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Analytical and Numerical Methods for some Problems related to Financial Option Pricing

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Para a minha família e amigos

Abstract

We investigate second-order PDE problems arising in financial option pricing.

Firstly, we consider a non-linear problem arising when transaction costs are included and investigate the existence, uniqueness, and localisation of the solution by using topological methods.

Secondly, we study the discretisation of a generalised version of the linear Black-Scholes PDE, where both the asset appreciation rate and the volatility are taken time and space-dependent and growing in the space variable, under the strong assumption that the PDE is nondegenerate. The PDE solvability is considered in the framework of the variational approach and the discretisation is obtained by using basic finite-difference methods. For the time variable, we consider a general evolution equation problem the PDE problem can be cast into, and discretise it by using both the implicit and the explicit schemes. We obtain a convergence result under a smoothness assumption weaker than the usual Hölder continuity. Furthermore, we investigate two main types of operator specification.

For the spatial variable, we consider the discretised version of the PDE problem in discrete weighted Sobolev spaces and obtain a convergence result stronger than we could find in the literature.

Finally, the rate of convergence for the approximation in space and time is computed.

Resumo

Neste estudo, investigamos EDPs de segunda ordem que surgem no apreçamento de opções financeiras.

Em primeiro lugar, consideramos um problema não-linear que surge quando são incluídos custos de transacção e investigamos a existência, unicidade e localização da solução, usando métodos topológicos.

Em segundo lugar, estudamos a discretização de um versão generalizada da EDP de Black-Scholes linear, em que a taxa de apreciação do activo e a volatilidade são dependentes do tempo e do espaço e crescentes na variável espacial, assumindo que a EDP é não degenerada.

A solvabilidade da EDP é considerada numa abordagem variacional e a discretização é obtida usando métodos básicos de diferenças finitas. Para a variável temporal, consideramos uma equação de evolução geral, de que a EDP é caso particular, que discretizamos usando os esquemas implícito e explícito. Obtemos um resultado de convergência sob uma hipótese de regularidade mais fraca que a usual continuidade de Hölder. Adicionalmente, investigamos dois tipos fundamentais de especificação do operador. Para a variável espacial, consideramos a versão discreta da EDP em espaços de Sobolev discretos e obtemos um resultado de convergência mais forte do que pudemos encontrar na literatura.

Finalmente, obtemos a taxa de convergência para a aproximação no espaço e no tempo.

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Introduction

The undeniable importance of the financial activity in the modern world justifies the need for advanced methods in the treatment of inherent problems.

Modelling of risky asset prices and modern option pricing techniques are often considered among the most mathematically complex of all applied areas of Finance. It became essential for the financial analyst to possess a high level of mathematical skills. Conversely, the complex challenges posed by the problems and models relevant to Finance have, for a long time, been an important source of new research topics for mathematicians.

In this work we are motivated by the investigation of second-order partial differential equation (PDE) problems arising in European financial option pricing.

We consider the inclusion of costs, such as taxes or fees, in the transactions that take place when an agent deals with a portfolio of assets. For a model where the cost of the transaction of each share of stock diminishes as the number of shares transactioned increases

$$h\left(\nu\right) = a - b\left|\nu\right|$$

where ν is the number of shares traded and $a, b > 0$, we arrive to the non-linear equation on the price V of the option

$$\frac{\partial V}{\partial t} + \frac{1}{2}\tilde{\sigma}^2 S^2 \frac{\partial^2 V}{\partial S^2} + b\sigma^2 S^3 \left(\frac{\partial^2 V}{\partial S^2}\right)^2 + r\left(\frac{\partial V}{\partial S} S - V\right) = 0,$$

where $\tilde{\sigma}$ is an adjusted volatility of the real volatility σ of the underlying S, and r is the short rate of return.

This equation is an extension of the most relevant option pricing formula: the Black-Scholes equation. The introduction of our particular model for the transaction costs in the option pricing market led us to a PDE that contains the Black-Scholes terms with an additional non-linear term modelling the presence of transaction costs.

We also proceed to the discretisation of PDE problems using finite-difference methods.

There has been a long and extensive research on the application of finite-difference methods to financial option pricing. We refer to [55] for a brief summary of the method's history, and also for the references of the seminal work by R. Courant, K. O. Friedrichs and H. Lewy, and further major contributions by many others.

Also in [55], we can find the numerical study, making use of finite differences, of the Cauchy problem for a general multidimensional linear parabolic PDE of order $m \geq 2$, with bounded time and space-dependent coefficients. This study is pursued in the framework of the classical approach.

Although the theory can be considered reasonably complete since three decades ago, some important research continues. We mention, just as an example, the recent works [29, 30].

The finite-difference method was early applied to financial option pricing, the pioneering work being due to M. Brennan and E. S. Schwartz in 1978, and was, since then, widely researched in the context of the financial application, and extensively used by practitioners. For the references of the original publications and further major research, we refer to the review paper [11].

Most studies concerning the discretisation of PDE problems in Finance consider the particular case where the PDE coefficients are constant (see, e.g., [6, 9, 19, 53]). This occurs, namely, in option pricing under Black-Scholes stochastic model (in one or several dimensions), when the asset application rate and volatility are taken constant.

Multidimensional PDE problems arise in Financial Mathematics and in Mathematical Physics. We are mainly motivated by the application to a class of stochastic models in Financial Mathematics, comprising the non path-dependent options, with fixed exercise, written on multiple assets (*basket options*, *exchange options*, *compound options*, European options on future contracts and foreign-exchange, and others), and also, to a particular type of path-dependent options, the Asian options (see, e.g., [34, 56]).

Let us consider the stochastic modelling of a multi-asset financial option of European type under the framework of a general version of Black-Scholes model, where the vector of asset appreciation rates and the volatility matrix are taken time and space-dependent. Owing to a Feynman-Kač type formula, pricing this option can be reduced to solving the Cauchy problem (with terminal condition) for a second-order linear parabolic PDE of nondivergent type, with null term and unbounded coefficients, degenerating in the space variables (see, e.g., [34]).

After a change of the time variable, the PDE problem is written

$$\frac{\partial u}{\partial t} = Lu + f \;\text{ in }\; [0,T] \times \mathbb{R}^d, \quad u(0,x) = g(x) \;\text{ in }\; \mathbb{R}^d, \tag{0.1}$$

where L is the second-order partial differential operator in the nondivergence form

$$L(t,x) = a^{ij}(t,x)\frac{\partial^2}{\partial x_i \partial x_j} + b^i(t,x)\frac{\partial}{\partial x_i} + c(t,x), \quad i,j = 1,\dots,d,$$

with real coefficients, f and g are given real-valued functions (the free term f is included to further improve generality), and $T \in (0,\infty)$ is a constant. For each $t \in [0,T]$ the operator $-L$ is degenerate elliptic, and the growth in the spatial variables of the coefficients a, b, and of the free data f, g is allowed.

When problem (0.1) is considered in connection with the Black-Scholes modelling of a financial option, we see that the growth of the vector stochastic differential equation (SDE) coefficients in the underlying financial model is appropriately matched. Also, by setting the problem with this generality, we cover the general case where the asset appreciation rate vector and the volatility matrix are taken time and space-dependent. Finally, by letting the initial data g non-specified, a large class of payoff functions can be considered in the underlying financial derivative modelling. The free term f is included to further improve generality.

One possible approach for the discretisation of the PDE problem (0.1) is to proceed to a two-stage discretisation. First, the problem is semi-discretised in space, and both the possible equation degeneracy and coefficient unboundedness are dealt with (see, e.g., [21, 22], where the spatial approximation is pursued in a variational framework, under the strong assumption that the PDE does not degenerate, and [20]). Subsequently, a time discretisation takes place.

For the time discretisation, it can be tackled by approximating the linear evolution equation problem in which the PDE problem (0.1) can be cast into

$$\frac{\mathrm{d}u}{\mathrm{d}t} = A(t)u + f(t) \;\text{ in }\; [0,T], \quad u(0) = g, \tag{0.2}$$

where, for every $t \in [0,T]$ with $T \in (0,\infty)$, $A(t)$ is a linear operator from a reflexive separable Banach space V to its dual V^*, $u : [0,T] \to V$ is an unknown function, $f : [0,T] \to V^*$, g belongs to a Hilbert space H, with f and g given, and V is continuously and densely embedded into H. We assume that operator $A(t)$ is continuous and impose a coercivity condition.

This simpler general approach, which we follow, is powerful enough to obtain the desired results. On the other hand, it covers a variety of problems, namely initial-value and initial boundary-value problems for linear parabolic PDEs of any order $m \geq 2$.

The numerical methods and possible approximation results are strongly linked to the theory on the solvability of the PDEs. In this work, we make use of the L^2 theory of solvability of linear PDEs in weighted Sobolev spaces. In particular, we consider the PDE solvability in the deterministic special case of a class of weighted Sobolev spaces introduced by O. G. Purtukhia [46, 47, 48, 49], and further generalised by I. Gyöngy and N. V. Krylov [27], for the treatment of linear stochastic partial differential equations (SPDE). By considering discrete versions of these spaces, we set a suitable discretised framework and investigate the PDE approximation.

The particularization of PDE (0.1) to one dimension, leads to the Cauchy problem

$$Lu - \frac{du}{dt} + f = 0 \quad \text{in} \quad Q, \quad u(0,x) = g(x) \quad \text{in} \quad \mathbb{R}, \tag{0.3}$$

where $Q = [0,T] \times \mathbb{R}$, with T a positive constant, L is the second-order partial differential operator with real coefficients

$$L(t,x) = a(t,x)\frac{\partial^2}{\partial x^2} + b(t,x)\frac{\partial}{\partial x} + c(t,x)$$

for each $t \in [0,T]$ uniformly elliptic in the space variable, and f and g are given real-valued functions. We allow the growth in space of the first and second-order coefficients in L (linear and quadratic growth, respectively), and of the data f and g (polynomial growth).

The motivation for the study of this case is obvious: the possibility of obtaining stronger results than for the multidimensional setting.

Now we summarise the contents of this work.

The basics of stochastic calculus theory is reviewed in Chapter 1. We refer the close connection existent between SDE and certain PDE, in particular the Black-Scholes equation.

The inclusion of transaction costs leads us to a problem approached in Chapter 2. We obtain existence, uniqueness and localisation results for the solution of the associated stationary problem.

We study, in Chapter 3, the discretisation of the linear parabolic equation (0.1) in abstract spaces making use of both the implicit and the explicit finite-difference schemes. Under a nondegeneracy assumption, we consider the PDE solvability in the framework of the variational approach. The stability of the explicit scheme is obtained, and the schemes' rates of convergence are estimated. Additionally, we study the special cases where A and f are approximated by integral averages and also by weighted arithmetic averages.

In Chapter 4 we deal with the challenge posed by the unboundedness of the coefficients of PDE in problem (0.3), under the strong assumption that the PDE does not degenerate. We use finite-difference methods to approximate in space the weak solution of the problem. We follow the previous work by Gonçalves and Grossinho ([20, 21]), where the same approach was used for the more general case of multidimensional PDEs. By considering the special case of one dimension in space, a stronger convergence result is obtained in this chapter. In particular, the same order of accuracy is obtained under regularity assumptions weaker than those required in [20, 21] for the corresponding convergence result.

In Chapter 5, we estimate the rate of convergence for the approximation in space and time in abstract spaces for general linear evolution equations, and then specified to the second-order PDE problem.

We conclude with some final comments and the direction of future research related to this work.

Chapter 1

Stochastic calculus applied to Finance

We present in this chapter the main guidelines of stochastic calculus. First, we summarise the main concepts and results associated to the stochastic processes, we introduce the stochastic integral, the Itô lemma or formula, and discuss the solution of a stochastic differential equation. The importance of Feynman-Kač formula and the Kolmogorov backward equation is evident with the results presented in Section 1.4, where we make the connection between the stochastic differential equations and the partial differential equations. One of the main applications of stochastic calculus to Finance is shown in Section 1.6, with the famous Black-Scholes equation.

The text presented follows Björk ([7]), Mikosch ([41]), and Shreve ([52]).

1.1 Stochastic processes

We start to summarise the basic theory of stochastic processes.

Definition 1.1. A *stochastic process* X is a collection of random variables

$$(X_t, t \in T) = (X_t(\omega), t \in T, \omega \in \Omega)$$

defined on some probability space $(\Omega, \mathcal{A}, P)$.

We will deal exclusively with stochastic processes $X = (X_t)_{t\in T}$ where T is an interval, usually $T = [0, \infty)$, and, for obvious reasons, the index t of X_t is frequently referred to as *time*. Then, we call X a *continuous-time stochastic process*, or, simply, a *continuous stochastic process*.

A very important stochastic process is the *Brownian motion*, which has been playing a central role in Finance.

Definition 1.2. A real-valued continuous stochastic process $W = (W_t)_{t\geq 0}$ in the probability space $(\Omega, \mathcal{A}, P)$ is called a (standard) *Brownian motion*, or *Wiener process*, if the following conditions are verified

- $W_0 = 0$.
- For all $0 = t_0 < t_1 < \cdots < t_m$, the increments

$$W_{t_1} = W_{t_1} - W_{t_0},\ W_{t_2} - W_{t_1},\ \ldots,\ W_{tm} - W_{t_{m-1}}$$

  are independent.
- For $s < t$, the random variable $W_t - W_s$ has the Gaussian distribution $N(0, \sqrt{t-s})$.
- W has continuous trajectories.

A *filtration* can be regarded as an increasing stream of information.

Definition 1.3. Assume that $(\mathcal{F}_t)_{t\geq 0}$ is a collection of σ-algebras on the same probability space $(\Omega, \mathcal{A}, P)$ and that $\mathcal{F}_t$, for all $t \geq 0$, is a subset of the larger σ-algebra $\mathcal{A}$.

The collection $(\mathcal{F}_t)_{t\geq 0}$ of σ-algebras on Ω is called a *filtration* if

$$\mathcal{F}_s \subset \mathcal{F}_t \quad \text{for all } 0 \leq s \leq t.$$

Representing by $\sigma(Y)$ the σ-algebra generated by the information given by the random variable Y, we define the concept of *adapted process.*

Definition 1.4. Let $(\Omega, \mathcal{A}, P)$ be a probability space.

- A stochastic process $X = (X_t)_{t\geq 0}$ is said to be *adapted to the filtration* $(\mathcal{F}_t)_{t\geq 0}$ if, for any t, X_t is $\mathcal{F}_t$-measurable, i.e.,

$$\sigma(X_t) \subset \mathcal{F}_t \quad \text{for all } t \geq 0.$$

- A stochastic process X is always adapted to the *natural filtration* generated by X

$$\mathcal{F}_t = \sigma(X_s, s \leq t).$$

- The process X is said to be *adapted to the Brownian motion* W if X is adapted to the natural Brownian filtration $(\mathcal{F}_t)_{t\geq 0}$. This means that X_t is a function of W_s, for $s \leq t$.

The notion of martingale is fundamental in the theory of pricing financial derivatives.

Definition 1.5. A stochastic process $X = (X_t)_{t\geq 0}$ is called an $\mathcal{F}_t$-martingale if the following conditions are verified

- For all $t \geq 0$, $E\left(|X_t|\right) < \infty$.
- X is adapted to the filtration $(\mathcal{F}_t)_{t\geq 0}$.
- For all $0 \leq s \leq t$, X_s is the best prediction of X_t given $\mathcal{F}_s$, i.e.

$$\mathbf{E}\left(X_t|\mathcal{F}_s\right) = X_s.$$

There are some interesting results about the martingales (see [41], pp.81–82).

Proposition 1.6. *The expectation function of a martingale is constant.*

Proposition 1.7. *The Brownian motion is a martingale.*

1.2 Stochastic integrals

We now discuss the concept of Itô stochastic integral. This one is used in Finance to model the value of a portfolio that results from trading assets in continuous time. Due to the fact that a Brownian path is nowhere differentiable and has unbounded variation, the calculus used to manipulate these integrals differs from ordinary calculus.

Thus we are interested in integrals of the form $\int_0^t X_s \, \mathrm{d}W_s$, where $(W_t)_{t\geq 0}$ is a given $\mathcal{F}_t$-Brownian motion and $X = (X_t)_{t\geq 0}$ is an $\mathcal{F}_t$-adapted process.

We begin to define the stochastic integral for a class of processes whose paths assume only a finite number of values.

Definition 1.8. The stochastic process $C = (C_t)_{t\geq 0}$ is said to be *simple* if the following properties are satisfied

- There exist a partition

$$\tau_n : 0 = t_0 < t_1 < \cdots < t_{n-1} < t_n = T,$$

and a sequence $(Z_i, i = 1, \ldots, n)$ of random variables such that

$$C_t = \begin{cases} Z_n & \text{if} \quad t = T \\ Z_i & \text{if} \quad t_{i-1} \le t < t_i, \quad i = 1, \ldots, n. \end{cases}$$

- The sequence (Z_i) is adapted to $(\mathcal{F}_{t_{i-1}})_{i=1,\ldots,n}$, i.e., Z_i is a function of the Brownian motion up to time t_{i-1}, and satisfies $\mathbf{E}(Z_i^2) < \infty$ for all i.

Then we define the stochastic integral as the obvious formula.

Definition 1.9. The Itô stochastic integral of a simple process C on $[0, T]$ is given by

$$\int_0^T C_s \, \mathrm{d}W_s := \sum_{i=1}^n C_{t_{i-1}}(W_{t_i} - W_{t_{i-1}}) = \sum_{i=1}^n Z_i(W_{t_i} - W_{t_{i-1}}).$$

The Itô stochastic integral of a simple process C on $[0, t]$, $t_{k-1} \le t \le t_k$, is given by

$$\int_0^t C_s \mathrm{d}W_s := \int_0^t C_s I_{[0,t]}(s) \mathrm{d}W_s = \sum_{i=1}^{k-1} Z_i(W_{t_i} - W_{t_{i-1}}) + Z_k(W_t - W_{t_{k-1}}),$$

where $\sum_{i=1}^0 Z_i(W_{t_i} - W_{t_{i-1}}) = 0$.

In the following proposition we present some fundamental properties of the stochastic integral of a simple process (see [41], pp.105–107). We consider

$$I_t(C) = \int_0^t C_s \mathrm{d}W_s, \text{ for } t \in [0, T].$$

Proposition 1.10. *If $C = (C_t)_{t \ge 0}$ is a simple process then*

1. *The stochastic process $(I_t(C))_{0 \le t \le T}$ is a $\mathcal{F}_t$-martingale.*

2. *The Itô stochastic integral has expectation zero.*

3. *The Itô stochastic integral satisfies the isometry property*

$$\mathbf{E}\left(\int_0^t C_s \, \mathrm{d}W_s\right)^2 = \int_0^t \mathbf{E}\left(C_s^2\right) \mathrm{d}s, \quad t \in [0, T].$$

4. The Itô stochastic integral is linear.

5. The process $(I_t(C))_{0\leq t\leq T}$ has continuous paths.

The definition of stochastic integral will be extended to a larger class of processes that we denote by $\mathcal{H}$.

Definition 1.11. We say that the process $C = (C_t)_{a\leq t\leq b}$ belongs to the class $\mathcal{H}[a,b]$ if the following conditions are satisfied

- The process C is adapted to the natural filtration.
- The integral $\int_a^b \mathbf{E}\left(C_s^2\right) \mathrm{d}s$ is finite.

We say that the process $C = (C_t)_{t\geq 0}$ belongs to the class $\mathcal{H}$ if $C \in \mathcal{H}[0,t]$ for all $t > 0$.

The Itô stochastic integral of a process $C \in \mathcal{H}$ is defined as the limit of the Itô stochastic integrals of a sequence of simple processes converging to C in some sense, whose existence is guaranteed by the following result (see [41], p.109).

Proposition 1.12. *Let $C = (C_t)_{0\leq t\leq T}$ be a process of the class $\mathcal{H}$.*

Then there exists a sequence $\left(C^{(n)}\right)_{n\in\mathbf{N}}$ of simple processes such that

$$\int_0^T \mathbf{E}\left[C_s - C_s^{(n)}\right]^2 \mathrm{d}s \to 0.$$

Writing $I(C^{(n)}) = \int_0^t C_s^{(n)} \mathrm{d}W_s$, we also have that the sequence $(I(C^{(n)}))$ of Itô stochastic integrals converges in a mean square sense to a unique limit process, i.e., there exists a unique process $I(C)$ on $[0,T]$ such that

$$\mathbf{E}\left\{\sup_{0\leq t\leq T}\left[I_t(C) - I_t\left(C^{(n)}\right)\right]^2\right\} \to 0.$$

The next definition follows naturally.

Definition 1.13. Let $C = (C_t)_{0\leq t\leq T}$ be a process of the class $\mathcal{H}$.

The unique mean square limit $I(C)$, guaranteed by Proposition 1.12, is called the *Itô stochastic integral of* C. It is denoted by

$$I_t(C) = \int_0^t C_s \mathrm{d}W_s, \quad t \in [0,T].$$

As for the Itô stochastic integral for a simple process, we state the following properties (see [41], pp.111-112).

Proposition 1.14. *If* $C = (C_t)_{0 \le t \le T}$ *is a process of the class* $\mathcal{H}$ *then*

1. *The stochastic process* $(I_t(C))_{0 \le t \le T}$ *is a* $\mathcal{F}_t$*-martingale.*
2. *The Itô stochastic integral has expectation zero.*
3. *The Itô stochastic integral satisfies the isometry property*

$$\mathbf{E}\left(\int_0^t C_s \, \mathrm{d}W_s\right)^2 = \int_0^t \mathbf{E}(C_s^2)\mathrm{d}s, \quad t \in [0, T].$$

4. *The Itô stochastic integral is linear.*
5. *The process* $I(C)$ *has continuous paths.*

Although the existence of a definition for a general Itô stochastic integral of the process $C \in \mathcal{H}$, we are not able to write the integral $\int_0^t C_s \mathrm{d}W_s$ in simple terms of the Brownian motion. However, using the Itô lemma presented later in this section it is possible to obtain explicit formulae for Itô stochastic integrals in some particular cases of the integrand process C.

First, we introduce the Itô process.

Definition 1.15. A process $X = (X_t)_{0 \le t \le T}$ that has the representation

$$X_t = X_0 + \int_0^t A_s^{(1)} \mathrm{d}s + \int_0^t A_s^{(2)} \mathrm{d}W_s \tag{1.1}$$

with $A^{(1)}$ and $A^{(2)}$ $\mathcal{F}_t$-adapted processes, and such that the above integrals are well defined in the Riemann and Itô senses, respectively, is called an *Itô process.*

The integral equation (1.1) can be represented in its differential form

$$\mathrm{d}X_t = A_t^{(1)} \mathrm{d}t + A_t^{(2)} \mathrm{d}W_t,$$

which describes the "dynamics" of the stochastic process X.

The following result states the uniqueness of the previous decomposition (see [41], p.119).

Proposition 1.16. *If a process $X = (X_t)_{0 \le t \le T}$ has the representation (1.1), then the processes $A^{(1)}$ and $A^{(2)}$ are uniquely determined in the sense that, if X has a representation (1.1), where the $A^{(i)}$ are replaced with adapted processes $D^{(i)}$, then $A^{(i)}$ and $D^{(i)}$ coincide a.e., for $i = 1, 2$.*

Now we state an important result in the theory of stochastic calculus known as the Itô lemma, or Itô formula (see [41], p.120).

Theorem 1.17. *(Itô lemma) Let $X = (X_t)_{0 \le t \le T}$ be an Itô process with representation (1.1) and $f(t, x)$ be a function whose second order partial derivatives are continuous. Then*

$$\begin{aligned} f(t, X_t) &= f(0, X_0) \\ &+ \int_0^t \left[\frac{\partial f}{\partial t}(s, X_s) + A_s^{(1)} \frac{\partial f}{\partial x}(s, X_s) + \frac{1}{2} \left(A_s^{(2)}\right)^2 \frac{\partial^2 f}{\partial x^2}(s, X_s) \right] \mathrm{d}s \\ &+ \int_0^t A_s^{(2)} \frac{\partial f}{\partial x}(s, X_s)\, \mathrm{d}W_s. \end{aligned} \tag{1.2}$$

Remark 1.1. Formula (1.2) is frequently given in the symbolic form

$$f(t, X_t) = f(0, X_0) + \int_0^t \left[\frac{\partial f}{\partial t}(s, X_s) + \frac{1}{2} \left(A_s^{(2)}\right)^2 \frac{\partial^2 f}{\partial x^2}(s, X_s) \right] \mathrm{d}s + \int_0^t \frac{\partial f}{\partial x}(s, X_s) \mathrm{d}X_s$$

where

$$\mathrm{d}X_s = A_s^{(1)} \mathrm{d}s + A_s^{(2)} \mathrm{d}W_s.$$

We state the integration by parts formula (see [41], p.122).

