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Research article

Hostile-based bipartite containment control of nonlinear fractional multi-agent systems with input delays: a signed graph approach under disturbance and switching networks

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Abstract: This article addresses the hostile-based bipartite containment control of nonlinear fractional multi-agent systems (FMASs) with input delays. Several fundamental algebraic criteria have been offered by the use of signed graph theory. To make the controller design more realistic, we assumed that the controller was under some disturbance. For the analysis of bipartite containment control, we used a fixed and switching signed network. The commonly used Lyapunov function approach and the Razumikhin technique were used. The use of these techniques can conquer the challenge brought on by switching, temporal delays, and fractional mathematics. To better elucidate the theoretical results, two examples are provided.

Keywords: bipartite containment control; Lyapunov function; signed digraph; switching network
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1. Introduction

The cooperative control of multi-agent systems (MASs) has sparked considerable attention in recent years due to its numerous applications in the formation of multi-vehicle [1], mobile sensors [2], swarms, and flocks [3, 4]. Consensus has attracted widespread attention as the most significant and elementary coordination control [5]. Multi-agent systems can either involve agents working together (cooperative) or against each other (hostile), and most current efforts focus on cooperative systems. Nonetheless, in certain practical settings, some agents are more likely to engage in competition, while others may lean towards collaboration. Networks containing hostile connections are widespread, like social networking [6], and such networks are sometimes referred to as signed networks and these networks can be represented using signed graphs. A signed graph's negative and positive weights indicate the hostile and cooperative relationship between two agents connected by an edge. If a set of MASs may be divided into two opposing subgroups, all agents tend to agree on values with the same modulus but distinct signs [7] this asymptotic behavior is known as bipartite consensus. A detailed examination is being conducted on both the first-order as well as high-order bipartite consensus [8–10]. In reality, multi-leader MASs could expand their useful applications, such as silkworm moth swarming [11] and unmanned aerial vehicle obstacle avoidance [12]. Bipartite fractional-order multi-agent systems can easily model the behavior of more complex systems as compared to traditional integer-order models. They can control large-scale sensor networks. They can also be used to control large groups of robots that are working together towards a goal. Fractional-order dynamics can easily capture complex inter-robot connections and environmental influences, providing more effective and adaptable behavior of swarms. Swarm robotics is the field that tells us how to manage large groups of low-complexity robots through the use of basic rules, and it is inspired by the capabilities of social insects to work together to implement complex tasks that are out of reach for any single insect. In [13] a detailed introduction is presented about swarm robotics.

Confinement command is the term used to describe these asymptotic actions [14]. It means that the followers of all leaders come together to form a convex hull. Some fascinating efforts on bipartite containment control were demonstrated in [15–17]. Ahsan and Ma [18] use a matrix transformation to convert bipartite containment control to general containment control. A strategy based on observers was proposed to solve bipartite containment control in [19]. By putting together different control protocols, a problem with bipartite output containment [20] examines this. Furthermore, some other methods, such as the fuzzy observer method [21] and the output feedback approach [22] together with the delayed event-triggered mechanism [23] are applied. The delay in time is a common issue in real dynamical systems that has a significant impact on the dynamics of systems and can even make systems unstable. Currently, frequency-domain analysis is an important method for exploring the consensus of fractional-order delayed systems; for instance, [24, 25] are considered, as they presented consensus on fractional-order systems with input delays. Shen et al. [26] investigate directed multi-agent systems with variable input and delay in communication. In [27], a multi-agent system with undirected input delays was taken into account. Researchers have recently investigated the potential of time-delayed complex-valued neural networks and also proposed stability analysis of these networks [28, 29]. In [30–32], the authors examined genetic regulatory networks, neural networks with time delays, and heterogeneous multi-agent systems. Notable papers explored stability in genetic networks with time-varying delays [33], analyzed neural network stability with delays [34], and proposed control methods for achieving consensus in multi-agent systems with dynamic communication structures. A new result on $H\infty$ state estimation is present in [35] which is based on convex inequality.

Despite the fact that the systems under consideration are switched, their topology is still fixed, so we present the flexible control strategy to obtain bipartite containment control on switching fractional-order multi-agent systems by applying Lyapunov stability theory of fractional order [36]. Time delay is often inevitable because of the finite propagation speed of the signal over long distances. During this time, the topology of multi-agent systems is generally dynamic over time because of changes in position. Interaction between agents can be best modeled utilizing network switching, but the presence of delays and switching connections leads to systems losing constancy and developing complex animated interactions. Based on the above study, we will analyze the results of hostile-based containment control for the non-linear FMAS with distribution and input delays. We will use fixed and switching signed networks for the analysis.

This strategy is designed to effectively deal with input delays (delays in the information or commands given to the system). The goal is to explore and propose solutions for effectively controlling complex systems in the presence of these challenges. The remaining part of the paper is structured as follows: Section 2 provides background information or preliminary details about fractional-order multi-agent systems. Section 3 presents the key findings or main results of the study. This is where the core contributions and outcomes of the research are discussed. Section 4 Provides an analysis of the main result. Section 5 includes a practical example through simulation to demonstrate how the theoretical results can be applied or how they work in real-world scenarios. Section 6 concludes the paper by summarizing the main, points and drawing overall insights or implications from the study.

2. Preliminaries

2.1. Graph theory

Let $\mathcal{G} = \{\mathcal{N}, \mathcal{L}\}$ be a weighted signed digraph in which $\mathcal{N} = \{n_1, n_2..., n_M\}$ is the set of nodes, $\mathcal{L} \subseteq \mathcal{N} \times \mathcal{N}$ is the set of edges, and $\mathcal{D} = [d_{pq}]_{M\times M}$ is the adjacency matrix such that $d_{pq} \neq 0$ if $(n_p, n_q) \in \mathcal{L}$, else $d_{pq} = 0$, $d_{pq} > 0$, *and* $d_{pq} < 0$ represent the cooperative and hostile relationship of nodes, respectively. Let $\mathbb{L} = [l_{pq}]_{M\times M}$ be the Laplacian matrix of graph $\mathcal{G}$ defined as $l_{pq} = -dpq$ if $p \neq q$, and if $p = q$ then $l_{pq} = \sum_{p=1,q\neq p}^{M} |d_{pq}|$. Hence, if the graph has both a hostile and cooperative relationship between nodes then the graph is called a signed graph. Furthermore, the graph $\mathcal{G}$ is structurally balanced if the node set $\mathcal{N}$ of a signed graph $\mathcal{G}$ may be divided into two separate subsets $\mathcal{N}_1$ and $\mathcal{N}_2$ where each element is exclusively part of one group and does not overlap with the other, such that $d_{pq} \geq 0$ for any $n_p, n_q \in \mathcal{N}_1(\mathcal{N}_2)$ and $d_{pq} \leq 0$ for any $n_p \in \mathcal{N}_1, n_q \in \mathcal{N}_2$.

2.2. Definition and important lemmas

In this section, some important definitions and lemmas are discussed.

Definition 2.1 ([37]). For a continuous differentiable function $s(\tau)$ of order k, the Caputo derivative of order ω is given as

$$D^{\omega} s(\tau) = \frac{1}{\Gamma(k-\omega)} \int_{\tau_0}^{\tau} \frac{s^k(u)}{(\tau-u)^{\omega-k+1}} du,$$

$0 \leq k-1 < \omega \leq k, k \in Z^+$.

