 + E_3 I_3$, $\alpha = h_2 + \left(\frac{h_1+h_3}{2}\right)$, $k = \frac{1}{h_2}\left(\frac{E_2}{2(1+\mu)}\right)$, where μ is the Poisson ratio $-1 < \mu < \frac{1}{2}$. The functions $\alpha_1(t)$, $\alpha_2(t)$, $\alpha_3(t)$ are the time dependent of the nonlinear frictional dampings $f_1(u_t)$, $f_2(v_t)$, $f_3(w_t)$, respectively. For the system (1.1), we consider the following Dirichlet-Neumann boundary conditions

$$
\begin{aligned}
u(0,t) = u(L,t) &= 0, \ \ t \in (0,\infty),\\
v(0,t) = v(L,t) &= 0, \ \ t \in (0,\infty),\\
w(0,t) = w_x(0,t) &= 0, \ \ t \in (0,\infty),\\
w(L,t) = w_x(L,t) &= 0, \ \ t \in (0,\infty),
\end{aligned}
\tag{1.2}
$$

and the initial conditions

$$
\begin{aligned}
u(x,0) = u_0, \ \ u_t(x,0) &= u_1 \ \text{ in } (0,L),\\
v(x,0) = v_0, \ \ v_t(x,0) &= v_1 \ \text{ in } (0,L),\\
w(x,0) = w_0, \ \ w_t(x,0) &= w_1 \ \text{ in } (0,L).
\end{aligned}
\tag{1.3}
$$

The system (1.1)–(1.3) consists of one Euler-Bernoulli beam equation for the transversal displacement, and two wave equations for the longitudinal displacements of the top and bottom layers.

Our aim is to investigate the asymptotic behavior of solutions for the system (1.1)–(1.3). We study the effect of the three nonlinear dampings on the asymptotic behavior of the energy function. Under nonrestrictive on the growth assumption on the frictional damping terms, we establish exponential and general energy decay rates for this system by using the multiplier approach. The results generalize some earlier decay results on the Rao-Nakra sandwich beam equation.

First, let's review some previous findings about multilayered sandwich beam models. By applying the Riesz basis approach, Wang et al. [1] studied a sandwich beam system with a boundary control and established the exponential stability as well as the exact controllability and observability of the system. The author of [2] employed the multiplier approach to determine the precise controllability of a Rao-Nakra sandwich beam with boundary controls rather than the Riesz basis approach. Hansen and Imanuvilov [3, 4] investigated a multilayer plate system with locally distributed control in the boundary and used Carleman estimations to determine the precise controllability results. Özer and Hansen [5, 6] succeeded in obtaining, for a multilayer Rao-Nakra sandwich beam, boundary feedback stabilization and perfect controllability. One viscous damping effect on either the beam equation or one of the wave equations was all that was taken into account by Liu et al. [7] when they established the polynomial decay rate using the frequency domain technique. The semigroup created by the system is polynomially stable of order 1/2, according to Wang [8], who analyzed a Rao-Nakra beam with boundary damping only on one end for two displacements using the same methodology. One can find additional results on the multilayer beam in [9–15].

In this paper, we consider a Rao-Nakra sandwich beam with time-varying weights and frictional dampings, i.e., system (1.1)–(1.3). The main results are twofold:
(I) We establish an exponential decay of the system in the case of linear frictional dampings by using the multiplier approach, and the decay result depends on the time-varying weights α_i.

(II) We establish more general decay of the system in the case of nonlinear frictional dampings by using the multiplier approach, and the decay result depends on the time-varying weights α_i and the frictional dampings f_i. To the best of our knowledge, there is no stability results on the Rao-Nakra sandwich beam with nonlinear frictional dampings. The remainder of the paper is as follows. In Section 2, we introduce some notations and preliminary results. In Section 3, we state the theorem of the stability and give a detailed proof.

2. Preliminaries

In this section, we present some materials needed in the proof of our results. Throughout this paper, c and ε are used to denote generic positive constants. We consider the following assumptions:

$(H1)$ For $(i = 1, 2, 3)$, the functions $f_i : \mathbb{R} \to \mathbb{R}$ are C^0 nondecreasing satisfying, for $c_1, c_2 > 0$,

$$\begin{aligned} & s^2 + f_i^2(s)| \le F_i^{-1}(s f_i(s)) \quad \text{for all } |s| \le r_i, \\ & c_1|s| \le |f_i(s)| \le c_2|s| \qquad \text{for all } |s| \ge r_i, \end{aligned} \tag{2.1}$$

where $F_i : (0, \infty) \to (0, \infty)$ $(i = 1, 2, 3)$ are C^1 functions, which are linear or strictly increasing and strictly convex C^2 functions on $(0, r_i]$ with $F_i(0) = F_i'(0) = 0$.

$(H2)$ For $(i = 1, 2, 3)$, the time dependent functions $\alpha_i : [0, \infty) \to (0, \infty)$ are C^1 functions satisfying $\int_0^\infty \alpha_i(t)dt = \infty$.

Remark 2.1. (1) Hypothesis $(H1)$ implies that $sf_i(s) > 0$, for all $s \neq 0$. This condition $(H1)$ was introduced and employed by Lasiecka and Tataru [16]. It was shown there that the monotonicity and continuity of f_i guarantee the existence of the function F_i with the properties stated in $(H1)$.

(2) For more results on the convexity properties on the nonlinear frictional dampings and sharp energy decay rates, we refer to the works by Boussouira and her co-authors [17–19].

3. Essential lemmas

The following lemmas will be of essential use in establishing our main results.