Theorem 1.18. *(Integration by parts formula) Let $X^{(1)}$ and $X^{(2)}$ be two Itô processes with respect to the same Brownian motion*

$$X_t^{(i)} = X_0^{(i)} + \int_0^t A_s^{(1,i)} \mathrm{d}s + \int_0^t A_s^{(2,i)} \mathrm{d}W_s, \quad i = 1, 2.$$

Then

$$X_t^{(1)} X_t^{(2)} = X_0^{(1)} X_0^{(2)} + \int_0^t X_s^{(2)} \mathrm{d}X_s^{(1)} + \int_0^t X_s^{(1)} \mathrm{d}X_s^{(2)} + \int_0^t A_s^{(2,1)} A_s^{(2,2)} \mathrm{d}s.$$

1.3 Stochastic differential equations

We are now interested in finding a stochastic process $X = (X_t)_{t \ge 0}$ which satisfies the differential equation

$$\mathrm{d}X_t = \mu(t, X_t)\mathrm{d}t + \sigma(t, X_t)\mathrm{d}W_t, \quad X_0 = Y, \tag{1.3}$$

where, as usual, $W = (W_t)_{t\geq 0}$ denotes the Brownian motion, and $\mu(t,x)$ and $\sigma(t,x)$ are deterministic functions, with $t \in \mathbb{R}^+$ and $x \in \mathbb{R}$.

The randomness of the solution X, if it exists, results, on one hand, from the initial condition, and, on the other hand, from the noise generated by the Brownian motion.

We should interpret the equation in (1.3) as the stochastic integral equation

$$X_t = X_0 + \int_0^t \mu(s, X_s)\mathrm{d}s + \int_0^t \sigma(s, X_s)\mathrm{d}W_s, \quad t \geq 0, \tag{1.4}$$

where the first integral on the right-hand side is a Riemann integral, and the second one is an Itô stochastic integral.

Equation (1.4) is called an *Itô stochastic differential equation*, or simply, *stochastic differential equation (SDE)*.

The Brownian motion W is called the *driving process* of the SDE (1.4).

Definition 1.19. A *strong solution to the SDE* (1.4) is a $\mathcal{F}_t$-adapted stochastic process $X = (X_t)_{t\geq 0}$ which satisfies the following conditions

- The integrals ocurring in (1.4) are well defined as Riemann or Itô stochastic integrals, respectively.

- X satisfies

$$X_t = Y + \int_0^t \mu(s, X_s)\mathrm{d}s + \int_0^t \sigma(s, X_s)\mathrm{d}W_s \quad a.s., \quad \forall t \geq 0.$$

While the strong solution of (1.4) is based on the path of the underlying Brownian motion, for the weak solutions the path behaviour is not essential, we are only interested in the distribution of X. Weak solutions X are sufficient in order to determine the distributional characteristics of X, such as the expectation, variance and covariance functions of the process.

A strong or weak solution X of the SDE (1.4) is called a *diffusion process*. In particular, taking $\mu(t,x) = 0$ and $\sigma(t,x) = 1$, we have that the Brownian motion is a diffusion process.

We establish sufficient conditions for the existence and uniqueness of a strong solution of the SDE with the following result (see [41], p.138).

Theorem 1.20. *If the coefficient functions $\mu(t,x)$ and $\sigma(t,x)$ satisfy the following conditions*

- *they are continuous, and*
- *they satisfy a Lipschitz condition with respect to the second variable, i.e., there exists a constant K such that, for all $t \in \mathbb{R}^+$ and $x, y \in \mathbb{R}$*

$$|\mu(t,x) - \mu(t,y)| + |\sigma(t,x) - \sigma(t,y)| \leq K|x-y|,$$

and if the initial condition Y

- *has a finite second moment: $\mathbf{E}(Y^2) < \infty$, and*
- *is independent of $W = (W_t)_{t\geq 0}$,*

then the SDE (1.4) has a unique strong solution X on $[0,T]$, $T > 0$.

Two important examples of SDEs are the following ones (see [41], p.155, p.139, respectively).

Example 1.1. The general linear stochastic differential equation

$$X_t = X_0 + \int_0^t [\mu_1(s)X_s + \mu_2(s)]\mathrm{d}s + \int_0^t [\sigma_1(s)X_s + \sigma_2(s)]\mathrm{d}W_s, \quad t \in [0,T]$$

where the (deterministic) coefficient functions μ_i and σ_i are continuous, $i = 1, 2$, has a unique strong solution on every interval $[0,T]$.

Example 1.2. The *geometric Brownian motion*, broadly used to model the price of an asset in Finance, is given by

$$X_t = X_0 \mathrm{e}^{(\mu - \sigma^2/2)t + \sigma W_t}, \quad t \in [0,T].$$

This process is the unique solution of the linear Itô stochastic differential equation

$$X_t = X_0 + \mu \int_0^t X_s \mathrm{d}s + \sigma \int_0^t X_s \mathrm{d}W_s, \quad t \in [0,T].$$

1.4 Partial differential equations

In this section, we intend to relate the solution X on the time interval $[t, T]$, for fixed T and $t < T$, of the SDE

$$\begin{aligned} dX_s &= \mu(s, X_s)ds + \sigma(s, X_s)dW_s \\ X_t &= x \end{aligned}$$

to the solution of the following boundary value problem

$$\begin{aligned} \frac{\partial F}{\partial t}(t,x) + \mu(t,x)\frac{\partial F}{\partial x} + \frac{1}{2}\sigma^2(t,x)\frac{\partial^2 F}{\partial x^2}(t,x) &= 0 \\ F(T,x) &= \Phi(x), \end{aligned}$$

where $\mu(t,x)$, $\sigma(t,x)$, and $\Phi(t,x)$ are deterministic functions.

We define the *infinitesimal operator* $\mathcal{A}$ of the process X for any function on $C^2(\mathbb{R})$ by

$$\mathcal{A} = \mu(t,x)\frac{\partial}{\partial x} + \frac{1}{2}\sigma^2(t,x)\frac{\partial^2}{\partial x^2}.$$

Thus in terms of the infinitesimal generator, the Itô formula (1.2) takes the form

$$f(t, X_t) = f(0, X_0) + \int_0^t \left[\frac{\partial f}{\partial t}(s, X_s) + \mathcal{A}f(s, X_s)\right] ds + \int_0^t \sigma(s, X_s)\frac{\partial f}{\partial x}(s, X_s)dW_s$$

Using $\mathcal{A}$, we may write the boundary value problem as

$$\begin{aligned} \frac{\partial F}{\partial t}(t,x) + \mathcal{A}F(t,x) &= 0 & (1.5) \\ F(T,x) &= \Phi(x). & (1.6) \end{aligned}$$

Applying the Itô formula to the process $F(s, X_s)$ on the time interval $[t, T]$, we obtain

$$\begin{aligned} F(T, X_T) &= F(t, X_t) + \int_t^T \left\{\frac{\partial F}{\partial t}(x, X_s) + \mathcal{A}F(s, X_s)\right\} ds \\ &\quad + \int_t^T \sigma(s, X_s)\frac{\partial F}{\partial x}(s, X_s)dW_s. \end{aligned}$$

Since F satisfies (1.5), the time integral vanishes. Taking expectations and assuming that the stochastic integral exists, then it vanishes as well, leaving us with the formula

$$F(t,x) = \mathbf{E}_{t,x}[\Phi(X_T)],$$

where the notation $\mathbf{E}_{t,x}$ emphasizes that the expected value is to be taken given the initial value $X_t = x$.

Thus we have proved the following result (see [7], p.69).

Proposition 1.21. *(Feynman-Kač stochastic representation formula) Assume that F is a solution to the boundary value problem*

$$\begin{aligned} \frac{\partial F}{\partial t}(t,x) + \mu(t,x)\frac{\partial F}{\partial x} + \frac{1}{2}\sigma^2(t,x)\frac{\partial^2 F}{\partial x^2}(t,x) &= 0 \\ F(T,x) &= \Phi(x). \end{aligned}$$

Assume furthermore that the process $\sigma(s,X_s) = \frac{\partial F}{\partial x}(s,X_s)$ is in $\mathcal{H}$ given by Definition 1.11, where X is defined below. Then F has the representation

$$F(t,x) = \mathbf{E}_{t,x}[\Phi(X_T)],$$

where X satisfies the SDE

$$\begin{aligned} \mathrm{d}X_s &= \mu(s,X_s)\mathrm{d}s + \sigma(s,X_s)\mathrm{d}W_s \\ X_t &= x. \end{aligned}$$

The following result deals with an important boundary value problem in financial theory (see [7], p.70).

Proposition 1.22. *Assume that F is a solution to the boundary value problem*

$$\begin{aligned} \frac{\partial F}{\partial t}(t,x) + \mu(t,x)\frac{\partial F}{\partial x} + \frac{1}{2}\sigma^2(t,x)\frac{\partial^2 F}{\partial x^2}(t,x) - rF(t,x) &= 0 \\ F(T,x) &= \Phi(x). \end{aligned}$$

Assume furthermore that the process $\sigma(s,X_s) = \frac{\partial F}{\partial x}(s,X_s)$ is in $\mathcal{H}$, where X is defined below. Then F has the representation

$$F(t,x) = \mathrm{e}^{-r(T-t)}\mathbf{E}_{t,x}[\Phi(X_T)],$$

where X satisfies the SDE

$$\begin{aligned} \mathrm{d}X_s &= \mu(s,X_s)\mathrm{d}s + \sigma(s,X_s)\mathrm{d}W_s \\ X_t &= x. \end{aligned}$$

Using Feynman-Kač we can obtain some classical results concerning the transition probabilities for the solution of a SDE (see [52], p.291).

Proposition 1.23. *(Kolmogorov backward equation) Let X be a solution to the SDE*

$$\mathrm{d}X_s = \mu(s, X_s)\mathrm{d}s + \sigma(s, X_s)\mathrm{d}W_s.$$

For $0 \le t < T$, let $p(t,T;x,y)$ be the transition density for the initial solution $X_t = x$ to this equation (i.e., if we solve the equation with the initial condition $X_t = x$, then the random variable X_T has density $p(t,T;x,y)$ in the y variable). Assume that $p(t,T;x,y) = 0$ for $0 \le t < T$ and $y \le 0$.

Then we have

$$-\frac{\partial p}{\partial t}(t,T;x,y) = \mu(t,x)\frac{\partial p}{\partial x}(t,T;x,y) + \frac{1}{2}\sigma^2(t,x)\frac{\partial^2 p}{\partial x^2}(t,T;x,y) \tag{1.7}$$

The reason that (1.7) is called backward equation is that the differential operator is working on the "backward variables" (t,x). The corresponding forward equation is also known as the *Fokker-Planck equation* (see [52], p.291).

Proposition 1.24. *(Kolmogorov forward equation or Fokker-Planck equation) Let X be a solution to the SDE*

$$\mathrm{d}X_s = \mu(s, X_s)\mathrm{d}s + \sigma(s, X_s)\mathrm{d}W_s.$$

For $0 \le t < T$, let $p(t,T;x,y)$ be the transition density for the initial solution $X_t = x$ to this equation. Assume that $p(t,T;x,y) = 0$ for $0 \le t < T$ and $y \le 0$.

Then we have

$$\frac{\partial p}{\partial T}(t,T;x,y) = -\frac{\partial}{\partial y}[\mu(t,y)p(t,T;x,y)] + \frac{1}{2}\frac{\partial^2}{\partial x^2}[\sigma^2(T,y)p(t,T;x,y)]. \tag{1.8}$$

In contrast to the Kolmogorov backward equation (1.7), where T and y were held constant and the variables were t and x, here t and x are held constant and the variables are T and y. The variables t and x are sometimes called the *backward variables*, and T and y are called the *forward variables.*

1.5 Change of measure

The main idea of the change of measure technique consists of introducing a new probability measure via a so-called density function which is in general not a probability density function. The importance of this theory becomes evident in Section 1.6.

Definition 1.25. Let P and Q be two probability measures on the σ-algebra $\mathcal{F}$. If there exists a non-negative function f_1 such that

$$Q(A) = \int_A f_1(\omega) \mathrm{d}P(\omega), \quad A \in \mathcal{F}$$

we say that f_1 is the density of Q with respect to P and we also say that Q is absolutely continuous with respect to P.

If P is absolutely continuous with respect to Q, and Q is absolutely continuous with respect to P, we say that P and Q are *equivalent probability measures.*

The famous Girsanov's theorem allows us to describe how the dynamics of a stochastic process changes when the original measure is changed to an equivalent probability measure (see [41], pp.178–179).

Theorem 1.26. *(Girsanov's theorem) The following statements hold*

- *The stochastic process*

$$M_t = \exp\left\{-qW_t - \frac{1}{2}q^2 t\right\}, \quad t \in [0, T],$$

is a martingale with respect to the natural Brownian filtration $\mathcal{F} = \sigma(W_s, s \leq t)$ under the probability measure P.

- *The relation*

$$Q(A) = \int_A M_T(\omega) \mathrm{d}P(\omega), \quad A \in \mathcal{F},$$

defines a probability measure Q on $\mathcal{F}$ which is equivalent to P.

- *Under the probability measure Q, the process*

$$\tilde{W} = W_t + qt, \quad t \in [0, T]$$

is a standard Brownian motion (is a Q-Brownian motion).

- *The process $\tilde{W}$ is adapted to the natural Brownian filtration.*

The probability measure Q is called an equivalent martingale measure.

1.6 Applications of stochastic calculus to Finance

In this section we present the Black-Scholes model and some famous results from the stochastic calculus in Finance problems (see [41], chapter 4, and [7], chapter 7).

We consider a *risky asset* (called *stock*) with price S_t at time t. We assume that the stock price is the solution of the SDE

$$S_t = S_0 + \int_0^t \mu(s, S_s) S_s \mathrm{d}s + \int_0^t \sigma(s, S_s) S_s \mathrm{d}W_s, \tag{1.9}$$

where W is a Brownian motion and μ and σ are given deterministic functions. The function σ is known as the *volatility* of S and μ is the *drift* of S.

The case where the functions μ and σ are constants was presented in Section 1.3, and we have that the unique strong solution of the linear stochastic differential equation in this case is the geometric Brownian motion

$$S_t = S_0 \mathrm{e}^{(\mu - \sigma^2/2)t + \sigma W_t}.$$

In our model we consider also a *riskless asset* such as a bank account. We assume that an investment of B_0 in this asset yields an amount of

$$B_t = B_0 \exp\left(\int_0^t r(s) \mathrm{d}s \right), \tag{1.10}$$

where we admit that the *short rate of interest* r changes with time.

If r is constant, we say that the asset B is a *bond*, and we have that

$$B_t = B_0 \mathrm{e}^{rt}$$

at time t. Note that B satisfies the deterministic integral equation

$$B_t = B_0 + r \int_0^t B_s \mathrm{d}s.$$

From now on, we consider that we are in presence of a market with a stock S and a bond B given by (1.9) and (1.10), respectively, where μ, σ and r are deterministic constants. This is called the *Black-Scholes model.*

We are interested in constitute a portfolio of a_t units of stock and b_t units of bond at time t. We assume that a_t and b_t are stochastic processes adapted to the Brownian motion and call the pair (a_t, b_t), $t \in [0, T]$, a *trading strategy.*

Notice that the value of the portfolio at time t is given by

$$V_t = a_t S_t + b_t B_t.$$

Our purpose is to choose a strategy (a_t, b_t) in a reasonable way, where we do not lose, and such that it is *self-financing*, i.e., the increments of our wealth V result only from changes of the prices of the stock and the bond. In terms of differentials, we write

$$\mathrm{d}V_t = \mathrm{d}(a_t S_t + b_t W_t) = a_t \mathrm{d}S_t + b_t \mathrm{d}B_t,$$

which we interpret in the Itô sense as the relation

$$V_t - V_0 = \int_0^t a_s \mathrm{d}S_s + \int_0^t b_s \mathrm{d}B_s.$$

Thus, the value of our portfolio at time t is equal to the initial investment V_0 plus capital gains from stock and bond up to time t.

Towards the main result in this section, we introduce one of the most important *derivatives* (i.e., a financial instrument whose value is defined in terms of the value of some underlying asset, such as a stock).

An *European option* is a contract that gives the purchaser of the option the right, but not the obligation, to exercise the option precisely at *time of maturity* T, for a fixed price K, called the *exercise price* or *strike price* of the option. For a *call option*, the purchaser will have the right to buy, and for a *put option*, he will have the right to sell.

Thus, if we have a European call option on a stock S, at time T the holder of the option is entitled to a *payoff* of

$$C_T = (S_T - K)^+ = \max(0, S_T - K),$$

and for an European put option the payoff at time T is

$$P_T = (K - S_T)^+ = \max(0, K - S_T).$$

Since the payoff for the European option only depends of the stock price at time of maturity T, we say that we have a *simple contingent claim*

$$\mathfrak{X} = \Phi(S(T)),$$

and the function Φ is called the *contract function*.

The natural problem is to find the "fair" price of the option at time $t = 0$ (time of purchase) since we do not know the price S_T at the time of the contract.

In order to present the main pricing equation of financial derivatives, we make the following assumptions.

Assumption 1.1. We assume that

1. The derivative instrument in question can be bought and sold on a market.
2. The market is *free of arbitrage possibilities*, i.e., any possible profit in the market is accompanied by a risk of loss.
3. The price process for the derivative asset with contingent claim $\mathfrak{X}$ is of the form

$$\Pi(t;\mathfrak{X}) = F(t, S_t),$$

where F is some smooth function.

We have the following result (see [7], p.97).

Theorem 1.27. *(Black-Scholes Equation) Assume that the market is specified by equations*

$$\mathrm{d}B_t = rB_t\mathrm{d}t \tag{1.11}$$

$$\mathrm{d}S_t = S_t\alpha(t, S_t)\mathrm{d}t + S_t\sigma(t, S_t)\mathrm{d}W_t, \tag{1.12}$$

and that we want to price a contingent claim of the form $\mathfrak{X} = \Phi(S_T)$.

Then the only pricing function of the form $\Pi(t) = F(t, S_t)$, *for some smooth function* F, *which is consistent with the absence of arbitrage is when* F *is the solution of the following boundary value problem in the domain* $[0, T] \times \mathbb{R}^+$

$$F_t(t, s) + rsF_s(t, s) + \frac{1}{2}s^2\sigma^2(t, s)F_{ss}(t, s) - rF(t, s) = 0 \tag{1.13}$$

$$F(T, s) = \Phi(s). \tag{1.14}$$

We now turn to the question of actually solving the pricing equation and we notice that this equation is precisely of the form which can be solved using a stochastic representation formula *à la* Feynman-Kač presented in Section 1.4.

In order to apply Proposition 1.22, we must define another probability measure Q under which the process S has a different probability distribution, such that the Q-dynamics of S is

$$\mathrm{d}S_t = S_t\alpha(t, S_t)\mathrm{d}t + S_t\sigma(t, S_t)\mathrm{d}\tilde{W}_t, \tag{1.15}$$

where $\tilde{W}$ is a Q-Brownian motion.

This is not the real dynamic of S. The real model is

$$\mathrm{d}S_t = S_t\alpha(t, S_t)\mathrm{d}t + S_t\sigma(t, S_t)\mathrm{d}W_t$$

with W a P-Brownian motion, where P denotes the "objective" probability measure.

The measure Q is often called the *martingale measure*. The reason for the name is explained by the following result (see [7], p.100).

Theorem 1.28. *(The Martingale Property) In the Black-Scholes model, the price process $\Pi(t)$ for every traded asset, be it the underlying or the derivative asset, has the property that the normalized process*

$$Z(t) = \frac{\Pi(t)}{B(t)}$$

is a martingale under the measure Q.

Let $\mathbf{E}$ denote expectations taken under the measure P whereas $\mathbf{E}^Q$ denotes expectations under the measure Q.

We state the following result, which establishes the price of a derivative as the discounted value to present of the expected payoff, where the expectation is taken under the measure Q (see [7], p.99).

Theorem 1.29. *(Risk Neutral Valuation) The arbitrage free price of the contingent claim $\Phi(S(T))$ is given by $\Pi(t; \Phi) = F(t, S(t))$, where F is given by the formula*

$$F(t, s) = e^{-r(T-t)}\mathbf{E}^Q_{t,s}\left[\Phi(S(T))\right],$$

where the Q-dynamics of S are

$$\mathrm{d}S_t = S_t\alpha(t, S_t)\mathrm{d}t + S_t\sigma(t, S_t)\mathrm{d}\tilde{W}_t$$

with $\tilde{W}$ a Q-Wiener process.

In order to obtain specific formulae for the European call option, we place ourselves again in a Black-Scholes model, which consists of two assets with dynamics given by

$$\begin{aligned}\mathrm{d}B_t &= rB_t\mathrm{d}t\\ \mathrm{d}S_t &= \alpha S_t\mathrm{d}t + \sigma S_t\mathrm{d}W_t\end{aligned}$$

where r, α, and σ are deterministic constants.

The following result is known as the Black-Scholes formula (see [7], p.101).

Proposition 1.30. *(Black-Scholes formula) The price of a European call option with strike price K and time of maturity T is given by the formula $\Pi(t) = F(t, S(t))$, where*

$$F(t,s) = sN[d_1(t,s)] - e^{-r(T-t)}KN[d_2(t,s)].$$

Here N is the cumulative distribution function for the $N(0,1)$ distribution and

$$\begin{aligned} d_1(t,s) &= \frac{1}{\sigma\sqrt{T-t}}\left\{\ln\left(\frac{s}{K}\right) + \left(r + \frac{1}{2}\sigma^2\right)(T-t)\right\}, \\ d_2(t,s) &= d_1(t,s) - \sigma\sqrt{T-t}. \end{aligned}$$

Chapter 2

A stationary problem for a Black-Scholes equation with transaction costs

2.1 An option valuation problem with transaction costs

In 1973, Fisher Black and Myron Scholes suggested a model that became fundamental for the valuation of financial derivatives in a complete frictionless market. Along with the no-arbitrage possibilities, the basic Black-Scholes model assumes that in order to replicate exactly the returns of a certain derivative, the hedging portfolio is continuously adjusted by transactioning the underlying asset of the derivative. The existence of costs, such as taxes or fees, for those transactions implies that, in opposition to the basic Black-Scholes model, the replication can not happen continuously otherwise those costs would be infinitely large.

Hence the introduction of transaction costs in the market is a problem that has been motivating the work of several authors.

One of the first main references is the work presented by Leland in [35], that suggests a market with proportional transaction costs. That is, considering ν the number of shares (it is positive if the agent buys or negative if the agent sells) and S the price of the asset at time t, the costs of the transaction of ν shares at time t are given by

$$kS\,|\nu|$$

where the constant $k > 0$ depends on the parts involved in the transaction.

Leland's replication strategy consisted in using the common Black-Scholes formulae in periodical revisions of the portfolio but with an appropriately enlarged volatility. This is a model widely accepted in the financial industry. However there are some mathematical problems with this approach, as referred by Kabanov and Safarian ([33]): the terminal value of the replicating portfolio does not converge to the terminal payoff of the derivative if the transaction costs do not depend on the number of revisions (tending to infinity), limiting discrepancy that can be calculated explicitly.

We also refer to Avellaneda and Parás ([4]) who obtained an extension of Leland's approach for non-convex payoff functions.

The original results presented in this chapter were motivated by the work of Amster *et al.* ([2]), who proposed a model where the transaction costs are not proportional to the amount of the transaction, but the individual cost of the transaction of each share diminishes as the number of shares transactioned increases. Therefore, the cost is the percentage of the transaction which is given by

$$h(\nu) = a - b|\nu|$$

where ν is the number of shares traded and $a, b > 0$. In the framework of this model, Amster *et al.* obtained a non-linear Black-Scholes type equation and studied the stationary problem associated with appropriated boundary conditions. The authors proved the existence and uniqueness of the solution of this problem, which may be obtained as a limit of a nonincreasing (nondecreasing) sequence of upper (respectively lower) solutions.

In the following section we detail the construction of a Black-Scholes' type equation in an option pricing problem with transaction costs in the same framework of Amster *et al.*. In section 2.3 we prove the existence of a stationary solution without imposing some Lipschitz as assumed in [2]. Moreover, we give some information on the localisation of the solution.

2.2 Evaluation of the model

The existence of additional costs in the transaction of the asset, requires us to reformulate the hedging strategy used in the option pricing problem.

Leland ([35]) suggested a new strategy using transaction costs proportional to the monetary value of any buy or sell of the asset. Thus if ν shares are bought ($\nu > 0$) or sold ($\nu < 0$) at a price S, then the transaction costs are

$$k|\nu|S,$$

where k is a constant depending on the individual investor.

We use here a slightly different structure for the transaction costs that was presented by Amster *et al.* in [2]. We assume that the individual cost of the transaction of each share diminishes as the number of shares transactioned increases, which is represented by considering the cost as the percentage given by

$$h(\nu) = a - b|\nu|$$

where ν still is the number of shares traded ($\nu > 0$ in a buy or $\nu < 0$ in a sell) and $a, b > 0$ are constants depending on the individual investor.

We consider the hedging portfolio consisting in an option of value V (short position) and Δ shares of the underlying asset of price S (long position). So, the portfolio's value at time $t \in (0, T)$, with $T > 0$, is given by

$$\Pi = V - \Delta S.$$

The adopted hedging strategy implies that the portfolio is reviewed every δt, where δt is a finite, fixed time step. The change in the value of the portfolio after each time step δt is given by

$$\delta\Pi = \delta V - \Delta\delta S - (a - b|\nu|) S|\nu| \tag{2.1}$$

where we subtract the costs, which are always positive, due to the transaction of $|\nu|$ shares of the asset.