Lemma 2.1 ([36]). *An absolutely continuous function $s(\tau) \in \mathbb{R}^n$ follows that*

$$\frac{1}{2} D^{\omega}(s^T(\tau)s(\tau))) \leq s^T(\tau) D^{\omega} s(\tau), \quad \omega \in (0,1).$$

Lemma 2.2 ([38]). *For any $\rho > 0$ and $C_1, C_2 \in \mathbb{R}^n$, the following relationship holds:*

$$2C_1^T C_2 \leq \rho C_1^T C_1 + \frac{1}{\rho} C_2^T C_2.$$

Let $\mathcal{B} = \mathbb{B}(([-x, 0] \to \mathbb{R}^n)$ be the Banach space of each function exhibiting continuity on $[-x, 0]$. Now, the fractional-order system under delay for $0 < \omega < 1$ is taken into account.

$$\begin{cases} D^{\omega} y(\tau) = H(\tau, y_t), \\ \quad y_{\tau_0} = \phi(\vartheta), \ \vartheta \in [-x, 0], \end{cases} \tag{2.1}$$

where $y_t(\theta) = y(\tau + \theta), \ \theta \in [-x, 0]$ for any $\tau \geq \tau_0$ and time delay $x > 0, and \phi \in \mathcal{B}$. H maps $\mathbb{R} \times$ (*bounded sets of* $\mathcal{B}$) to $\mathbb{R}^n$, which is also a bounded set in which $H(\tau, 0) = 0$.

Lemma 2.3. *Consider a quadratic Lyapunov function* $G : \mathbb{R}^n \to \mathbb{R}$*. If there are positive numbers* $b_1, b_2, andb_3$*, then for some* $\eta > 1$ *the following relationship holds:*

$$b_1 \|s(\tau)\|^2 \leq G(s(\tau)) \leq b_2 \|s(\tau)\|^2$$

and whenever

$$G(s(\tau + \theta)) \leq \eta G(s(\tau)), \ \ \theta \in [-x, 0], \ \ D^{\omega} z(s(\tau)) \leq -b_3 \|s(\tau)\|^2,$$

then the long-term stability of the required solution of system (2.1) *can be obtained.*

3. Problem formulation

Consider a fractional order multi-agent system having F followers and $M - F$ leaders, marked as $\overline{F} = \{1, 2, 3..., F\}$ and $\overline{B} = \{F + 1, F + 2..., M\}$, respectively. The *pth* agent state $z_p(\tau) \in \mathbb{R}^n$ is described by the system of fractional differential equations

$$\begin{cases} D^{\omega} z_p(\tau) = V z_p(\tau) + h(\tau, z_p(\tau)) + \sigma_p(\tau - x) + \Delta\sigma_p(\tau), & p \in \overline{F}, \\ D^{\omega} z_p(\tau) = V z_p(\tau) + h(\tau, z_p(\tau)), & p \in \overline{B}, \end{cases} \tag{3.1}$$

where $\sigma_p(\tau)$ is the control input of agent p, $\Delta\sigma_p(\tau)$ is the disturbance in protocol, x represents input delays V$\in \mathbb{R}^{n \times n}$, and $h : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ is an odd function that is also continuous on $z_p(\tau)$ with $h(\tau, 0) = 0$.

Definition 3.1 ([14])**.** For any $s_1, s_2 \in \zeta$ and $0 < \delta < 1$, if $(1 - \delta)s_1 + \delta s_2 \in \zeta$, then $\zeta \in \mathbb{R}^n$ is called convex function and

$$co\{s_1, s_2...., s_n\} = \left\{ \sum_{p=1}^{n} \delta_p s_p | \delta_p \geq 0, \sum_{p=1}^{n} \delta_p = 1 \right\}$$

is called a convex hull formed by $s_1, s_2, ...s_n \in \mathbb{R}^n$.

Definition 3.2. For FMAS (3.1), the bipartite containment control will be achieved if some followers converge to $co\{z_i, i \in \overline{B}\}$ and others converge to -$co\{z_i, i \in \overline{B}\}$.

The following hypotheses are needed to obtain the main result.

$\mathbb{H}1$. For arbitrary $s, s_p \in \mathbb{R}^n, \delta_p$ with $\sum_{p=1}^{M-F} \delta_p = 1$, and $k > 0$, the nonlinear function g satisfies the following:

$$\|h(\tau, s) - \sum_{p=1}^{M-F} \delta_p h(\tau, s_p)\| \leq k \|s - \sum_{p=1}^{M-F} \delta_p s_p\|. \tag{3.2}$$

$\mathbb{H}2$. If $\mathcal{G}$ is a signed directed network, it is also structurally balanced.

$\mathbb{H}3$. In the signed directed network $\mathcal{G}$, each follower has a directed edge to at least one leader.

If hypothesis $\mathbb{H}2$ holds, then all followers will be divided into two subsets such that there exist cooperative interactions between agents in subsets $\overline{F}_1$ and $\overline{F}_2$, while these subsets have antagonistic interactions with different subsets. Now, according to Definition 3.2, the FMAS realizes bipartite containment control if

$$\begin{cases} \lim_{\tau\to\infty} \|z_p(\tau) - co\{z_i(\tau),\ \ i \in \overline{B}\}\| = 0, \quad \forall p \in \overline{F}_1, \\ \lim_{\tau\to\infty} \|z_p(\tau) + co\{z_i(\tau),\ \ i \in \overline{B}\}\| = 0, \quad \forall p \in \overline{F}_2. \end{cases}$$

Remark 3.1. *Containment control refers to the idea that all followers converge to the leader, as opposed to the consensus tracking concerns to many leaders, for instance with formation control and the set tracking leader-composed convex hull. The definition of set tracking in mathematics is equivalent to formation control and confinement control. The following suggests that one does more than just follow the convex hull shaped by leaders and yet keeps the intended configuration. While there are several leaders involved in consensus tracking, the leaders may be exposed to control inputs to accomplish some preferred tasks [39–41], but in terms of the containment control problem. Other agents will not have an impact on any one leader, so, typically, the leaders get no control input: refer to [18–20].*

4. Analysis of bipartite containment control

4.1. Bipartite containment control under fixed signed digraph

Under fixed topology, we make the following disturbed delayed control protocol for achieving bipartite containment control of the system (3.1):

$$\sigma_p(\tau - x) = \xi \sum_{q=1}^{M} |d_{pq}|(sgn(d_{pq}))z_q(\tau - x) - z_p(\tau - x) + \Delta\sigma_p(\tau), \quad p \in \overline{F} \tag{4.1}$$

where $\xi > 0$ is the gain constant and d_{pq} are the members of the D (adjacency matrix). $\Delta\sigma_p(\tau)$ represents disturbance in the control protocol. Since $u_p = 0$ for $p \in \overline{B}$, matrix D can be defined as

$$D = \begin{pmatrix} D'_{F\times F} & D''_{F\times(M-F)} \\ 0_{(M-F)\times F} & 0_{(M-F)\times(M-F)} \end{pmatrix}. \tag{4.2}$$

Therefore, the Laplacian matrix $\mathbb{L}$ can be written as

$$\mathbb{L} = \begin{pmatrix} \mathbb{L}_F & \mathbb{L}_B \\ 0_{(M-F)\times F} & 0_{(M-F)\times(M-F)} \end{pmatrix}, \tag{4.3}$$

where $\mathbb{L}_F \in \mathbb{R}^{F\times F}$ and $\mathbb{L}_B \in \mathbb{R}^{F\times(M-F)}$. If $\mathbb{H}2$, then we may select a diagonal matrix $\Omega = diag(\Omega_1, \Omega_2, ..., \Omega_F, \Omega_{F+1}, ..., \Omega_M)$ in which

$$\begin{cases} \Omega_p = 1, & \text{if} \quad p \in \overline{F}_1 \cup \overline{B}, \\ \Omega_p = -1, & \text{if} \quad p \in \overline{F}_2, \end{cases} \tag{4.4}$$

such that $\overline{\mathbb{L}} = \Omega \mathbb{L} \Omega$,

$$\overline{\mathbb{L}} = \begin{pmatrix} \overline{\mathbb{L}}_F & \overline{\mathbb{L}}_B \\ 0_{(M-F)\times F} & 0_{(M-F)\times(M-F)} \end{pmatrix} \tag{4.5}$$

where

$$\overline{\mathbb{L}}_F = [\bar{l}_{pq}]_{F\times F} = \begin{cases} -|d_{pq}|, & p \neq q, \\ \sum_{q=1, q\neq p}^{M} |d_{pq}|, & p = q, \end{cases} \tag{4.6}$$

and $\overline{\mathbb{L}}_B = [\bar{l}pq]_{F\times(M-F)}$ with $\bar{l}_{pq} = -|d_{pq}| \leq 0$ where $p = 1, ...F, q = F+1, F+2, ..., M$. See [18] for more details.