Lemma 3.1. *[20] Let $E : \mathbb{R}^+ \to \mathbb{R}^+$ be a nonincreasing function and $\gamma : \mathbb{R}^+ \to \mathbb{R}^+$ be a strictly increasing C^1-function, with $\gamma(t) \to +\infty$ as $t \to +\infty$. Assume that there exists $c > 0$ such that*

$$\int_S^\infty \gamma'(t)E(t)dt \le cE(S), \qquad 1 \le S < +\infty,$$

then there exist positive constants k and ω such that

$$E(t) \le ke^{-\omega\gamma(t)}.$$

Lemma 3.2. *Let $E : \mathbb{R}^+ \to \mathbb{R}^+$ be a differentiable and nonincreasing function and let $\chi : \mathbb{R}^+ \to \mathbb{R}^+$ be a convex and increasing function such that $\chi(0) = 0$. Assume that*

$$\int_s^{+\infty} \chi\left(E(t)\right) dt \le E(s), \qquad \forall s \ge 0, \tag{3.1}$$

then E satisfies the estimate

$$E(t) \leq \psi^{-1}\left(h(t) + \psi\left(E(0)\right)\right), \quad \forall t \geq 0, \tag{3.2}$$

where $\psi(t) = \int_t^1 \frac{1}{\chi(s)} ds$ *for* $t > 0$*, and*

$$\begin{cases} h(t) = 0, & 0 \leq t \leq \dfrac{E(0)}{\chi(E(0))}, \\ h^{-1}(t) = t + \dfrac{\psi^{-1}\left(t + \psi(E(0))\right)}{\chi\left(\psi^{-1}(t + \psi(E(0)))\right)}, & t > 0. \end{cases}$$

Proof. Since $E'(t) \leq 0$, this implies $E(0) \leq E(t_0)$ for all $t \geq t_0 \geq 0$. If $E(t_0) = 0$ for $t_0 \geq 0$, then $E(t) = 0$ for all $t \geq t_0 \geq 0$, and there is nothing to prove in this case. As in [21], we assume that $E(t) > 0$ for $t \geq 0$ without loss of generality. Let

$$L(s) = \int_s^\infty \chi\left(E(t)\right) dt, \ \forall \ s \geq 0.$$

We have $L(s) \leq E(s)$, $\forall \ s \geq 0$. The functional L is positive, decreasing, and of class $C^1(0, \infty)$ satisfying

$$-L'(s) = \chi\left(E(s)\right) \geq \chi\left(L(s)\right), \ \forall \ s \geq 0.$$

Since the functional χ is decreasing, we have

$$\chi\left(L(s)\right)' = -\frac{L'(s)}{\chi\left(L(s)\right)} \geq 1, \ \forall \ s \geq 0.$$

Integrating this differential equation over $(0, t)$, we get

$$\chi\left(L(t)\right) \geq t + \psi\left(E(0)\right), \ \forall \ t \geq 0. \tag{3.3}$$

Since χ is convex and $\chi(0) = 0$, we have

$$\chi(s) \leq s\chi(1), \ \forall \ s \in [0, 1] \ \text{ and } \ \chi(s) \geq s\chi(1), \ \forall \ s \geq 1.$$

We find $\lim_{t \to 0} \psi(t) = \infty$ and $[\psi\left(E(0)\right), \infty)) \subset \text{Image}\left(\psi\right)$, then (3.3) imply that

$$L(t) \leq \psi^{-1}\left(t + \psi\left(E(0)\right)\right), \ \forall \ t \geq 0. \tag{3.4}$$

Now, using the properties of χ and E, we have

$$L(s) \geq \int_s^t \chi\left(E(\tau)\right) d\tau \geq (t - s)\chi\left(E(t)\right), \ \forall \ t \geq s \geq 0. \tag{3.5}$$

Since $\lim_{t \to 0} \chi(t) = \infty$, $\chi(0) = 0$, and χ is increasing, (3.4) and (3.5) imply that

$$E(t) \geq \chi^{-1}\left(\inf_{s \in [0,t)} \frac{\psi^{-1}\left(t + \psi\left(E(0)\right)\right)}{t - s}\right), \ \forall \ t > 0. \tag{3.6}$$

Now, let $t > \frac{E(0)}{\chi(E(0))}$ and $J(s) = \dfrac{\psi^{-1}\left(t + \psi\left(E(0)\right)\right)}{t - s}$, $s \in [0, t)$.

The function J is differentiable, then we have

$$J'(s) = (t-s)^{-2}\left[\psi^{-1}\left(s + \psi\left(E(0)\right)\right) - (t-s)\chi\left(\psi^{-1}\left(s + \psi\left(E(0)\right)\right)\right)\right]. \tag{3.7}$$

Thus, $J'(s) = 0 \Leftrightarrow K(s) = t$ and $J'(s) < 0 \Leftrightarrow K(s) < t$, where

$$K(t) = t + \frac{\psi^{-1}\left(t + \psi\left(E(0)\right)\right)}{\chi\left(\psi^{-1}\left(t + \psi\left(E(0)\right)\right)\right)}.$$

Since $K(0) = \frac{E(0)}{\chi(E(0))}$ and K is increasing (because ψ^{-1} is decreasing and $s \to \frac{s}{\chi(s)}, s > 0$ is nonincreasing thanks to the fact χ is convex), for $t > \frac{E(0)}{\chi(E(0))}$, we have

$$\inf_{s\in[0,t)} J\left(K^{-1}(t)\right) = J\left(h(t)\right).$$

Since h satisfies $J'\left(h(t)\right) = 0$, we conclude from (3.6) our desired estimate (3.2). □

We define the energy associated to the problem (1.1)–(1.3) by the following formula

$$\begin{aligned} E(t) =& \frac{1}{2}\Big[\rho_1 h_1 \int_0^L u_t^2\,dx + E_1 h_1 \int_0^L u_x^2\,dx + \rho_3 h_3 \int_0^L v_t^2\,dx + E_3 h_3 \int_0^L v_x^2\,dx \\ & + \rho h \int_0^L w_t^2\,dx + EI \int_0^L w_{xx}^2\,dx + k \int_0^L (-u + v + \alpha w_x)^2\,dx\Big]. \end{aligned} \tag{3.8}$$

Lemma 3.3. *The energy $E(t)$ satisfies*

$$E'(t) \le -\alpha_1(t) \int_0^L u_t f_1\left(u_t\right) dx - \alpha_2(t) \int_0^L v_t f_2\left(v_t\right) dx - \alpha_3(t) \int_0^L w_t f_3\left(w_t\right) dx \qquad \le 0. \tag{3.9}$$

Proof. Multiplying $(1.1)_1$ by u_t, $(1.1)_2$ by v_t, $(1.1)_3$ by w_t and integrating each of them by parts over $(0, L)$, we get the desired result. □

4. Stability result

In this section, we state and prove our main result. For this purpose, we establish some lemmas. From now on, we denote by c various positive constants, which may be different at different occurrences. For simplicity, we consider the case $\alpha(t) = \alpha_1(t) = \alpha_2(t) = \alpha_3(t)$.