It is assumed that the value of the underlying follows the random walk

$$\delta S = \mu S\delta t + \sigma S\phi\sqrt{\delta t}, \tag{2.2}$$

where ϕ is the standard Gaussian distribution, μ is the drift coefficient and σ is the volatility.

Noticing that

- $\dfrac{\partial\Pi}{\partial t} = \dfrac{\partial V}{\partial t}$,
- $\dfrac{\partial\Pi}{\partial S} = \dfrac{\partial V}{\partial S} - \Delta$,
- $\dfrac{\partial^2\Pi}{\partial S^2} = \dfrac{\partial^2 V}{\partial S^2}$,

using Itô's Lemma we obtain

$$\delta\Pi = \sigma S\left(\frac{\partial V}{\partial S} - \Delta\right)\phi\sqrt{\delta t} \quad + \quad \left(\frac{1}{2}\sigma^2 S^2\frac{\partial^2 V}{\partial S^2}\phi^2 + \mu S\frac{\partial V}{\partial S} + \frac{\partial V}{\partial t} - \mu\Delta S\right)\delta t$$
$$- \quad (a - b|\nu|)\, S\,|\nu|\,.$$

We follow the Black-Scholes replication strategy, thus we choose the number of shares of the underlying asset held at time t to be

$$\varDelta = \frac{\partial V}{\partial S}(S,t)\,,$$

which has been evaluated at time t and asset price S. So, the number of shares of the asset traded after time δt is given by

$$\nu = \frac{\partial V}{\partial S}(S + \delta S, t + \delta t) - \frac{\partial V}{\partial S}(S,t)\,.$$

Expanding the first term in the right side of the previous equation in a Taylor series for small δS and δt, we get

$$\frac{\partial V}{\partial S}(S + \delta S, t + \delta t) = \frac{\partial V}{\partial S}(S,t) + \frac{\partial^2 V}{\partial S^2}(S,t)\,\delta S + \frac{\partial^2 V}{\partial t\partial S}(S,t)\,\delta t + \cdots$$

Hence

$$\nu = \frac{\partial^2 V}{\partial S^2}(S,t)\,\delta S + \frac{\partial^2 V}{\partial t\partial S}(S,t)\,\delta t + \cdots$$

and using 2.2

$$\nu = \frac{\partial^2 V}{\partial S^2}\sigma S\phi\sqrt{\delta t} + O(\delta t)$$

following

$$\nu \approx \frac{\partial^2 V}{\partial S^2}\sigma S\phi\sqrt{\delta t}.$$

Thus the expected transaction cost in a time step δt is

$$\begin{aligned}
\mathbf{E}\left[(a - b|\nu|)\,S\,|\nu|\right] &= aS\,\mathbf{E}(|\nu|) - bS\,\mathbf{E}\left(\nu^2\right) \\
&= a\sigma S^2\left|\frac{\partial^2 V}{\partial S^2}\right|\sqrt{\delta t}\mathbf{E}(|\phi|) - b\sigma^2 S^2\mathbf{E}\left(\phi^2\right)\delta t \\
&= \left(a\sigma S^2\left|\frac{\partial^2 V}{\partial S^2}\right|\sqrt{\frac{2}{\pi\delta t}} - b\sigma^2 S^3\left(\frac{\partial^2 V}{\partial S^2}\right)^2\right)\delta t \qquad (2.3)
\end{aligned}$$

where we use the fact that $\mathbf{E}(|\phi|) = \sqrt{2/\pi}$.

So the expected change in the value of the portfolio is

$$\begin{aligned}\mathbf{E}(\delta\Pi) &= \mathbf{E}\left[\frac{\partial V}{\partial t}+\left(\frac{1}{2}\sigma^2S^2\frac{\partial^2 V}{\partial S^2}\phi^2\right)\delta t-(a-b\left|\nu\right|)S\left|\nu\right|\right]\\ &= \left(\frac{\partial V}{\partial t}+\frac{1}{2}\sigma^2S^2\frac{\partial^2 V}{\partial S^2}-a\sigma S^2\left|\frac{\partial^2 V}{\partial S^2}\right|\sqrt{\frac{2}{\pi\delta t}}+b\sigma^2S^3\left(\frac{\partial^2 V}{\partial S^2}\right)^2\right)\delta t.\end{aligned} \tag{2.4}$$

In order to guarantee that the market doesn't admit opportunities of arbitrage, the expected return of the portfolio must be the same of a risk-free bond

$$\mathbf{E}\left(\delta\Pi\right)=r\left(V-\Delta S\right)\delta t=r\left(V-\frac{\partial V}{\partial S}S\right)\delta t \tag{2.5}$$

where r is the riskless interest rate.

Applying (2.5) to (2.4) we obtain the equation

$$\frac{\partial V}{\partial t}+\frac{1}{2}\sigma^2S^2\frac{\partial^2 V}{\partial S^2}-a\sigma S^2\left|\frac{\partial^2 V}{\partial S^2}\right|\sqrt{\frac{2}{\pi\delta t}}+b\sigma^2S^3\left(\frac{\partial^2 V}{\partial S^2}\right)^2+rS\frac{\partial V}{\partial S}-rV=0.$$

Considering a small enough such that

$$\tilde{\sigma}^2=\sigma^2\left[1-\operatorname{sign}\left(\frac{\partial^2 V}{\partial S^2}\right)\frac{2a}{\sigma}\sqrt{\frac{2}{\pi\delta t}}\right]>0 \tag{2.6}$$

the following non-linear equation is obtained

$$\frac{\partial V}{\partial t}+\frac{1}{2}\tilde{\sigma}^2S^2\frac{\partial^2 V}{\partial S^2}+b\sigma^2S^3\left(\frac{\partial^2 V}{\partial S^2}\right)^2+r\left(\frac{\partial V}{\partial S}S-V\right)=0. \tag{2.7}$$

Remark 2.1. Equation (2.7) is clearly an extension of the Black-Scholes equation. The introduction of our particular model for the transaction costs in the option pricing market led us to a partial differential equation that contains the Black-Scholes terms with an additional non-linear term modelling the presence of transaction costs. We also use an adjusted volatility in the model, not the real volatility.

Remark 2.2. A particular case of (2.7) is the one introduced by Leland where the transaction costs are proportional to the amount of the transaction made when reviewing the hedging portfolio. Hoggard, Whalley and Wilmott (see [32]) developed a model with proportional transaction costs $k\left|\nu\right|S$, and reached the equation

$$\frac{\partial V}{\partial t}+\frac{1}{2}\sigma^2S^2\frac{\partial^2 V}{\partial S^2}-k\sigma S^2\sqrt{\frac{2}{\pi\delta t}}\left|\frac{\partial^2 V}{\partial S^2}\right|+r\left(\frac{\partial V}{\partial S}S-V\right)=0$$

known as the Hoggard-Whalley-Wilmott Equation.

2.3 Stationary solutions localisation

In this section we state our main result concerning the existence of convex stationary solutions for (2.7) with Dirichlet boundary conditions

$$V(c) = V_c \quad \text{and} \quad V(d) = V_d \tag{2.8}$$

where $0 < c < d$ are fixed real numbers and V_c and V_d are such that $V_c \leq V_d$, which turns out to be quite natural in some financial settings, for instance, if we are dealing with call options.

These stationary solutions give the option value as a function of the stock price. This feature can be interesting when dealing with a model where the time does not play a relevant role such as, for instance, in perpetual options.

More precisely, we consider the following non-linear Dirichlet boundary value problem

$$\begin{cases} \frac{1}{2}\tilde{\sigma}^2 S^2 \frac{\partial^2 V}{\partial S^2} + b\sigma^2 S^3 \left(\frac{\partial^2 V}{\partial S^2}\right)^2 + r\left(S\frac{\partial V}{\partial S} - V\right) = 0, \\ V\left(c\right) = V_c, \qquad V\left(d\right) = V_d. \end{cases} \tag{2.9}$$

Since we are concerned with convex solutions for this problem, the modified volatility (2.6) of the model is in this case given by

$$\tilde{\sigma}^2 = \sigma^2 \left(1 - \frac{2a}{\sigma}\sqrt{\frac{2}{\pi \delta t}}\right) > 0. \tag{2.10}$$

The following result we state not only concerns the existence of solution but also gives information about its localisation. As we will see, the localisation statement will be a consequence of the use of the method of upper and lower solutions technique in the proof. We present now our main result (see [26]).

Theorem 2.1. *Consider the non-linear Dirichlet boundary value problem* (2.9). *The following assertions hold:*

1. *The function* $V\left(S\right) = \frac{V_c}{c}S$ *is a (linear) solution of the problem* (2.9) *if and only if* $\frac{V_d}{d} = \frac{V_c}{c}$.

2. *Let* $\frac{V_d}{d} < \frac{V_c}{c}$. *Then the problem* (2.9) *has a convex solution* V *such that*

$$\frac{V_d}{d}S \leq V(S) \leq \frac{V_d - V_c}{d-c}S + \frac{dV_c - cV_d}{d-c}, \tag{2.11}$$

$$\frac{V_d - V_c}{d-c} < V'(d) \leq \frac{V_d}{d} \text{ and } V'(c) < \frac{V_d - V_c}{d-c}. \tag{2.12}$$

3. Moreover, in both cases, V is the unique convex solution of (2.9).

The differential equation of problem (2.9) can be written using a simpler notation which we will use from now on in the form

$$b\sigma^2 S^3 (V'')^2 + \frac{1}{2}\tilde{\sigma}^2 S^2 V'' + r(V'S - V) = 0, \quad \text{in} \quad]c, d[.$$

Observe that this equation is from the algebraic point of view a second order equation in the variable $X = V''$. So, solving it algebraically in order of V'', we obtain the equivalent form

$$V'' = \frac{-\tilde{\sigma}^2 \frac{S^2}{2} \pm \sqrt{\tilde{\sigma}^4 \frac{S^4}{4} - 4bS^3\sigma^2 r(V'S - V)}}{2b\sigma^2 S^3},$$

which leads us to consider the equation

$$V'' + g_{\pm}(S, V, V') = 0$$

where

$$g_{\pm}(S, V, V') = \frac{\tilde{\sigma}^2 \frac{S^2}{2} \mp \sqrt{\tilde{\sigma}^4 \frac{S^4}{4} + 4bS^3\sigma^2 r(V'S - V)}}{2b\sigma^2 S^3}.$$

This fact suggests the study of an auxiliary problem as a mean to prove the above stated theorem.

Consider the auxiliary problem

$$\begin{cases} V'' + g(S, V, V') = 0 \\ V(c) = V_c, \quad V(d) = V_d \end{cases} \tag{2.13}$$

where

$$g(S, V, V') = \frac{\tilde{\sigma}^2 \frac{S^2}{2} - \sqrt{\tilde{\sigma}^4 \frac{S^4}{4} + 4bS^3\sigma^2 r\,|V'S - V|}}{2b\sigma^2 S^3}.$$

We will deduce the existence of a solution V of the problem (2.13) and prove then that it is a convex solution of the problem (2.9). Moreover, we will obtain information about the localisation of V. The main argument to solve the problem (2.13) relies on the method of upper and lower solutions. In fact, if we prove that problem (2.13) has a lower solution $\alpha \in C^2$ and an upper solution $\beta \in C^2$

satisfying $\alpha \leq \beta$, and that some Nagumo condition is satisfied, then the thesis will follow (see result in [15], p.77), as referred above.

We recall that $\alpha \in C^2$ is a *lower solution* of (2.13) if

$$\begin{cases} \alpha'' + g(S, \alpha, \alpha') \geq 0 \\ \alpha(c) \leq V_c, \quad \alpha(d) \leq V_d . \end{cases} \tag{2.14}$$

Similarly, an *upper solution* $\beta \in C^2$ of (2.13) is defined by reversing the inequalities in (2.14), that is,

$$\begin{cases} \beta'' + g(S, \beta, \beta') \leq 0 \\ \beta(c) \geq V_c, \quad \alpha(d) \geq V_d . \end{cases} \tag{2.15}$$

A *solution* of (2.13) is a function u which is simultaneously a lower and an upper solution.

A function f is said to satisfy the Nagumo on some given subset $E \subset I \times \mathbb{R}^2$ if there exists a positive continuous function $\varphi \in C\left(\mathbb{R}_0^+, [\varepsilon, +\infty[\right)$, $\varepsilon > 0$, such that

$$|f(x, y, z)| \leq \varphi(|z|), \quad \forall (x, y, z) \in E,$$

and

$$\int_0^{+\infty} \frac{s}{\varphi(s)} ds = +\infty. \tag{2.16}$$

Now we state an existence and localisation result for the problem (2.13).

Theorem 2.2. *Suppose that*

$$\frac{V_d}{d} \leq \frac{V_c}{c}. \tag{2.17}$$

Then the problem (2.13) *has a solution* V *such that*

$$\frac{V_d}{d} S \leq V(S) \leq \frac{V_d - V_c}{d - c} S + \frac{dV_c - cV_d}{d - c}.$$

Proof. Consider the following functions, defined in $[c, d]$,

$$\begin{aligned} \alpha(S) &= \frac{V_d}{d} S \quad , \\ \beta(S) &= \frac{V_d - V_c}{d - c} S + \frac{dV_c - cV_d}{d - c}. \end{aligned}$$

Observe that $\alpha(\cdot)$ and $\beta(\cdot)$ are lower and upper solutions of (2.13), respectively. In fact, we have that

$$\begin{aligned} \alpha'' + g(S, \alpha, \alpha') &= 0 + \frac{\tilde{\sigma}^2 \frac{S^2}{2} - \sqrt{\tilde{\sigma}^4 \frac{S^4}{4} + 4bS^3\sigma^2 r \left| \frac{V_d}{d} \cdot S - \frac{V_d}{d} \cdot S \right|}}{2b\sigma^2 S^3} \\ &= \frac{\tilde{\sigma}^2 \frac{S^2}{2} - \sqrt{\tilde{\sigma}^4 \frac{S^4}{4} + 0}}{2b\sigma^2 S^3} = 0, \end{aligned}$$

and

$$\alpha(c) \leq V_c, \quad \alpha(d) = V_d.$$

We also have

$$\begin{aligned} \beta'' + g(S, \beta, \beta') &= \frac{\tilde{\sigma}^2 \frac{S^2}{2} - \sqrt{\tilde{\sigma}^4 \frac{S^4}{4} + 4bS^3\sigma^2 r \left| \frac{V_d - V_c}{d-c} S - \frac{V_d - V_c}{d-c} S - \frac{dV_c - cV_d}{d-c} \right|}}{2b\sigma^2 S^3} \\ &\leq 0 \end{aligned}$$

and

$$\beta(c) = V_c, \quad \beta(d) = V_d.$$

Moreover $\alpha \leq \beta$. In fact, algebraic computations show easily that

$$\frac{V_d}{d} S \leq \frac{V_d - V_c}{d - c} S + \frac{dV_c - cV_d}{d - c}$$

is equivalent to

$$0 \leq (d - S)(-cV_d + dV_c)$$

and then to

$$0 \leq (-cV_d + dV_c)$$

which holds by (2.17.)

Now we consider the set:

$$E = \left\{ (S, x, y) \in [c, d] \times \mathrm{IR}^2 \quad : \quad \frac{V_d}{d} S \leq x \leq \frac{V_d - V_c}{d - c} S + \frac{dV_c - cV_d}{d - c} \right\}$$

We observe that g satisfies the Nagumo condition in E. In fact,

$$\begin{aligned} |g(S, x, y)| &= \left| \frac{\tilde{\sigma}^2 \frac{S^2}{2} - \sqrt{\tilde{\sigma}^4 \frac{S^4}{4} + 4bS^3\sigma^2 r |yS - x|}}{2b\sigma^2 S^3} \right| \\ &\leq \frac{\tilde{\sigma}^2 S^2 + \sqrt{4bS^3\sigma^2 r |yS|} + \sqrt{4bS^3\sigma^2 r |x|}}{2b\sigma^2 S^3} \\ &\leq \frac{\tilde{\sigma}^2}{2b\sigma^2 c} + \frac{1}{\sigma c\sqrt{c\, b}} \sqrt{r \left(\frac{V_d - V_c}{d - c} d + \frac{dV_c - cV_d}{d - c} \right)} + \frac{\sqrt{r}}{\sigma c\sqrt{b}} \sqrt{|y|}, \end{aligned}$$

that is, for some positive constants k_1, k_2,

$$|g(S,x,y)| \leq k_1 + k_2\sqrt{|y|}.$$

Put $\varphi(y) = k_1 + k_2\sqrt{|y|}$. Then

$$\int_0^{+\infty} \frac{y}{k_1 + k_2\sqrt{|y|}} dy = +\infty.$$

Therefore the function g satisfies the Nagumo condition in E. So, by the result contained in [15], we can derive that there exists a solution V_* of (2.13) such that

$$\alpha(S) = \frac{V_d}{d}S \leq V_*(S) \leq \frac{V_d - V_c}{d-c}S + \frac{dV_c - cV_d}{d-c} = \beta(S).$$

□

We shall prove now that the solution of the auxiliary problem given by theorem 2.2 is in fact a solution of problem (2.9).

Proof. (Theorem 2.1)

1. It is clear that $V(S) = \frac{V_c}{c}S$ satisfies the equation of (2.9) and also the boundary condition $V(c) = V_c$. It is easy to see that V satisfies the boundary condition $V(d) = V_d$ if and only if $\frac{V_d}{d} = \frac{V_c}{c}$, which finishes the proof.

2. Consider a solution of (2.13), whose existence was proved in Theorem 2.2, and denote it by V_*. Recall that

$$\alpha(S) = \frac{V_d}{d}S \leq V_*(S) \leq \frac{V_d - V_c}{d-c}S + \frac{dV_c - cV_d}{d-c} = \beta(S).$$

   We organize the proof in three steps (that sistematize some ideas that can be found in [2]). The first two steps point out two properties of V_* which will be used in step 3 to conclude that V_* is a solution of (2.9).

   Step 1. *V_* is convex.*

   Observe that g is a nonpositive function. In fact

$$g(S, V_*, V_*') = \frac{\tilde{\sigma}^2\frac{S^2}{2} - \sqrt{\tilde{\sigma}^4\frac{S^4}{4} + 4bS^3\sigma^2 r\,|V_*'S - V_*|}}{2b\sigma^2 S^3} \leq \frac{\tilde{\sigma}^2\frac{S^2}{2} - \sqrt{\tilde{\sigma}^4\frac{S^4}{4}}}{2b\sigma^2 S^3} = 0.$$

   Therefore, V_* is convex since

$$V_*'' = -g(S, V_*, V_*') \geq 0.$$

Step 2. *The following inequality holds*

$$V_*'S - V_* \leq 0.$$

To prove this inequality we will make use of the upper and lower solutions, α and β, of the previous section. Since

$$\alpha(d) = V_*(d) = \beta(d),$$

and, for $S \in [c,d]$,

$$\alpha(S) \leq V_*(S) \leq \beta(S),$$

it follows that

$$\beta'(d) \leq V_*'(d) \leq \alpha'(d), \tag{2.18}$$

that is,

$$\frac{V_d - V_c}{d-c} \leq V_*'(d) \leq \frac{V_d}{d}.$$

In particular, we have

$$V_*'(d)\,d \leq V_d = V_*(d). \tag{2.19}$$

As seen in step 1, V_*'' is nonnegative. Then

$$(V_*'(S)\,S - V_*(S))' = V_*''(S)\,S + V_*'(S) - V_*'(S) = V_*''(S)\,S \geq 0$$

and so $V_*'(S)\,S - V_*(S)$ is nondecreasing in S. Therefore, by (2.19),

$$V_*'(S)\,S - V_*(S) \leq V_*'(d)\,d - V_*(d) \leq 0.$$

Step 3 V_* *is a solution of the problem* (2.9).
This statement follows easily from the above comments. In fact, since

$$V_*'S - V_* \leq 0$$

then

$$\begin{aligned} g(S, V_*, V_*') &= \frac{\tilde{\sigma}^2 \frac{S^2}{2} - \sqrt{\tilde{\sigma}^4 \frac{S^4}{4} + 4bS^3\sigma^2 r \left|V_*'S - V_*\right|}}{2b\sigma^2 S^3} \\ &= \frac{\tilde{\sigma}^2 \frac{S^2}{2} - \sqrt{\tilde{\sigma}^4 \frac{S^4}{4} - 4bS^3\sigma^2 r \left(V_*'S - V_*\right)}}{2b\sigma^2 S^3}. \end{aligned}$$

So, V_* is a convex solution of

$$V'' = \frac{-\tilde{\sigma}^2 \frac{S^2}{2} + \sqrt{\tilde{\sigma}^4 \frac{S^4}{4} - 4bS^3\sigma^2 r\,(V'S - V)}}{2b\sigma^2 S^3},$$

and, therefore, of

$$b\sigma^2 S^3 \left(V''\right)^2 + \frac{1}{2}\tilde{\sigma}^2 S^2 V'' + r\,(V'S - V) = 0 \quad \text{in} \quad]c, d[\,,$$

satisfying the boundary conditions

$$V(c) = V_c, \quad V(d) = V_d$$

and (2.11). As for (2.12), as seen in (2.18)

$$\beta'(d) \leq V_*'(d) \leq \alpha'(d)\,,$$

that is,

$$\frac{V_d - V_c}{d - c} \leq V_*'(d) \leq \frac{V_d}{d}.$$

Since $V_*(S) \leq \beta(S)$ in $[c, d]$ with $V_d = \beta(d)$, the fact that V_* is convex implies that the inequality $\beta'(d) \leq V_*'(d)$ must be strict, that is, $\beta'(d) < V_*'(d)$. Thus

$$\frac{V_d - V_c}{d - c} < V_*'(d) \leq \frac{V_d}{d}.$$

Analogous arguments applied to c show that

$$V_*'(c) < \frac{V_d - V_c}{d - c}.$$

The proof of assertion 2. is finished.

3. The uniqueness result follows immediately from Theorem 2.1 of [2].

□

As we saw, the lower and upper bounds for the solution of (2.9) referred in (2.11) of Theorem 2.1 were precisely the lower and upper solutions of the auxiliary problem (2.13). We can even add that they are, in fact, the minimal and maximal linear upper and lower solutions, respectively, of the auxiliary problem. We state that fact in the following proposition for the sake of completeness.

Proposition 2.3. *Let* $\frac{V_d}{d} \leq \frac{V_c}{c}$. *Then the upper and lower bounds for the convex solution of problem* (2.9) *given by Theorem 2.1 are such that*

1. $\beta(S) = \dfrac{V_d - V_c}{d-c} S + \dfrac{dV_c - cV_d}{d-c}$ *is the minimal linear upper solution of problem (2.13);*

2. $\alpha(S) = \dfrac{V_d}{d} S$ *is the maximal linear lower solution of problem (2.13) such that* $\alpha(d) = V_d$.

Proof. Any linear upper solution $\tilde{\beta}$ of problem (2.13) must satisfy

$$\tilde{\beta}(c) \geq V_c \;, \quad \tilde{\beta}(d) \geq V_d.$$

Since β is linear and satisfies $\beta(c) = V_c$, $\beta(d) = V_d$, it is obvious that

$$\beta \leq \tilde{\beta} \qquad \text{in } [c, d].$$

On the other hand, any linear lower solution $\tilde{\alpha}$ of problem (2.13) must satisfy

$$\tilde{\alpha}(c) \leq V_c \;, \quad \tilde{\alpha}(d) \leq V_d \;.$$

Suppose that $\tilde{\alpha}(d) = V_d$ and $\tilde{\alpha} > \alpha$ for some $S \in [c, d]$. Then, we have that

$$\tilde{\alpha}(S) = V_d + \gamma(S - d) \quad \text{for some} \quad \gamma \in \left[\frac{V_d - V_c}{d-c}, \frac{V_d}{d}\right].$$

Plugging $\tilde{\alpha}$ in the first member of equation of the auxiliar problem we obtain

$$\begin{aligned} \tilde{\alpha}'' &+ g\left(S, \tilde{\alpha}, \tilde{\alpha}'\right) = \\ &= 0 + g\left(S, \tilde{\alpha}, \tilde{\alpha}'\right) \\ &= \frac{\tilde{\sigma}^2 \frac{S^2}{2} - \sqrt{\tilde{\sigma}^4 \frac{S^4}{4} + 4bS^3\sigma^2 r \left|\gamma S - V_d - \gamma(S-d)\right|}}{2b\sigma^2 S^3} \\ &= \frac{\tilde{\sigma}^2 \frac{S^2}{2} - \sqrt{\tilde{\sigma}^4 \frac{S^4}{4} + 4bS^3\sigma^2 r \left|-V_d + d\right|}}{2b\sigma^2 S^3} \\ &< 0 \end{aligned}$$

which yields a contradiction as we assumed that $\tilde{\alpha}$ was a lower solution, if $V_d \neq d$. This ends the proof.

□

Now we improve the localisation result for Problem (2.9) for situations where $\frac{r}{b\sigma^2 c^3}$ is small enough.

