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Forbidden subgraphs and the König–Egerváry property

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ABSTRACT

The *matching number* of a graph is the maximum size of a set of vertex-disjoint edges. The *transversal number* is the minimum number of vertices needed to meet every edge. A graph *has the König–Egerváry property* if its matching number equals its transversal number. Lovász proved a characterization of graphs having the König–Egerváry property by means of forbidden subgraphs within graphs with a perfect matching. Korach, Nguyen, and Peis proposed an extension of Lovász's result to a characterization of all graphs having the König–Egerváry property in terms of forbidden configurations (which are certain arrangements of a subgraph and a maximum matching). In this work, we prove a characterization of graphs having the König–Egerváry property by means of forbidden subgraphs which is a strengthened version of the characterization by Korach et al. Using our characterization of graphs with the König–Egerváry property, we also prove a forbidden subgraph characterization for the class of edge-perfect graphs.

1. Introduction

The *matching number* $\nu(G)$ of a graph G is the maximum size of a set of vertex-disjoint edges of G. The *transversal number* $\tau(G)$ is the minimum number of vertices necessary to meet every edge of G. Clearly, $\nu(G) \leq \tau(G)$ for every graph G. In 1931, Kőnig [12] and Egerváry [9] proved independently that every bipartite graph B satisfies $\nu(B) = \tau(B)$. Graphs G satisfying $\nu(G) = \tau(G)$ are called after them *König–Egerváry graphs* or said to *have the König–Egerváry property*. Graphs having the König–Egerváry property have been extensively studied [4,6,13,14,17–20,23–27]. In 1979, Deming [6] and Sterboul [27] independently gave the first structural characterization of graphs having the König–Egerváry property. Moreover, in [6] also a polynomial-time recognition algorithm for graphs having the König–Egerváry property was devised. In 1983, Lovász [23] introduced the notion of *nice subgraphs* and characterized graphs having the König–Egerváry property by forbidden nice subgraphs within graphs with a perfect matching. We will show that it is not possible to extend his result to a characterization of all graphs having the König–Egerváry property by forbidden nice subgraphs. In this work, we introduce the notion of *strongly splitting subgraphs*, providing a suitable extension of Lovász's nice subgraphs in the sense that all graphs having the König–Egerváry property can be characterized by forbidden strongly splitting subgraphs.

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Our result relies on a characterization by Korach, Nguyen, and Peis [14] of graphs having the König–Egerváry property by means of what we call *forbidden configurations* (certain arrangements of a subgraph and a maximum matching) which is itself an extension of Lovász's characterization within graphs with a perfect matching. The main result of this work is a characterization of all graphs having the König–Egerváry property by forbidden strongly splitting subgraphs which is a strengthened version of the characterization due to Korach et al. by forbidden configurations: (1) first, we show that one of their forbidden configurations is redundant and can be omitted; (2) then, we reformulate the resulting characterization in terms of forbidden subgraphs; (3) finally, we strengthen the formulation by restricting the way in which the forbidden subgraphs may occur.

Using our main result we also prove a characterization of *edge-perfect graphs*, which are defined similarly to the well-known class of perfect graphs. Perfect graphs were introduced by Berge in the early 1960s by means of the equality between the *clique number* ω (the size of the largest set of pairwise adjacent vertices) and the *chromatic number* χ (the minimum number of colors needed to color the vertices such that adjacent vertices receive different colors); namely, a graph G is *perfect* if $\omega(H) = \chi(H)$ for each induced subgraph H of G. In 1961, Berge [1] conjectured a characterization of perfect graphs by forbidden induced subgraphs which was settled few years ago and is now known as the *Strong Perfect Graph Theorem* [5]. The class of *edge-perfect* graphs [10] can be defined similarly but by means of the equality between ν and τ; i.e., by means of the König–Egerváry property. Imposing the König–Egerváry property to each induced subgraph of a graph can easily be seen to coincide with requiring the graph to be bipartite, because odd chordless cycles do not have the König–Egerváry property and graphs without odd chordless cycles are precisely the bipartite graphs. Instead, a graph G is called *edge-perfect* if each of its *edge-subgraphs* has the König–Egerváry property, where an edge-subgraph is any induced subgraph that arises by removing a (possibly empty) set of edges together with their endpoints. Using our characterization of graphs with the König–Egerváry property, we state and prove a simple characterization of edge-perfect graphs by forbidden edge-subgraphs. Unfortunately, this result does not lead to a polynomial-time recognition algorithm. In fact, although the problem of recognizing edge-perfect graphs is known to be polynomial-time solvable when restricted to certain graph classes [7], it is NP-hard for the general class of graphs [8].

This work is organized as follows. In Section 2, we give some basic definitions. In Section 3, we discuss the characterization by Lovász and that of Korach et al. and we present our characterization of graphs with the König–Egerváry property by forbidden strongly splitting subgraphs. In Section 4, we use our characterization of graphs having the König–Egerváry property in order to prove a characterization of edge-perfect graphs by forbidden edge-subgraphs.