Lemma 4.1. *[14] If $\mathbb{H}3$ holds, then $\overline{\mathbb{L}}_F$ is a non-singular matrix and -$\overline{\mathbb{L}}_F^{-1}\overline{\mathbb{L}}_B$ is a non-negative matrix whose row sums are equal to 1.*

Theorem 4.2. *The bipartite containment control of fractional order multi-agent system (3.1) using controller (4.1) will be achieved under $\mathbb{H}1$–$\mathbb{H}3$ if there exists a symmetric matrix $U > 0$ satisfying the following inequality for $\lambda_1 = \lambda_{max}\{\overline{\mathbb{L}}_F \overline{\mathbb{L}}_F^T\}$:*

$$\begin{pmatrix} UV + V^T U + k^2 I_n + \xi U - \xi\lambda_1 U + 4\frac{\sigma_F(\tau)}{\nu} U & U \\ U & -I_n \end{pmatrix} < 0. \tag{4.7}$$

Proof. Consider $g_p(.) = \Omega_p z_p(.)$, the coordinate transformation for any $p = 1, 2, ..., M$ in which (4.4) holds. Now, substituting control protocol (4.1) into system (3.1) yields following result:

$$\begin{cases} D^\omega g_p(\tau) = V g_p(\tau) + h(\tau, g_p(\tau)) + \xi\Omega_p \sum_{q=1}^{M} |d_{pq}|(sgn(d_{pq})\Omega_q g_q(\tau - x) - \Omega_p g_p(\tau - x)) + 2\Delta\sigma_p(\tau), & p \in \overline{F}, \\ D^\omega g_p(\tau) = V g_p(\tau) + h(\tau, g_p(\tau)), & p \in \overline{B}. \end{cases} \tag{4.8}$$

Observe that

$$\begin{aligned} &\sum_{q=1}^{M} |d_{pq}| sgn(d_{pq}) g_q(\tau - x) - g_p(\tau - x) \\ &= \sum_{q=1, q\neq p}^{M} |d_{pq}| sgn(d_{pq}) g_q(\tau - x) - \sum_{q=1, q\neq p}^{M} |d_{pq}| g_p(\tau - x) \\ &= \sum_{q=1, q\neq p}^{M} |d_{pq}| g_q(\tau - x) - \sum_{q=1, q\neq p}^{M} |d_{pq}| g_p(\tau - x) \\ &= -\sum_{q=1, q\neq p}^{M} l_{pq} g_q(\tau - x) - l_{pp} g_p(\tau - x) \\ &= -\sum_{q=1}^{M} l_{pq} g_q(\tau - x). \end{aligned}$$

Therefore, system (4.8) becomes

$$\begin{cases} D^{\omega} g_p(\tau) = V g_p(\tau) + h(\tau, g_p(\tau)) - \xi \sum_{q=1}^{M} \Omega_p l_{pq} \Omega_q g_q(\tau - x) + 2\Delta\sigma_p(\tau), & p \in \overline{\mathbb{F}}, \\ D^{\omega} g_p(\tau) = V g_p(\tau) + h(\tau, g_p(\tau)), & p \in B. \end{cases} \tag{4.9}$$

Let

$$G_F(.) = (g_1^T(.), g_2^T(.), ...g_F^T(.))^T, \;\; G_B(.) = (g_{F+1}^T(.), g_{F+2}^T(.), ...g_M^T(.))^T,$$
$$H(\tau, g_F) = (h^T(\tau, g_1), ...h^T(\tau, g_F))^T, H(\tau, g_B) = (h^T(\tau, g_{F+1}), ...h^T(\tau, g_M))^T,$$

then system (4.9) becomes

$$\begin{cases} D^{\omega} G_F(\tau) = (I_F \otimes V) G_F(\tau) + H(\tau, G_F(\tau)) - \xi(\overline{\mathbb{L}}_F \otimes I_n) G_F(\tau - x) - \xi(\overline{\mathbb{L}}_B \otimes I_n) G_B(\tau - x) + 2\Delta\sigma_F(\tau), \\ D^{\omega} G_B(\tau) = (I_{M-F} \otimes V) G_B(\tau) + H(\tau, G_B(\tau)). \end{cases} \tag{4.10}$$

The error system is defined as

$$\alpha(.) = G_F(.) - (-\overline{\mathbb{L}}_F^{-1} \overline{\mathbb{L}}_B \otimes I_n) G_B(.),$$

and then system (4.10) becomes as

$$\begin{aligned} D^{\omega}\alpha(\tau) =& (I_F \otimes V) G_F(\tau) + (\overline{\mathbb{L}}_F^{-1} \overline{\mathbb{L}}_B \otimes V) G_B(\tau) + H(\tau, G_F(\tau)) + (\overline{\mathbb{L}}_F^{-1} \overline{\mathbb{L}}_B \otimes I_n) H(\tau, G_B(\tau)) \\ & - (\xi\overline{\mathbb{L}}_F \otimes I_n)(I_F \otimes I_n) G_F(\tau - x) - (\xi\overline{\mathbb{L}}_F \otimes I_n)(\overline{\mathbb{L}}_F^{-1} \overline{\mathbb{L}}_B \otimes I_n) G_B(\tau - x) + 2\Delta\sigma_F(\tau), \\ D^{\omega}\alpha(\tau) =& (I_F \otimes V)\alpha(\tau) + H(\tau, G_F(\tau)) + (\overline{\mathbb{L}}_F^{-1} \overline{\mathbb{L}}_B \otimes I_n) H(\tau, G_B(\tau)) \\ & - (\xi\overline{\mathbb{L}}_F \otimes I_n)\alpha(\tau - x) + 2\Delta\sigma_F(\tau). \end{aligned} \tag{4.11}$$

Now, we construct a Lyapunov function such that

$$S(\alpha(\tau)) = \alpha^T(\tau)(I_F \otimes U)\alpha(\tau).$$

Now, applying Lemmas 2.1 and 2.2 on Eq (4.11),

$$\begin{aligned} D^{\omega} S(\alpha(\tau)) &= D^{\omega}[\alpha^T(\tau)](I_F \otimes U)\alpha(\tau) \\ &\leq 2\alpha^T(\tau)(I_F \otimes U)[D^{\omega}\alpha(\tau)] = 2\alpha^T(\tau)(I_F \otimes U)[(I_F \otimes V)\alpha(\tau) \\ &+ H(\tau, G_F(\tau)) + (\overline{\mathbb{L}}_F^{-1} \overline{\mathbb{L}}_B \otimes I_n) H(\tau, G_B(\tau)) - (\xi\overline{\mathbb{L}}_F \otimes I_n)\alpha(\tau - x) + 2\Delta\sigma_F(\tau)] \\ &= \alpha^T(\tau)[I_F \otimes (UV + V^T U)]\alpha(\tau) + 2\alpha^T(\tau)(I_F \otimes U)[H(\tau, G_F(\tau)) + (\overline{\mathbb{L}}_F^{-1} \overline{\mathbb{L}}_B \otimes I_n) H(\tau, G_B(\tau))] \\ &- 2\alpha^T(\tau)(\xi\overline{\mathbb{L}}_F \otimes U)\alpha(\tau - x) + 2\alpha^T(\tau)(I_F \otimes U)[2\Delta\sigma_F(\tau)] \\ &\leq \alpha^T(\tau)[I_F \otimes (UV + V^T U)]\alpha(\tau) - 2\alpha^T(\tau)(\xi\overline{\mathbb{L}}_F \otimes U)\alpha(\tau - x) + \alpha^T(\tau)(I_F \otimes U^2)\alpha(\tau) \\ &\leq \alpha^T(\tau)[I_F \otimes (UV + V^T U)]\alpha(\tau) - 2\alpha^T(\tau)(\xi\overline{\mathbb{L}}_F \otimes U)\alpha(\tau - x) + \alpha^T(\tau)(I_F \otimes U^2)\alpha(\tau) \\ &+ [H(\tau, G_F(\tau)) + (\overline{\mathbb{L}}_F^{-1} \overline{\mathbb{L}}_B \otimes I_n) H(\tau, G_B(\tau))]^T [[H(\tau, G_F(\tau)) + (\overline{\mathbb{L}}_F^{-1} \overline{\mathbb{L}}_B \otimes I_n) H(\tau, G_B(\tau))] \\ &+ 4\alpha^T(\tau)(I_F \otimes U)\Delta\sigma_F(\tau)]. \end{aligned} \tag{4.12}$$