Lemma 4.1. ***(Case: F_i is linear)*** *For $T > S \ge 0$, the energy functional of the system* (1.1)–(1.3) *satisfies*

$$\int_S^T \alpha(t)E(t)dt \le cE(S). \tag{4.1}$$

Proof. Multiplying $(1.1)_1$ by αu and integrating over $(S, T) \times (0, L)$, we obtain

$$\int_S^T \alpha(t) \int_0^L u\left[\rho_1 h_1 u_{tt} - E_1 h_1 u_{xx} - k\left(-u + v + \alpha w_x\right) + \alpha(t) f_1\left(u_t\right)\right] dx\,dt = 0.$$

Notice that

$$u_{tt}u = (u_t u)_t - u_t^2.$$

Using integration by parts and the boundary conditions, we get

$$\begin{aligned}
&- \int_S^T \alpha(t) \int_0^L \rho_1 h_1 u_t^2 \, dx \, dt + \int_S^T \alpha(t) \int_0^L E_1 h_1 u_x^2 \, dx \, dt - \int_S^T \alpha(t) \int_0^L ku(-u + v + \alpha w_x) \, dx \, dt \\
&= -\rho_1 h_1 \left[\alpha(t) \int_0^L u u_t \, dx \right]_S^T - \int_S^T \alpha^2(t) \int_0^L u f_1 (u_t) \, dx \, dt. \qquad (4.2)
\end{aligned}$$

Adding $2 \int_S^T \alpha(t) \int_0^L \rho_1 h_1 u_t^2 dxdt$ to both sides of the above equation, we have

$$\begin{aligned}
&\int_S^T \alpha(t) \int_0^L \rho_1 h_1 u_t^2 \, dx \, dt + \int_S^T \alpha(t) \int_0^L E_1 h_1 u_x^2 \, dx \, dt - \int_S^T \alpha(t) \int_0^L ku(-u + v + \alpha w_x) \, dx \, dt \\
&= -\rho_1 h_1 \left[\alpha(t) \int_0^L u u_t \, dx \right]_S^T - \int_S^T \alpha^2(t) \int_0^L u f_1 (u_t) \, dx \, dt + 2 \int_S^T \alpha(t) \int_0^L \rho_1 h_1 u_t^2 dxdt. \qquad (4.3)
\end{aligned}$$

Similarly, multiplying $(1.1)_2$ by $\alpha(t)v$ and $(1.1)_3$ by $\alpha(t)w$, and integrating each of them over $(S, T) \times (0, L)$, we obtain

$$\begin{aligned}
&\int_S^T \alpha(t) \int_0^L \rho_3 h_3 v_t^2 \, dx \, dt + \int_S^T \alpha(t) \int_0^L E_3 h_3 v_x^2 \, dx \, dt + \int_S^T \alpha(t) \int_0^L kv(-u + v + \alpha w_x) \, dx \, dt \\
&= -\rho_3 h_3 \left[\int_0^L \alpha(t) v v_t \, dx \right]_S^T - \int_S^T \alpha^2(t) \int_0^L v f_2 (v_t) \, dx \, dt + 2 \int_S^T \alpha(t) \int_0^L \rho_3 h_3 v_t^2 \, dx \, dt, \qquad (4.4)
\end{aligned}$$

and

$$\begin{aligned}
&\int_S^T \alpha(t) \int_0^L \rho h w_t^2 \, dx \, dt + \int_S^T \alpha(t) \int_0^L EI w_{xx}^2 \, dx \, dt - \int_S^T \alpha(t) \int_0^L k\alpha w(-u + v + \alpha w_x)_x \, dx \, dt \\
&= -\rho h \left[\int_0^L \alpha(t) w w_t \, dx \right]_S^T - \int_S^T \alpha^2(t) \int_0^L w f_3 (w_t) \, dx \, dt + 2 \int_S^T \alpha(t) \int_0^L \rho h w_t^2 \, dx \, dt. \qquad (4.5)
\end{aligned}$$

Recalling the definition of E, and from (4.3)–(4.5), we get

$$\begin{aligned}
&2 \int_S^T \alpha(t) E(t) dt \\
&\leq -\rho_1 h_1 \left[\alpha(t) \int_0^L u u_t \, dx \right]_S^T - \rho_3 h_3 \left[\alpha(t) \int_0^L v v_t \, dx \right]_S^T - \rho h \left[\alpha(t) \int_0^L w w_t \, dx \right]_S^T \\
&\quad - \int_S^T \alpha^2(t) \int_0^L u f_1 (u_t) \, dxdt - \int_S^T \alpha^2(t) \int_0^L v f_2 (v_t) \, dxdt - \int_S^T \alpha^2(t) \int_0^L w f_3 (w_t) \, dxdt \\
&\quad + 2 \int_S^T \alpha(t) \int_0^L \rho_1 h_1 u_t^2 dxdt + 2 \int_S^T \alpha(t) \int_0^L \rho_3 h_3 v_t^2 dxdt + 2 \int_S^T \alpha(t) \int_0^L \rho h w_t^2 dxdt. \qquad (4.6)
\end{aligned}$$

Now, using Young's and Poincaré inequalities, we get for any $\varepsilon_i > 0$ $(i = 1, 2, 3)$,

$$\begin{aligned}
&\int_0^L uu_t dx \le \varepsilon_1 \int_0^L u^2 dx + \frac{1}{4\varepsilon_1}\int_0^L u_t^2 dx \le c_p\varepsilon_1 \int_0^L u_x^2 dx + \frac{1}{4\varepsilon_1}\int_0^L u_t^2 dx \le cE(t),\\
&\int_0^L zz_t dx \le \varepsilon_2 \int_0^L z^2 dx + \frac{1}{4\varepsilon_2}\int_0^L z_t^2 dx \le c_p\varepsilon_2 \int_0^L z^2 dx + \frac{1}{4\varepsilon_2}\int_0^L z_t^2 dx \le cE(t),\\
&\int_0^L ww_t dx \le \varepsilon_3 \int_0^L w^2 dx + \frac{1}{4\varepsilon_3}\int_0^L w_t^2 dx \le c_p\varepsilon_3 \int_0^L w^2 dx + \frac{1}{4\varepsilon_3}\int_0^L w_t^2 dx \le cE(t),
\end{aligned} \tag{4.7}$$

which implies that

$$-\left[\alpha(t)\int_0^L uu_t\,dx\right]_S^T \le c\alpha(S)E(S) - c\alpha(S)E(T) \le cE(S),$$

$$-\left[\alpha(t)\int_0^L vv_t\,dx\right]_S^T \le cE(S),$$

and

$$-\left[\alpha(t)\int_0^L ww_t\,dx\right]_S^T \le cE(S).$$

Using $(H1)$, the fact that F_1 is linear, Hölder and Poincaré inequalities, we obtain

$$\begin{aligned}
\alpha^2(t)\int_0^L uf_1(u_t)\,dx &\le \alpha^2(t)\left(\int_0^L |u|^2 dx\right)^{\frac{1}{2}}\left(\int_0^L |f_1(u_t)|^2 dx\right)^{\frac{1}{2}}\\
&\le \alpha^{\frac{3}{2}}\|u\|_2\left(\alpha\int_0^L u_t f_1(u_t)\,dx\right)^{\frac{1}{2}} \le cE^{\frac{1}{2}}(t)\left(-E'(t)\right)^{\frac{1}{2}}.
\end{aligned} \tag{4.8}$$