2. Definitions

All graphs in this work are finite, undirected, without loops, and without multiple edges. Let G be a graph. The vertex set of G will be denoted by $V(G)$ and the edge set by $E(G)$. A *subgraph* of G is a graph H such that $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$. A subgraph H of G is *spanning* if $V(H) = V(G)$. If $X \subseteq V(G)$, the subgraph of G *induced* by X is the subgraph $G[X]$ whose vertex set is X and whose edge set is $\{uv \in E(G) : u, v \in X\}$. A vertex is *isolated* if it has no neighbors and *pendant* if it has exactly one neighbor. A graph is *complete* if every pair of different vertices are adjacent. The complete graph on n vertices will be denoted by K_n. For any set S, $|S|$ denotes its cardinality. For any sets A and B, $A \triangle B$ denotes the *symmetric difference* $(A \setminus B) \cup (B \setminus A)$.

Paths and cycles have no repeated vertices (apart from the starting and ending vertices in the case of cycles). Trivial paths consisting of only one vertex (and no edges) will be allowed, but cycles must have at least three vertices. Let Z be a path or a cycle. Then, $E(Z)$ denotes the set of edges joining two consecutive vertices of Z and the *length* of Z is $|E(Z)|$. Z is *odd* if its length is odd, and *even* otherwise. A *chord* of Z is any edge joining two nonadjacent vertices of Z. Z is *chordless* if it has no chords. The chordless cycle of length n is denoted by C_n. Let $P = x_1x_2 \cdots x_n$ and $Q = y_1y_2 \cdots y_m$ be two paths (where the x_i's and y_j's are vertices). If P and Q are vertex-disjoint except for $x_n = y_1$, then PQ denotes the concatenated path $x_1x_2 \cdots x_ny_2 \cdots y_m$. If z is adjacent to x_1, zP denotes the path $zx_1x_2 \cdots x_n$.

Let G and H be graphs with $V(G) \cap V(H) = \emptyset$. The *disjoint union* of G and H is the graph $G \cup H$ whose vertex set is $V(G) \cup V(H)$ and whose edge set is $E(G) \cup E(H)$. For each nonnegative integer t, tG denotes the disjoint union of t copies of G.

A *matching* of a graph G is a set of vertex-disjoint edges of G. Let M be a matching of G. The endpoints of the edges belonging to M are called *M-saturated* and the remaining vertices of G are called *M-unsaturated*. M is *maximum* if it is of maximum size; i.e., $|M| = \nu(G)$ (where $\nu(G)$ denotes the matching number as defined in the Introduction). M is *perfect* if it saturates every vertex of G and *near-perfect* if it saturates all but one vertex of G. Clearly, graphs with a perfect matching have an even number of vertices, while graphs with a near-perfect matching have an odd number of vertices. Perfect and near-perfect matchings are trivially maximum. A path is *M-alternating* if, for each two consecutive edges of the path, exactly one of them belongs to M. An *M-augmenting path* is an M-alternating path starting and ending in M-unsaturated vertices. Notice that if P is an M-augmenting path then $M' = M \triangle E(P)$ is also a matching and $|M'| = |M| + 1$. Indeed, a matching M is maximum if and only if there are no M-augmenting paths [2].

Let G be a graph. If X is a subset of $V(G)$, $G - X$ denotes the subgraph of G induced by $V(G) \setminus X$. If F is any set of edges, we will denote by $V(F)$ the set of endpoints of the edges belonging to F; i.e., $V(F) = \bigcup_{e \in F} e$ by regarding each edge e as the set of its endpoints. With this notation, the edge-subgraphs of a graph G are those induced subgraphs $G - V(F)$ for some $F \subseteq E(G)$. Edge-perfect graphs are customarily defined in terms of the stability number $\alpha(G)$ and the edge covering number $\rho(G)$ (cf. [8]). The equivalence between that definition and ours follows from the result by Gallai [11] stating that if a graph

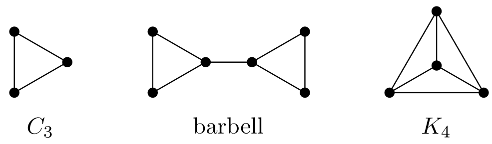

Fig. 1. C_3, barbell, and K_4.

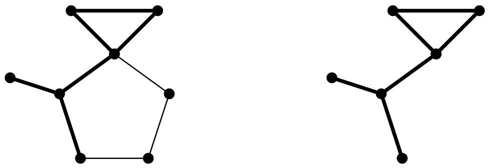

Fig. 2. The König–Egerváry property is not always inherited by the nice subgraphs. The graph on the left has the König–Egerváry property while its bold edges correspond to a nice subgraph of it (depicted also on the right) that does not have the König–Egerváry property.

G has no isolated vertices then $\alpha(G) + \tau(G) = |V(G)| = \rho(G) + \nu(G)$. Edge-perfect graphs form a superclass of the class of bipartite graphs and a subclass of the class of graphs having the König–Egerváry property. Both inclusions are proper, as shown by the graph that arises from C_3 by attaching a pendant vertex, known as the paw (which is edge-perfect but not bipartite) and the graph that arises from C_6 by adding a chord joining two vertices at distance two within the cycle (which has the König–Egerváry property but is not edge-perfect). The classes of perfect graphs and of edge-perfect graphs are not comparable. For instance, C_3 is perfect but not edge-perfect and the graph that arises from C_5 by attaching a pendant vertex is edge-perfect but not perfect.