Theorem 2.4. *Suppose that*

$$\frac{V_d}{d} \leq \frac{V_c}{c}. \tag{2.20}$$

If $k = \frac{r}{b\sigma^2 c^3}$ *is small enough, then the convex solution* V *of problem* (2.9) *given by Theorem* 2.1 *satisfies the following localisation statement*

$$\max\left\{\frac{V_d}{d}S, \frac{V_c - V_d}{c^2 - d^2}S^2 + \frac{c^2V_d - d^2V_c}{c^2 - d^2}\right\} \leq V(S) \leq \frac{V_d - V_c}{d - c}S + \frac{dV_c - cV_d}{d - c}. \tag{2.21}$$

Proof. Consider the following function, defined in $[c, d]$,

$$\tilde{\alpha}(S) = \frac{V_c - V_d}{c^2 - d^2}S^2 + \frac{c^2V_d - d^2V_c}{c^2 - d^2}.$$

We will prove that $\tilde{\alpha}$ is a lower solution of the auxiliary problem (2.13). We have that

$$\tilde{\alpha}(c) = \frac{V_c - V_d}{c^2 - d^2}c^2 + \frac{c^2V_d - d^2V_c}{c^2 - d^2} = V_c$$

and

$$\tilde{\alpha}(d) = \frac{V_c - V_d}{c^2 - d^2}d^2 + \frac{c^2V_d - d^2V_c}{c^2 - d^2} = V_d.$$

On the other hand, observe that

$$\begin{aligned}
g(S, V, V') &= \frac{\tilde{\sigma}^2\frac{S^2}{2} - \sqrt{\tilde{\sigma}^4\frac{S^4}{4} + 4bS^3\sigma^2 r\,|V'S - V|}}{2b\sigma^2 S^3} \\
&\geq \frac{\tilde{\sigma}^2\frac{S^2}{2} - \sqrt{\tilde{\sigma}^4\frac{S^4}{4}} - \sqrt{4bS^3\sigma^2 r\,|V'S - V|}}{2b\sigma^2 S^3} \\
&= -\sqrt{\frac{bS^3\sigma^2 r}{b^2\sigma^4 S^6}}\sqrt{|V'S - V|} \\
&= -\sqrt{\frac{r}{b\sigma^2 S^3}}\sqrt{|V'S - V|} \\
&\geq -\sqrt{\frac{r}{b\sigma^2 c^3}}\sqrt{|V'S - V|} \\
&= -\tilde{k}\sqrt{|V'S - V|}
\end{aligned}$$

where

$$\tilde{k} = \sqrt{\frac{r}{b\sigma^2 c^3}}.$$

So

$$\begin{aligned}
\tilde{\alpha}'' + g(S, \tilde{\alpha}, \tilde{\alpha}') &\geq 2\frac{V_c - V_d}{c^2 - d^2} - \tilde{k}\sqrt{\left|2S^2\frac{V_c - V_d}{c^2 - d^2} - \frac{V_c - V_d}{c^2 - d^2}S^2 - \frac{c^2V_d - d^2V_c}{c^2 - d^2}\right|} \\
&\geq 2\frac{V_c - V_d}{c^2 - d^2} - \tilde{k}\sqrt{\left|S^2\frac{V_c - V_d}{c^2 - d^2} - \frac{c^2V_d - d^2V_c}{c^2 - d^2}\right|}
\end{aligned}$$

and, for k small enough

$$\tilde{\alpha}'' + g(S, \tilde{\alpha}, \tilde{\alpha}') \geq 0$$

therefore $\tilde{\alpha}$ is a lower solution of the auxiliary problem.

Since α and $\tilde{\alpha}$ are both lower solutions of the problem that together with the upper solution β satisfy the conditions of the upper and lower solutions method, the unique solution V provided must satisfy

$$\alpha \leq V \qquad \text{and} \qquad \tilde{\alpha} \leq V.$$

So, in $[c, d]$,

$$\max\left\{\frac{V_d}{d}S, \frac{V_c - V_d}{c^2 - d^2}S^2 + \frac{c^2 V_d - d^2 V_c}{c^2 - d^2}\right\} \leq V(S).$$

The upper bound is given by the upper solution β as previously stated in the proof of Theorem 2.2. In fact, as shown in the proof of the main theorem 2.1, this solution V of the auxiliary problem (2.13) is also the solution of the problem (2.9) and so the thesis holds. □

Chapter 3

Discretisation of abstract linear evolution equations of parabolic type

3.1 Introduction

In this chapter we investigate the discretisation of multidimensional PDE problems arising in European financial option pricing. Let us consider the stochastic modelling of a multi-asset financial option of European type under the framework of a general version of Black-Scholes model, where the vector of asset appreciation rates and the volatility matrix are taken time and space-dependent. Owing to a Feynman-Kač type formula, pricing this option can be reduced to solving the Cauchy problem (with terminal condition) for a second-order linear parabolic PDE of nondivergent type, with null term and unbounded coefficients, degenerating in the space variables (see, e.g., [34]).

After a change of the time variable, the PDE problem is written

$$\frac{\partial u}{\partial t} = Lu + f \quad \text{in} \quad [0,T] \times \mathbb{R}^d, \quad u(0,x) = g(x) \quad \text{in} \quad \mathbb{R}^d, \tag{3.1}$$

where L is the second-order partial differential operator in the nondivergence form

$$L(t,x) = a^{ij}(t,x)\frac{\partial^2}{\partial x_i \partial x_j} + b^i(t,x)\frac{\partial}{\partial x_i} + c(t,x), \quad i,j = 1,\dots,d,$$

with real coefficients, f and g are given real-valued functions (the free term f is included to further improve generality), and $T \in (0,\infty)$ is a constant. For each $t \in [0,T]$ the operator $-L$ is degenerate elliptic, and the growth in the spatial variables of the coefficients a, b, and of the free data f, g is allowed.

One possible approach for the discretisation of the PDE problem (3.1) is to proceed to a two-stage discretisation. First, the problem is semi-discretised in space, and both the possible equation degeneracy and coefficient unboundedness are dealt with (see, e.g., [21, 22], where the spatial approximation is pursued in a variational framework, under the strong assumption that the PDE does not degenerate, and [20]). Subsequently, a time discretisation takes place.

For the time discretisation, the topic of the present chapter, it can be tackled by approximating the linear evolution equation problem in which the PDE problem (3.1) can be cast into

$$\frac{\mathrm{d}u}{\mathrm{d}t} = A(t)u + f(t) \quad \text{in} \quad [0,T], \qquad u(0) = g, \tag{3.2}$$

where, for every $t \in [0,T]$ with $T \in (0,\infty)$, $A(t)$ is a linear operator from a reflexive separable Banach space V to its dual V^*, $u : [0,T] \to V$ is an unknown function, $f : [0,T] \to V^*$, g belongs to a Hilbert space H, with f and g given, and V is continuously and densely embedded into H. We assume that operator $A(t)$ is continuous and impose a coercivity condition.

This simpler general approach, which we follow, is powerful enough to obtain the desired results. On the other hand, it covers a variety of problems, namely initial-value and initial boundary-value problems for linear parabolic PDEs of any order $m \geq 2$.

In the present chapter, we study the discretisation, using both the implicit and the explicit finite-difference methods schemes, in time of problem (3.2) with time-dependent operator A in a general setting. To further improve generality, we proceed to the study leaving the discretised versions of A and f nonspecified. Also, in order to obtain the convergence of the schemes, we need to assume that the solution of (3.2) satisfies a smoothness condition but weaker than the usual Hölder-continuity.

It is well known that, to guarantee the explicit scheme stability, an additional assumption has to be made, usually involving an inverse inequality between V and H (see, e.g., [31]). In our study, the explicit discretisation is investigated by assuming instead a not usual inverse inequality between H and V^*.

In addition, we illustrate our study by exploring two fundamental types for the discretised versions of A and f. First, we consider the approximation of A and f by integral averages. We show that the standard smoothness and coercivity assumptions for problem (3.2) induce correspondent properties for the discretised problem, so that stability results can be proved. Moreover, the rate of convergence we obtain is optimal. Then, we study the alternative approximation of A and f

by weighted arithmetic averages of their respective values at consecutive time-grid points. In this case, stronger smoothness assumptions are needed in order to obtain the scheme convergence.

The chapter is organized as follows. In Section 3.2, we set an abstract framework for a linear parabolic evolution equation and present a solvability classical result. In the following two sections, we study the discretisation of the evolution equation with the use of the Euler's implicit scheme (Section 3.3) and the Euler's explicit scheme (Section 3.4). Finally, in Sections 3.5 and 3.6, we discuss two main types of the discrete operator and free term, for the implicit and the explicit discretisation schemes, respectively.

3.2 Preliminaries

We establish some facts on the solvability of linear evolution equations of parabolic type.

Let V be a reflexive separable Banach space embedded continuously and densely into a Hilbert space H with inner product $(\cdot,\cdot)$. Then H^*, the dual space of H, is also continuously and densely embedded into V^*, the dual of V. Let us use the notation $\langle\cdot,\cdot\rangle$ for the dualization between V and V^*. Let H^* be identified with H in the usual way, by the Riesz isomorphism. Then we have the so called normal (or Gelfand) triple

$$V \hookrightarrow H \equiv H^* \hookrightarrow V^*,$$

with continuous and dense embeddings. It follows that $\langle u, v\rangle = (u, v)$, for all $u \in H$ and for all $v \in V$. Furthermore, $|\langle u, v\rangle| \leq \|u\|_{V^*}\|v\|_V$, for all $u \in V^*$ and for all $v \in V$, where the notation $\|\cdot\|_X$ stands for the Banach space X norm.

Let us consider the Cauchy problem for an evolution equation

$$\frac{\mathrm{d}u}{\mathrm{d}t} = A(t)u + f(t) \quad \text{in} \quad [0,T], \qquad u(0) = g, \tag{3.3}$$

with $T \in (0,\infty)$, where $A(t)$ is a linear operator from V to V^* for every $t \in [0,T]$ and $A(\cdot)v : [0,T] \to V^*$ is measurable for fixed $v \in V$, $u : [0,T] \to V$ is an unknown differentiable function, $f : [0,T] \to V^*$ is a measurable given function, $\mathrm{d}/\mathrm{d}t$ is the standard derivative with respect to the time variable t, and $g \in H$ is given.

We assume that the operator $A(t)$ is continuous and impose a coercivity condition, as well as some regularity on the free data f and g.

Assumption 3.1. Suppose that there exist constants $\lambda > 0$, K, M, and N such that

1. $\langle A(t)v, v\rangle + \lambda\|v\|_V^2 \le K\|v\|_H^2, \quad \forall v \in V$ and $\forall t \in [0,T]$;
2. $\|A(t)v\|_{V^*} \le M\|v\|_V, \quad \forall v \in V$ and $\forall t \in [0,T]$;
3. $\int_0^T \|f(t)\|_{V^*}^2 \mathrm{d}t \le N$ and $\|g\|_H \le N$.

Let X be a Banach space with norm $\|\cdot\|_X$. We denote by $C([0,T];X)$ the space of all continuous X-valued functions z on $[0,T]$ such that

$$\|z\|_{C([0,T];X)} := \max_{0\le t\le T} \|z(t)\|_X < \infty$$

and by $L^2([0,T];X)$ the space comprising all strongly measurable functions $w : [0,T] \to X$ such that

$$\|w\|_{L^2([0,T];X)} := \left(\int_0^T \|w(t)\|_X^2 \mathrm{d}t\right)^{1/2} < \infty.$$

We define the generalised solution of problem (3.3).

Definition 3.1. We say that $u \in C([0,T];H)$ is a generalised solution of (3.3) on $[0,T]$ if

1. $u \in L^2([0,T];V)$;
2. $(u(t),v) = (g,v) + \int_0^t \langle A(s)u(s), v\rangle \mathrm{d}s + \int_0^t \langle f(s), v\rangle \mathrm{d}s, \ \forall v \in V, \ \forall t \in [0,T]$.

The following well-known result states the existence and uniqueness of the generalised solution of problem (3.3) (see, e.g., [38]).

Theorem 3.2. *Under conditions (1)–(3) of Assumption 3.1, problem (3.3) has a unique generalised solution on* $[0,T]$. *Moreover*

$$\sup_{t\in[0,T]} \|u(t)\|_H^2 + \int_0^T \|u(t)\|_V^2 \mathrm{d}t \le N\left(\|g\|_H^2 + \int_0^T \|f(t)\|_{V^*}^2 \mathrm{d}t\right),$$

where N is a constant.

3.3 Implicit discretisation

We will now study the time discretisation of problem (3.3) making use of an implicit finite-difference scheme. We begin by constructing an appropriate discrete framework.

Take a number $T \in (0, \infty)$, a non-negative integer n such that $T/n \in (0, 1]$, and define the n-grid on $[0, T]$

$$T_n = \{t \in [0, T] : t = jk, \quad j = 0, 1, \dots, n\}, \tag{3.4}$$

where $k := T/n$. Denote $t_j = jk$ for $j = 0, 1, \dots, n$.

For all $z \in V$, we consider the backward difference quotient

$$\Delta^- z(t_{j+1}) = k^{-1}(z(t_{j+1}) - z(t_j)), \quad j = 0, 1, \dots, n-1.$$

Let A_k, f_k be some time-discrete versions of A and f, respectively, i.e., $A_k(t_j)$ is a linear operator from V to V^* for every $j = 0, 1, \dots, n$ and $f_k : T_n \to V^*$ a function. For all $z \in V$, denote $A_{k,j+1}z = A_k(t_{j+1})z$, $f_{k,j+1} = f_k(t_{j+1})$, $j = 0, 1, \dots, n-1$.

For each $n \geq 1$ fixed, we define $v_j = v(t_j)$, $j = 0, 1, \dots, n$, a vector in V satisfying

$$\Delta^- v_{i+1} = A_{k,i+1}v_{i+1} + f_{k,i+1} \quad \text{for } i = 0, 1, \dots, n-1, \quad v_0 = g. \tag{3.5}$$

Problem (3.5) is a time-discrete version of problem (3.3).

Assumption 3.2. Suppose that

1. $\langle A_{k,j+1}v, v\rangle + \lambda\|v\|_V^2 \leq K\|v\|_H^2, \quad \forall v \in V,\ j = 0, 1, \dots, n-1,$
2. $\|A_{k,j+1}v\|_{V^*} \leq M\|v\|_V, \quad \forall v \in V,\ j = 0, 1, \dots, n-1,$
3. $\sum_{j=0}^{n-1} \|f_{k,j+1}\|_{V^*}^2 k \leq N$ and $\|g\|_H \leq N,$

where λ, K, M, and N are the constants in Assumption 3.1.

Remark 3.1. Note that as problem (3.5) is a time-discrete version of problem (3.3) and g denotes the same function in both problems, under Assumption 3.1 we have that $g \in H$ and $\|g\|_H \leq N$.

Under the above assumption, we establish the existence and uniqueness of the solution of problem (3.5).

Theorem 3.3. *Let Assumption 3.2 be satisfied and the constant K be such that $Kk \leq 1$. Then for all $n \in \mathbb{N}$ there exists a unique vector $v_0, v_1, \dots, v_n$ in V satisfying (3.5).*

To prove this result, we consider the following well known lemma (see, e.g., [38, 39]).

Lemma 3.4 (Lax-Milgram). *Let $B : V \to V^*$ be a bounded linear operator. Assume there exists $\lambda > 0$ such that $\langle Bv, v\rangle \geq \lambda \|v\|_V^2$, for all $v \in V$. Then $Bv = v^*$ has a unique solution $v \in V$ for every given $v^* \in V^*$.*

Proof. (Theorem 3.3)

From (3.5), we have that $(I - kA_{k,1})v_1 = g + f_{k,1}k$ and $(I - kA_{k,i+1})v_{i+1} = v_i + f_{k,i+1}k$, for $i = 0, 1, \dots, n-1$, with I the identity operator on V.

We first check that the operators $I - kA_{k,j+1}$, $j = 0, 1, \dots, n-1$, satisfy the hypotheses of Lemma 3.4. These operators are obviously bounded. We have to show that there exists $\lambda > 0$ such that $\langle (I - kA_{k,j+1})v, v\rangle \geq \lambda\|v\|_V^2$, for all $v \in V$, $j = 0, 1, \dots, n-1$. Owing to (1) in Assumption 3.2, we have

$$\begin{aligned}\langle (I - kA_{k,j+1})v, v\rangle &= \langle Iv - kA_{k,j+1}v, v\rangle = \|v\|_H^2 - k\langle A_{k,j+1}v, v\rangle \\ &\geq \|v\|_H^2 - kK\|v\|_H^2 + k\lambda\|v\|_V^2.\end{aligned}$$

Then, as $Kk \leq 1$, we have that $\langle (I - kA_{k,j+1})v, v\rangle \geq k\lambda\|v\|_V^2$ and the hypotheses of Lemma 3.4 are satisfied.

For v_1, we have that $(I - kA_{k1})v_1 = g + f_{k,1}k$. This equation has a unique solution by Lemma 3.4. Suppose now that equation $(I - kA_{k,i})v_i = v_{i-1} + f_{k,i}k$ has a unique solution. Then equation $(I - kA_{k,i+1})v_{i+1} = v_i + f_{k,i+1}k$ has also a unique solution, again by Lemma 3.4. The result is obtained by induction. □

Next, we prove an auxiliary result and then obtain a version of the discrete Gronwall's lemma convenient for our purposes.

Lemma 3.5. *Let $a_1^n, a_2^n, \dots, a_n^n$ be a finite sequence of numbers for every integer $n \geq 1$ such that $0 \leq a_j^n \leq c_0 + C\sum_{i=1}^{j-1} a_i^n$, for all $j = 1, 2, \dots, n$, where C is a positive constant and $c_0 \geq 0$ is some real number. Then $a_j^n \leq (C+1)^{j-1}c_0$, for all $j = 1, 2, \dots, n$.*

Proof. Let $b_j^n := c_0 + C\sum_{i=1}^{j-1} b_i^n$, $j = 1, 2, \dots, n$. Then $a_j^n \leq b_j^n$ for all $j \geq 1$. Indeed for $j = 1$, we have that $a_1^n \leq b_1^n = c_0$. Assume now that $a_i^n \leq b_i^n$ for all $i \leq j$. Then

$$b_{j+1}^n = c_0 + C\sum_{i=1}^{j} b_i^n \geq c_0 + C\sum_{i=1}^{j} a_i^n \geq a_{j+1}^n$$

and, by induction, $a_j^n \leq b_j^n$ for all $j \geq 1$. It is easy to see that $b_{j+1}^n - b_j^n = Cb_j^n$, $j \geq 1$, giving

$$a_{j+1}^n \leq b_{j+1}^n = (C+1)b_j^n = (C+1)^2 b_{j-1}^n = \ldots = (C+1)^j b_1^n = (C+1)^j c_0,$$

and the result is proved. □

Lemma 3.6 (Discrete Gronwall's inequality). *Let $a_0^n, a_1^n, \ldots, a_n^n$ be a finite sequence of numbers for every integer $n \geq 1$ such that*

$$0 \leq a_j^n \leq a_0^n + K\sum_{i=1}^{j} a_i^n k \tag{3.6}$$

holds for every $j = 1, 2, \ldots, n$, with $k := T/n$, and K a positive number such that $Kk =: q < 1$, with q a fixed constant. Then

$$a_j^n \leq a_0^n e^{K_q T},$$

for all integers $n \geq 1$ and $j = 1, 2, \ldots, n$, where $K_q := -K\ln(1-q)/q$.

Proof. The result is obtained by using standard discrete Gronwall arguments. From (3.6), as $Kk < 1$ we have

$$(1 - Kk)a_j^n \leq a_0^n + K\sum_{i=1}^{j-1} a_i^n k \Leftrightarrow a_j^n \leq \frac{a_0^n}{1-Kk} + \frac{Kk}{1-Kk}\sum_{i=1}^{j-1} a_i^n, \tag{3.7}$$

for every $j = 1, 2, \ldots, n$. Owing to Lemma 3.5, with $c_0 = a_0^n/(1-Kk)$ and $C = Kk/(1-Kk)$, from the right inequality in (3.7) we obtain

$$a_j^n \leq \left(\frac{Kk}{1-Kk} + 1\right)^{j-1} \frac{a_0^n}{1-Kk} = \frac{a_0^n}{(1-Kk)^j} \leq \frac{a_0^n}{(1-Kk)^n}.$$

Noting that

$$(1-Kk)^n = \exp(n\ln(1-Kk)) = \exp\left(nKk\frac{\ln(1-q)}{q}\right)$$
$$= \exp\left(KT\frac{\ln(1-q)}{q}\right),$$

the result is proved. □

We are now able to prove that the scheme (3.5) is stable, that is, the solution of the discrete problem remains bounded independently of k.

Theorem 3.7. *Let Assumption 3.2 be satisfied and assume further that constant K satisfies: $2Kk < 1$. Denote $v_{k,j}$, with $j = 0, 1, \dots, n$, the unique solution of problem (3.5) in Theorem 3.3. Then there exists a constant N independent of k such that*

1. $\max_{0\le j\le n} \|v_{k,j}\|_H^2 \le N\left(\|g\|_H^2 + \sum_{j=1}^{n} \|f_{k,j}\|_{V^*}^2 k\right)$;

2. $\sum_{j=0}^{n} \|v_{k,j}\|_V^2 k \le N\left(\|g\|_H^2 + \sum_{j=1}^{n} \|f_{k,j}\|_{V^*}^2 k\right)$.

Remark 3.2. Owing to (3) in Assumption 3.2, the estimates (1) and (2) above can be written, respectively,

$$\sup_{n\ge 1}\left(\max_{0\le j\le n} \|v_{k,j}\|_H^2 \le N\right) \quad \text{and} \quad \sup_{n\ge 1}\left(\sum_{j=0}^{n} \|v_{k,j}\|_V^2 k \le N\right).$$

Remark 3.3. Under Assumption 3.2, with K satisfying $2Kk < 1$, Theorem 3.3 obviously holds so that problem (3.5) has a unique solution.

Proof. (Theorem 3.7)

For $i = 0, 1, \dots, n-1$, we have that

$$\|v_{k,i+1}\|_H^2 - \|v_{k,i}\|_H^2 = 2\langle v_{k,i+1} - v_{k,i}, v_{k,i+1}\rangle - \|v_{k,i+1} - v_{k,i}\|_H^2 \tag{3.8}$$

and, summing up both members of equation (3.8), we obtain, for $j = 1, 2, \dots, n$,

$$\|v_{k,j}\|_H^2 = \|v_{k,0}\|_H^2 + \sum_{i=0}^{j-1} 2\langle v_{k,i+1} - v_{k,i}, v_{k,i+1}\rangle - \sum_{i=0}^{j-1} \|v_{k,i+1} - v_{k,i}\|_H^2.$$

Hence

$$\begin{aligned} \|v_{k,j}\|_H^2 &\le \|v_{k,0}\|_H^2 + \sum_{i=0}^{j-1} 2\langle v_{k,i+1} - v_{k,i}, v_{k,i+1}\rangle \\ &= \|v_{k,0}\|_H^2 + \sum_{i=0}^{j-1} 2\langle A_{k,i+1} v_{k,i+1} k + f_{k,i+1} k, v_{k,i+1}\rangle. \end{aligned}$$

As, by Cauchy's inequality,

$$2\langle f_{k,i+1}, v_{k,i+1}\rangle k \le \lambda \|v_{k,i+1}\|_V^2 k + \frac{1}{\lambda}\|f_{k,i+1}\|_{V^*}^2 k,$$

with $\lambda > 0$, owing to (1) in Assumption 3.2 we obtain

$$\|v_{k,j}\|_H^2 \le \|v_{k,0}\|_H^2 + 2K\sum_{i=0}^{j-1}\|v_{k,i+1}\|_H^2 k - \lambda\sum_{i=0}^{j-1}\|v_{k,i+1}\|_V^2 k + \frac{1}{\lambda}\sum_{i=0}^{j-1}\|f_{k,i+1}\|_{V^*}^2 k,$$

and then

$$\|v_{k,j}\|_H^2 + \lambda\sum_{i=1}^{j}\|v_{k,i}\|_V^2 k \le \|v_{k,0}\|_H^2 + 2K\sum_{i=1}^{j}\|v_{k,i}\|_H^2 k + \frac{1}{\lambda}\sum_{i=1}^{n}\|f_{k,i}\|_{V^*}^2 k. \quad (3.9)$$

In particular,

$$\|v_{k,j}\|_H^2 \le \|v_{k,0}\|_H^2 + 2K\sum_{i=1}^{j}\|v_{k,i}\|_H^2 k + \frac{1}{\lambda}\sum_{i=1}^{n}\|f_{k,i}\|_{V^*}^2 k, \quad (3.10)$$

and, using Lemma 3.6,

$$\|v_{k,j}\|_H^2 \le \left(\|v_{k,0}\|_H^2 + \frac{1}{\lambda}\sum_{i=1}^{n}\|f_{k,i}\|_{V^*}^2 k\right) e^{2K_qT}, \quad (3.11)$$

where K_q is the constant defined in the Lemma. Estimate (1) follows.