Applying Young's inequality on the term $E^{\frac{1}{2}}(t)(-E'(t))^{\frac{1}{2}}$, we obtain for $\varepsilon > 0$

$$\alpha^2(t)\int_0^L uf_1(u_t)\,dx \le c\alpha(t)\left(\varepsilon E(t) - C_\varepsilon E'(t)\right) \le c\varepsilon\alpha(t)E(t) - C_\varepsilon E'(t), \tag{4.9}$$

which implies that

$$\int_S^T \alpha^2(t)\left(\int_0^L (-uf_1(u_t))dx\right)dt \le c\varepsilon\int_S^T \alpha(t)E(t)dt + C_\varepsilon E(S). \tag{4.10}$$

In the same way, we have

$$\int_S^T \alpha^2(t)\left(\int_0^L (-vf_2(v_t))dx\right)dt \le c\varepsilon\int_S^T \alpha(t)E(t)dt + C_\varepsilon E(S), \tag{4.11}$$

and

$$\int_S^T \alpha^2(t)\left(\int_0^L (-wf_3(w_t))dx\right)dt \le c\varepsilon\int_S^T \alpha(t)E(t)dt + C_\varepsilon E(S). \tag{4.12}$$

Finally, using $(H1)$, and the fact that F_1 is linear, we find that

$$2\int_S^T \alpha(t)\int_0^L \rho_1 h_1 u_t^2 dxdt \le c\int_S^T \alpha(t)\int_0^L u_t f_1(u_t)\, dxdt \le c\int_S^T (-E'(t))\, dt \le cE(S). \tag{4.13}$$

Similarly, we get

$$2\int_S^T \alpha(t)\int_0^L \rho_3 h_3 v_t^2 dxdt \le cE(S), \text{ and } 2\int_S^T \alpha(t)\int_0^L \rho h w_t^2 dxdt \le cE(S). \tag{4.14}$$

We combine the above estimates and take ε small enough to get the estimate (4.1). □

Lemma 4.2. ***(Case:** F_i **are nonlinear)*** *For* $T > S \ge 0$*, the energy functional of the system* (1.1)–(1.3) *satisfies*

$$\begin{aligned}\int_S^T \alpha(t)\Lambda(E(t))\, dt \le c\Lambda(E(S)) &+ c\int_S^T \alpha(t)\frac{\Lambda(E)}{E}\int_0^L \left(|u_t|^2 + |uf_1(u_t)|\right) dxdt\\ &+ c\int_S^T \alpha(t)\frac{\Lambda(E)}{E}\int_0^L \left(|v_t|^2 + |vf_2(v_t)|\right) dxdt\\ &+ c\int_S^T \alpha(t)\frac{\Lambda(E)}{E}\int_0^L \left(|v_t|^2 + |vf_3(w_t)|\right) dxdt,\end{aligned} \tag{4.15}$$

where Λ *is convex, increasing, and of class* $C^1[0,\infty)$ *such that* $\Lambda(0) = 0$.

Proof. We multiply $(1.1)_1$ by $\alpha(t)\frac{\Lambda(E)}{E}u$ and integrate over $(0, L) \times (S, T)$ to get

$$\begin{aligned}&\rho_1 h_1\int_S^T \alpha(t)\frac{\Lambda(E)}{E}\int_0^L u_t^2 dxdt + E_1 h_1\int_S^T \alpha(t)\frac{\Lambda(E)}{E}\int_0^L u_x^2 dxdt\\ &- k\int_S^T \alpha(t)\frac{\Lambda(E)}{E}\int_0^L u(-u + v + \alpha w_x) dxdt\\ &= -\rho_1 h_1\int_S^T \alpha(t)\frac{\Lambda(E)}{E}\int_0^L (uu_t)_t\, dxdt - \int_S^T \alpha^2(t)\frac{\Lambda(E)}{E}\int_0^L uf_1(u_t)\, dxdt\\ &+ 2\rho_1 h_1\int_S^T \alpha(t)\frac{\Lambda(E)}{E}\int_0^L u_t^2 dxdt.\end{aligned} \tag{4.16}$$

Also, we multiply $(1.1)_2$ by $\alpha(t)\frac{\Lambda(E)}{E}v$ and integrate over $(0, L) \times (S, T)$ to get

$$\begin{aligned}&\rho_3 h_3\int_S^T \alpha(t)\frac{\Lambda(E)}{E}\int_0^L v_t^2 dxdt + E_3 h_3\int_S^T \alpha(t)\frac{\Lambda(E)}{E}\int_0^L v_x^2 dxdt\\ &+ k\int_S^T \alpha(t)\frac{\Lambda(E)}{E}\int_0^L v(-u + v + \alpha w_x) dxdt\\ &= -\rho_3 h_3\int_S^T \alpha(t)\frac{\Lambda(E)}{E}\int_0^L (vv_t)_t\, dxdt - \int_S^T \alpha^2(t)\frac{\Lambda(E)}{E}\int_0^L vf_2(v_t)\, dxdt\\ &+ 2\rho_3 h_3\int_S^T \alpha(t)\frac{\Lambda(E)}{E}\int_0^L v_t^2 dxdt.\end{aligned} \tag{4.17}$$