3. The König–Egerváry property in terms of forbidden subgraphs

An *even subdivision* of an edge uv consists in replacing the edge uv by two new vertices w_1 and w_2 together with three edges uw_1, w_1w_2, and w_2v. An *even subdivision* of a graph G is either the graph G itself or any of the graphs that arise from G by successive application of even subdivisions. A subgraph H of a graph G is *nice* if $G - V(H)$ has a perfect matching. The theorem below, due to Lovász, characterizes graphs with the König–Egerváry property by forbidden nice subgraphs within graphs with a perfect matching. For the *barbell* graph, see Fig. 1.

Theorem 1 ([23]). *A graph with a perfect matching has the König–Egerváry property if and only if it has no even subdivision of barbell or K_4 as a nice subgraph.*

We will show that it is not possible to characterize the whole class of graphs having the König–Egerváry property by forbidden nice subgraphs. That is, we cannot drop the hypothesis that G has a perfect matching by adding some extra forbidden nice subgraphs. It is not possible to do so because, while the relation "is a nice subgraph of" is clearly transitive, the König–Egerváry property is not always inherited by the nice subgraphs (as the example given in Fig. 2 shows). Suppose, by the way of contradiction, that it were possible to characterize the whole class of graphs having the König–Egerváry property by forbidden nice subgraphs. Consider Fig. 2, where a graph is displayed on the left and a nice subgraph of it on the right. Since the graph on the right does not have the König–Egerváry property, it should have some nice subgraph Φ which is forbidden in the characterization whose existence we are assuming. Then, by transitivity, the forbidden nice subgraph Φ would also be a nice subgraph of the graph on the left, which would contradict the fact that the graph on the left does have the König–Egerváry property. This contradiction proves that Theorem 1 cannot be extended to a characterization by forbidden nice subgraphs of all graphs having the König–Egerváry property. Instead, our approach toward obtaining a similar result holding for all graphs will be to replace nice subgraphs by *splitting subgraphs* (to be defined after Lemma 2) and later by *strongly splitting subgraphs* (to be defined on page 10).

Let G be a graph and let X be a subset of $V(G)$. We say that X is a *splitting set* of G if and only if there is some maximum matching M of G such that no edge of M joins a vertex of X to a vertex of $G - X$. If so, we say that *M is split by X*. The next lemma gives a sufficient condition for a subgraph of a graph having the König–Egerváry property to also have the König–Egerváry property. Recall that the König–Egerváry property is not inherited by subgraphs in general, as the example of Fig. 2 shows.

Lemma 2. *Let G be a graph having the König–Egerváry property and let H be a subgraph of G. If $V(H)$ is a splitting set of G and $\nu(H) = \nu(G[V(H)])$, then H also has the König–Egerváry property.*

Proof. Suppose that $V(H)$ is a splitting set of G and $\nu(H) = \nu(G[V(H)])$. Let M be a maximum matching of G split by $V(H)$; i.e., there is no edge of M joining a vertex of H to a vertex of $G - V(H)$. Let M_H be the set of edges of M joining two vertices of $V(H)$ and let $M_{G-V(H)}$ be the set of edges of M joining two vertices of $G - V(H)$. Since M is a maximum matching of G and M is split by $V(H)$, M_H is a maximum matching of $G[V(H)]$. Since $\nu(H) = \nu(G[V(H)])$, there is maximum matching M'_H of

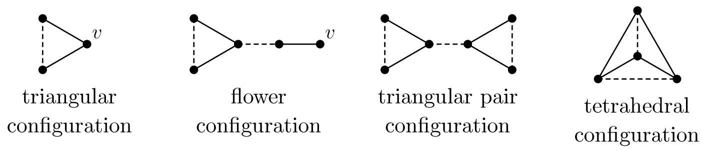

Fig. 3. Forbidden configurations for graphs having the König–Egerváry property.

H such that $|M'_H| = |M_H|$. Therefore, $M' = M'_H \cup M_{G-V(H)}$ is a maximum matching of G. Then,

$$\nu(G) = \nu(H) + \nu(G - V(H)) \leq \tau(H) + \tau(G - V(H)) \leq \tau(G). \tag{1}$$

Since G has the König–Egerváry property, both inequalities in (1) hold with equality and, necessarily, $\nu(H) = \tau(H)$ and $\nu(G - V(H)) = \tau(G - V(H))$. This proves that H has the König–Egerváry property. □

The above lemma leads us to introduce the notion of *splitting subgraphs* as follows. Let G be a graph and let H be a subgraph of G. We will say that H is a *splitting subgraph* of G if and only if $V(H)$ is a splitting set of G and H has a perfect or near-perfect matching. Notice that if H has a perfect or near-perfect matching, $\nu(H) = \nu(G[V(H)])$ holds trivially. Therefore, we have the following corollary of Lemma 2 showing that, contrary to the case of nice subgraphs, the König–Egerváry property is always inherited by the splitting subgraphs.

Corollary 3. *If a graph has the König–Egerváry property, then each of its splitting subgraphs has the König–Egerváry property.*

Notice that if G has a perfect matching, then H is a splitting subgraph of G if and only if H has a perfect matching and H is a nice subgraph of G. Since all the graphs involved in Theorem 1 have perfect matchings, the result still holds if we replace 'nice subgraphs' by 'splitting subgraphs'.