Now, if we suppose that $-\overline{\mathbb{L}}_F^{-1}\overline{\mathbb{L}}_B = [\bar{l}_{pq}]_{F\times(M-F)}$, from hypothesis $\mathbb{H}1$ we have

$$[H(\tau, G_F(\tau)) - (-\overline{\mathbb{L}}_F^{-1}\overline{\mathbb{L}}_B \otimes I_n)H(\tau, G_B(\tau))]^T[[H(\tau, G_F(\tau)) - (-\overline{\mathbb{L}}_F^{-1}\overline{\mathbb{L}}_B \otimes I_n)H(\tau, G_B(\tau))]$$

$$= \begin{pmatrix} h(\tau, g_1) - \sum_{q=1}^{M-F} \bar{l}_{1q} h(\tau, g_{F+q}) \\ h(\tau, g_2) - \sum_{q=1}^{M-F} \bar{l}_{2q} h(\tau, g_{F+q}) \\ . \\ . \\ . \\ h(\tau, g_F) - \sum_{q=1}^{M-F} \bar{l}_{Fq} h(\tau, g_{F+q}) \end{pmatrix}^T \times \begin{pmatrix} h(\tau, g_1) - \sum_{q=1}^{M-F} \bar{l}_{1q} h(\tau, g_{F+q}) \\ h(\tau, g_2) - \sum_{q=1}^{M-F} \bar{l}_{2q} h(\tau, g_{F+q}) \\ . \\ . \\ . \\ h(\tau, g_F) - \sum_{q=1}^{M-F} \bar{l}_{Fq} h(\tau, g_{F+q}) \end{pmatrix}$$

$$\begin{aligned} &= \sum_{p=1}^{F}[h(\tau, g_p) - \sum_{q=1}^{M-F} \bar{l}_{pq} h(\tau, g_{F+q})]^T [\sum_{p=1}^{F} h(\tau, g_p) - \sum_{q=1}^{M-F} \bar{l}_{pq} h(\tau, g_{F+q})] \\ &\leq \sum_{p=1}^{F} k^2 [g_p - \sum_{q=1}^{M-F} \bar{l}_{pq} g_{F+q}]^T [\sum_{p=1}^{F} g_p - \sum_{q=1}^{M-F} \bar{l}_{pq} g_{F+q}] \\ &= k^2 [G_F(\tau) - (-\overline{\mathbb{L}}_F^{-1}\overline{\mathbb{L}}_B \otimes I_n) G_B(\tau)]^T [G_F(\tau) - (-\overline{\mathbb{L}}_F^{-1}\overline{\mathbb{L}}_B \otimes I_n) G_B(\tau)] \\ &= k^2 \alpha^T(\tau)\alpha(\tau). \end{aligned} \tag{4.13}$$

Now, consider

$$\begin{aligned} 2\alpha^T(\tau)(\xi\overline{\mathbb{L}}_F \otimes U)\alpha(\tau - x) &= \xi.2\alpha^T(\tau)(\overline{\mathbb{L}}_F \otimes U)(I_F \otimes U^{\frac{-1}{2}}(I_F \otimes U^{\frac{1}{2}})\alpha(\tau - x) \\ &\leq \xi.\alpha^T(\tau)(\overline{\mathbb{L}}_F \otimes U)(I_F \otimes U^{\frac{-1}{2}})(I_F \otimes U^{\frac{-1}{2}})(\overline{\mathbb{L}}_F^{-T} \otimes U)\alpha(\tau) \\ &+ \xi.\alpha^T(\tau - x)(\overline{\mathbb{L}}_F \otimes U)(I_F \otimes U^{\frac{1}{2}}(I_F \otimes U^{\frac{1}{2}})\alpha(\tau - x) \\ &= \alpha^T(\tau)(\xi\overline{\mathbb{L}}_F\overline{\mathbb{L}}_F^{-T} \otimes U)\alpha(\tau) + \xi.\alpha^T(\tau - x)(I_F \otimes U)\alpha(\tau - x). \end{aligned}$$

Now, substituting Eq (4.13) and the above equation into Eq (4.12):

$$\begin{aligned} \leq \alpha^T(\tau)[I_F \otimes (UV + V^T U + U^2 + k^2 I_n]\alpha(\tau) - \alpha^T(\tau)(\xi\overline{\mathbb{L}}_F\overline{\mathbb{L}}_F^{-T} \otimes U)\alpha(\tau) \\ -\xi.\alpha^T(\tau - x)(I_F \otimes U)\alpha(\tau - x) + 4\Delta\sigma_F(\tau)(I_F \otimes U) \end{aligned}$$

since $\overline{\mathbb{L}}_F$ is invertible from Lemma 4.1, $\overline{\mathbb{L}}_F\overline{\mathbb{L}}_F^{-T}$ is positive definite matrix, so we let

$$\begin{aligned} \lambda_1 &= \lambda_{max}\{\overline{\mathbb{L}}_F\overline{\mathbb{L}}_F^{-T}\} \\ &\leq \alpha^T(\tau)[I_F \otimes (UV + V^T U + U^2 + k^2 I_n + 4\frac{\sigma_F(\tau)}{\nu} U \\ &- \xi\lambda_1 U]\alpha(\tau) - \xi.\alpha^T(\tau - x)(I_F \otimes U)\alpha(\tau - x), \end{aligned} \tag{4.14}$$

where $\nu = \alpha(\tau)$. Now, for some $\varrho>1$, whenever $S(\alpha(\tau + \vartheta)) \leq \varrho S(\alpha(\tau))$, $\vartheta \in [-x, 0]$. That is

$$\alpha^T(\tau - x)(I_F \otimes U)\alpha(\tau - x) \leq \varrho\alpha^T(\tau)(I_F \otimes U)\alpha(\tau). \tag{4.15}$$

Now, substituting Eq (4.15) into (4.14) yields, for sufficiently small $\varrho > 0$, $\varrho = 1 + \varrho$,

$$D^{\omega}S(\alpha(\tau)) \leq \alpha^T(\tau)[I_F \otimes (UV + V^T U + U^2 + k^2 I_n + \xi U - \xi\lambda_1 U + 4\frac{\sigma_F(\tau)}{\nu}U)]\alpha(\tau). \tag{4.16}$$

Inequality (4.7) means $D^{\omega}S(\alpha(\tau)) < 0$ from (4.16), thus there exists a scalar $\mu > 0$ satisfying

$$D^{\omega}S(\alpha(\tau)) \leq -\mu S(\alpha(\tau)) \leq -\mu\lambda_{min}(U)\|\alpha(\tau)\|^2,$$

which indicates that error system (4.11) is asymptotically stable by using Lemma 2.3, and hence the bipartite containment control of FMAS (3.1) with controller (4.1) is realized.