Similarly, we multiply $(1.1)_3$ by $\alpha(t)\frac{\Lambda(E)}{E}w$ and integrate over $(0, L) \times (S, T)$ to get

$$\rho h \int_S^T \alpha(t)\frac{\Lambda(E)}{E}\int_0^L w_t^2 dxdt + Eh\int_S^T \alpha(t)\frac{\Lambda(E)}{E}\int_0^L w_{xx}^2 dxdt$$
$$+ \alpha k \int_S^T \alpha(t)\frac{\Lambda(E)}{E}\int_0^L w_x(-u+v+\alpha w_x)dxdt$$
$$= -\rho h \int_S^T \alpha(t)\frac{\Lambda(E)}{E}\int_0^L (ww_t)_t\, dxdt - \int_S^T \alpha^2(t)\frac{\Lambda(E)}{E}\int_0^L wf_3(w_t)\, dxdt \tag{4.18}$$
$$+ 2\rho h \int_S^T \alpha(t)\frac{\Lambda(E)}{E}\int_0^L w_t^2 dxdt.$$

Integrating by parts in the first term of the righthand side of the Eq (4.16), we find that Eq (4.16) becomes

$$\rho_1 h_1 \int_S^T \alpha(t)\frac{\Lambda(E)}{E}\int_0^L u_t^2 dxdt + E_1 h_1 \int_S^T \alpha(t)\frac{\Lambda(E)}{E}\int_0^L u_x^2 dxdt$$
$$- k\int_S^T \alpha(t)\frac{\Lambda(E)}{E}\int_0^L u(-u+v+\alpha w_x)dxdt$$
$$= -\rho_1 h_1 \left[\alpha(t)\frac{\Lambda(E)}{E}\int_0^L uu_t dx\right]_S^T$$
$$+ \rho_1 h_1 \int_S^T \int_0^L u_t\left(\alpha'(t)\frac{\Lambda(E)}{E}u + \alpha(t)\left(\frac{\Lambda(E)}{E}\right)' u\right)dxdt$$
$$+ 2\rho_1 h_1 \int_S^T \alpha(t)\frac{\Lambda(E)}{E}\int_0^L u_t^2 dxdt - \int_S^T \alpha^2(t)\frac{\Lambda(E)}{E}\int_0^L uf_1(u_t)\, dxdt.$$

Using the fact that $\int_0^L uu_t dx \le cE(t)$, the properties of $\alpha(t)$, and the facts that the function $s \to \frac{\Lambda(s)}{s}$ is nondecreasing and E is nonincreasing, we have

$$\int_S^T \alpha'(t)\frac{\Lambda(E)}{E}\int_0^L uu_t dxdt \le c\int_S^T \alpha'(t)\frac{\Lambda(E)}{E}E(t)dt$$
$$\le c\Lambda(E(S))\int_S^T \alpha'(t)dt \le c\Lambda(E(S)). \tag{4.19}$$

Similarly, we get

$$\int_S^T \alpha(t)\left(\frac{\Lambda(E)}{E}\right)'\int_0^L uu_t dxdt \le E(S)\int_S^T \alpha(t)\left(\frac{\Lambda(E)}{E}\right)' dt$$
$$\le E(S)\left[\alpha(t)\frac{\Lambda(E)}{E}\right]_S^T - E(S)\int_S^T \alpha'(t)\frac{\Lambda(E)}{E}dt$$
$$\le E(S)\left(\alpha(T)\frac{\Lambda(E(T))}{E(T)} - \alpha(S)\frac{\Lambda(E(S))}{E(S)}\right) - E(S)\frac{\Lambda(E(S))}{E(S)}\int_S^T \alpha'(t)dt \tag{4.20}$$
$$\le E(S)\alpha(T)\frac{\Lambda(E(T))}{E(T)} - \Lambda(E(S))\,(\alpha(T)-\alpha(S))$$
$$\le E(S)\alpha(S)\frac{\Lambda(E(S))}{E(S)} + \Lambda(E(S))\alpha(S) \le c\Lambda(E(S)).$$

Combining (4.19)–(4.20), we have

$$\begin{aligned}
&\rho_1 h_1 \int_S^T \alpha(t)\frac{\Lambda(E)}{E}\int_0^L u_t^2 dxdt + E_1 h_1 \int_S^T \alpha(t)\frac{\Lambda(E)}{E}\int_0^L u_x^2 dxdt \\
&- k\int_S^T \alpha(t)\frac{\Lambda(E)}{E}\int_0^L u(-u+v+\alpha w_x)dxdt \\
&\leq c\Lambda(E(S)) + 2\rho_1 h_1 \int_S^T \alpha(t)\frac{\Lambda(E)}{E}\int_0^L u_t^2 dxdt - \int_S^T \alpha^2(t)\frac{\Lambda(E)}{E}\int_0^L u f_1(u_t)\, dxdt.
\end{aligned}$$

Similarly, we have

$$\begin{aligned}
&\rho_3 h_3 \int_S^T \alpha(t)\frac{\Lambda(E)}{E}\int_0^L v_t^2 dxdt + E_3 h_3 \int_S^T \alpha(t)\frac{\Lambda(E)}{E}\int_0^L v_x^2 dxdt \\
&+ k\int_S^T \alpha(t)\frac{\Lambda(E)}{E}\int_0^L v(-u+v+\alpha w_x)dxdt \\
&\leq c\Lambda(E(S)) + 2\rho_3 h_3 \int_S^T \alpha(t)\frac{\Lambda(E)}{E}\int_0^L v_t^2 dxdt - \int_S^T \alpha^2(t)\frac{\Lambda(E)}{E}\int_0^L v f_2(v_t)\, dxdt,
\end{aligned}$$

and

$$\begin{aligned}
&\rho h \int_S^T \alpha(t)\frac{\Lambda(E)}{E}\int_0^L w_t^2 dxdt + E h \int_S^T \alpha(t)\frac{\Lambda(E)}{E}\int_0^L w_{xx}^2 dxdt \\
&+ \alpha k\int_S^T \alpha(t)\frac{\Lambda(E)}{E}\int_0^L w_x(-u+v+\alpha w_x)dxdt \\
&\leq c\Lambda(E(S)) + 2\rho h \int_S^T \alpha(t)\frac{\Lambda(E)}{E}\int_0^L w_t^2 dxdt - \int_S^T \alpha^2(t)\frac{\Lambda(E)}{E}\int_0^L w f_3(w_t)\, dxdt.
\end{aligned}$$

Gathering all the above estimations by using (3.8), we obtain

$$\begin{aligned}
\int_S^T \alpha(t)\Lambda(E(t))\,dt \leq c\Lambda(E(S)) &+ c\int_S^T \alpha^2(t)\frac{\Lambda(E)}{E}\int_0^L \left(|u_t|^2 + |u f_1(u_t)|\right) dxdt \\
&+ c\int_S^T \alpha^2(t)\frac{\Lambda(E)}{E}\int_0^L \left(|v_t|^2 + |v f_2(v_t)|\right) dxdt \\
&+ c\int_S^T \alpha^2(t)\frac{\Lambda(E)}{E}\int_0^L \left(|w_t|^2 + |w f_3(w_t)|\right) dxdt.
\end{aligned} \tag{4.21}$$

This completes the proof of the estimate (4.15). □

In order to finalize the proof of our result, we let

$$\Lambda(s) = 2\varepsilon_0 s F_i'(\varepsilon_0^2 s), \qquad \Psi_i(s) = F_i(s^2),$$

where F_i^* and Ψ_i^* denote the dual functions of the convex functions F_i and Ψ_i, respectively, in the sense of Young (see Arnold [22], pp. 64).