Theorem 1 in terms of splitting subgraphs ([23]). *A graph with a perfect matching has the König–Egerváry property if and only if it has no even subdivision of barbell or K_4 as a splitting subgraph.*

We will show that, contrary to the case of nice subgraphs, the whole class of graph having the König–Egerváry property can be characterized by means of splitting subgraphs. That is, when Theorem 1 is reformulated in terms of forbidden splitting subgraphs as above, the hypothesis that G has a perfect matching can be dropped by simply adding some extra forbidden splitting subgraphs. The characterization of the graphs having the König–Egerváry property by forbidden splitting subgraphs will follow from a characterization due to Korach, Nguyen, and Peis [14] by what we call *forbidden configurations*. A *configuration* of a graph G is an ordered pair $\xi = (S, M)$ where S is a subgraph of G, M is a maximum matching of G, and S belongs to one of the four families of graphs represented in Fig. 3, where dashed edges stand for M-alternating paths starting and ending in edges of M, solid edges stand for M-alternating paths starting and ending in edges not belonging to M, and the vertex v is M-unsaturated. The graph S is said to be the *underlying graph* of ξ. Korach et al. proposed the following characterization of graphs with the König–Egerváry property by forbidden configurations.

Theorem 4 ([14]). *A graph has the König–Egerváry property if and only if it has none of the configurations in Fig. 3.*

The lemma below shows that it is not essential to forbid the flower configurations in Theorem 4 because forbidding triangular configurations prevents both triangular and flower configurations from occurring.

Lemma 5. *If a graph has a flower configuration, then it also has a triangular configuration.*

Proof. Assume that a graph G has some flower configuration $\xi = (S, M)$. Let v be the M-unsaturated vertex of S and let w be the vertex of S of degree 3 in S. Let P be the path of S joining v to w and let C be the only cycle of S. Notice that $M' = M \triangle E(P)$ is also a maximum matching of G because P is an M-alternating even path of G and v is M-unsaturated. Therefore, (C, M') is a triangular configuration of G, which completes the proof. □

Next we observe that the occurrence of the three remaining configurations coincides with the occurrence of their underlying graphs as splitting subgraphs.

Lemma 6. *Let G be a graph and let S be a subgraph of G. Then, S is the underlying graph of a triangular, triangular pair, or tetrahedral configuration of G if and only if S is a splitting subgraph of G which is an even subdivision of C_3, barbell, or K_4, respectively.*

Proof. Assume that there is some splitting subgraph S of G which is an even subdivision of C_3, barbell, or K_4. By definition, $V(S)$ is a splitting set of G; i.e., there is a maximum matching M of G such that no edge of M joins a vertex of S with a vertex of $G - V(S)$. Let M_S be the set of edges of M that join two vertices of S and let $M_{G-V(S)}$ be the set of edges of M that join two vertices of $G - V(S)$. By construction, M_S is a maximum matching of $G[V(S)]$. Since S is an even subdivision of C_3, barbell, or K_4, there is a perfect or near-perfect matching R_S of S. Notice that R_S is unique up to isomorphisms of S. As S is a spanning subgraph of $G[V(S)]$, $|R_S| = |M_S|$. Then, $M' = R_S \cup M_{G-V(S)}$ is a maximum matching of G. By construction, (S, M') is a triangular, triangular pair, or tetrahedral configuration of G depending on whether S is an even subdivision of C_3, barbell, or K_4, respectively.

Conversely, assume that S is the underlying graph of a triangular, triangular pair, or tetrahedral configuration $\xi = (S, M)$ of G. By definition, $V(S)$ is a splitting set of G and S has a perfect or near-perfect matching. Thus, S is a splitting subgraph of G. We conclude that S is a splitting subgraph of G which is an even subdivision of C_3, barbell, or K_4 depending on whether ξ is a triangular, triangular pair, or tetrahedral configuration, respectively. □

Therefore, the characterization by Korach et al. can be reformulated in terms of splitting subgraphs.

Theorem 4 in terms of splitting subgraphs. *A graph has the König–Egerváry property if and only if it has no even subdivision of any of the graphs in Fig. 1 as a splitting subgraph.*

Notice that the above statement is precisely a characterization of the whole class of graphs having the König–Egerváry property in terms of splitting subgraphs of the kind that we were looking for. Indeed, it arises from the reformulation of Theorem 1 in terms of forbidden splitting subgraphs by dropping the hypothesis that G has a perfect matching and adding the even subdivisions of C_3 as the extra forbidden splitting subgraphs.

Finally, we will prove Theorem 7, which is a strengthened characterization of graphs with the König–Egerváry property obtained by restricting the way in which the forbidden subgraphs may occur. For the purpose of formulating our characterization, we introduce the notion of *strongly splitting subgraphs* as follows. Let G be a graph. A subset X of $V(G)$ is a *strongly splitting set* if there is a maximum matching M of G such that no edge of M joins a vertex of X to a vertex of $G - X$ and *no vertex of X is adjacent to any M-unsaturated vertex of $G - X$*. A subgraph H of G is a *strongly splitting subgraph* if $V(H)$ is a strongly splitting set of G and H has a perfect or near-perfect matching.