Remark 4.3. *Comparison of spatial non-uniformity and limited singularity of derivatives of fractional order, with integer-order MASs, pose numerous challenges in the exploration of FMASs, particularly when encountering temporal delays. When studying the asymptotic performance of integer-order MASs under delays, a Lyapunov function incorporating an integral expression is typically employed. However, in the case of delayed FMASs, this Lyapunov function becomes invalid in the sense of Caputo derivatives, given that fractional operators lack a composition properly, that is,* $D^m(D^n x(\tau)) \neq D^{m+n}x(\tau)$. *In this instance, using signed graph theory and the fractional Razumikhin method, a useful approach is created to address delayed FMASs bipartite containment control. Our technique effectively addresses the issues brought on by delays and fractional derivatives by deciding on a straightforward quadratic Lyapunov function. The same method can also be used for fractional order multi-agent systems under time delays as well as distributed types of delays [42, 43].*

4.2. Bipartite containment control under switching signed digraph

Consider a switching signed directed network $\mathcal{G}^{\ell(\tau)} = (\mathcal{N}, \mathcal{L}^{\ell(\tau)})$ and switching point τ_J where the piecewise switching signal ℓ assigns $[\tau_0, \infty)$ into set N ={1,2,3,....n} and $J \in Z^+ \cup \{0\}$. $\forall \tau_J$ $\exists$ an arbitrary point t which is small and also satisfies inequality $\tau_{J+1} - \tau_J \geq t$, which can avoid Zeno behavior. The nth topology is activated if for $\tau \in [\tau_J, \tau_{J+1})$, $\ell(\tau) = n \in N$. $\mathcal{D}^{\ell(\tau)} = [d_{pq}^{\ell(\tau)}]_{M\times M}$ is the adjacency matrix and $\mathbb{L}^{\ell(\tau)} = [l_{pq}^{\ell(\tau)}]_{M\times M}$ represent, the Laplacian matrices of $\mathcal{G}^{\ell(\tau)}$. Also, for any $\ell(\tau) = \imath \in N$, $\mathcal{G}^{\imath} = \mathcal{G}$, and we need some hypotheses to attain bipartite containment control of fractional order multi-agent system (3.1) with switching signed network.

$\mathbb{H}$4. Signed directed network $\mathcal{G}^{\ell(\tau)}$ for any $\ell(\tau) \in N$ is structurally balanced.

$\mathbb{H}$5. In $\mathcal{G}^{\ell(\tau)}$ every follower has a directed connection to at least one leader.

Under switching topology we make the following disturbed delayed control protocol for achieving bipartite containment control of the system (3.1).

$$\sigma_p(\tau - x_2) = \xi \sum_{q=1}^{M} |d_{pq}^{\ell(\tau)}|(sgn(d_{pq}^{\ell(\tau)}))z_q(\tau - x_2) - z_p(\tau - x_2) + \Delta\sigma_p(\tau), \quad p \in \overline{F}. \tag{4.17}$$

Remark 4.4. *Using (4.2) and (4.4) for* $\ell(\tau) \in M$ *the Laplacian matrix of* $\mathcal{G}^{\ell(\tau)}$ *is defined by*

$$\mathbb{L}^{\ell(\tau)} = \begin{pmatrix} \mathbb{L}_F^{\ell(\tau)} & \mathbb{L}_B^{\ell(\tau)} \\ 0_{(M-F)\times F} & 0_{(M-F)\times(M-F)} \end{pmatrix}, \tag{4.18}$$

where $\mathbb{L}_F^{\ell(\tau)} \in \mathbb{R}^{F\times F}$ *and* $\mathbb{L}_B^{\ell(\tau)} \in \mathbb{R}^{F\times(M-F)}$. *If* $\mathbb{H}4$, *then we may select a diagonal matrix* $\Omega^{\ell(\tau)} = diag(\Omega_1, \Omega_2, ..., \Omega_F, \Omega_{F+1}, ..., \Omega_M)$ *in which*

$$\begin{cases} \Omega_p = 1, & \text{if } p \in \overline{F}_1 \cup \overline{B}, \\ \Omega_p = -1, & \text{if } p \in \overline{F}_2, \end{cases} \tag{4.19}$$

such that

$$\overline{\mathbb{L}}^{\ell(\tau)} = \Omega^{\ell(\tau)} \mathbb{L}^{\ell(\tau)} \Omega^{\ell(\tau)},$$
$$\overline{\mathbb{L}}^{\ell(\tau)} = \begin{pmatrix} \overline{\mathbb{L}}_F^{\ell(\tau)} & \overline{\mathbb{L}}_B^{\ell(\tau)} \\ 0_{(M-F)\times F} & 0_{(M-F)\times(M-F)} \end{pmatrix}, \tag{4.20}$$

where

$$\overline{\mathbb{L}}_F^{\ell(\tau)} = [\overline{l}_{pq}^{\ell(\tau)}]_{F\times F} = \begin{cases} -|d_{pq}^{\ell(\tau)}|, & p \neq q, \\ \sum_{q=1, q\neq p}^{M} |d_{pq}^{\ell(\tau)}|, & p = q, \end{cases} \tag{4.21}$$

and

$$\overline{\mathbb{L}}_B^{\ell(\tau)} = [\overline{l}_{pq}^{\ell(\tau)}]_{F\times(M-F)} \text{ with } \overline{l}_{pq}^{\ell(\tau)} = -|d_{pq}^{\ell(\tau)}| \leq 0,$$

where $p = 1, ...F, q = F + 1, F + 2, ..., M$.

Remark 4.5. *If* $\mathbb{H}5$ *holds, then for any* $\ell(\tau) \in N$, $\overline{\mathbb{L}}_F^{\ell(\tau)}$ *is a matrix which is not singular and every element of* -$(\overline{\mathbb{L}}_F^{\ell(\tau)})^{-1}\overline{\mathbb{L}}_B^{\ell(\tau)}$ *is non negative with row sums equal to 1.*

Theorem 4.6. *The bipartite containment control of fractional multi-agent system (3.1) using the switching disturbed controller (4.17) for arbitrary switching signal* $\ell(\tau)$ *shall be achieved under* $\mathbb{H}1$–$\mathbb{H}3$, *if there exists a symmetric matrix* $U > 0$ *satisfying the following inequality for*

$$\lambda_2 = max_{\ell(\tau)\in N}[\lambda_{max}\{\overline{\mathbb{L}}_F^{\ell(\tau)}(\overline{\mathbb{L}}_F^{\ell(\tau)})^T\}],$$

$$\begin{pmatrix} UV + V^T U + k^2 I_n + \xi U - \xi\lambda_2 U + 4\frac{\sigma_F(\tau)}{\nu} U & U \\ U & -I_n \end{pmatrix} < 0. \tag{4.22}$$

Proof. For error system is defined as

$$\alpha(.) = G_F(.) - ((-\overline{\mathbb{L}}_F^{\ell(\tau)})^{-1}\overline{\mathbb{L}}_B^{\ell(\tau)} \otimes I_n)G_B(.),$$

then the system under controller (4.17) gives the following result:

$$\begin{aligned} D^{\omega}\alpha(\tau) = {} & (I_F \otimes V)\alpha(\tau) + H(\tau, G_F(\tau)) + ((\overline{\mathbb{L}}_F^{\ell(\tau)})^{-1}\overline{\mathbb{L}}_B^{\ell(\tau)} \otimes I_n)H(\tau, G_B(\tau)) \\ & - (\xi\overline{\mathbb{L}}_F^{\ell(\tau)} \otimes I_n)\alpha(\tau - x) + 2\Delta\sigma_F(\tau). \end{aligned} \tag{4.23}$$

Now, we construct a Lyapunov function such that

$$S(\alpha(\tau)) = \alpha^T(\tau)(I_F \otimes U)\alpha(\tau). \tag{4.24}$$

Now, by using Theorem 4.2, we can get

$$D^{\omega}\alpha(\tau) \leq \alpha^{T}(\tau)[I_F \otimes (UV + V^{T}U + U^{2} + k^{2}I_n + \xi I_n + 4\frac{\sigma_F(\tau)}{\nu}U]\alpha(\tau)$$
$$- \alpha^{T}(\tau)[\xi\overline{\mathbb{L}}_F^{\ell(\tau)}(\overline{\mathbb{L}}_F^{\ell(\tau)})^{T} \otimes U\}]\alpha(\tau)$$
$$\leq \alpha^{T}(\tau)[UV + V^{T}U + k^{2}I_n + \xi U + U^{2} - \xi\lambda_2 U + 4\frac{\sigma_F(\tau)}{\nu}U]\alpha(\tau),$$

where

$$\lambda_2 = max_{\ell(\tau)\in N}[\lambda_{max}\{\overline{\mathbb{L}}_F^{\ell(\tau)}(\overline{\mathbb{L}}_F^{\ell(\tau)})^{T}\}]$$

and $\nu = \alpha(\tau)$.