From (3.9), (3.10), and (3.11) we finally obtain

$$\|v_{k,j}\|_H^2 + \lambda\sum_{i=1}^{j}\|v_{k,i}\|_V^2 k \le \left(\|v_{k,0}\|_H^2 + \frac{1}{\lambda}\sum_{i=1}^{n}\|f_{k,i}\|_{V^*}^2 k\right) e^{2K_qT}$$

and

$$\sum_{i=1}^{j}\|v_{k,i}\|_V^2 k \le \left(\|v_{k,0}\|_H^2 + \frac{1}{\lambda}\sum_{i=1}^{n}\|f_{k,i}\|_{V^*}^2 k\right) \frac{1}{\lambda}e^{2K_qT}.$$

Estimate (2) follows. □

We will now study the convergence properties of the scheme we have constructed. We impose stronger regularity on the solution $u = u(t)$ of problem (3.3).

Assumption 3.3. Let u be the solution of problem (3.3) in Theorem 3.2. We suppose that there exist a fixed number $\delta \in (0, 1]$ and a constant C such that

$$\frac{1}{k}\int_{t_i}^{t_{i+1}} \|u(t_{i+1}) - u(s)\|_V \mathrm{d}s \le Ck^\delta,$$

for all $i = 0, 1, \ldots, n-1$.

Remark 3.4. Assume that u satisfies the following condition: "There exist a fixed number $\delta \in (0,1]$ and a constant C such that $\|u(t) - u(s)\|_V \le C|t-s|^\delta$, for all $s, t \in [0,T]$". Then Assumption 3.3 obviously holds.

By assuming this stronger regularity of the solution u of (3.3), we can prove the convergence of the solution of problem (3.5) to the solution of problem (3.3) and determine the convergence rate. The accuracy we obtain is of order δ.

Theorem 3.8. *Let Assumptions 3.1 and 3.2 be satisfied and assume further that constant K satisfies: $2Kk < 1$. Denote $u(t)$ the unique solution of (3.3) in Theorem 3.2 and $v_{k,j}$, $j = 0, 1, \ldots, n$, the unique solution of (3.5) in Theorem 3.3. Let also Assumption 3.3 be satisfied. Then there exists a constant N independent of k such that*

1. $$\max_{0\le j\le n} \|v_{k,j} - u(t_j)\|_H^2 \le N\left(k^{2\delta} + \sum_{j=1}^n \frac{1}{k}\left\|A_{k,j}u(t_j)k - \int_{t_{j-1}}^{t_j} A(s)u(t_j)\mathrm{d}s\right\|_{V^*}^2 + \sum_{j=1}^n \frac{1}{k}\left\|f_{k,j}k - \int_{t_{j-1}}^{t_j} f(s)\mathrm{d}s\right\|_{V^*}^2\right);$$
2. $$\sum_{j=0}^n \|v_{k,j} - u(t_j)\|_V^2 k \le N\left(k^{2\delta} + \sum_{j=1}^n \frac{1}{k}\left\|A_{k,j}u(t_j)k - \int_{t_{j-1}}^{t_j} A(s)u(t_j)\mathrm{d}s\right\|_{V^*}^2 + \sum_{j=1}^n \frac{1}{k}\left\|f_{k,j}k - \int_{t_{j-1}}^{t_j} f(s)\mathrm{d}s\right\|_{V^*}^2\right).$$

Proof. Define $w(t_i) := v_{k,i} - u(t_i)$, $i = 0, 1, \ldots, n$. For $i = 0, 1, \ldots, n-1$,

$$\begin{aligned} w(t_{i+1}) - w(t_i) &= A_{k,i+1}w(t_{i+1})k + f_{k,i+1}k - u(t_{i+1}) + u(t_i) + A_{k,i+1}u(t_{i+1})k \\ &= A_{k,i+1}w(t_{i+1})k + \varphi(t_{i+1}), \end{aligned}$$

where $\varphi(t_{i+1}) := f_{k,i+1}k - u(t_{i+1}) + u(t_i) + A_{k,i+1}u(t_{i+1})k$.

Owing to (1) in Assumption 3.2, we obtain

$$\begin{aligned} \|w(t_{i+1})\|_H^2 - \|w(t_i)\|_H^2 =& 2\langle w(t_{i+1}) - w(t_i), w(t_{i+1})\rangle - \|w(t_{i+1}) - w(t_i)\|_H^2 \\ \le& 2\langle A_{k,i+1}w(t_{i+1}), w(t_{i+1})\rangle k + 2\langle \varphi(t_{i+1}), w(t_{i+1})\rangle \\ \le& -2\lambda\|w(t_{i+1})\|_V^2 k + 2K\|w(t_{i+1})\|_H^2 k \\ &+ 2|\langle \varphi(t_{i+1}), w(t_{i+1})\rangle|. \end{aligned} \tag{3.12}$$

Noting that $\varphi(t_{i+1})$ can be written

$$\varphi(t_{i+1}) = \int_{t_i}^{t_{i+1}} A(s)(u(t_{i+1}) - u(s))\mathrm{d}s + \varphi_1(t_{i+1}) + \varphi_2(t_{i+1}),$$

where

$$\varphi_1(t_{i+1}) := A_{k,i+1}u(t_{i+1})k - \int_{t_i}^{t_{i+1}} A(s)u(t_{i+1})\mathrm{d}s$$

and

$$\varphi_2(t_{i+1}) := f_{k,i+1}k - \int_{t_i}^{t_{i+1}} f(s)\mathrm{d}s,$$

for the last term in (3.12) we have the estimate

$$\begin{aligned} 2|\langle\varphi(t_{i+1}), w(t_{i+1})\rangle| \leq 2\left|\left\langle \int_{t_i}^{t_{i+1}} A(s)(u(t_{i+1}) - u(s))\mathrm{d}s, w(t_{i+1})\right\rangle\right| \\ + 2|\langle\varphi_1(t_{i+1}), w(t_{i+1})\rangle| + 2|\langle\varphi_2(t_{i+1}), w(t_{i+1})\rangle|. \end{aligned} \tag{3.13}$$

Let us estimate separately each one of the three terms in (3.13).

For the first term, owing to (2) in Assumption 3.1 and using Cauchy's inequality, we obtain

$$\begin{aligned} &2\left|\left\langle \int_{t_i}^{t_{i+1}} A(s)(u(t_{i+1}) - u(s))\mathrm{d}s, w(t_{i+1})\right\rangle\right| \\ &\leq 2\int_{t_i}^{t_{i+1}} |\langle A(s)(u(t_{i+1}) - u(s)), w(t_{i+1})\rangle|\mathrm{d}s \\ &\leq 2M\|w(t_{i+1})\|_V \int_{t_i}^{t_{i+1}} \|u(t_{i+1}) - u(s)\|_V\mathrm{d}s \\ &\leq \frac{\lambda}{3}\|w(t_{i+1})\|_V^2 k + \frac{3M^2}{\lambda k}\left(\int_{t_i}^{t_{i+1}} \|u(t_{i+1}) - u(s)\|_V\mathrm{d}s\right)^2, \end{aligned} \tag{3.14}$$

with $\lambda > 0$.

For the two remaining terms, we have the estimates

$$2|\langle\varphi_1(t_{i+1}), w(t_{i+1})\rangle| \leq \frac{\lambda}{3}\|w(t_{i+1})\|_V^2 k + \frac{3}{\lambda k}\|\varphi_1(t_{i+1})\|_{V^*}^2 \tag{3.15}$$

and

$$2|\langle\varphi_2(t_{i+1}), w(t_{i+1})\rangle| \leq \frac{\lambda}{3}\|w(t_{i+1})\|_V^2 k + \frac{3}{\lambda k}\|\varphi_2(t_{i+1})\|_{V^*}^2, \tag{3.16}$$

with $\lambda > 0$, using Cauchy's inequality.

Therefore, from (3.14), (3.15), and (3.16) we get the following estimate for (3.13)

$$2|\langle\varphi(t_{i+1}),w(t_{i+1})\rangle| \le \lambda\|w(t_{i+1})\|_V^2 k + \frac{3M^2}{\lambda k}\left(\int_{t_i}^{t_{i+1}} \|u(t_{i+1}) - u(s)\|_V \mathrm{d}s\right)^2 + \frac{3}{\lambda k}\|\varphi_1(t_{i+1})\|_{V^*}^2 + \frac{3}{\lambda k}\|\varphi_2(t_{i+1})\|_{V^*}^2. \tag{3.17}$$

Putting estimates (3.12) and (3.17) together and summing up, owing to Assumption 3.3 we obtain, for $j = 1, 2, \ldots, n$,

$$\|w(t_j)\|_H^2 + \lambda\sum_{i=0}^{j-1}\|w(t_{i+1})\|_V^2 k \le 2K\sum_{i=0}^{j-1}\|w(t_{i+1})\|_H^2 k + \frac{3C^2M^2}{\lambda}\sum_{i=0}^{j-1} k^{2\delta+1} + \frac{3}{\lambda k}\sum_{i=0}^{j-1}\|\varphi_1(t_{i+1})\|_{V^*}^2 + \frac{3}{\lambda k}\sum_{i=0}^{j-1}\|\varphi_2(t_{i+1})\|_{V^*}^2.$$

Hence

$$\begin{aligned}\|w(t_j)\|_H^2 + \lambda\sum_{i=1}^{j}\|w(t_i)\|_V^2 k \le{}& 2K\sum_{i=1}^{j}\|w(t_i)\|_H^2 k + Nk^{2\delta}\\ &+ N\sum_{i=1}^{n}\frac{1}{k}\left\|A_{k,i}u(t_i)k - \int_{t_{i-1}}^{t_i} A(s)u(t_i)\mathrm{d}s\right\|_{V^*}^2\\ &+ N\sum_{i=1}^{n}\frac{1}{k}\left\|f_{k,i}k - \int_{t_{i-1}}^{t_i} f(s)\mathrm{d}s\right\|_{V^*}^2,\end{aligned}$$

with N a constant. Following the same steps as in the proof of Theorem 3.7, estimates (1) and (2) follow. □

Next result is an immediate consequence of Theorem 3.8.

Corollary 3.9. *Let the hypotheses of Theorem 3.8 be satisfied and denote $u(t)$ the unique solution of (3.3) in Theorem 3.2 and $v_{k,j}$, $j = 0, 1, \ldots, n$, the unique solution of (3.5) in Theorem 3.3. If there exists a constant N' independent of k such that*

$$\left\|A_{k,j}u(t_j) - \frac{1}{k}\int_{t_{j-1}}^{t_j} A(s)u(t_j)\mathrm{d}s\right\|_{V^*}^2 + \left\|f_{k,j} - \frac{1}{k}\int_{t_{j-1}}^{t_j} f(s)\mathrm{d}s\right\|_{V^*}^2 \le N'k^{2\delta},$$

for $j = 1, 2, \ldots, n$, then

$$\max_{0\le j\le n}\|v_{k,j} - u(t_j)\|_H^2 \le Nk^{2\delta} \text{ and } \sum_{j=0}^{n}\|v_{k,j} - u(t_j)\|_V^2 k \le Nk^{2\delta},$$

with N be a constant independent of k.

3.4 Explicit discretisation

We now approach the time-discretisation with the use of an explicit finite-difference scheme. As in the previous section, we begin by setting a suitable discrete framework and then investigate the stability and convergence properties of the scheme.

Observe that, when using the explicit scheme, a previous "discretisation in space" has to be assumed. Therefore, we will consider the following version of problem (3.3) in the spaces V_h, H_h, and V_h^*, "space-discrete versions" of V, H, and V^*, respectively,

$$\frac{\mathrm{d}u}{\mathrm{d}t} = A_h(t)u + f_h(t) \ \text{ in } \ [0,T], \quad u(0) = g_h, \tag{3.18}$$

with $A_h(t)$, $f_h(t)$, and g_h "space-discrete versions" of $A(t)$, $f(t)$, and g, and $h \in (0,1]$ a constant. We will use the notation $(\cdot,\cdot)_h$ for the inner product in H_h and $\langle\cdot,\cdot\rangle_h$ for the duality between V_h^* and V_h.

Let the time-grid T_n as defined in (3.4). For all $z \in V_h$, consider the forward difference quotient in time

$$\Delta^+ z(t_j) = k^{-1}(z(t_{j+1}) - z(t_j)), \quad j = 0, 1, \ldots, n-1.$$

Let A_{hk}, f_{hk} be some time-discrete versions of A_h and f_h, respectively, and denote, for all $z \in V_h$,

$$A_{hk,j}z = A_{hk}(t_j)z, \quad f_{hk,j} = f_{hk}(t_j),$$

with $j = 0, 1, \ldots, n-1$.

For each $n \geq 1$ fixed, we consider the time-discrete version of (3.18),

$$\Delta^+ v_i = A_{hk,i}v_i + f_{hk,i} \quad \text{for } i = 0, 1, \ldots, n-1, \quad v_0 = g_h, \tag{3.19}$$

with $v_j = v(t_j)$, $j = 0, 1, \ldots, n$, in V_h.

Problem (3.19) can be solved uniquely by recursion

$$v_j = g_h + \sum_{i=0}^{j-1} A_{hk,i}v_i k + \sum_{i=0}^{j-1} f_{hk,i}k \quad \text{for } j = 1, \ldots, n, \quad v_0 = g_h.$$

We make some assumptions.

Assumption 3.4. Suppose that

1. $\langle A_{hk,j}v, v\rangle_h + \lambda\|v\|^2_{V_h} \leq K\|v\|^2_{H_h}, \quad \forall v \in V_h,\ j = 0, 1, \ldots, n-1,$
2. $\|A_{hk,j}v\|_{V_h^*} \leq M\|v\|_{V_h}, \quad \forall v \in V_h,\ j = 0, 1, \ldots, n-1,$
3. $\sum_{j=0}^{n-1} \|f_{hk,j}\|^2_{V_h^*} k \leq N$ and $\|g_h\|_{H_h} \leq N,$

where λ, K, M, and N are the constants in Assumption 3.1.

Remark 3.5. We refer to Remark 3.1 and note that, under Assumption 3.1, $g_h \in H_h$ and $\|g_h\|_{H_h} \leq N$.

The following version of the discrete Gronwall's inequality is an immediate consequence of Lemma 3.6.

Lemma 3.10. *Let $a_0^n, a_1^n, \ldots, a_n^n$ be a finite sequence of numbers for every integer $n \geq 1$ such that*

$$0 \leq a_j^n \leq a_0^n + K\sum_{i=0}^{j-1} a_i^n k, \tag{3.20}$$

holds for every $j = 0, 1, \ldots, n$, with $k := T/n$ and K a positive number such that $Kk =: q < 1$, with q a fixed constant. Then

$$a_j^n \leq a_0^n e^{K_q T},$$

for all integers $n \geq 1$ and $j = 0, 1, \ldots, n$, where $K_q := -K\ln(1-q)/q$.

Proof. From (3.20), owing to Lemma 3.6 we have

$$(1+Kk)a_j^n \leq (1+Kk)a_0^n + K\sum_{i=1}^{j} a_i^n k \leq (1+Kk)a_0^n e^{K_q T},$$

for $j = 1, 2, \ldots, n$. The result follows. □

In order to obtain stability for the scheme (3.19) we make an additional assumption, involving an inverse inequality between H_h and V_h^*. We note that, for the case of the implicit scheme, there was no such need: the implicit scheme's stability was met unconditionally.

Assumption 3.5. Suppose that there exists a constant C_h, dependent of h, such that

$$\|z\|_{H_h} \leq C_h\|z\|_{V_h^*}, \quad \text{for all } z \in V_h. \tag{3.21}$$

Remark 3.6. The usual assumption involves instead an inverse inequality between V_h and H_h:

$$\|z\|_{V_h} \leq C_h \|z\|_{H_h}, \quad \text{for all } z \in V_h. \tag{3.22}$$

It can be easily checked that (3.22) implies (3.21). In fact, for all $z \in V_h$, $z \neq 0$,

$$\|z\|_{V_h^*} = \sup_{\substack{u \in V_h \\ u \neq 0}} \frac{|(z,u)_h|}{\|u\|_{V_h}} \geq \frac{|(z,z)_h|}{\|z\|_{V_h}} = \frac{\|z\|^2_{H_h}}{\|z\|_{V_h}} \geq \frac{\|z\|^2_{H_h}}{C_h \|z\|_{H_h}} = \frac{\|z\|_{H_h}}{C_h},$$

with the last inequality above due to (3.22).

Remark 3.7. Assumption 3.5 is not void. For example, when the solvability of a multidimensional linear PDE of parabolic type is considered in Sobolev spaces, and its discretised version solvability in discrete counterparts of those spaces (see [21]), (3.21) is satisfied with C_h such that $C_h^2 - 1 \geq Ch^{-2}$, with C a constant independent of h.

Theorem 3.11. *Let Assumptions 3.4 and 3.5 be satisfied and λ, K, M, and C_h the constants defined in the Assumptions. Denote by $v_{hk,j}$, with $j = 0, 1, \ldots, n$, the unique solution of problem (3.19). Assume that constant K is such that $2Kk < 1$. If there exists a number p such that $M^2 C_h^2 k \leq p < \lambda$ then there exists a constant N, independent of k and h, such that*

1. $\max_{0 \leq j \leq n} \|v_{hk,j}\|^2_{H_h} \leq N \left(\|g_h\|^2_{H_h} + \sum_{j=0}^{n-1} \|f_{hk,j}\|^2_{V_h^*} k \right)$;

2. $\sum_{j=0}^{n-1} \|v_{hk,j}\|^2_{V_h} k \leq N \left(\|g_h\|^2_{H_h} + \sum_{j=0}^{n-1} \|f_{hk,j}\|^2_{V_h^*} k \right)$.

Remark 3.8. Remark 3.2 applies to the above theorem with the obvious adaptations.

Proof. (Theorem 3.11)

For $i = 0, 1, \ldots, n-1$, we have

$$\|v_{hk,i+1}\|^2_{H_h} - \|v_{hk,i}\|^2_{H_h} = 2\langle v_{hk,i+1} - v_{hk,i}, v_{hk,i}\rangle_h + \|v_{hk,i+1} - v_{hk,i}\|^2_{H_h} \tag{3.23}$$

and, summing up both members of equation (3.23), for $j = 1, 2, \ldots, n$, we get

$$\begin{aligned}\|v_{hk,j}\|_{H_h}^2 &= \|v_{hk,0}\|_{H_h}^2 + \sum_{i=0}^{j-1} 2\langle v_{hk,i+1} - v_{hk,i}, v_{hk,i}\rangle_h + \sum_{i=0}^{j-1} \|v_{hk,i+1} - v_{hk,i}\|_{H_h}^2 \\ &= \|v_{hk,0}\|_{H_h}^2 + \sum_{i=0}^{j-1} 2\langle A_{hk,i} v_{hk,i}, v_{hk,i}\rangle_h k + \sum_{i=0}^{j-1} 2\langle f_{hk,i}, v_{hk,i}\rangle_h k \\ &\quad + \sum_{i=0}^{j-1} \|A_{hk,i} v_{hk,i} + f_{hk,i}\|_{H_h}^2 k^2.\end{aligned} \tag{3.24}$$

Owing to (1) in Assumption 3.4 and using Cauchy's inequality, from (3.24) we obtain the estimate

$$\begin{aligned}\|v_{hk,j}\|_{H_h}^2 &\le \|v_{hk,0}\|_{H_h}^2 + 2K \sum_{i=0}^{j-1} \|v_{hk,i}\|_{H_h}^2 k - \lambda \sum_{i=0}^{j-1} \|v_{hk,i}\|_{V_h}^2 k \\ &+ \frac{1}{\lambda} \sum_{i=0}^{j-1} \|f_{hk,i}\|_{V_h^*}^2 k + \sum_{i=0}^{j-1} \|A_{hk,i} v_{hk,i} + f_{hk,i}\|_{H_h}^2 k^2,\end{aligned} \tag{3.25}$$

with $\lambda > 0$.

For the last term in the above estimate (3.25), owing to (2) in Assumption 3.4 and to Assumption 3.5, and using Cauchy's inequality we obtain

$$\begin{aligned}&\sum_{i=0}^{j-1} \|A_{hk,i} v_{hk,i} + f_{hk,i}\|_{H_h}^2 k^2 \\ &\quad \le C_h^2 k \sum_{i=0}^{j-1} \|A_{hk,i} v_{hk,i} + f_{hk,i}\|_{V_h^*}^2 k \\ &\quad \le (1+\mu) C_h^2 k \sum_{i=0}^{j-1} \|A_{hk,i} v_{hk,i}\|_{V_h^*}^2 k + \left(1 + \frac{1}{\mu}\right) C_h^2 k \sum_{i=0}^{j-1} \|f_{hk,i}\|_{V_h^*}^2 k \\ &\quad \le (1+\mu) M^2 C_h^2 k \sum_{i=0}^{j-1} \|v_{hk,i}\|_{V_h}^2 k + \left(1 + \frac{1}{\mu}\right) C_h^2 k \sum_{i=0}^{j-1} \|f_{hk,i}\|_{V_h^*}^2 k,\end{aligned} \tag{3.26}$$

with $\mu > 0$.

Finally, putting estimates (3.25) and (3.26) together, we get

$$\begin{aligned}\|v_{hk,j}\|_{H_h}^2 \leq &\|v_{hk,0}\|_{H_h}^2 + 2K\sum_{i=0}^{j-1}\|v_{hk,i}\|_{H_h}^2 k \\ &+ \left((1+\mu)M^2C_h^2k - \lambda\right)\sum_{i=0}^{j-1}\|v_{hk,i}\|_{V_h}^2 k \\ &+ \left(\frac{1}{\lambda} + \left(1+\frac{1}{\mu}\right)C_h^2 k\right)\sum_{i=0}^{j-1}\|f_{hk,i}\|_{V_h^*}^2 k.\end{aligned} \tag{3.27}$$

Now, if there is a constant p such that

$$M^2C_h^2k \leq p < \lambda,$$

implying that, for μ sufficiently small,

$$(1+\mu)M^2C_h^2k - \lambda \leq (1+\mu)p - \lambda < 0,$$

then from (3.27) we obtain the estimate

$$\begin{aligned}&\|v_{hk,j}\|_{H_h}^2 + (\lambda - (1+\mu)p)\sum_{i=0}^{j-1}\|v_{hk,i}\|_{V_h}^2 k \\ &\leq \|v_{hk,0}\|_{H_h}^2 + 2K\sum_{i=0}^{j-1}\|v_{hk,i}\|_{H_h}^2 k + L\sum_{i=0}^{n-1}\|f_{hk,i}\|_{V_h^*}^2 k,\end{aligned} \tag{3.28}$$

where $L := (\mu M^2 + \lambda(1+\mu)p)/\lambda\mu M^2$.

In particular,

$$\|v_{hk,j}\|_{H_h}^2 \leq \|v_{hk,0}\|_{H_h}^2 + 2K\sum_{i=0}^{j-1}\|v_{hk,i}\|_{H_h}^2 k + L\sum_{i=0}^{n-1}\|f_{hk,i}\|_{V_h^*}^2 k \tag{3.29}$$

and, using Lemma 3.10,

$$\|v_{hk,j}\|_{H_h}^2 \leq \left(\|v_{hk,0}\|_{H_h}^2 + L\sum_{i=0}^{n-1}\|f_{hk,i}\|_{V_h^*}^2 k\right)e^{2K_qT}, \tag{3.30}$$

where K_q is the constant defined in Lemma 3.10. (1) follows.

From (3.28), (3.29), and (3.30) we finally obtain

$$\begin{aligned}&\|v_{hk,j}\|_{H_h}^2 + (\lambda-(1+\mu)p)\sum_{i=0}^{j-1}\|v_{hk,i}\|_{V_h}^2 k \\ &\qquad \leq \left(\|v_{hk,0}\|_{H_h}^2 + L\sum_{i=0}^{n-1}\|f_{hk,i}\|_{V_h^*}^2 k\right)e^{2K_qT}\end{aligned}$$

and (2) follows. □

Finally, we prove the convergence of the scheme and determine the convergence rate. The accuracy obtained is of order δ, with δ given by Assumption 3.3.

Theorem 3.12. *Let Assumptions 3.1, 3.4, and 3.5 be satisfied and λ, K, M, and C_h the constants defined in the Assumptions. Denote by $u_h(t)$ the unique solution of problem (3.18) in Theorem 3.2 and by $v_{hk,j}$, with $j = 0, 1, \ldots, n$, the unique solution of problem (3.19). Assume that constant K is such that $2Kk < 1$ and that Assumption 3.3 is satisfied. If there exists a number p such that $M^2C_h^2k \leq p < \lambda$ then there exists a constant N, independent of k and h, such that*

1. $$\max_{0\leq j\leq n} \|v_{hk,j} - u_h(t_j)\|_{H_h}^2 \leq N\left(k^{2\delta} + \sum_{j=0}^{n-1} \frac{1}{k}\left\|A_{hk,j}u_h(t_j)k - \int_{t_j}^{t_{j+1}} A_h(s)u_h(t_j)\mathrm{d}s\right\|_{V_h^*}^2 + \sum_{j=0}^{n-1} \frac{1}{k}\left\|f_{hk,j}k - \int_{t_j}^{t_{j+1}} f_h(s)\mathrm{d}s\right\|_{V_h^*}^2\right);$$

2. $$\sum_{j=0}^{n-1} \|v_{hk,j} - u_h(t_j)\|_{V_h}^2 k \leq N\left(k^{2\delta} + \sum_{j=0}^{n-1} \frac{1}{k}\left\|A_{hk,j}u_h(t_j)k - \int_{t_j}^{t_{j+1}} A_h(s)u_h(t_j)\mathrm{d}s\right\|_{V_h^*}^2 + \sum_{j=0}^{n-1} \frac{1}{k}\left\|f_{hk,j}k - \int_{t_j}^{t_{j+1}} f_h(s)\mathrm{d}s\right\|_{V_h^*}^2\right).$$

Proof. Define $w(t_i) := v_{hk,i} - u_h(t_i)$, $i = 0, 1, \ldots, n$. For $i = 0, 1, \ldots, n-1$

$$\begin{aligned} w(t_{i+1}) - w(t_i) &= A_{hk,i}w(t_i)k + f_{hk,i}k - u_h(t_{i+1}) + u_h(t_i) + A_{hk,i}u_h(t_i)k \\ &= A_{hk,i}w(t_i)k + \varphi(t_i), \end{aligned}$$

where $\varphi(t_i) := f_{hk,i}k - u_h(t_{i+1}) + u_h(t_i) + A_{hk,i}u_h(t_i)k$.