Clearly, strongly splitting sets are splitting sets, and strongly splitting subgraphs are splitting subgraphs. Moreover, the notion of strongly splitting subgraphs is indeed more restrictive than that of splitting subgraphs. For instance, K_5 has K_4 as a splitting subgraph but not as a strongly splitting subgraph. More generally, if H has a perfect matching, then $H + K_1$ has H as a splitting subgraph but not as a strongly splitting subgraph.

The theorem below is our main result and shows that the forbidden splitting subgraphs can be forced to occur as strongly splitting subgraphs.

Theorem 7. *A graph has the König–Egerváry property if and only if it has no even subdivision of any of the graphs in Fig. 1 as a strongly splitting subgraph.*

Proof. Since strongly splitting subgraphs are splitting subgraphs, Corollary 3 implies that if G has the König–Egerváry property then no strongly splitting subgraph of G is an even subdivision of any of the graphs in Fig. 1. Therefore, it suffices to prove that if G does not have the König–Egerváry property then G has a strongly splitting subgraph which is an even subdivision of one of the graphs in Fig. 1.

Suppose that G does not have the König–Egerváry property. By Theorem 4 and Lemma 5, G has a triangular, triangular pair, or tetrahedral configuration $\xi = (S, M)$. Denote by U the set of M-unsaturated vertices of $G - V(S)$.

Case 1: Assume first that $\xi = (S, M)$ is a triangular configuration and let v be the M-unsaturated vertex of S. Suppose, by the way of contradiction, that there is a vertex $s \in V(S)$ adjacent to some vertex $u \in U$. Since M is maximum and u is M-unsaturated, s is M-saturated. In particular, $s \neq v$. Since S is a chordless odd cycle, there is exactly one even path P in S joining s to v. By construction, uP is an M-alternating path joining the M-unsaturated vertices u and v; i.e., uP is an M-augmenting path, a contradiction with the fact that M is maximum. This contradiction proves that there is no edge joining a vertex of S and a vertex of U. We conclude that if G has a triangular configuration $\xi = (S, M)$ then S is a strongly splitting subgraph of G which is an even subdivision of C_3. From now on, we assume, without loss of generality, that G has no triangular configuration.

Case 2: Assume that $\xi = (S, M)$ is a triangular pair configuration. Suppose, by the way of contradiction, that there is a vertex $s \in V(S)$ adjacent to some vertex $u \in U$. Let w_1 and w_2 be the two vertices of S of degree 3 in S. Let P be the path in S joining w_1 to w_2 and let C^i be the cycle of S through w_i for $i = 1, 2$. If $s \in V(P)$, let Q be the subpath of P that joins s to w_1 and, by symmetry, we can assume that Q is odd. If, on the contrary, $s \in V(S) \setminus V(P)$, we can assume without loss of generality that $s \in V(C^2) \setminus \{w_2\}$ and let Q be the odd path in S joining s to w_1 (which exists because C^2 is odd). In both cases, uQ is an M-alternating even path of G where u is not saturated by M. Therefore, $M' = M \bigtriangleup E(uQ)$ is a maximum matching of G and (C^1, M') is a triangular configuration of G, a contradiction. This contradiction proves that there is no edge joining a vertex of S and a vertex of U. We conclude that if G has a triangular pair configuration $\xi = (S, M)$ and G has no triangular configuration, then S is a strongly splitting subgraph of G which is an even subdivision of barbell.

Case 3: Assume that $\xi = (S, M)$ is a tetrahedral configuration. Let w_1, w_2, w_3, w_4 be the set of vertices of S of degree 3 in S. For each $i, j \in \{1, 2, 3, 4\}$ such that $i \neq j$, let $P^{i,j}$ be the path of S joining w_i to w_j but not passing through w_k for any $k \neq i, j$. Without loss of generality, we assume that the vertices w_1, w_2, w_3, w_4 are labeled in such a way that the path $P^{i,i+1}$ starts and ends in edges not belonging to M for each $i = 1, 2, 3, 4$ (superindices should be understood modulo 4). For each pairwise different $i, j, k \in \{1, 2, 3, 4\}$, let $C^{i,j,k}$ be the cycle of S passing through w_i, w_j, and w_k but not passing through w_ℓ where $\ell \neq i, j, k$. Suppose, by the way of contradiction, that there is a vertex $s \in V(S)$ that is adjacent to some vertex $u \in U$. By symmetry, we can assume that $s \in V(P^{1,2})$ or $s \in V(P^{1,3})$. Suppose first that $s \in V(P^{1,2})$. Since $P^{1,2}$ is odd, there is an even subpath Q of $P^{1,2}$ joining s to w_j for $j = 1$ or $j = 2$. (Eventually P is the empty path starting and ending in w_j.) Without loss of generality, assume that Q joins s to w_1. Since $uQP^{1,3}$ is an M-alternating even path and u is M-unsaturated, $M' = M \triangle E(uQP^{1,3})$ is a maximum matching of G and $(C^{2,3,4}, M')$ is a triangular configuration of G, a contradiction. Necessarily, $s \in V(P^{1,3})$. Since $P^{1,3}$ is odd, there is an odd subpath Q of P joining s to w_1 or w_3. Without loss of generality assume that Q joins s to w_1. Since uQ is an M-alternating even path and u is M-unsaturated, $M' = M \triangle E(uQ)$ is a maximum matching of G and $(C^{1,2,4}, M')$ is a triangular configuration of G, a contradiction. This contradiction proves that there is no edge between $V(S)$ and U. We conclude that if G has a tetrahedral configuration (S, M) and G has no triangular configuration, then S is a strongly splitting subgraph of G which is an even subdivision of K_4.