Lemma 4.3. *Suppose $F_i (i = 1, 2, 3)$ are nonlinear, then the following estimates*

$$F_i^*\left(\frac{\Lambda(s)}{s}\right) \leq \frac{\Lambda(s)}{s}(F_i')^{-1}\left(\frac{\Lambda(s)}{s}\right) \tag{4.22}$$

and

$$\Psi_i^*\left(\frac{\Lambda(s)}{\sqrt{s}}\right) \leq \varepsilon_0\Lambda(\sqrt{s}), \tag{4.23}$$

hold.

Proof. We prove for $i = 1$, and the remaining are similar. Since F_1^* and Ψ_1^* are the dual functions of the convex functions F_1 and Ψ_1, respectively, then

$$F_1^*(s) = s(F_1')^{-1}(s) - F_1\left[(F_1')^{-1}(s)\right] \leq s(F_1')^{-1}(s) \tag{4.24}$$

and

$$\Psi_1^*(s) = s(\Psi_1')^{-1}(s) - \Psi_1\left[(\Psi_1')^{-1}(s)\right] \leq s(\Psi_1')^{-1}(s). \tag{4.25}$$

Using (4.24) and the definition of Λ, we obtain (4.22).

For the proof of (4.23), we use (4.25) and the definitions of Ψ_1 and Λ to obtain

$$\begin{aligned}
\frac{\Lambda(s)}{\sqrt{s}}(\Psi_1')^{-1}\left(\frac{\Lambda(s)}{\sqrt{s}}\right) &\leq 2\varepsilon_0\sqrt{s}F_1'(\varepsilon_0^2 s)(\Psi_1')^{-1}\left(2\varepsilon_0\sqrt{s}F_1'(\varepsilon_0^2 s)\right) \\
&= 2\varepsilon_0\sqrt{s}F_1'(\varepsilon_0^2 s)(\Psi_1')^{-1}\left(\Psi_1'(\varepsilon_0\sqrt{s})\right) \\
&= 2\varepsilon_0^2 sF_1'(\varepsilon_0^2 s) \\
&= \varepsilon_0\Lambda(\sqrt{s}).
\end{aligned} \tag{4.26}$$

□

Now, we state and prove our main decay results.

Theorem 4.4. *Let $(u_0, u_1) \times (z_0, z_1) \in \left[H_0^2(0, L) \times L^2(0, L)\right]^2$. Assume that $(H1)$ and $(H2)$ hold, then there exist positive constants k and c such that, for t large, the solution of the system* (1.1)–(1.3) *satisfies*

$$E(t) \leq ke^{-c\int_0^t \alpha(s)ds}, \qquad \textit{if } F_i \textit{ is linear,} \tag{4.27}$$

$$E(t) \leq \psi^{-1}\left(h(\tilde{\alpha}(t)) + \psi(E(0))\right), \quad \forall t \geq 0, \qquad \textit{if } F_i \textit{ are nonlinear,} \tag{4.28}$$

where $\tilde{\alpha}(t) = \int_0^t \alpha(t)dt$, $\psi(t) = \int_t^1 \frac{1}{\chi(s)}ds$, $\chi(t) = c\Lambda(t)$, and

$$\begin{cases}
h(t) = 0, \quad 0 \leq t \leq \dfrac{E(0)}{\chi(E(0))}, \\[2ex]
h^{-1}(t) = t + \dfrac{\psi^{-1}\left(t + \psi(E(0))\right)}{\chi\left(\psi^{-1}(t + \psi(E(0)))\right)}, \quad t > 0.
\end{cases}$$

Proof. To establish (4.27), we use (4.1), and Lemma 3.1 for $\gamma(t) = \int_0^t \alpha(s)ds$. Consequently, the result follows.

For the proof of (4.28), we re-estimate the terms of (4.15) as follows:

We consider the following partition of the domain $(0, L)$:

$$\Omega_1 = \{x \in (0, L) : |u_t| \geq \varepsilon_1\}, \quad \Omega_2 = \{x \in (0, L) : |u_t| \leq \varepsilon_1\}.$$

So,

$$\begin{aligned}
&\int_S^T \alpha(t)\frac{\Lambda(E)}{E}\int_{\Omega_1}\left(|u_t|^2 + |uf_1(u_t)|\right)dxdt \\
&\qquad = \int_S^T \alpha(t)\frac{\Lambda(E)}{E}\int_{\Omega_1}|u_t|^2 dxdt + \int_S^T \alpha(t)\frac{\Lambda(E)}{E}\int_{\Omega_1}|uf_1(u_t)|dxdt \\
&\qquad := I_1 + I_2.
\end{aligned}$$

Using the definition of Ω_1, condition $(H1)$, and (3.9), we have

$$\begin{aligned}
I_1 &\leq c\int_S^T \alpha(t)\frac{\Lambda(E)}{E}\int_{\Omega_1} u_t f_1(u_t)\,dxdt \\
&\leq c\int_S^T \frac{\Lambda(E)}{E}\left(-E'(t)\right)dt \leq c\Lambda(E(S)).
\end{aligned} \tag{4.29}$$

After applying Young's inequality, and the condition $(H1)$, we obtain

$$I_2 \leq \varepsilon\int_S^T \alpha(t)\frac{\Lambda^2(E)}{E}dt + c(\varepsilon)\int_S^T \alpha(t)\int_{\Omega_1}|f_1(u_t)|^2 dt. \tag{4.30}$$