Inequality (4.22) means $D^{\omega}S(\alpha(\tau)) < 0$, and thus there exists a scalar $\mu > 0$ satisfying $D^{\omega}S(\alpha(\tau)) \leq -\mu S(\alpha(\tau))$. We can say that $D^{\omega}S(\alpha(\tau)) \leq -\mu S(\alpha(\tau))$ is true for every topology, which concludes that the Lyapunov function described by Eq (4.24) is common for all types of topologies. The error system (4.23) under any switching signal is asymptotically stable using Lemma 2.3. Hence, bipartite containment control under any switching for FMAS (3.1) is achieved with the disturbed controller (4.17).

5. Numerical examples

5.1. Fixed signed network

Example 5.1. We consider a fractional order multi-agent system (3.1) as having 5 followers and 2 leaders. In Figure 1, the signed digraph of the problem is present. $\overline{F_1} = \{1, 2\}$ and $\overline{F_2} = \{3, 4, 5\}$ are bipartite subgroups of the signed directed network $\mathcal{G}$ which is structurally balanced. $h(\tau, z_p(\tau)) = \frac{3}{5}sin(z_p(\tau)), p = 1, 2, 3...7$ with $\sigma_F(\tau) = cos(0.1)$, $\nu = 0.21$, and we can take k=0.6, which satisfies (3.2).

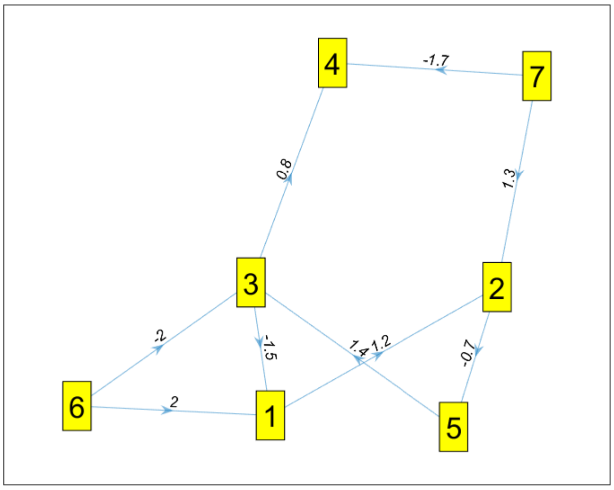

Figure 1. The signed directed network of $\mathcal{G}$.

Now, let $V = \begin{pmatrix} -6.3 & 0 \\ 4 & -8 \end{pmatrix}$. From Figure 1, the adjacency matrix

$$D = \begin{pmatrix} 0 & 0 & -1.5 & 0 & 0 & 2 & 0 \\ 1.2 & 0 & 0 & 0 & 0 & 0 & 1.3 \\ 0 & 0 & 0 & 0 & 1.4 & -2 & 0 \\ 0 & 0 & 0.8 & 0 & 0 & 0 & -1.7 \\ 0 & -0.7 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

$$\Omega = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix},$$

$$\mathbb{L} = \begin{pmatrix} 3.5 & 0 & 1.5 & 0 & 0 & -2 & 0 \\ -1.2 & 2.5 & 0 & 0 & 0 & 0 & -1.3 \\ 0 & 0 & 3.4 & 0 & -1.4 & 2 & 0 \\ 0 & 0 & -0.8 & 2.5 & 0 & 0 & 1.7 \\ 0 & 0.7 & 0 & 0 & 0.7 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

and

$$\overline{\mathbb{L}} = \Omega\mathbb{L}\Omega = \begin{pmatrix} 3.5 & 0 & -1.5 & 0 & 0 & -2 & 0 \\ -1.2 & 2.5 & 0 & 0 & 0 & 0 & -1.3 \\ 0 & 0 & 3.4 & 0 & -1.4 & -2 & 0 \\ 0 & 0 & -0.8 & 2.5 & 0 & 0 & -1.7 \\ 0 & -0.7 & 0 & 0 & 0.7 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

where

$$\overline{\mathbb{L}}_F = \begin{pmatrix} 3.5 & 0 & -1.5 & 0 & 0 \\ -1.2 & 2.5 & 0 & 0 & 0 \\ 0 & 0 & 3.4 & 0 & -1.4 \\ 0 & 0 & -0.8 & 2.5 & 0 \\ 0 & -0.7 & 0 & 0 & 0.7 \end{pmatrix},$$

and

$$\overline{\mathbb{L}}_B = \begin{pmatrix} -2 & 0 \\ 0 & -1.3 \\ -2 & 0 \\ 0 & -1.7 \\ 0 & 0 \end{pmatrix},$$

$$\lambda_1 = \lambda_{max}[\overline{\mathbb{L}}_F \overline{\mathbb{L}}_F^T] = 20.0032.$$

Now, by solving inequality (4.7), we can get $U = \begin{pmatrix} 1.21 & 0.02 \\ 0.02 & 1.21 \end{pmatrix}$ and $\xi = 1.14$, which satisfies inequality (4.7). The state of agents and error trajectories of FMAS (3.1) under the fixed signed network is described in Figures 2 and 3, respectively, in which we take $\omega = 0.7$ and x=1.5.

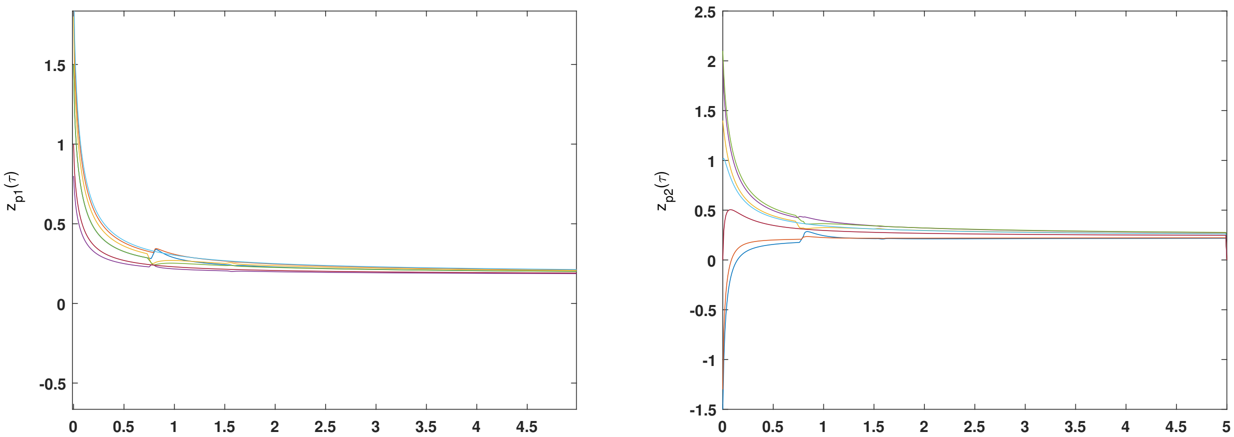

Figure 2. $z_{p1}(\tau)$ and $z_{p2}(\tau)$ represent state of agents that are described by FMAS(3.1) where p = {1,2,3,4,5,6,7}.