We proved that if G does not have the König–Egerváry property then G has a strongly splitting subgraph which is an even subdivision of C_3, barbell, or K_4, which concludes the proof. □

Notice that if G is a graph having a perfect matching and H is a strongly splitting subgraph of G, then H is a nice subgraph of G and H has a perfect matching. In addition, the even subdivisions of C_3 clearly do not have perfect matchings (because they have an odd number of vertices). Therefore, for graphs with a perfect matching, Theorem 7 reduces precisely to Lovász's characterization (Theorem 1).

The aim of our characterization is not to address the recognition problem, which was already addressed in [6]. Instead, the usefulness of our characterization is on the structural side: given that a graph does not have the König–Egerváry property, our result ensures that an even subdivision of C_3, barbell, or K_4 occurs as a *strongly splitting subgraph*. As an example of this, in the next section, we use Theorem 7 to derive a characterization of edge-perfect graphs by forbidden edge-subgraphs.

4. Edge-perfectness and forbidden edge-subgraphs

Recall from the Introduction that a graph is edge-perfect if and only if each of its edge-subgraphs has the König–Egerváry property, where the edge-subgraphs of a graph G are the induced subgraphs $G - V(F)$ for each $F \subseteq E(G)$. Notice that the class of edge-perfect graphs is not closed by taking induced subgraphs. Indeed, the paw (the graph that arises from C_3 by attaching a pendant vertex) is edge-perfect but contains an induced C_3 which is not edge-perfect. This simple example shows that the class of edge-perfect graphs cannot be characterized by forbidden induced subgraphs. Instead, we will characterize edge-perfect graphs by forbidden edge-subgraphs. Before turning into the proof of the characterization, we observe the following two facts.

Lemma 8. *If F is an edge-subgraph of H and H is an edge-subgraph of G, then F is an edge-subgraph of G.*

Proof. Let E_1 be a set of edges of H such that $H - V(E_1) = F$ and let E_2 be a set of edges of G such that $G - V(E_2) = H$. Then, $G - V(E_1 \cup E_2) = F$ where $E_1 \cup E_2$ is a set of edges of G because H is a subgraph of G. □

Lemma 9. *Let G be a graph. If G has an odd cycle whose vertex set induces an edge-subgraph of G, then G has an edge-subgraph which is a chordless odd cycle.*

Proof. Suppose that G has an odd cycle whose vertex set induces an edge-subgraph of G and let C be the shortest such odd cycle. It suffices to prove that C is chordless. Suppose, by the way of contradiction, that C has some chord $e = xy$. Since C is odd, its vertices can be labeled in such a way that $C = v_1v_2\cdots v_{2k+1}v_1$, where $v_1 = x$ and $v_{2p+1} = y$ for some $p \in \{1, 2, 3, \ldots, k-1\}$. Now $C' = v_1v_2v_3\cdots v_{2p+1}v_1$ is an odd cycle of G and $G[V(C')]$ is an edge-subgraph of $G[V(C)]$ because $G[V(C')] = G[V(C)] - V(\{x_jx_{j+1} \mid 2p+2 \leq j \leq 2k\})$. Since $G[V(C')]$ is an edge-subgraph of $G[V(C)]$ and $G[V(C)]$ is an edge-subgraph of G, by Lemma 8, $G[V(C')]$ is an edge-subgraph of G. Therefore, C' is an odd cycle of G that induces an edge-subgraph of G and C' is shorter than C, a contradiction with the choice of C. This contradiction arose by assuming that C had some chord. So, $G[V(C)]$ is an edge-subgraph of G which is a chordless odd cycle, which completes the proof. □

The chordless odd cycles and K_4 are not edge-perfect because they do not even have the König–Egerváry property. Therefore, these graphs cannot be edge-subgraphs of any edge-perfect graph. The following result shows that, conversely, if a graph without isolated vertices is not edge-perfect, it is because it contains a chordless odd cycle or K_4 as an edge-subgraph.

Theorem 10. *A graph with no isolated vertices is edge-perfect if and only if it has neither a chordless odd cycle nor K_4 as an edge-subgraph.*

Proof. As we have just discussed, if a graph is edge-perfect then no edge-subgraph of it can be a chordless odd cycle or K_4. Conversely, let G be a graph with no isolated vertices that is not edge-perfect. Then, G has at least one edge-subgraph that does not have the König–Egerváry property. Let H be an edge-subgraph of G with a minimum number of vertices that does not have the König–Egerváry property. As H does not have the König–Egerváry property, there is some connected component H' of H that does not have the König–Egerváry property. In particular, H' consists of at least two vertices.

We claim that H' is the only connected component of H having at least two vertices. Suppose, by the way of contradiction, that H has some other connected component H'' having at least two vertices. If $E_{H''}$ is the set of edges of H joining vertices of H'', then $H - V(E_{H''}) = H - V(H'')$ is an edge-subgraph of H that does not have the König–Egerváry property because one of its connected components is still H'. By Lemma 8, $H - V(H'')$ is also an edge-subgraph of G. Since $H - V(H'')$ does not have the König–Egerváry property and has less vertices than H, this contradicts the minimality of H. This contradiction shows that H' is the only connected component of H having at least two vertices.