We have that

$$\begin{aligned} \|w(t_{i+1})\|_{H_h}^2 - \|w(t_i)\|_{H_h}^2 &= 2\langle w(t_{i+1}) - w(t_i), w(t_i)\rangle_h + \|w(t_{i+1}) - w(t_i)\|_{H_h}^2 \\ &\leq 2\langle A_{hk,i}w(t_i), w(t_i)\rangle_h k + 2|\langle \varphi(t_i), w(t_i)\rangle_h| \\ &\quad + \|A_{hk,i}w(t_i)k + \varphi(t_i)\|_{H_h}^2. \end{aligned} \tag{3.31}$$

We want to estimate each one of the three terms in (3.31). For the first term in (3.31), owing to (1) in Assumption 3.4, we obtain

$$2\langle A_{hk,i}w(t_i), w(t_i)\rangle_h k \leq -2\lambda\|w(t_i)\|_{V_h}^2 k + 2K\|w(t_i)\|_{H_h}^2 k. \tag{3.32}$$

Noting that $\varphi(t_i)$ can be written

$$\varphi(t_i) = \int_{t_i}^{t_{i+1}} A_h(s)(u_h(t_i) - u_h(s))\mathrm{d}s + \varphi_1(t_i) + \varphi_2(t_i),$$

where

$$\varphi_1(t_i) := A_{hk,i}u_h(t_i)k - \int_{t_i}^{t_{i+1}} A_h(s)u_h(t_i)\mathrm{d}s \quad \text{and} \quad \varphi_2(t_i) := f_{hk,i}k - \int_{t_i}^{t_{i+1}} f_h(s)\mathrm{d}s,$$

for the second term in (3.31) we have

$$\begin{aligned} 2|\langle\varphi(t_i), w(t_i)\rangle_h| \leq & 2\left|\left\langle \int_{t_i}^{t_{i+1}} A_h(s)(u_h(t_i) - u_h(s))\mathrm{d}s, w(t_i)\right\rangle_h\right| \\ & + 2|\langle\varphi_1(t_i), w(t_i)\rangle_h| + 2|\langle\varphi_2(t_i), w(t_i)\rangle_h| \end{aligned} \tag{3.33}$$

and, following the same steps as in the proof of Theorem 3.8, we obtain the estimate

$$\begin{aligned} 2|\langle\varphi(t_i), w(t_i)\rangle_h| \leq \; & \lambda\|w(t_i)\|_{V_h}^2 k + \frac{3M^2}{\lambda k}\left(\int_{t_i}^{t_{i+1}} \|u_h(t_i) - u_h(s)\|_{V_h}\mathrm{d}s\right)^2 \\ & + \frac{3}{\lambda k}\|\varphi_1(t_i)\|_{V_h^*}^2 + \frac{3}{\lambda k}\|\varphi_2(t_i)\|_{V_h^*}^2. \end{aligned} \tag{3.34}$$

Next, we estimate the last term in (3.31). Owing to (2) in Assumption 3.4 and to Assumption 3.5, and using Cauchy's inequality,

$$\begin{aligned} \|A_{hk,i}w(t_i)k + \varphi(t_i)\|_{H_h}^2 & \leq C_h^2\|A_{hk,i}w(t_i)k + \varphi(t_i)\|_{V_h^*}^2 \\ & \leq (1+\mu)C_h^2\|A_{hk,i}w(t_i)\|_{V_h^*}^2 k^2 + \left(1 + \frac{1}{\mu}\right) C_h^2\|\varphi(t_i)\|_{V_h^*}^2 \\ & \leq (1+\mu)M^2C_h^2 k\|w(t_i)\|_{V_h}^2 k + \left(1 + \frac{1}{\mu}\right) C_h^2\|\varphi(t_i)\|_{V_h^*}^2, \end{aligned} \tag{3.35}$$

with $\mu > 0$. As, owing to (2) in Assumption 3.1 and to Cauchy's inequality, $\|\varphi(t_i)\|_{V_h^*}^2$ in (3.35) can be estimated by

$$\begin{aligned} \|\varphi(t_i)\|_{V_h^*}^2 & = \left\|\int_{t_i}^{t_{i+1}} A_h(s)(u_h(t_i) - u_h(s))\mathrm{d}s + \varphi_1(t_i) + \varphi_2(t_i)\right\|_{V_h^*}^2 \\ & \leq \left(1 + \nu + \frac{1}{\nu}\right)\left\|\int_{t_i}^{t_{i+1}} A_h(s)(u_h(t_i) - u_h(s))\mathrm{d}s\right\|_{V_h^*}^2 \\ & \quad + \left(1 + \nu + \frac{1}{\nu}\right)\|\varphi_1(t_i)\|_{V_h^*}^2 + \left(1 + \nu + \frac{1}{\nu}\right)\|\varphi_2(t_i)\|_{V_h^*}^2 \\ & \leq \left(1 + \nu + \frac{1}{\nu}\right) M^2\left(\int_{t_i}^{t_{i+1}} \|u_h(t_i) - u_h(s)\|_{V_h}\mathrm{d}s\right)^2 \\ & \quad + \left(1 + \nu + \frac{1}{\nu}\right)\|\varphi_1(t_i)\|_{V_h^*}^2 + \left(1 + \nu + \frac{1}{\nu}\right)\|\varphi_2(t_i)\|_{V_h^*}^2, \end{aligned} \tag{3.36}$$

with $\nu > 0$, from (3.35) and (3.36), we obtain the following estimate for the last term in (3.31)

$$\begin{aligned}\|A_{hk,i}w(t_i)k + \varphi(t_i)\|_{H_h}^2 &\leq (1+\mu)M^2C_h^2k\|w(t_i)\|_{V_h}^2 k\\ &+ \left(1+\frac{1}{\mu}\right)\left(1+\nu+\frac{1}{\nu}\right)M^2C_h^2\left(\int_{t_i}^{t_{i+1}}\|u_h(t_i)-u_h(s)\|_{V_h}\mathrm{d}s\right)^2\\ &+ \left(1+\frac{1}{\mu}\right)\left(1+\nu+\frac{1}{\nu}\right)C_h^2\|\varphi_1(t_i)\|_{V_h^*}^2 + \left(1+\frac{1}{\mu}\right)\left(1+\nu+\frac{1}{\nu}\right)C_h^2\|\varphi_2(t_i)\|_{V_h^*}^2.\end{aligned} \tag{3.37}$$

Putting estimates (3.32), (3.34), and (3.37) together and summing up, owing to Assumption 3.3, we have, for $j = 0, 1, \ldots, n$,

$$\begin{aligned}\|w(t_j)\|_{H_h}^2 \leq & 2K\sum_{i=0}^{j-1}\|w(t_i)\|_{H_h}^2 k + ((1+\mu)M^2C_h^2k - \lambda)\sum_{i=0}^{j-1}\|w(t_i)\|_{V_h}^2 k\\ &+ M^2C^2\left(\left(1+\frac{1}{\mu}\right)\left(1+\nu+\frac{1}{\nu}\right)C_h^2k + \frac{3}{\lambda}\right)\sum_{i=0}^{j-1}k^{2\delta+1}\\ &+ \left(\left(1+\frac{1}{\mu}\right)\left(1+\nu+\frac{1}{\nu}\right)C_h^2k + \frac{3}{\lambda}\right)\sum_{i=0}^{j-1}\frac{1}{k}\|\varphi_1(t_i)\|_{V_h^*}^2\\ &+ \left(\left(1+\frac{1}{\mu}\right)\left(1+\nu+\frac{1}{\nu}\right)C_h^2k + \frac{3}{\lambda}\right)\sum_{i=0}^{j-1}\frac{1}{k}\|\varphi_2(t_i)\|_{V_h^*}^2.\end{aligned} \tag{3.38}$$

As we assume that there is a constant p such that

$$M^2C_h^2k \leq p < \lambda,$$

we have that, for μ sufficiently small,

$$(1+\mu)M^2C_h^2k - \lambda \leq (1+\mu)p - \lambda < 0.$$

Then, from (3.38),

$$\begin{aligned}&\|w(t_j)\|_{H_h}^2 + (\lambda - (1+\mu)p)\sum_{i=0}^{j-1}\|w(t_i)\|_{V_h}^2 k\\ &\leq 2K\sum_{i=0}^{j-1}\|w(t_i)\|_{H_h}^2 k + M^2C^2TLk^{2\delta}\\ &\quad + L\sum_{i=0}^{n-1}\frac{1}{k}\left\|A_{hk,i}u_h(t_i)k - \int_{t_i}^{t_{i+1}}A_h(s)u_h(t_i)\mathrm{d}s\right\|_{V_h^*}^2\\ &\quad + L\sum_{i=0}^{n-1}\frac{1}{k}\left\|f_{hk,i}k - \int_{t_i}^{t_{i+1}}f_h(s)\mathrm{d}s\right\|_{V_h^*}^2,\end{aligned}$$

where $L := ((3M^2 + \lambda p + \nu\lambda p)\mu\nu + (1 + \mu + \nu + \nu^2)\lambda p)/\mu\nu\lambda M^2$. Estimates (1) and (2) are obtained following the same steps as in Theorem 3.11. $\square$

Next result follows immediately from Theorem 3.12.

Corollary 3.13. *Assume that the hypotheses of Theorem 3.12 are satisfied. Denote by $u_h(t)$ the unique solution of problem (3.18) in Theorem 3.2 and by $v_{hk,j}$, with $j = 0, 1, \ldots, n$, the unique solution of problem (3.19). If there exists a constant N', independent of k, such that*

$$\left\| A_{hk,j} u_h(t_j) - \frac{1}{k} \int_{t_j}^{t_{j+1}} A_h(s) u_h(t_j) \mathrm{d}s \right\|_{V_h^*}^2 + \left\| f_{hk,j} - \frac{1}{k} \int_{t_j}^{t_{j+1}} f_h(s) \mathrm{d}s \right\|_{V_h^*}^2 \leq N' k^{2\delta},$$

for $j = 0, 1, \ldots, n-1$, then

$$\max_{0 \leq j \leq n} \|v_{hk,j} - u_h(t_j)\|_{H_h}^2 \leq N k^{2\delta} \quad \textit{and} \quad \sum_{j=0}^{n-1} \|v_{hk,j} - u_h(t_j)\|_{V_h}^2 k \leq N k^{2\delta},$$

with N a constant independent of k.

3.5 Types of operator specification - implicit scheme

In this Section, we investigate two possible types of discretising operator A_k and function f_k, under the framework of the implicit scheme.

We begin by considering the particular case where A_k and f_k in problem (3.5) are specified, respectively, by the integral averages

$$\bar{A}_k(t_{j+1}) z := \frac{1}{k} \int_{t_j}^{t_{j+1}} A(s) z \mathrm{d}s \quad \text{and} \quad \bar{f}_k(t_{j+1}) := \frac{1}{k} \int_{t_j}^{t_{j+1}} f(s) \mathrm{d}s, \qquad (3.39)$$

for all $z \in V$, $j = 0, 1, \ldots, n-1$.

For all $z \in V$, we denote

$$\bar{A}_{k,j+1} z = \bar{A}_k(t_{j+1}) z, \quad \bar{f}_{k,j+1} = \bar{f}_k(t_{j+1}), \quad j = 0, 1, \ldots, n-1.$$

We prove that, under Assumption 3.1, $\bar{A}_k$ and $\bar{f}_k$ satisfy Assumption 3.2.

Proposition 3.14. *Under Assumption 3.1, operator $\bar{A}_k$ and function $\bar{f}_k$ satisfy*

1. $\langle \bar{A}_{k,j+1}v, v\rangle + \lambda\|v\|_V^2 \le K\|v\|_H^2, \quad \forall v \in V,\ j = 0, 1, \ldots, n-1,$
2. $\|\bar{A}_{k,j+1}v\|_{V^*} \le M\|v\|_V, \quad \forall v \in V,\ j = 0, 1, \ldots, n-1,$
3. $\sum_{j=0}^{n-1} \|\bar{f}_{k,j+1}\|_{V^*}^2 k \le N,$

where λ, K, M, and N are the constants in Assumption 3.1.

Proof. For all $v \in V$, owing to (1) in Assumption 3.1,

$$\begin{aligned}\langle \bar{A}_{k,j+1}v, v\rangle &= \left\langle \frac{1}{k}\int_{t_j}^{t_{j+1}} A(s)v\mathrm{d}s, v\right\rangle = \frac{1}{k}\int_{t_j}^{t_{j+1}} \langle A(s)v, v\rangle \mathrm{d}s \\ &\le \frac{1}{k}\int_{t_j}^{t_{j+1}} \left(K\|v\|_H^2 - \lambda\|v\|_V^2\right)\mathrm{d}s \\ &= K\|v\|_H^2 - \lambda\|v\|_V^2,\end{aligned}$$

with $j = 0, 1, \ldots, n-1$, and (1) is proved.

For all $v \in V$, owing to (2) in Assumption 3.1,

$$\begin{aligned}\|\bar{A}_{k,j+1}v\|_{V^*} &= \left\|\frac{1}{k}\int_{t_j}^{t_{j+1}} A(s)v\mathrm{d}s\right\|_{V^*} \le \frac{1}{k}\int_{t_j}^{t_{j+1}} \|A(s)v\|_{V^*}\mathrm{d}s \\ &\le \frac{1}{k}\int_{t_j}^{t_{j+1}} M\|v\|_V\mathrm{d}s = M\|v\|_V,\end{aligned}$$

with $j = 0, 1, \ldots, n-1$, and (2) is proved.

For (3), we have

$$\begin{aligned}\sum_{j=0}^{n-1} \|\bar{f}_{k,j+1}\|_{V^*}^2 k &= \sum_{j=0}^{n-1} \left\|\frac{1}{k}\int_{t_j}^{t_{j+1}} f(s)\mathrm{d}s\right\|_{V^*}^2 k \le \sum_{j=0}^{n-1} \frac{1}{k}\int_{t_j}^{t_{j+1}} \|f(s)\|_{V^*}^2\mathrm{d}s k \\ &= \int_0^T \|f(s)\|_{V^*}^2\mathrm{d}s \le N,\end{aligned}$$

using Jensen's inequality and owing to (3) in Assumption 3.1. □

As an immediate consequence of Proposition 3.14, the existence and uniqueness and the stability results, Theorems 3.3 and 3.7, respectively, hold for this particular scheme under Assumption 3.1 instead of Assumption 3.2.

For the scheme's convergence, we state a new result.

Theorem 3.15. *Let Assumption 3.1 be satisfied and assume that constant K satisfies: $2Kk < 1$. Denote by $u(t)$ the unique solution of problem (3.3) in Theorem 3.2. Assume that A_k and f_k in problem (3.5) are specified, respectively, by $\bar{A}_k$ and $\bar{f}_k$ in (3.39) and denote by $v_{k,j}$, $j = 0, 1, \dots, n$, the unique solution of problem (3.5) in Theorem 3.3. Let Assumption 3.3 be satisfied. Then there exists a constant N independent of k such that*

$$\max_{0\le j\le n} \|v_{k,j} - u(t_j)\|_H^2 \le Nk^{2\delta} \quad \textit{and} \quad \sum_{j=0}^{n} \|v_{k,j} - u(t_j)\|_V^2 k \le Nk^{2\delta}.$$

Proof. The estimates in Theorem 3.8 are obtained as an immediate consequence of Proposition 3.14. Additionally, due to the particular form of operator $\bar{A}_k$ and function $\bar{f}_k$, we have

$$\begin{aligned}
&\sum_{j=1}^{n} \frac{1}{k} \left\| \bar{A}_{k,j} u(t_j) k - \int_{t_{j-1}}^{t_j} A(s) u(t_j) \mathrm{d}s \right\|_{V^*}^2 \\
&\quad = \sum_{j=1}^{n} \frac{1}{k} \left\| \frac{1}{k} \int_{t_{j-1}}^{t_j} A(s) u(t_j) \mathrm{d}s k - \int_{t_{j-1}}^{t_j} A(s) u(t_j) \mathrm{d}s \right\|_{V^*}^2 = 0
\end{aligned}$$

and

$$\sum_{j=1}^{n} \frac{1}{k} \left\| \bar{f}_{k,j} k - \int_{t_{j-1}}^{t_j} f(s) \mathrm{d}s \right\|_{V^*}^2 = \sum_{j=1}^{n} \frac{1}{k} \left\| \frac{1}{k} \int_{t_{j-1}}^{t_j} f(s) \mathrm{d}s k - \int_{t_{j-1}}^{t_j} f(s) \mathrm{d}s \right\|_{V^*}^2 = 0.$$

The result follows. □

From Theorem 3.15, we see that the rate of convergence is optimal when A and f are approximated by the integral averages $\bar{A}_k$ and $\bar{f}_k$, respectively.

Moreover, it can be easily checked that any operator $\hat{A}_k$ and function $\hat{f}_k$ optimizing the rate of convergence coincide with $\bar{A}_k$ and $\bar{f}_k$, in the sense that

$$\left\| \hat{A}_{k,j+1} z - \bar{A}_{k,j+1} z \right\|_{V^*} = 0 \quad \text{and} \quad \left\| \hat{f}_{k,j+1} - \bar{f}_{k,j+1} \right\|_{V^*} = 0$$

for all $z \in V$, $j = 0, 1, \dots, n-1$. In fact,

$$\begin{aligned}
k^2 \left\| \hat{A}_{k,j} u(t_j) - \bar{A}_{k,j} u(t_j) \right\|_{V^*}^2 \le & 2 \left\| \hat{A}_{k,j} u(t_j) k - \int_{t_{j-1}}^{t_j} A(s) u(t_j) \mathrm{d}s \right\|_{V^*}^2 \\
& + 2 \left\| \bar{A}_{k,j} u(t_j) k - \int_{t_{j-1}}^{t_j} A(s) u(t_j) \mathrm{d}s \right\|_{V^*}^2 = 0
\end{aligned}$$

and

$$k^2 \left\| \hat{f}_{k,j} - \bar{f}_{k,j} \right\|_{V^*}^2 \le 2 \left\| \hat{f}_{k,j} k - \int_{t_{j-1}}^{t_j} f(s) \mathrm{d}s \right\|_{V^*}^2 + 2 \left\| \bar{f}_{k,j} k - \int_{t_{j-1}}^{t_j} f(s) \mathrm{d}s \right\|_{V^*}^2 = 0,$$

for all $j = 1, 2, \ldots, n$.

Next, we investigate a different type of specification for A_k and f_k in problem (3.5).

Consider the pairs of discrete weight functions

$$\left(\rho^{j+1}(t_j), \rho^{j+1}(t_{j+1})\right), \quad \left(r^{j+1}(t_j), r^{j+1}(t_{j+1})\right), \quad j = 0, 1, \ldots, n-1$$

such that

$$\rho^{j+1}(t_j), \rho^{j+1}(t_{j+1}), r^{j+1}(t_j), r^{j+1}(t_{j+1}) \ge 0$$

and

$$\rho^{j+1}(t_j) + \rho^{j+1}(t_{j+1}) = r^{j+1}(t_j) + r^{j+1}(t_{j+1}) = 1,$$

for all $j = 0, 1, \ldots, n-1$.

We define the discrete operator

$$\tilde{A}_k(t_{j+1})z := \rho^{j+1}(t_j) A(t_j) z + \rho^{j+1}(t_{j+1}) A(t_{j+1}) z \tag{3.40}$$

and the discrete function

$$\tilde{f}_k(t_{j+1}) := r^{j+1}(t_j) f(t_j) + r^{j+1}(t_{j+1}) f(t_{j+1}), \tag{3.41}$$

for all $z \in V$, $j = 0, 1, \ldots, n-1$. Denote

$$\tilde{A}_{k,j+1} z = \tilde{A}_k(t_{j+1}) z, \quad \tilde{f}_{k,j+1} = \tilde{f}_k(t_{j+1})$$

and

$$\rho_j^{j+1} = \rho^{j+1}(t_j), \quad \rho_{j+1}^{j+1} = \rho^{j+1}(t_{j+1}), \quad r_j^{j+1} = r^{j+1}(t_j), \quad r_{j+1}^{j+1} = r^{j+1}(t_{j+1}),$$

for all $z \in V$, $j = 0, 1, \ldots, n-1$.

We prove that, in this particular case, under Assumption 3.1, Assumption 3.2 is satisfied.

Proposition 3.16. *Under Assumption 3.1, $\tilde{A}_k$ and $\tilde{f}_k$ satisfy*

1. $\langle \tilde{A}_{k,j+1}v, v \rangle + \lambda \|v\|_V^2 \le K\|v\|_H^2, \quad \forall v \in V,\ j = 0, 1, \ldots, n-1,$
2. $\|\tilde{A}_{k,j+1}v\|_{V^*} \le M\|v\|_V, \quad \forall v \in V,\ j = 0, 1, \ldots, n-1,$
3. $\sum_{j=0}^{n-1} \|\tilde{f}_{k,j+1}\|_{V^*}^2 k \le N,$

where λ, K, M, and N are constants, with λ, K, and M the constants in Assumption 3.1.

Proof. For all $v \in V$, owing to (1) in Assumption 3.1,

$$\begin{aligned}
\langle \tilde{A}_{k,j+1}v, v \rangle &= \langle \rho_j^{j+1} A(t_j)v + \rho_{j+1}^{j+1} A(t_{j+1})v, v \rangle \\
&= \rho_j^{j+1} \langle A(t_j)v, v \rangle + \rho_{j+1}^{j+1} \langle A(t_{j+1})v, v \rangle \\
&\le \left(\rho_j^{j+1} + \rho_{j+1}^{j+1}\right) \left(K\|v\|_H^2 - \lambda\|v\|_V^2\right) = K\|v\|_H^2 - \lambda\|v\|_V^2,
\end{aligned}$$

with $j = 0, 1, \ldots, n-1$, and (1) is proved.

For all $v \in V$, owing to (2) in Assumption 3.1, we have

$$\begin{aligned}
\|\tilde{A}_{k,j+1}v\|_{V^*} &= \|\rho_j^{j+1} A(t_j)v + \rho_{j+1}^{j+1} A(t_{j+1})v\|_{V^*} \\
&\le \rho_j^{j+1} \|A(t_j)v\|_{V^*} + \rho_{j+1}^{j+1} \|A(t_{j+1})v\|_{V^*} \\
&\le M \left(\rho_j^{j+1} + \rho_{j+1}^{j+1}\right) \|v\|_V = M\|v\|_V,
\end{aligned}$$

with $j = 0, 1, \ldots, n-1$, and (2) is proved.

Inequality (3) is satisfied trivially and the result is proved. □

For this particular scheme, the existence and uniqueness and the stability results, respectively, Theorems 3.3 and 3.7, hold under Assumption 3.1 instead of Assumption 3.2 as an immediate consequence of Proposition 3.16.

In order to prove a result on the scheme's convergence, we assume further smoothness. Denote by $\mathcal{B}(V, V^*)$ the Banach space of all bounded linear operators from V into V^*. Also, denote by $\mathrm{Lip}([0,T]; X)$ the space of Lipschitz-continuous X-valued functions on $[0,T]$, with X a Banach space. Let both spaces be endowed with the usual norms.

Assumption 3.6. Suppose that

1. $A \in \mathrm{Lip}([0,T]; \mathcal{B}(V, V^*))$;
2. $f \in \mathrm{Lip}([0,T]; V^*)$.

Remark 3.9. (1) and (2) in Assumption 3.6 could be replaced, respectively, by the weaker conditions

$$A \in C^{\alpha}([0,T]; \mathcal{B}(V,V^*)) \quad \text{and} \quad f \in C^{\alpha}([0,T]; V^*),$$

where $0 < \delta \leq \alpha \leq 1$, with δ the constant defined in Assumption 3.3.