The definition of Ω_1, the condition $(H1)$, (3.8), (3.9), and (4.30) lead to

$$\begin{aligned}
I_2 &\leq \varepsilon\int_S^T \alpha(t)\frac{\Lambda^2(E)}{E}dt + c(\varepsilon)\int_S^T \alpha(t)\int_{\Omega_1} u_t f_1(u_t)dxdt \\
&\leq \varepsilon\int_S^T \alpha(t)\frac{\Lambda^2(E)}{E}dt + c(\varepsilon)E(S).
\end{aligned} \tag{4.31}$$

Using the definition of Λ, and the convexity of F_1, (4.31) becomes

$$\begin{aligned}
I_2 &\leq \varepsilon\int_S^T \alpha(t)\frac{\Lambda^2(E)}{E}dt + c\varepsilon E(S) \\
&= 2\varepsilon\varepsilon_0\int_S^T \alpha(t)\Lambda(E)F_1'\left(\varepsilon_0^2 E(t)\right)dt + c\varepsilon E(S) \\
&\leq 2\varepsilon\varepsilon_0\int_S^T \alpha(t)\Lambda(E)F_1'\left(\varepsilon_0^2 E(0)\right)dt + c\varepsilon E(S) \\
&\leq 2c\varepsilon\varepsilon_0\int_S^T \alpha(t)\Lambda(E)dt + c\varepsilon E(S).
\end{aligned} \tag{4.32}$$

Using Young's inequality, Jensen's inequality, condition $(H1)$, and (3.8), we get

$$\begin{aligned}\int_S^T \alpha(t)\frac{\Lambda(E)}{E}\int_{\Omega_2}\left(|u_t|^2+|uf_1(u_t)|\right)dxdt &\le \int_S^T \alpha(t)\frac{\Lambda(E)}{E}\int_{\Omega_2} F_1^{-1}\left(u_t f_1(u_t)\right)dxdt\\ &+\int_S^T \alpha(t)\frac{\Lambda(E)}{E}\|u\|^{\frac{1}{2}}\left(\int_{\Omega_2} F_1^{-1}(u_t f_1(u_t))dx\right)^{\frac{1}{2}} dxdt\\ &\le L\int_S^T \alpha(t)\frac{\Lambda(E)}{E}F_1^{-1}\left(\frac{1}{L}\int_0^L u_t f_1(u_t)dx\right)dt\\ &+\int_S^T \alpha(t)\frac{\Lambda(E)}{E}\sqrt{E}\sqrt{LF_1^{-1}\left(\frac{1}{L}\int_0^L u_t f_1(u_t)dx\right)}dt.\end{aligned} \tag{4.33}$$

We apply the generalized Young inequality

$$AB \le F^*(A) + F(B)$$

to the first term of (4.33) with $A = \frac{\Lambda(E)}{E}$ and $B = F_1^{-1}\left(\frac{1}{L}\int_0^L u_t f_1(u_t)dx\right)$ to get

$$\frac{\Lambda(E)}{E}F_1^{-1}\left(\frac{1}{L}\int_0^L u_t f_1(u_t)dx\right) \le F_1^*\left(\frac{\Lambda(E)}{E}\right) + \frac{1}{L}\int_0^L u_t f_1(u_t)dx. \tag{4.34}$$

We then apply it to the second term of (4.33) with $A = \frac{\Lambda(E)}{E}\sqrt{E}$ and $B = \sqrt{LF_1^{-1}\left(\frac{1}{L}\int_0^L u_t f_1(u_t)dx\right)}$ to obtain

$$\frac{\Lambda(E)}{E}\sqrt{E}\sqrt{LF_1^{-1}\left(\frac{1}{L}\int_0^L u_t f_1(u_t)dx\right)} \le F_1^*\left(\frac{\Lambda(E)}{E}\sqrt{E}\right) + LF_1^{-1}\left(\frac{1}{L}\int_0^L u_t f_1(u_t)dx\right). \tag{4.35}$$

Combining (4.33)–(4.35), using (4.22), and (4.23), we arrive at

$$\begin{aligned}&\int_S^T \alpha(t)\frac{\Lambda(E)}{E}\int_{\Omega_2}\left(|u_t|^2+|uf_1(u_t)|\right)dxdt\\ &\le c\int_S^T \alpha(t)\left(F_1^*\left(\frac{\Lambda(E)}{E}\sqrt{E}\right)+F_1^*\left(\frac{\Lambda(E)}{E}\right)\right)dt + c\int_S^T \alpha(t)\int_\Omega u_t f_1(u_t)dxdt\\ &\le c\int_S^T \alpha(t)\left(\varepsilon_0+\frac{(F_1')^{-1}\left(\frac{\Lambda(E)}{E}\right)}{E}\right)\Lambda(E)dt + cE(S).\end{aligned} \tag{4.36}$$

Using the fact that $s \to (F_1')^{-1}(s)$ is nondecreasing, we deduce that, for $0<\varepsilon_0\le \sqrt{E(0)}$,

$$\int_S^T \alpha(t)\frac{\Lambda(E)}{E}\int_{\Omega_2}\left(|u_t|^2+|uf_1(u_t)|\right)dxdt \le c\varepsilon_0\int_S^T \alpha(t)\Lambda(E)dt + cE(S).$$

Therefore, combining (4.15), (4.29), and (4.32), we find that

$$\int_S^T \alpha(t)\frac{\Lambda(E)}{E}\int_0^L\left(|u_t|^2+|uf_1(u_t)|\right)dxdt \le c\Lambda\left(E(S)\right) + c\varepsilon\varepsilon_0\int_S^T \alpha(t)\Lambda(E)dt + c\varepsilon E(S),$$

and similarly,

$$\int_S^T \alpha(t)\frac{\Lambda(E)}{E}\int_0^L \left(|v_t|^2 + |vf_2(v_t)|\right)dxdt \le c\Lambda\left(E(S)\right) + c\varepsilon\varepsilon_0\int_S^T \alpha(t)\Lambda(E)dt + c\varepsilon E(S),$$

and

$$\int_S^T \alpha(t)\frac{\Lambda(E)}{E}\int_0^L \left(|w_t|^2 + |wf_3(w_t)|\right)dxdt \le c\Lambda\left(E(S)\right) + c\varepsilon\varepsilon_0\int_S^T \alpha(t)\Lambda(E)dt + c\varepsilon E(S),$$