We now show that the fact that G has no isolated vertices implies that H is connected; i.e., $H = H'$. Indeed, since G has no isolated vertices, for each isolated vertex v of H (i.e., $v \in V(H) \setminus V(H')$) there is some edge $e_v \in E(G)$ that is incident to v. If e_v were incident to some vertex of H' then e_v would be an edge of H, which would contradict the fact that v does not belong to the connected component H' of H. Therefore, e_v is not incident to any vertex of H' for any $v \in V(H) \setminus V(H')$. Since H is an edge-subgraph of G, there is some $E_H \subseteq E(G)$ such that $G - V(E_H) = H$. So, if $E_I = \{e_v : v \in V(H) \setminus V(H')\}$ then $G - V(E_H \cup E_I) = H'$, which proves that H' is an edge-subgraph of G. Since H' does not have the König–Egerváry property, the minimality of H implies that $H = H'$, as claimed.

Since H does not have the König–Egerváry property, Theorem 7 ensures that there is a strongly splitting subgraph S of H which is an even subdivision of C_3, barbell, or K_4. We claim that $H[V(S)]$ is an edge-subgraph of H. Indeed, since S is a strongly splitting subgraph of H, there is a maximum matching M of H such that no edge of M joins a vertex of S with a vertex of $H - V(S)$ and such that no vertex of S is adjacent to an M-unsaturated vertex of $H - V(S)$. Let E_1 be the set of edges of M joining two vertices of $H - V(S)$, and let E_2 be the set of edges of H incident to some M-unsaturated vertex of $H - V(S)$. Since H is connected, for each M-unsaturated vertex of $H - V(S)$ there is at least one edge incident to it in E_2. Also notice that since S is strongly splitting subgraph, no edge of E_2 is incident to a vertex of S. We conclude that $H[V(S)] = H - V(E_1 \cup E_2)$, which shows that $H[V(S)]$ is an edge-subgraph of H, as claimed.

Finally, we claim that H has a chordless odd cycle or K_4 as an edge-subgraph.

Case 1: Suppose first that S is an even subdivision of C_3. Then, S is an odd cycle of H whose vertex set induces an edge-subgraph of H. By Lemma 9, H has an edge-subgraph which is a chordless odd cycle, as claimed.

Case 2: Suppose now that S is an even subdivision of barbell. Let w_1 and w_2 be the vertices of S of degree 3 in S, let C^i be the cycle of S through w_i for $i = 1, 2$ and let P be the path of S joining w_1 to w_2. Let $P = x_1x_2x_3 \cdots x_{2k+1}$ where $x_1 = w_1$ and $x_{2k+1} = w_2$. Let $E_3 = E(C^2)$ and let $E_4 = \{x_jx_{j+1} \mid 2 \le j \le 2k\}$. Then, $H[V(C^1)]$ is an edge-subgraph of $H[V(S)]$ because $H[V(C^1)] = H[V(S)] - V(E_3 \cup E_4)$. Since $H[V(S)]$ is an edge-subgraph of H, by Lemma 8, $H[V(C^1)]$ is an edge-subgraph of H. Thus, C^1 is an odd cycle of H whose vertex set induces an edge-subgraph of H and, by Lemma 9, H has an edge-subgraph which is a chordless odd cycle, as claimed.

Case 3: Finally, suppose that S is an even subdivision of K_4. Let W be the set of vertices of S of degree 3 in S. For each $w, w' \in W$, let $P^{w,w'}$ be the path in S joining w to w' and not passing through any vertex of $W \setminus \{w, w'\}$. If $P^{w,w'}$ has length 1 for each $w, w' \in W$, then $S = H[V(S)]$ is an edge-subgraph of H which is a K_4, and the claim holds. Therefore, we assume without loss of generality that there are two vertices $w_1, w_2 \in W$ such that P^{w_1,w_2} has length greater than 1. Let w_3 and w_4 be the remaining two vertices of W. Let C be the cycle of S through w_2, w_3, and w_4, but not through w_1. For each $i = 2, 3, 4$, let $P^{w_1w_i} = y^i_1y^i_2y^i_3 \cdots y^i_{2k_i+1}$ where $y^i_1 = w_1$ and $y^i_{2k_i+1} = w_i$ and let $F_i = \{y^i_jy^i_{j+1} \mid 1 \le j \le 2k_i - 1\}$. Notice that $H[V(C)]$ is an edge-subgraph of $H[V(S)]$ because $H[V(C)] = H[V(S)] - V(F_2 \cup F_3 \cup F_4)$. Since $H[V(S)]$ is an edge-subgraph of H, by Lemma 8, $H[V(C)]$ is an edge-subgraph of H and, by Lemma 9, H has an edge-subgraph which is a chordless odd cycle, as claimed.