Theorem 3.17. *Let Assumption 3.1 be satisfied and assume further that constant K satisfies: $2Kk < 1$. Denote by $u(t)$ the unique solution of problem (3.3) in Theorem 3.2. Assume that A_k and f_k in problem (3.5) are specified, respectively, by $\tilde{A}_k$ and $\tilde{f}_k$ in (3.40), (3.41) and denote by $v_{k,j}$, $j = 0, 1, \ldots, n$, the unique solution of problem (3.5) in Theorem 3.3. Let Assumptions 3.3 and 3.6 are satisfied. Then there exists a constant N independent of k such that*

$$\max_{0 \leq j \leq n} \|v_{k,j} - u(t_j)\|_H^2 \leq Nk^{2\delta} \quad \textit{and} \quad \sum_{j=0}^{n} \|v_{k,j} - u(t_j)\|_V^2 k \leq Nk^{2\delta}.$$

Proof. The estimates in Theorem 3.8 are obtained as an immediate consequence of Proposition 3.16. Due to the particular form of operator $\tilde{A}_k$ and function $\tilde{f}_k$, we have

$$\begin{aligned}
&\sum_{j=1}^{n} \frac{1}{k} \left\| \tilde{A}_{k,j} u(t_j) k - \int_{t_{j-1}}^{t_j} A(s) u(t_j) \mathrm{d}s \right\|_{V^*}^2 \\
&= \sum_{j=1}^{n} \frac{1}{k} \left\| \left(\rho_{j-1}^j A(t_{j-1}) u(t_j) + \rho_j^j A(t_j) u(t_j)\right) k - \int_{t_{j-1}}^{t_j} A(s) u(t_j) \mathrm{d}s \right\|_{V^*}^2 \\
&= \sum_{j=1}^{n} \frac{1}{k} \left\| \int_{t_{j-1}}^{t_j} \left(\rho_{j-1}^j A(t_{j-1}) + \rho_j^j A(t_j) - \left(\rho_{j-1}^j + \rho_j^j\right) A(s)\right) u(t_j) \mathrm{d}s \right\|_{V^*}^2 \\
&\leq \sum_{j=1}^{n} \int_{t_{j-1}}^{t_j} \left\| \left(\rho_{j-1}^j \left(A(t_{j-1}) - A(s)\right) + \rho_j^j (A(t_j) - A(s))\right) u(t_j) \right\|_{V^*}^2 \mathrm{d}s,
\end{aligned} \tag{3.42}$$

with the inequality obtained by the use of Jensen's inequality.

For the argument of the integral in (3.42),

$$\begin{aligned}
&\left\| \left(\rho_{j-1}^j (A(t_{j-1}) - A(s)) + \rho_j^j (A(t_j) - A(s))\right) u(t_j) \right\|_{V^*}^2 \\
&\leq 2 \left(\rho_{j-1}^j\right)^2 \|(A(t_{j-1}) - A(s)) u(t_j)\|_{V^*}^2 + 2 \left(\rho_j^j\right)^2 \|(A(t_j) - A(s)) u(t_j)\|_{V^*}^2 \\
&\leq 2 \left(N|t_{j-1} - s| \cdot \|u(t_j)\|_V\right)^2 + 2 \left(N|t_j - s| \cdot \|u(t_j)\|_V\right)^2 \\
&\leq Nk^2 \|u(t_j)\|_V^2 \leq Nk^2 \leq Nk^{2\delta},
\end{aligned} \tag{3.43}$$

owing to (1) in Assumption 3.6.

Finally, from (3.42), (3.43),

$$\sum_{j=1}^{n} \frac{1}{k} \left\| \tilde{A}_{k,j} u(t_j) k - \int_{t_{j-1}}^{t_j} A(s) u(t_j) \mathrm{d}s \right\|_{V^*}^2 \leq N \sum_{j=1}^{n} \int_{t_{j-1}}^{t_j} k^{2\delta} \mathrm{d}s \leq N k^{2\delta}.$$

Following the same steps, owing to (2) in Assumption 3.6, we also obtain

$$\sum_{j=1}^{n} \frac{1}{k} \left\| \tilde{f}_{k,j} k - \int_{t_{j-1}}^{t_j} f(s) \mathrm{d}s \right\|_{V^*}^2 \leq N k^{2\delta},$$

and the result follows. □

Remark 3.10. For $j = 1, \ldots, n$, the two-point closed Newton-Cotes quadrature formulas for the integrals

$$\int_{t_{j-1}}^{t_j} A(s) u(t_j) \mathrm{d}s \quad \text{and} \quad \int_{t_{j-1}}^{t_j} f(s) \mathrm{d}s$$

are written, respectively,

$$\int_{t_{j-1}}^{t_j} A(s) u(t_j) \mathrm{d}s \simeq \tilde{A}_{k,j} u(t_j) k \quad \text{and} \quad \int_{t_{j-1}}^{t_j} f(s) \mathrm{d}s \simeq \tilde{f}_{k,j} k,$$

with the weights

$$\left(\rho^j(t_{j-1}), \rho^j(t_j)\right) = \left(r^j(t_{j-1}), r^j(t_j)\right) = \left(\frac{1}{2}, \frac{1}{2}\right).$$

It can be easily shown that in the particular case where $A(s)u(t_j)$ and $f(s)$, with $t_{j-1} \leq s \leq t_j$ and $j = 1, \ldots, n$, are real-valued polynomials of degree 1 the approximation error is null.

3.6 Types of operator specification - explicit scheme

In this section, we investigate the same types of specification for the discretised operator A_{hk} and function f_{hk} but now under the framework of the explicit scheme.

We begin by considering the particular case, where A_{hk} and f_{hk} in problem (3.19) are specified, respectively, by the integral averages

$$\bar{A}_{hk}(t_j) z := \frac{1}{k} \int_{t_j}^{t_{j+1}} A_h(s) z \mathrm{d}s \quad \text{and} \quad \bar{f}_{hk}(t_j) := \frac{1}{k} \int_{t_j}^{t_{j+1}} f_h(s) \mathrm{d}s, \tag{3.44}$$

for all $z \in V_h$, $j = 0, 1, \ldots, n-1$. For all $z \in V_h$, denote

$$\bar{A}_{hk,j} z = \bar{A}_{hk}(t_j) z, \quad \bar{f}_{hk,j} = \bar{f}_{hk}(t_j), \qquad j = 0, 1, \ldots, n-1.$$

For this particular scheme, under Assumption 3.1, Assumption 3.4 is satisfied.

Proposition 3.18. *Under Assumption* 3.1, *operator* $\bar{A}_{hk}$ *and function* $\bar{f}_{hk}$ *satisfy*

1. $\langle \bar{A}_{hk,j} v, v \rangle_h + \lambda \|v\|_{V_h}^2 \leq K \|v\|_{H_h}^2, \quad \forall v \in V_h, \quad j = 0, 1, \ldots, n-1,$
2. $\|\bar{A}_{hk,j} v\|_{V_h^*} \leq M \|v\|_{V_h}, \quad \forall v \in V_h, \quad j = 0, 1, \ldots, n-1,$
3. $\sum_{j=0}^{n-1} \|\bar{f}_{hk,j}\|_{V_h^*}^2 k \leq N,$

where λ, K, M, *and* N *are the constants in Assumption* 3.1.

Proof. Operator $\bar{A}_{hk,j}$ and function $\bar{f}_{hk,j}$ coincide, respectively, with $\bar{A}_{k,j+1}$ and $\bar{f}_{k,j+1}$ in (3.39), for $j = 0, 1, \ldots, n-1$, replacing A and f by their versions A_h and f_h in the integrals' arguments. The result follows from Proposition 3.14. □

Owing to Proposition 3.18, the stability result, Theorem 3.11, holds for this particular scheme under Assumption 3.1 instead of Assumption 3.4.

As for the implicit scheme, an optimal rate of convergence is obtained when A_h and f_h are discretised, respectively, by the integral averages $\bar{A}_{hk}$ and $\bar{f}_{hk}$. The proof is the same as for Theorem 3.15.

Theorem 3.19. *Let Assumptions 3.1 and 3.5 be satisfied, and* λ, K, M, *and* C_h *the constants there defined. Denote by* $u_h(t)$ *the unique solution of problem (3.18) in Theorem 3.2. Assume that* A_{hk} *and* f_{hk} *in problem (3.19) are specified, respectively, by* $\bar{A}_{hk}$ *and* $\bar{f}_{hk}$ *in (3.44) and denote by* $v_{hk,j}$, *with* $j = 0, 1, \ldots, n$, *the unique solution of problem (3.19). Assume that constant* K *is such that* $2Kk < 1$ *and that Assumption 3.3 is satisfied. If there exists a number* p *such that* $M^2 C_h^2 k \leq p < \lambda$ *then there exists a constant* N, *independent of* k *and* h, *such that*

$$\max_{0 \leq j \leq n} \|v_{hk,j} - u_h(t_j)\|_{H_h}^2 \leq N k^{2\delta} \quad \textit{and} \quad \sum_{j=0}^{n-1} \|v_{hk,j} - u_h(t_j)\|_{V_h}^2 \, k \leq N k^{2\delta}.$$

Similarly to what we have done in Section 3.5, we study an alternative discretisation for A_h and f_h in problem (3.18). Consider the pairs of discrete weight functions

$$\left(\rho^j(t_j), \rho^j(t_{j+1})\right), \quad \left(r^j(t_j), r^j(t_{j+1})\right), \quad j = 0, 1, \dots, n-1$$

such that

$$\rho^j(t_j), \rho^j(t_{j+1}), r^j(t_j), r^j(t_{j+1}) \geq 0$$

and

$$\rho^j(t_j) + \rho^j(t_{j+1}) = r^j(t_j) + r^j(t_{j+1}) = 1,$$

for all $j = 0, 1, \dots, n-1$.

We define the discrete operator

$$\tilde{A}_{hk}(t_j)z := \rho^j(t_j)A_h(t_j)z + \rho^j(t_{j+1})A_h(t_{j+1})z \tag{3.45}$$

and the discrete function

$$\tilde{f}_{hk}(t_j) := r^j(t_j)f_h(t_j) + r^j(t_{j+1})f_h(t_{j+1}), \tag{3.46}$$

for all $z \in V$, $j = 0, 1, \dots, n-1$. We denote

$$\tilde{A}_{hk,j}z = \tilde{A}_{hk}(t_j)z, \quad \tilde{f}_{hk,j} = \tilde{f}_{hk}(t_j)$$

and

$$\rho^j_j = \rho^j(t_j), \quad \rho^j_{j+1} = \rho^j(t_{j+1}), \quad r^j_j = r^j(t_j), \quad r^j_{j+1} = r^j(t_{j+1}),$$

for all $z \in V$, $j = 0, 1, \dots, n-1$.

We prove that, under Assumption 3.1, Assumption 3.4 is satisfied.

Proposition 3.20. *Under Assumption 3.1, $\tilde{A}_{hk}$ and $\tilde{f}_{hk}$ satisfy*

1. $\langle \tilde{A}_{hk,j}v, v\rangle_h + \lambda\|v\|^2_{V_h} \leq K\|v\|^2_{H_h}, \quad \forall v \in V_h, \quad j = 0, 1, \dots, n-1,$
2. $\|\tilde{A}_{hk,j}v\|_{V_h^*} \leq M\|v\|_{V_h}, \quad \forall v \in V_h, \quad j = 0, 1, \dots, n-1,$
3. $\sum_{j=0}^{n-1} \|\tilde{f}_{hk,j}\|^2_{V_h^*}k \leq N,$

where λ, K, M, and N are constants, with λ, K, and M the constants in Assumption 3.1.

Proof. Operator $\tilde{A}_{hk,j}$ and function $\tilde{f}_{hk,j}$ coincide, respectively, with $\tilde{A}_{k,j+1}$ and $\tilde{f}_{k,j+1}$ in (3.40), (3.41), for $j = 0, 1, \dots, n-1$, after replacing A and f by A_h and f_h in their analytic expressions. The result follows from Proposition 3.16. □

Owing to Proposition 3.20, Theorem 3.11 on the scheme's stability holds under Assumption 3.1 instead of Assumption 3.4.

Finally, we state a result on the scheme's convergence. The proof is the same as for Theorem 3.17.

Theorem 3.21. *Let Assumptions 3.1 and 3.5 be satisfied, and λ, K, M, and C_h the constants there defined. Denote by $u_h(t)$ the unique solution of problem (3.18) in Theorem 3.3. Assume that A_{hk} and f_{hk} in problem (3.19) are specified, respectively, by $\tilde{A}_{hk}$ and $\tilde{f}_{hk}$ in (3.45), (3.46) and denote by $v_{hk,j}$, $j = 0, 1, \ldots, n$, the unique solution of problem (3.19). Assume that constant K is such that $2Kk < 1$ and that Assumptions 3.3 and 3.6 are satisfied. If there exists a number p such that $M^2C_h^2k \leq p < \lambda$ then there exists a constant N, independent of k and h, such that*

$$\max_{0\leq j\leq n} \|v_{hk,j} - u_h(t_j)\|_{H_h}^2 \leq Nk^{2\delta} \quad \text{and} \quad \sum_{j=0}^{n-1} \|v_{hk,j} - u_h(t_j)\|_{V_h}^2 k \leq Nk^{2\delta}.$$

Remark 3.11. Remark 3.10 still applies here with the obvious modifications.

Chapter 4

Discretisation of PDEs with unbounded coefficients for one spatial dimension

4.1 Introduction

In this chapter we investigate the discretisation of second order PDE of parabolic type with unbounded coefficients by using finite-difference methods.

We deal with the challenge posed by the unboundedness of the PDE coefficients, under the strong assumption that the PDE does not degenerate.

We make use of finite-difference methods to approximate in space the weak solution of the Cauchy problem

$$Lu - \frac{\partial u}{\partial t} + f = 0 \quad \text{in} \quad Q, \quad u(0,x) = g(x) \quad \text{in} \quad \mathbb{R}, \tag{4.1}$$

where $Q = [0,T] \times \mathbb{R}$, with T a positive constant, L is the second-order partial differential operator with real coefficients

$$L(t,x) = a(t,x)\frac{\partial^2}{\partial x^2} + b(t,x)\frac{\partial}{\partial x} + c(t,x) \tag{4.2}$$

for each $t \in [0,T]$ uniformly elliptic in the space variable, and f and g are given real-valued functions. We allow the growth in space of the first and second-order coefficients in L (linear and quadratic growth, respectively), and of the data f and g (polynomial growth).

We follow the previous works ([20, 21]), where the same approach of the more general case of multidimensional PDEs. By considering the special case of one dimension in space, a stronger convergence result is obtained in this chapter. In particular, the same order of accuracy is obtained under regularity assumptions weaker than those required in [20, 21] for the corresponding convergence result.

The chapter is organized as follows. In Section 4.2 we review the L^2 theory of solvability of linear PDEs in weighted Sobolev spaces and, in particular, consider the deterministic one spatial dimension special case of a class of weighted Sobolev spaces introduced by Purtukhia ([46]), and further generalised by Gyöngy and Krylov ([27]), for the treatment of linear SPDEs. We consider problem (4.1) in the framework of the variational approach, and impose weak regularity over the operator's coefficients and the data f and g. In Section 4.3 we consider discrete versions of these spaces and we set an appropriate discretised framework and investigate the spatial approximation of the PDE problem's weak solution. In order to facilitate the study, we make use of basic one-step finite-difference schemes.

4.2 Preliminaries

We consider the particular PDE problem

$$\frac{\partial u}{\partial t} = Lu + f \text{ in } Q, \quad u(0,x) = g(x) \text{ in } \mathbb{R}, \tag{4.3}$$

where L is the second-order operator with real coefficients

$$L(t,x) = a(t,x)\frac{\partial^2}{\partial x^2} + b(t,x)\frac{\partial}{\partial x} + c(t,x), \tag{4.4}$$

where $Q = [0,T] \times \mathbb{R}$, with $T \in (0,\infty)$, and f and g are given functions. We allow the growth, in the spatial variable, of the coefficients a and b, and of the free data f and g.

Under a suitable framework, problem (4.3) can be cast into problem (3.3) studied in Chapter 3 such that existence and uniqueness of its solution hold.

In order to set the framework for problem (4.3), we introduce the so-called well-weighted Sobolev spaces (we refer to [27] for a comprehensive description of these spaces).

We briefly review some facts on the solvability of problem (4.3). We first introduce the above mentioned class of weighted Sobolev spaces. A complete description of this class of spaces can be found in [27].

Let U be a domain in $\mathbb{R}$, i.e., an open set of $\mathbb{R}$. Let $r > 0, \rho > 0$ be smooth functions in U and $m \geq 0$ an integer. The weighted Sobolev space $W^{m,2}(r,\rho)(U)$ is the Banach space of all locally integrable functions $v : U \to \mathbb{R}$ such that for each integer $\alpha \geq 0$, with $\alpha \leq m$, $D^\alpha v$ exists in the weak sense, and

$$|v|_{W^{m,2}(r,\rho)(U)} := \left(\sum_{\alpha \leq m} \int_U r^2 \left|\rho^\alpha D^\alpha v\right|^2 \mathrm{d}x \right)^{1/2}$$

is finite. Endowed with the inner product

$$(v, w)_{W^{m,2}(r,\rho)(U)} := \sum_{\alpha \leq m} \int_U r^2 \rho^{2\alpha} D^\alpha v D^\alpha w \mathrm{d}x$$

for all $v, w \in W^{m,2}(r, \rho)(U)$, which generates the norm, $W^{m,2}(r, \rho)(U)$ is a Hilbert space.

Notation. In the sequel, when $U = \mathbb{R}$, the argument in the function space notation is dropped. For instance, we denote $W^{m,2}(r, \rho)(\mathbb{R}) =: W^{m,2}(r, \rho)$.

We make some assumptions on the behaviour of the weight functions r and ρ (see [27]).

Assumption 4.1. Let $m > 0$ be an integer, and $r > 0$ and $\rho > 0$ smooth functions on $\mathbb{R}$. There exists a constant K such that

1. $|D^\alpha \rho| \leq K\rho^{1-\alpha}$ for all α such that $\alpha \leq m - 1$ if $m \geq 2$;
2. $|D^\alpha r| \leq K \dfrac{r}{\rho^\alpha}$ for all α such that $\alpha \leq m$;
3. $\sup\limits_{|x-y|<\varepsilon} \left(\dfrac{r(x)}{r(y)} + \dfrac{\rho(x)}{\rho(y)} \right) = K$ for some $\varepsilon > 0$, $x, y \in \mathbb{R}$.

Remark 4.1. In (1) in the above Assumption 4.1, if $m < 2$ nothing is required.

Example 4.1. The following functions (taken from [27], citing O. G. Purtukhia [48]) satisfy (1) – (3) Assumption 4.1.

1. $r(x) = (1 + |x|^2)^\beta$, $\beta \in \mathbb{R}$; $\rho(x) = (1 + |x|^2)^\gamma$, $\gamma \leq \frac{1}{2}$;
2. $r(x) = \exp(\pm(1 + |x|^2)^\beta)$, $0 \leq \beta \leq \frac{1}{2}$; $\rho(x) = (1 + |x|^2)^\gamma$, $\gamma \leq \frac{1}{2} - \beta$;
3. $r(x) = (1 + |x|^2)^\beta$, $\beta \in \mathbb{R}$; $\rho(x) = \ln^\gamma(2 + |x|^2)$, $\gamma \in \mathbb{R}$;
4. $r(x) = (1 + |x|^2)^\beta \ln^\mu(2 + |x|^2)$, $\beta \geq 0$, $\mu \geq 0$; $\rho(x) = (1 + |x|^2)^\gamma$, $\gamma \leq \frac{1}{2}$;
5. $r(x) = (1 + |x|^2)^\beta \ \ln^\mu(2 + |x|^2)$, $\beta \geq 0$, $\mu \geq 0$; $\rho(x) = \ln^\gamma(2 + |x|^2)$, $\gamma \geq 0$;

6. $\rho(x) = \exp(-(1+|x|^2)^\gamma)$, $\gamma \geq 0$; each weight function $r(x)$ in examples (1) – (5).

Now, we switch our point of view and consider the functions $w : Q \to \mathbb{R}$ as mappings of t into certain spaces of functions of x we precise below such that, for all $t \in [0,T]$, $x \in \mathbb{R}$, $(w(t))(x) := w(t,x)$.

We impose a coercivity condition over the operator (4.4) and make assumptions on the growth and regularity of the operator's coefficients and also on the regularity of the free data f and g (see [27]).

Assumption 4.2. Let $r > 0$ and $\rho > 0$ be smooth functions on $\mathbb{R}$, and $m \geq 0$ an integer.

(1) There exists a constant $\lambda > 0$ such that $a(t,x) \geq \lambda\rho^2(x)$, for all $t \geq 0, x \in \mathbb{R}$;

(2) The coefficients in L and their derivatives in x up to the order m are measurable functions in $[0,T] \times \mathbb{R}$ such that

- $|D_x^\alpha a| \leq K\rho^{2-\alpha}, \quad \forall \alpha \leq m \vee 1$
- $|D_x^\alpha b| \leq K\rho^{1-\alpha}, \quad \forall \alpha \leq m$
- $|D_x^\alpha c| \leq K, \quad \forall \alpha \leq m$

for any $t \in [0,T], x \in \mathbb{R}$, with K a constant and D_x^α denoting the α^{th} partial derivative operator with respect to x;

(3) $f \in L^2([0,T]; W^{m-1,2}(r,\rho))$ and $g \in W^{m,2}(r,\rho)$.

Notation. We use the notation $W^{-1,2}(r,\rho) := (W^{1,2}(r,\rho))^*$, where $(W^{1,2}(r,\rho))^*$ is the dual of $W^{1,2}(r,\rho)$.

We define the generalised solution of problem (4.3).

Definition 4.1. We say that $u \in C([0,T]; W^{0,2}(r,\rho))$ is a generalised solution of (4.3) on $[0,T]$ if

(1) $u \in L^2([0,T]; W^{1,2}(r,\rho))$;

(2) For every $t \in [0,T]$

$$\begin{aligned}(u(t),\varphi) = (g,\varphi) + \int_0^t \{&-(a(s)D_xu(s), D_x\varphi)\\ &+ (b(s)D_xu(s) - D_xa(s)D_xu(s),\varphi) + (c(s)u(s),\varphi) + \langle f(s),\varphi\rangle\}\,\mathrm{d}s\end{aligned}$$

holds for all $\varphi \in C_0^\infty$.

Notation. The notation (,) in the above definition stands for the inner product in $W^{0,2}(r,\rho)$.

With the following result we state the existence and uniqueness of the solution of problem (4.3) (see [27, 39]).

Theorem 4.2. *Under (1)-(2) in Assumption 4.1, with $m+1$ in place of m, and (1)-(3) in Assumption 4.2, problem (4.3) admits a unique generalised solution u on $[0,T]$. Moreover*

$$u \in C\left([0,T]; W^{m,2}(r,\rho)\right) \cap L^2\left([0,T]; W^{m+1,2}(r,\rho)\right)$$

and

$$\sup_{0\le t\le T} |u(t)|^2_{W^{m,2}(r,\rho} + \int_0^T |u(t)|^2_{W^{m+1,2}(r,\rho)}\,\mathrm{d}t \le N\left(|g|^2_{W^{m,2}(r,\rho)} + \int_0^T |f(t)|^2_{W^{m-1,2}(r,\rho)}\,\mathrm{d}t\right),$$

with N a constant.

4.3 The discrete framework

We now proceed to the discretisation of problem (4.3) in the space-variable. We set a suitable discrete framework with the use of a finite-difference scheme and, by showing that discretised problem can be cast into the general problem (3.3), we prove an existence and uniqueness result for the discretised problem's generalised solution.

This study mirrors the study of problem (4.3), in the sense that the framework we now set is a discrete version of the framework set for problem (4.3), and the techniques used for proving the existence and uniqueness results are the same for both problems.

We define the h-grid on $\mathbb{R}$, with $h \in (0,1]$,

$$Z_h = \{x \in \mathbb{R} : x = nh,\ n = 0, \pm 1, \pm 2, \dots\}. \tag{4.5}$$

Denote

$$\partial^+ u = \partial^+ u(t,x) = h^{-1}\left(u(t,x+h) - u(t,x)\right) \tag{4.6}$$

and

$$\partial^- u = \partial^- u(t,x) = h^{-1}\left(u(t,x) - u(t,x-h)\right) \tag{4.7}$$

for every $x \in Z_h$, the forward and backward difference quotients in space, respectively. Define the discrete operator

$$L_h(t,x) = a(t,x)\partial^-\partial^+ + b(t,x)\partial^+ + c(t,x). \tag{4.8}$$

We consider the discrete problem

$$L_h u - u_t + f_h = 0 \text{ in } Q(h), \quad u(0,x) = g_h(x) \text{ in } Z_h, \tag{4.9}$$

where $Q(h) = [0,T] \times Z_h$, with $T \in (0,\infty)$, and f_h and g_h are functions such that $f_h : Q(h) \to \mathbb{R}$ and $g_h : Z_h \to \mathbb{R}$.

For functions $v : Z_h \to \mathbb{R}$, we introduce the discrete version of the weighted Sobolev space $W^{0,2}(r,\rho)$:

$$l^{0,2}(r) = \{v : |v|_{l^{0,2}(r)} < \infty\}$$

where the norm $|v|_{l^{0,2}(r)}$ is defined by

$$|v|_{l^{0,2}(r)} = \left(\sum_{x\in Z_h} r^2(x)|v(x)|^2 h\right)^{1/2}.$$

Define the inner product

$$(v,w)_{l^{0,2}(r)} = \sum_{x\in Z_h} r^2(x)v(x)w(x)h$$

for any $v, w \in l^{0,2}(r)$, which induces the above norm.