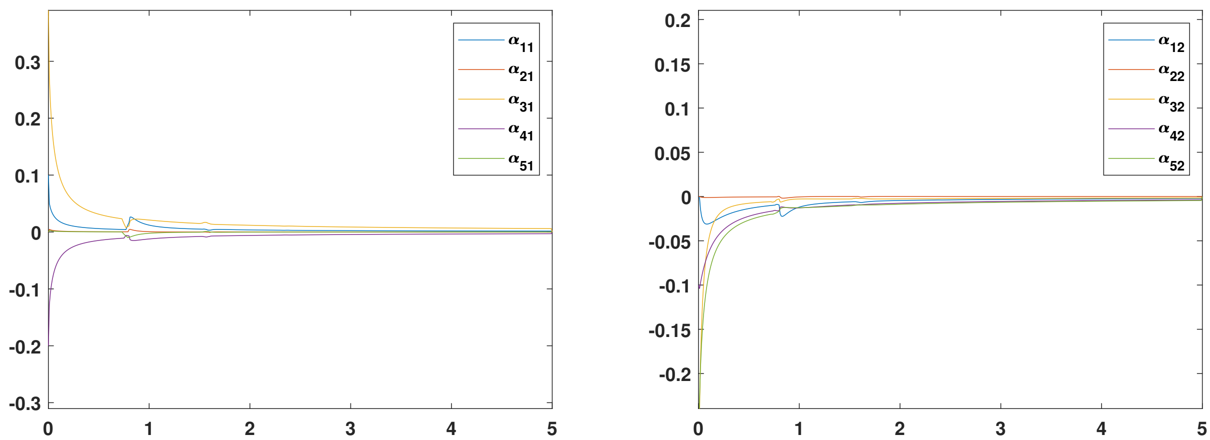

Figure 3. Error trajectories of FMAS (3.1) under the fixed signed network.

In error trajectories

$$\alpha_{p1}(\tau) = z_{p1}(\tau) - co\{z_{61}, z_{71}\},$$

$$\alpha_{p2}(\tau) = z_{p2}(\tau) - co\{z_{61}, z_{71}\},$$

and

$$\alpha_{p1}(\tau) = -(z_{p1}(\tau) + co\{z_{61}, z_{71}\}),$$

$$\alpha_{p2}(\tau) = -(z_{p2}(\tau) + co\{z_{61}, z_{71}\}),$$

now we could say that

$$\begin{cases} \lim_{\tau\to\infty} \|z_p(\tau) - co\{z_i(\tau), i \in \overline{B}\}\| = 0, & p \in \{1, 2\}, \\ \lim_{\tau\to\infty} \|z_p(\tau) + co\{z_i(\tau), i \in \overline{B}\}\| = 0, & p \in \{3, 4, 5\}. \end{cases}$$

This concludes that the system is still working under delays and with small disturbances in the controller, but larger delays and disturbances can slow the convergence speed of the system.

5.2. Switching signed network

Example 5.2. We consider a fractional order multi-agent system (3.1) as having 2 leaders and 5 followers. The switching topologies of graphs $\mathcal{G}_1, \mathcal{G}_2, \mathcal{G}_3$ are described in Figures 4–6, respectively. $\overline{F_1} = \{1, 2\}$ and $\overline{F_2} = \{3, 4, 5\}$ are bipartite subgroups of the signed digraph $\mathcal{G}_1$, $\mathcal{G}_2$, $\mathcal{G}_3$, which is structurally balanced.

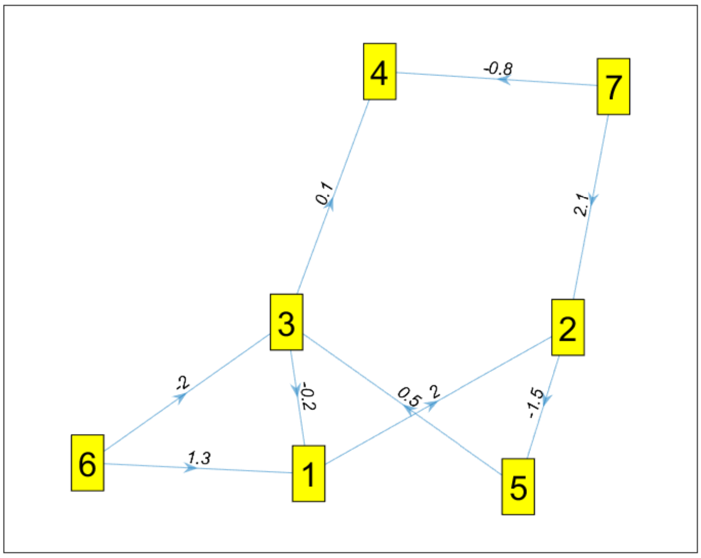

Figure 4. The signed directed network of $\mathcal{G}_1$.

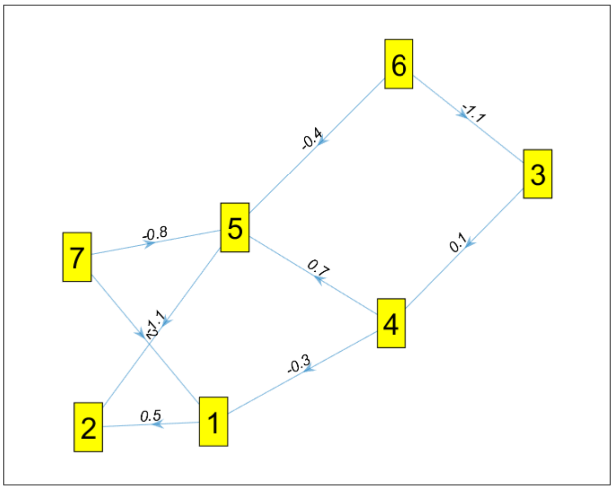

Figure 5. The signed directed network of $\mathcal{G}_2$.

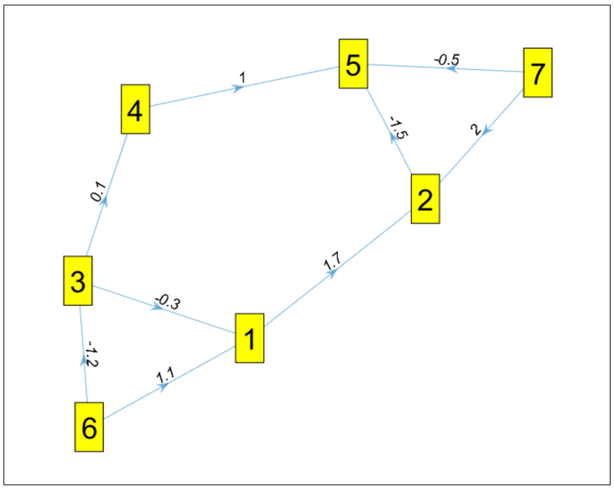

Figure 6. The signed directed network of $\mathcal{G}_3$.

$$D^1 = \begin{pmatrix} 0 & 0 & -0.2 & 0 & 0 & 1.3 & 0 \\ 2 & 0 & 0 & 0 & 0 & 0 & 2.1 \\ 0 & 0 & 0 & 0 & 0.5 & -2 & 0 \\ 0 & 0 & 0.1 & 0 & 0 & 0 & -0.8 \\ 0 & -1.5 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

$$D^2 = \begin{pmatrix} 0 & 0 & 0 & -0.3 & 0 & 0 & 2 \\ 0.5 & 0 & 0 & 0 & -1.1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1.1 & 0 \\ 0 & 0 & 0.1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.7 & 0 & -0.4 & -0.8 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

$$D^3 = \begin{pmatrix} 0 & 0 & -0.3 & 0 & 0 & 1.1 & 0 \\ 1.7 & 0 & 0 & 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 & -1.2 & 0 \\ 0 & 0 & 0.1 & 0 & 0 & 0 & 0 \\ 0 & -1.5 & 0 & 1 & 0 & 0 & -0.5 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

$$\Omega^1 = \Omega^2 = \Omega^3 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix},$$

then

$$\overline{\mathbb{L}}_F^1 = \begin{pmatrix} 1.5 & 0 & -0.2 & 0 & 0 \\ -2 & 4.1 & 0 & 0 & 0 \\ 0 & 0 & 2.5 & 0 & -0.5 \\ 0 & 0 & -0.1 & 0.9 & 0 \\ 0 & -1.5 & 0 & 0 & 1.5 \end{pmatrix},$$