Thus, we proved that H has a chordless odd cycle or K_4 as an edge-subgraph. Since H is an edge-subgraph of G, Lemma 8 implies that G has a chordless odd cycle or K_4 as edge-subgraphs, which completes the proof. □

We would like to draw attention to the role played by our characterization of graphs having the König–Egerváry property (Theorem 7) in the above proof. Indeed, the fact that S is a strongly splitting subgraph of H was key in the proof of the claim that $H[V(S)]$ is an edge-subgraph of H, because it guarantees that there is no M-unsaturated vertex of S such that each edge incident to it were also incident to some vertex of S and, in particular, no edge of E_2 is incident to a vertex of S.

Finally, we present the characterization of edge-perfectness by forbidden edge-subgraphs also for graphs that may have isolated vertices. Notice that when taking an edge-subgraph H of a graph G, the isolated vertices of G are never removed. Therefore, H has at least as many isolated vertices as G. That explains why in the theorem below we must forbid edge-subgraphs with an arbitrary number of isolated vertices.

Theorem 11. *A graph is edge-perfect if and only it has neither $K_4 \cup tK_1$ nor $C_{2k+1} \cup tK_1$ as an edge-subgraph for any $k \ge 1$ and any $t \ge 0$.*

Proof. If G is edge-perfect then all its edge-subgraphs have the König–Egerváry property and, in particular, G has neither $K_4 \cup tK_1$ nor $C_{2k+1} \cup tK_1$ as an edge-subgraph for any $k \ge 1$ and any $t \ge 0$.

Conversely, assume that G is not edge-perfect. Let t be the number of isolated vertices of G. Then, the graph G' that arises from G by removing its t isolated vertices is also not edge-perfect. By Theorem 11, G' has K_4 or C_{2k+1} for some $k \ge 1$ as an edge-subgraph. So, G has $K_4 \cup tK_1$ or $C_{2k+1} \cup tK_1$ for some $k \ge 1$ as an edge-subgraph. □

5. Further remarks

Our main result, Theorem 7, is a characterization of the graphs having the König–Egerváry property by means of a new type of forbidden subgraphs that we call *strongly splitting subgraphs*. Our characterization is a strengthened version of a characterization by Korach, Nguyen, and Peis [14] by forbidden configurations (Theorem 4) which in turn, generalizes the characterizations due to Deming [6] and Sterboul [27] and, when restricted to graphs having a perfect matching, reduces precisely to Lovász's characterization by nice subgraphs (Theorem 1). As an application of our main result, we derived a characterization of edge-perfect graphs by forbidden edge-subgraphs (Theorem 11).

In [17], Larson gave a different characterization of König–Egerváry graphs by relying on the notion of *critical stable sets*. A *stable set* of a graph is a set of pairwise nonadjacent vertices. A stable set I of a graph G is *critical* (resp. *maximum*) if $|I| - |N(I)| \geq |J| - |N(J)|$ (resp. $|I| \geq |J|$) for every stable set J of G, where $N(S)$ denotes $\bigcup_{s\in S} N(s)$. Larson proved that a graph has the König–Egerváry property if and only if it has a maximum independence set which is critical. This result was strengthened by Levit and Mandrescu [20], as follows.

Theorem 12 ([20]). *For every graph G, the following statements are equivalent:*

(i) *G has the König–Egerváry property.*
(ii) *Some maximum stable set of G is critical.*
(iii) *Every maximum stable set of G is critical.*

Larson [17] observed that there is no obvious connection between the non-existence of critical maximum stable sets and the presence of the forbidden configurations in the characterizations of Deming [6] and Sterboul [27]. Theorems 12 and 7 are strengthened versions of the characterizations by Larson and by Deming and Sterboul, respectively, but still, as far as we can see, there is no obvious direct connection between the existence of some non-critical maximum stable set and the presence of an even subdivision of one of the graphs in Fig. 1 as a strongly splitting subgraph. It is an interesting problem for future research that of looking for such direct connection.

A connection was found in [21] between the König–Egerváry property and the structure of the so-called *local maximum stable sets* of a graph. There are three main concepts involved in this characterization: *local maximum stable sets*, *uniquely restricted matchings*, and *greedoids*. A matching is *uniquely restricted* if its saturated vertices induce a subgraph which has a unique perfect matching and a stable set S is a *local maximum stable set* if S is a maximum stable set in the subgraph induced by its *closed neighborhood* $S \cup N(S)$. *Greedoids* are mathematical structures that are a generalization of matroids and were introduced by Korte and Lovász [15]. They observed that the optimality of the 'greedy algorithm' in many instances could be proved even if the underlying combinatorial structure was not necessarily a matroid. For the formal definitions of greedoids and matroids and an account of related results, see [3,16]. In [21], Levit and Mandrescu proved the following characterization of those triangle-free graphs whose family of local maximum stable sets is the family of feasible sets of a greedoid.

Theorem 13 ([21]). *For every triangle-free graph G, the following assertions are equivalent:*

(i) *The family of local maximum stable sets of G is a greedoid.*
(ii) *All the maximum matchings are uniquely restricted and the closed neighborhood of every local maximum stable set of G induces a König–Egerváry graph.*

Moreover, in [22], it was proved that the family $\Psi(G)$ of local maximum stable sets of a graph G forms a greedoid if and only if $\Psi(G)$ is *accessible*, which means that, for each non-empty $S \in \Psi(G)$, there is a vertex $v \in S$ such that $S - \{x\} \in \Psi(G)$.

As a future research, it would be interesting to analyze more connections between the results of this paper and those of [21,20,22] presented in this section.

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