Combining all the above estimations with choosing ε and ε_0 small enough, we arrive at

$$\int_S^T \alpha(t)\Lambda(E(t))dt \le c\left(1 + \frac{\Lambda(E(S))}{E(S)}\right)E(S).$$

Using the facts that E is nonincreasing and $s \to \frac{\Lambda(s)}{s}$ is nondecareasing, we can deduce that

$$\int_S^{+\infty} \alpha(t)\Lambda(E(t))dt \le cE(S).$$

Now, let $\tilde{E} = E \circ \tilde{\alpha}^{-1}$, where $\tilde{\alpha}(t) = \int_0^t \alpha(s)ds$, then we deduce from this inequality that

$$\begin{aligned}\int_S^\infty \Lambda(\tilde{E}(t))dt &= \int_S^\infty \Lambda(E(\tilde{\alpha}^{-1}(t)))dt\\ &= \int_{\tilde{\alpha}^{-1}(S)}^\infty \alpha(\eta)\Lambda(E(\eta))d\eta\\ &\le cE\left(\tilde{\alpha}^{-1}(S)\right)\\ &\le c\tilde{E}(S).\end{aligned}$$

Using Lemma 3.1 for $\tilde{E}$ and $\chi(s) = c\Lambda(s)$, we deduce from (3.1) the following estimate

$$\tilde{E}(t) \le \psi^{-1}\left(h(t) + \psi\left(E(0)\right)\right)$$

which, using the definition of $\tilde{E}$ and the change of variables, gives (4.28). □

Remark 4.1. The stability result (4.28) is a decay result. Indeed,

$$\begin{aligned}h^{-1}(t) &= t + \frac{\psi^{-1}\left(t + \psi(E(0))\right)}{\chi\left(\psi^{-1}(t + \psi(E(0)))\right)}\\ &= t + \frac{c}{2\varepsilon_0 cF'\left(\varepsilon_0^2\psi^{-1}(t + r)\right)}\\ &\ge t + \frac{c}{2\varepsilon_0 cF'\left(\varepsilon_0^2\psi^{-1}(r)\right)}\\ &\ge t + \tilde{c},\end{aligned}$$

where $F = \min\{F_i\}$ and $i = 1, 2, 3$. Hence, $\lim_{t\to\infty} h^{-1}(t) = \infty$, which implies that $\lim_{t\to\infty} h(t) = \infty$. Using the convexity of F, we have

$$\psi(t) = \int_t^1 \frac{1}{\chi(s)}ds = \int_t^1 \frac{c}{2\varepsilon_0 sF'\left(\varepsilon_0^2 s\right)} \ge \int_t^1 \frac{c}{sF'\left(\varepsilon_0^2\right)} \ge c\left[\ln|s|\right]_t^1 = -c\ln t,$$

where $F = \min\{F_i\}$ and $i = 1, 2, 3$. Therefore, $\lim_{t\to 0^+} \psi(t) = \infty$, which leads to $\lim_{t\to\infty} \psi^{-1}(t) = 0$.

5. Examples

Example 1. Let $f_i(s) = s^m$, $(i = 1, 2, 3)$, where $m \geq 1$, then the function F ($F = \min\{F_i\}$) is defined in the neighborhood of zero by

$$F(s) = cs^{\frac{m+1}{2}}$$

which gives, near zero,

$$\chi(s) = \frac{c(m+1)}{2} s^{\frac{m+1}{2}}.$$

So,

$$\psi(t) = c\int_t^1 \frac{2}{(m+1)s^{\frac{m+1}{2}}} ds = \begin{cases} \frac{c}{t^{\frac{m-1}{2}}}, & if\ m > 1; \\ -c\ln t, & if\ m = 1, \end{cases}$$

then in the neighborhood of ∞,

$$\psi^{-1}(t) = \begin{cases} ct^{-\frac{2}{m-1}}, & if\ m > 1; \\ ce^{-t}, & if\ m = 1. \end{cases}$$

Using the fact that $h(t) = t$ as t goes to infinity, we obtain from (4.27) and (4.28):

$$E(t) \leq \begin{cases} c\left(\int_0^t \alpha(s)ds\right)^{-\frac{2}{m-1}}, & if\ m > 1; \\ ce^{-\int_0^t \alpha(s)ds}, & if\ m = 1. \end{cases}$$

Example 2. Let $f_i(s) = s^m\sqrt{-\ln s}$, $(i = 1, 2, 3)$, where $m \geq 1$, then the function F is defined in the neighborhood of zero by

$$F(s) = cs^{\frac{m+1}{2}}\sqrt{-\ln\sqrt{s}},$$

which gives, near zero,

$$\chi(s) = cs^{\frac{m+1}{2}}\left(-\ln\sqrt{s}\right)^{-\frac{1}{2}}\left(\frac{m+1}{2}\left(-\ln\sqrt{s}\right) - \frac{1}{4}\right).$$

Therefore,

$$\begin{aligned} \psi(t) =& c\int_t^1 \frac{1}{s^{\frac{m+1}{2}}\left(-\ln\sqrt{s}\right)^{-\frac{1}{2}}\left(\frac{m+1}{2}\left(-\ln\sqrt{s}\right) - \frac{1}{4}\right)} ds \\ =& c\int_1^{\frac{1}{\sqrt{t}}} \frac{\tau^{m-2}}{(\ln\tau)^{-\frac{1}{2}}\left(\frac{m+1}{2}\ln\tau - \frac{1}{4}\right)} d\tau \\ =& \begin{cases} \frac{c}{t^{\frac{m-1}{2}}\sqrt{-\ln t}}, & if\ m > 1; \\ c\sqrt{-\ln t}, & if\ m = 1, \end{cases} \end{aligned}$$

then in the neighborhood of ∞,

$$\psi^{-1}(t) = \begin{cases} ct^{-\frac{2}{m-1}} (\ln t)^{-\frac{1}{m-1}}, & if\ m > 1; \\ \\ ce^{-t^2}, & if\ m = 1. \end{cases}$$

Using the fact that $h(t) = t$ as t goes to infinity, we obtain

$$E(t) \leq \begin{cases} c\left(\int_0^t \alpha(s)ds\right)^{-\frac{2}{m-1}} \left(\ln\left(\int_0^t \alpha(s)ds\right)\right)^{-\frac{1}{m-1}}, & if\ m > 1; \\ \\ ce^{-\left(\int_0^t \alpha(s)ds\right)^2}, & if\ m = 1. \end{cases}$$

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare that there is no conflict of interest.

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