$$\overline{\mathbb{L}}_B^1 = \begin{pmatrix} -1.3 & 0 \\ 0 & -2.1 \\ -2 & 0 \\ 0 & -0.8 \\ 0 & 0 \end{pmatrix},$$

$$\overline{\mathbb{L}}_F^2 = \begin{pmatrix} 2.3 & 0 & 0 & -0.3 & 0 \\ -0.5 & 1.6 & 0 & 0 & -1.1 \\ 0 & 0 & 1.1 & 0 & 0 \\ 0 & 0 & -0.1 & 0.1 & 0 \\ 0 & 0 & 0 & -0.7 & 1.9 \end{pmatrix},$$

$$\overline{\mathbb{L}}_B^2 = \begin{pmatrix} 0 & -2 \\ 0 & 0 \\ -1.1 & 0 \\ 0 & 0 \\ -0.4 & -0.8 \end{pmatrix},$$

$$\overline{\mathbb{L}}_F^3 = \begin{pmatrix} 1.4 & 0 & -0.3 & 0 & 0 \\ -1.7 & 1.9 & 0 & 0 & 0 \\ 0 & 0 & 1.2 & 0 & 0 \\ 0 & 0 & -0.1 & 0.1 & 0 \\ 0 & -1.5 & 0 & -1 & 3 \end{pmatrix},$$

$$\overline{\mathbb{L}}_B^3 = \begin{pmatrix} -1.1 & 0 \\ 0 & -2 \\ -1.2 & 0 \\ 0 & 0 \\ 0 & -0.5 \end{pmatrix},$$

$$\lambda_{max}[\overline{\mathbb{L}}_F^1(\overline{\mathbb{L}}_F^1)^T] = 23.2612,$$

$$\lambda_{max}[\overline{\mathbb{L}}_F^2(\overline{\mathbb{L}}_F^2)^T] = 6.8258,$$

$$\lambda_{max}[\overline{\mathbb{L}}_F^3(\overline{\mathbb{L}}_F^3)^T] = 13.4871,$$

from these three values we have $\lambda_2 = 23.2612$. Now, by solving inequalities (4.22), we can get $U = \begin{pmatrix} 1.22 & 0.01 \\ 0.01 & 1.22 \end{pmatrix}$ and $\xi = 1.04$, which satisfy (4.22). The state of agents and error trajectories of FMAS (3.1) under switching signed network is described in Figures 7 and 8 respectively, in which we take $\omega = 0.7$ and $x = 1.5$.

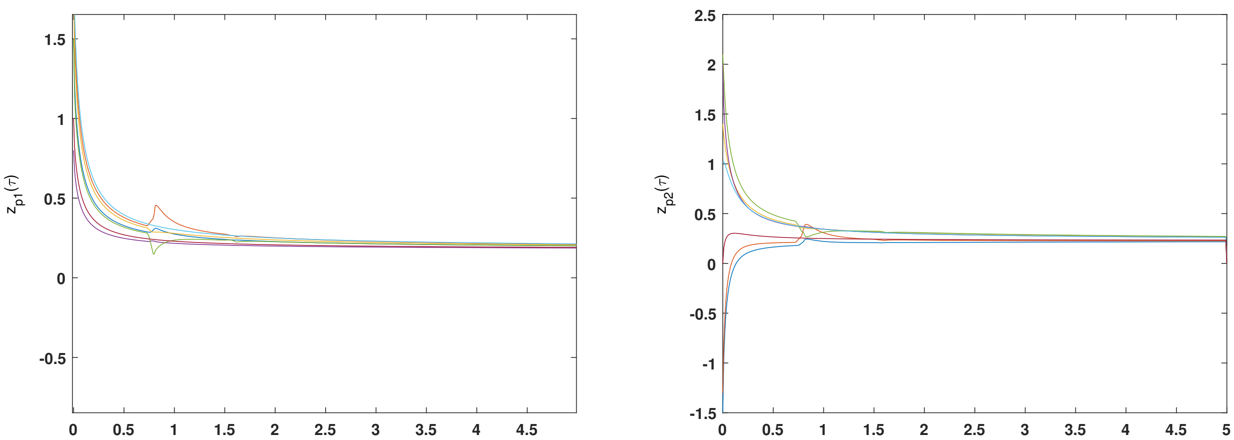

Figure 7. $z_{p1}(\tau)$ and $z_{p2}(\tau)$ represent the state of agents that are described by FMAS (3.1) where $p = \{1, 2, 3, 4, 5, 6, 7\}$.

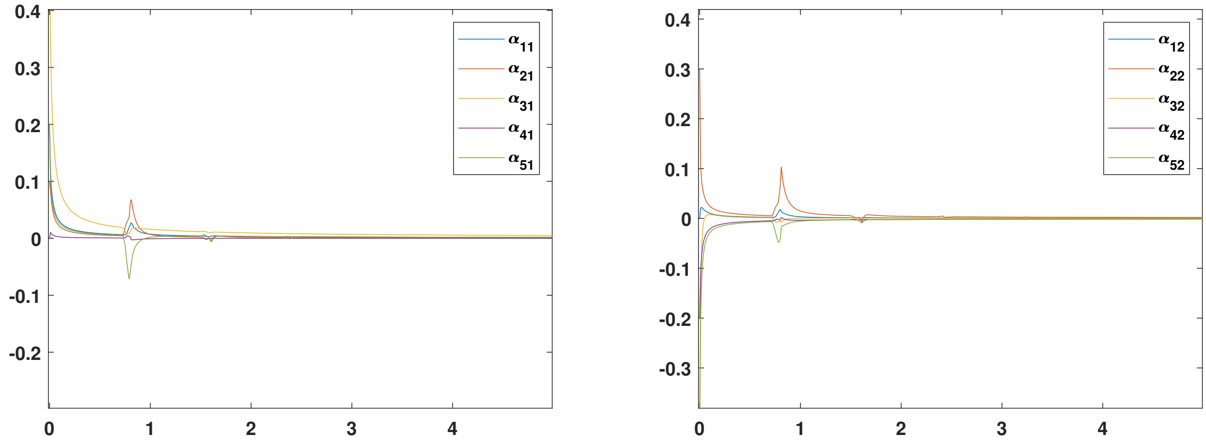

Figure 8. Error trajectories of FMAS (3.1) under the switching signed directed network.

Remark 5.3. *The graphs of the bipartite structure give us a clear picture of the connection and separation between sets of agents. Positive and negative values in signed digraphs show the cooperative and antagonistic behavior of agents. We can easily see from the digraph in Figure 1 that nodes 1 and 2, which are followers of node 7, cooperate with each other as the link between them is positive, but these nodes have a negative link with other nodes, like 3, 4, and 5, which are followers of node 6. Bipartite structure combined with fractional order dynamics can offer unique challenges in designing control strategies for achieving containment control and consensus.*

6. Conclusions

In networks that have a combination of fixed and switching attributes, along with signed directed links, where both cooperative and hostile agent interaction is present, bipartite containment control of nonlinear FMASs has been studied. Based on the presumptions that the associated signed digraph is structurally balanced, and at least one leader has a directed link to each follower the fixed and switching signed directed systems, delayed control methods have been developed to address bipartite containment control networks. With regard to the typical Lyapunov function approach and fractional Razumikhin technique, a trustworthy and practical solution has been put out to address the issues raised through switching topologies, fractional calculus, and delay, and multiple bipartite containment control has been guaranteed by the presentation of basic matrix inequalities. Concrete numerical examples elucidate the validity and viability of the primary findings.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Data availability statement

The code is considered an intellectual property of the Guangzhou Government project, and therefore not publicly available. The data that support the findings of this study are available from the corresponding author A. U. K. Niazi, upon reasonable request.

Conflict of interest

The authors declare that they have no conflicts of interest.

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