Endowed with the inner product, the space $l^{0,2}(r)$ is clearly a Hilbert space.

We also introduce for functions $w : Z_h \to \mathbb{R}$ the discrete version of the weighted Sobolev space $W^{1,2}(r,\rho)$

$$l^{1,2}(r,\rho) = \{w : |w|_{l^{1,2}(r,\rho)} < \infty\},$$

with the norm $|w|_{l^{1,2}(r,\rho)}$ defined by

$$|w|^2_{l^{1,2}(r,\rho)} = |w|^2_{l^{0,2}(r)} + |\rho\partial^+ w|^2_{l^{0,2}(r)}.$$

The above norm is the induced norm when we endow $l^{1,2}(r,\rho)$ with the inner product

$$(w,z)_{l^{1,2}(r,\rho)} = (w,z)_{l^{0,2}(r)} + \left(\rho\partial^+ w, \rho\partial^+ z\right)_{l^{0,2}(r)},$$

for any functions w, z in $l^{1,2}(r,\rho)$.

We want to show that the discrete framework we have set is a particular case of the general framework considered in Section 3.2.

It can be easily checked that $l^{1,2}(r,\rho)$ is a reflexive and separable Banach space, continuously and densely embedded into the Hilbert space $l^{0,2}(r)$ (we refer to [21], where this is proved for the more general case where $l^{0,2}(r)$ and $l^{1,2}(r,\rho)$ are spaces of real-valued functions on a $d-$dimensional grid).

As $l^{1,2}(r,\rho)$, endowed with the inner product $(\ ,\ )_{l^{1,2}(r,\rho)}$, is clearly a Hilbert space therefore it is reflexive, and the proof for the separability is trivial. The continuity of the embedding follows immediately from $|v|_{l^{0,2}(r)} \leq |v|_{l^{1,2}(r,\rho)}$, for all $v \in l^{1,2}(r,\rho)$. Finally, the denseness can be checked by noticing that, for an arbitrary function $w \in l^{0,2}(r)$, and B a ball in Z_h, the function z defined by

$$z(x) = \begin{cases} w(x), & x \in B \\ 0, & \text{otherwise} \end{cases}$$

belongs obviously to $l^{1,2}(r,\rho)$, and that, for any given $\varepsilon > 0$, $|w - z|_{l^{0,2}(r)} < \varepsilon$ if the diameter of B is chosen sufficiently large.

As in the previous Section, we switch our viewpoint and consider the functions $z : Q(h) \to \mathbb{R}$ as mappings of t into certain spaces of functions of x, defined by $(z(t))(x) := z(t,x)$, for all $t \in [0,T]$ and for all $x \in Z_h$.

For these functions, we consider the space $C([0,T]; l^{0,2}(r))$ of continuous $l^{0,2}(r)$-valued functions on $[0,T]$, and the spaces

$$L^2\left([0,T]; l^{m,2}(r,\rho)\right) = \left\{ z : [0,T] \to l^{m,2}(r,\rho) : \int_0^T |z(t)|^2_{l^{m,2}(r,\rho)}\, \mathrm{d}t < \infty \right\},$$

with $m = 0, 1$.

Notation. We identify $l^{0,2}(r,\rho)$ with $l^{0,2}(r)$.

We make some assumptions over the regularity of the data f_h and g_h in problem (4.9).

Assumption 4.3. Let $r > 0$ be a smooth function on $\mathbb{R}$.

(1) $f_h \in L^2\left([0,T]; l^{0,2}(r)\right)$;

(2) $g_h \in l^{0,2}(r)$.

We also define the generalised solution of problem (4.9).

Definition 4.3. We say that $u \in C([0,T]; l^{0,2}(r)) \cap L^2([0,T]; l^{1,2}(r,\rho))$ is a generalised solution of (4.9) if, for all $t \in [0,T]$,

$$\begin{aligned}(u(t),\varphi) =& (g_h,\varphi) + \int_0^t \{-\left(a(s)\partial^+ u(s), \partial^+\varphi\right) \\ &+ \left(b(s)\partial^+ u(s) - \partial^+ a(s)\partial^+ u(s), \varphi\right) + (c(s)u(s),\varphi) + \langle f_h(s),\varphi\rangle\}\, \mathrm{d}s\end{aligned}$$

holds for all $\varphi \in l^{1,2}(r,\rho)$.

Notation. In the above definition $(\ ,\ )$ denotes the inner product in $l^{0,2}(r)$.

Finally, we prove an existence and uniqueness result for the solution of the discrete problem (4.9). With this result we show that the numerical scheme is stable, that is that, informally, the discrete problem's solution remains bounded independently of the space-step h. The result is obtained as a consequence of Theorem 4.2, remaining only to prove that, under the discrete framework we constructed, $(1)-(2)$ in Assumption 4.1 hold. The result is proved in [22] for the more general multidimensional case. Here, we give that proof's particularization for the case of one dimension in space just to keep the chapter reasonably self-contained.

Theorem 4.4. *Under (1)-(2) in Assumption 4.2 and Assumption 4.3, problem (4.9) has a unique generalised solution u in $[0,T]$. Moreover*

$$\sup_{0\le t\le T} |u(t)|^2_{l^{0,2}(r)} + \int_0^T |u(t)|^2_{l^{1,2}(r,\rho)}\, \mathrm{d}t \le N\left(|g_h(t)|^2_{l^{0,2}(r)} + \int_0^T |f_h(t)|^2_{l^{0,2}(r)}\, \mathrm{d}t\right)$$

with N a constant independent of h.

Proof. Let $L_h(s) : l^{1,2}(r,\rho) \to (l^{1,2}(r,\rho))^*$, for every $s \in [0,T]$. We define

$$\langle L_h(s)\psi,\varphi\rangle := -(a(s)\partial^+\psi, \partial^+\varphi) + (b(s)\partial^+\psi - \partial^+ a(s)\partial^+\psi, \varphi) + (c(s)\psi,\varphi),$$

for all $s \in [0,T]$, $\varphi,\psi \in l^{1,2}(r,\rho)$.

It suffices to prove that the following properties hold

1. $\exists K, \lambda > 0$ constants : $\langle L_h(s)\psi,\psi\rangle + \lambda|\psi|_{l^{1,2}(r,\rho)} \le K|\psi|_{l^{0,2}(r)}$
2. $\exists K$ constant : $|\langle L_h(s)\psi,\varphi\rangle| \le K|\psi|_{l^{1,2}(r,\rho)} \cdot |\varphi|_{l^{1,2}(r,\rho)}$,

for all $s \in [0,T]$, $\varphi, \psi \in l^{1,2}(r,\rho)$.

For the first property, owing to (1) and (2) in Assumption 4.2, we have

$$\begin{aligned}\langle L_h(s)\psi,\psi\rangle =& -\sum_{x\in Z_h} r^2 a(s)\partial^+\psi\partial^+\psi h + \sum_{x\in Z_h} r^2(b(s)-\partial^+ a(s))\partial^+\psi\psi h \\ &+\sum_{x\in Z_h} r^2 c(s)\psi\psi h \\ \le& -\lambda\sum_{x\in Z_h} r^2|\rho\partial^+\psi|^2 h + 2K\sum_{x\in Z_h} r^2\rho|\partial^+\psi\psi|h \\ &+K\sum_{x\in Z_h} r^2|\psi|^2 h \\ =& -\lambda|\rho\partial^+\psi|^2_{l^{0,2}(r)} + 2K\sum_{x\in Z_h} r^2\rho|\partial^+\psi\psi|h + K|\psi|^2_{l^{0,2}(r)},\end{aligned} \tag{4.10}$$

where the variable x is omitted. We use the Cauchy's inequality on the second term in estimate (4.10), and obtain

$$\begin{aligned}&\langle L_h(s)\psi,\psi\rangle \\ &\le -\lambda|\rho\partial^+\psi|^2_{l^{0,2}(r)} + \varepsilon K\sum_{x\in Z_h} r^2|\rho\partial^+\psi|^2 h + \frac{K}{\varepsilon}\sum_{x\in Z_h} r^2|\psi|^2 h + K|\psi|^2_{l^{0,2}(r)} \\ &= -\lambda|\rho\partial^+\psi|^2_{l^{0,2}(r)} - \lambda|\psi|^2_{l^{0,2}(r)} + \varepsilon K|\rho\partial^+\psi|^2_{l^{0,2}(r)} + \frac{K}{\varepsilon}|\psi|^2_{l^{0,2}(r)} + (K+\lambda)|\psi|^2_{l^{0,2}(r)} \\ &\le -\lambda|\psi|^2_{l^{1,2}(r,\rho)} + K|\psi|^2_{l^{0,2}(r)},\end{aligned}$$

with $\lambda > 0$, K constants, by taking ε sufficiently small. The first property is proved.

The second property follows from (2) in Assumption 4.2 and Cauchy-Schwarz inequality

$$\begin{aligned}&\big|\langle L_h(s)\psi,\varphi\rangle\big| \\ &= \Big| -\sum_{x\in Z_h} r^2 a(s)\partial^+\psi\partial^+\varphi h + \sum_{x\in Z_h} r^2 b(s)\partial^+\psi\varphi h - \sum_{x\in Z_h} r^2\partial^+ a(s)\partial^+\psi\varphi h \\ &\quad + \sum_{x\in Z_h} r^2 c(s)\psi\varphi h\Big| \\ &\le K\sum_{x\in Z_h} r^2|\rho^2\partial^+\psi\partial^+\varphi|h + K\sum_{x\in Z_h} r^2|\rho\partial^+\psi\varphi|h + K\sum_{x\in Z_h} r^2|\psi\varphi|h \\ &\le K|\rho\partial^+\psi|_{l^{0,2}(r)}|\rho\partial^+\varphi|_{l^{0,2}(r)} + K|\rho\partial^+\psi|_{l^{0,2}(r)}|\varphi|_{l^{0,2}(r)} + K|\psi|_{l^{0,2}(r)}|\varphi|_{l^{0,2}(r)} \\ &\le K|\psi|_{l^{1,2}(r,\rho)}\cdot|\varphi|_{l^{1,2}(r,\rho)},\end{aligned}$$

where the same writing convention is kept.

Owing to Theorem 4.2 the result follows. □

4.4 Approximation results

In this Section, we study the approximation properties of the numerical scheme (4.9). We begin by investigating the consistency of the numerical scheme, and prove that the difference quotients approximate the partial derivatives (with accuracy of order 1). In addition, we estimate the rate of convergence of the difference quotients to the partial derivatives.

The result is obtained under the assumptions that the weights ρ are bounded from below by a positive constant. Notice that this amounts to assuming that the weights ρ are increasing functions of $|x|$, which is precisely the case we are interested in.

Observe also that the way we set our discrete framework, in strong connection with the framework for problem (4.3), plays a crucial role in obtaining the convergence rate.

We emphasize that, by considering the special case of one dimension in space, we can prove a result stronger than the corresponding result in [22] for the more general multidimensional case (see Remark 4.3 below).

Theorem 4.5. *Let $r > 0$ and $\rho > 0$ be functions on $\mathbb{R}$, and assume that $\rho(x) \geq C$ on $\mathbb{R}$, with $C > 0$ a constant. Assume also that (1)–(3) in Assumption 4.1 are satisfied. Let $u(t) \in W^{2,2}(r,\rho)$, $v(t) \in W^{3,2}(r,\rho)$, for all $t \in [0,T]$. Then there exists a constant N independent of h such that*

(1) $$\sum_{x \in Z_h} r^2(x) \left| \frac{\partial}{\partial x} u(t,x) - \partial^+ u(t,x) \right|^2 \rho^2(x) h \leq h^2 N |u(t)|^2_{W^{2,2}(r,\rho)}$$

(2) $$\sum_{x \in Z_h} r^2(x) \left| \frac{\partial^2}{\partial x^2} v(t,x) - \partial^- \partial^+ v(t,x) \right|^2 \rho^4(x) h \leq h^2 N |v(t)|^2_{W^{3,2}(r,\rho)},$$

for all $t \in [0,T]$.

Remark 4.2. Under the conditions of this theorem, function $u(t)$ (function $v(t)$) has a modification in x which is continuously differentiable in x up to the order 1 (up to the order 2), for every $t \in [0,T]$. Also, the partial derivatives in x up to the order 2 (up to the order 3) equal the weak derivatives a.e., for every $t \in [0,T]$.

These assertions can be proved by Sobolev's embedding of $W^{m,2}(B)$ into $C^n(\overline{B})$, with B a ball in $\mathbb{R}$, if $m > \frac{1}{2} + n$, and by Morrey's inequality (see, e.g., [17, 38]). We consider these modifications in the theorem's proof.

Remark 4.3. When particularizing the hypotheses of the corresponding multi-dimensional result in [22] to the case of one spatial dimension, we obtain that $u(t) \in W^{3,2}(r,\rho)$ and $v(t) \in W^{4,2}(r,\rho)$, for all $t \in [0,T]$, which is stronger than assumed in Theorem 4.5.

In fact, in [22], the result is obtained making use of a Sobolev's embedding, while this can be avoided in the proof of Theorem 4.5 due to the particular geometry of $\mathbb{R}$.

Proof. (Theorem 4.5) Let us prove (1). Observe that the forward difference quotient can be written

$$\partial^+ u(t,x) = h^{-1}(u(t,x+h) - u(t,x)) = \int_0^1 \frac{\partial}{\partial x} u(t,x+hq)\mathrm{d}q.$$

Thus

$$\begin{aligned} \left(\frac{\partial}{\partial x} - \partial^+\right) u(t,x) &= \int_0^1 \left(\frac{\partial}{\partial x} u(t,x) - \frac{\partial}{\partial x} u(t,x+hq)\right) \mathrm{d}q \\ &= h \int_0^1 \int_0^1 q \frac{\partial^2}{\partial x^2} u(t,x+hqs)\mathrm{d}s\mathrm{d}q. \end{aligned} \tag{4.11}$$

From (4.11), using Jensen's inequality, we obtain

$$\begin{aligned} \left|\left(\frac{\partial}{\partial x} - \partial^+\right) u(t,x)\right|^2 &\le h^2 \int_0^1 \int_0^1 q^2 \left|\frac{\partial^2}{\partial x^2} u(t,x+hqs)\right|^2 \mathrm{d}s\mathrm{d}q \\ &= h \int_0^1 \int_0^{hq} q \left|\frac{\partial^2}{\partial x^2} u(t,x+v)\right|^2 \mathrm{d}v\mathrm{d}q \\ &\le h \int_0^1 q\mathrm{d}q \int_0^h \left|\frac{\partial^2}{\partial x^2} u(t,x+v)\right|^2 \mathrm{d}v \\ &= \frac{h}{2} \int_0^h \left|\frac{\partial^2}{\partial x^2} u(t,x+v)\right|^2 \mathrm{d}v \\ &= \frac{h}{2} \int_x^{x+h} \left|\frac{\partial^2}{\partial z^2} u(t,z)\right|^2 \mathrm{d}z. \end{aligned} \tag{4.12}$$

Observe also that from (4.12), using (3) in Assumption 4.1 we have, for any $\theta \in (0,1)$,

$$\begin{aligned} &r^2(x) \left|\left(\frac{\partial}{\partial x} - \partial^+\right) u(t,x)\right|^2 \rho^2(x) \\ &\quad \le hNr^2(x+\theta h)\rho^2(x+\theta h) \int_x^{x+h} \left|\frac{\partial^2}{\partial z^2} u(t,z)\right|^2 \mathrm{d}z. \end{aligned} \tag{4.13}$$

As, by the mean value theorem for integration, for some $\theta \in (0,1)$,

$$r^2(x+\theta h)\rho^2(x+\theta h)\int_x^{x+h}\left|\frac{\partial^2}{\partial z^2}u(t,z)\right|^2 \mathrm{d}z = \int_x^{x+h} r^2(z)\left|\frac{\partial^2}{\partial z^2}u(t,z)\right|^2 \rho^2(z)\mathrm{d}z, \tag{4.14}$$

from(4.13) and (4.14), using Hölder inequality, we obtain

$$\begin{aligned} r^2(x)&\left|\left(\frac{\partial}{\partial x}-\partial^+\right)u(t,x)\right|^2\rho^2(x) \\ &\le hN\int_x^{x+h} r^2(z)\left|\frac{\partial^2}{\partial z^2}u(t,z)\right|^2\rho^4(z)\mathrm{d}z\cdot \sup_{z\in[x,x+h]}\left|\rho^{-2}(z)\right| \quad (4.15)\\ &\le hN\int_x^{x+h} r^2(z)\left|\frac{\partial^2}{\partial z^2}u(t,z)\right|^2\rho^4(z)\mathrm{d}z, \end{aligned}$$

owing to the hypotheses on the weights ρ.

Finally, summing up (4.15) over Z_h, we get

$$\sum_{x\in Z_h} r^2(x)\left|\left(\frac{\partial}{\partial x}-\partial^+\right)u(t,x)\right|^2\rho^2(x)h \le h^2N|u(t)|^2_{W^{2,2}(r,\rho)},$$

with N a constant independent of h, and (1) is proved.

We now prove (2). By writing the forward and backward difference quotients

$$\partial^+v(t,x) = h^{-1}(v(t,x+h)-v(t,x)) = \int_0^1 \frac{\partial}{\partial x}v(t,x+hq)\mathrm{d}q$$

and

$$\partial^-v(t,x) = h^{-1}(v(t,x)-v(t,x-h)) = \int_0^1 \frac{\partial}{\partial x}v(t,x-hs)\mathrm{d}s,$$

respectively, we have for the second-order difference quotient

$$\begin{aligned}\partial^-\partial^+v(t,x) = \partial^-\int_0^1\frac{\partial}{\partial x}v(t,x+hq)\mathrm{d}q &= \int_0^1\left(\frac{\partial}{\partial x}\int_0^1\frac{\partial}{\partial x}v(t,x+hq-hs)\mathrm{d}q\right)\mathrm{d}s \\ &= \int_0^1\int_0^1\frac{\partial^2}{\partial x^2}v(t,x+h(q-s))\mathrm{d}s\mathrm{d}q.\end{aligned}$$

Thus

$$\begin{aligned}\left(\frac{\partial^2}{\partial x^2}-\partial^-\partial^+\right)v(t,x) &= \int_0^1\int_0^1\left(\frac{\partial^2}{\partial x^2}(t,x)-\frac{\partial^2}{\partial x^2}v(t,x+h(q-s))\right)\mathrm{d}s\mathrm{d}q \\ &= h\int_0^1\int_0^1\int_0^1(q-s)\frac{\partial^3}{\partial x^3}v(t,x+hv(q-s))\mathrm{d}v\mathrm{d}s\mathrm{d}q.\end{aligned} \tag{4.16}$$

From (4.16), by Jensen's inequality,

$$\begin{aligned}\left|\left(\frac{\partial^2}{\partial x^2}-\partial^-\partial^+\right)v(t,x)\right|^2 &\le h^2\int_0^1\int_0^1\int_0^1|q-s|^2\left|\frac{\partial^3}{\partial x^3}v(t,x+hv(q-s))\right|^2\mathrm{d}v\mathrm{d}s\mathrm{d}q\\ &= h\int_0^1\int_0^1\int_0^{h(q-s)}(q-s)\left|\frac{\partial^3}{\partial x^3}v(t,x+w)\right|^2\mathrm{d}w\mathrm{d}s\mathrm{d}q\\ &\le h\int_0^1\int_0^1|q-s|\mathrm{d}s\mathrm{d}q\int_0^h\left|\frac{\partial^3}{\partial x^3}v(t,x+w)\right|^2\mathrm{d}w\\ &\le h\int_0^h\left|\frac{\partial^3}{\partial x^3}v(t,x+w)\right|^2\mathrm{d}w = h\int_x^{x+h}\left|\frac{\partial^3}{\partial z^3}v(t,z)\right|^2\mathrm{d}z,\end{aligned}$$

and, following the same steps as in the proof of (1), we finally obtain

$$\sum_{x\in Z_h} r^2(x)\left|\left(\frac{\partial^2}{\partial x^2}-\partial^-\partial^+\right)v(t,x)\right|^2\rho^4(x)h\le h^2N|v(t)|^2_{W^{3,2}(r,\rho)},$$

with N a constant independent of h, and (2) is proved. □

Finally, owing to the stability and consistency properties of the numerical scheme (Theorems 4.4 and 4.5, respectively), we prove the convergence of the discrete problem's solution to the PDE problem's solution, and compute a rate of convergence. The accuracy obtained is of order 1.

The result is obtained by imposing additional regularity on the exact solution of problem (4.3) so that Theorem 4.5 holds, but lesser than it is assumed in the corresponding result in [22], for the multidimensional case (see Remark 4.3).

Theorem 4.6. *Let $r>0$ and $\rho>0$ be functions on $\mathbb{R}$, and assume that $\rho(x)\ge C$ on $\mathbb{R}$, with $C>0$ a constant. Assume that the hypotheses of Theorems 4.2 and 4.4 are satisfied. Denote u the solution of problem (4.3) in Theorem 4.2 and u_h the solution of problem (4.9) in Theorem 4.4. Assume additionally that $u\in L^2([0,T];W^{3,2}(r,\rho))$, and that (3) in Assumption 4.1 holds. Then*

$$\begin{aligned}&\sup_{0\le t\le T}|u(t)-u_h(t)|^2_{l^{0,2}(r)}+\int_0^T|u(t)-u_h(t)|^2_{l^{1,2}(r,\rho)}\mathrm{d}t\\ &\le h^2N\int_0^T|u(t)|^2_{W^{3,2}(r,\rho)}\mathrm{d}t+N\left(|g-g_h|^2_{l^{0,2}(r)}+\int_0^T|f(t)-f_h(t)|^2_{l^{0,2}(r)}\mathrm{d}t\right),\end{aligned}$$

with N a constant independent of h.

Remark 4.4. Under the conditions of the above Theorem 4.6, there are modifications in x such that the data $f(t)$ and g are continuous in x, for every $t\in[0,T]$ (see Remark 4.2). We will consider these modifications in the theorem's proof.

Proof. (Theorem 4.6) From (4.3) and (4.9), we have that $u - u_h$ satisfies the problem

$$\begin{cases} (u-u_h)_t = L_h(u-u_h) + (L-L_h)u + (f-f_h) & \text{in } Q(h) \\ (u-u_h)(0,x) = (g-g_h)(x) & \text{in } Z_h. \end{cases} \tag{4.17}$$

Taking in mind Remark 4.4, we see clearly that $f - f_h \in L^2([0,T]; l^{0,2}(r))$ and $g - g_h \in l^{0,2}(r)$.

With respect to the term $(L - L_h)u$, note that if $u(t) \in W^{3,2}(r,\rho)$, for all $t \in [0,T]$,

$$\begin{aligned} &\sum_{x\in Z_h} r^2(x)|(L-L_h)(t)u(t)|^2 h \\ &= \sum_{x\in Z_h} r^2(x)\left|a(t,x)\left(\frac{\partial^2}{\partial x^2} - \partial^-\partial^+\right)u(t,x) + b(t,x)\left(\frac{\partial}{\partial x} - \partial^+\right)u(t,x)\right|^2 h \\ &\le h^2 N |u(t)|^2_{W^{3,2}(r,\rho)} < \infty, \end{aligned}$$

owing to (2) in Assumption 4.2 and to Theorem 4.5. Thus $(L-L_h)(t)u(t) \in l^{0,2}(r)$, for every $t \in [0,T]$. Moreover, as by assumption $u \in L^2([0,T]; W^{3,2}(r,\rho))$, we obtain immediately $(L - L_h)u \in L^2([0,T]; l^{0,2}(r))$.

We have shown that problem (4.17) satisfies the hypotheses of Theorem 4.4, therefore holding the estimate

$$\begin{aligned} &\sup_{0\le t\le T} |u(t) - u_h(t)|^2_{l^{0,2}(r)} + \int_0^T |u(t) - u_h(t)|^2_{l^{1,2}(r,\rho)} dt \\ &\le N\left(|g - g_h|^2_{l^{0,2}(r)} + \int_0^T |f(t) - f_h(t)|^2_{l^{0,2}(r)} dt + \int_0^T |(L-L_h)(t)u(t)|^2_{l^{0,2}(r)} dt\right). \end{aligned}$$

Owing again to (2) in Assumption 4.2 and to Theorem 4.5, the result follows.

□

The following result is an immediate consequence of Theorem 4.6.

Corollary 4.7. *Let the hypotheses of Theorem 4.6 be satisfied, and denote u the solution of (4.3) in Theorem 4.2 and u_h the solution of (4.9) in Theorem 4.4. If there is a constant N independent of h such that*

$$|g-g_h|^2_{l^{0,2}(r)}+\int_0^T |f(t)-f_h(t)|^2_{l^{0,2}(r)}\mathrm{d}t \le h^2 N\left(|g|^2_{W^{0,2}(r,\rho)}+\int_0^T |f(t)|^2_{W^{-1,2}(r,\rho)}\mathrm{d}t\right)$$

then

$$\sup_{0\le t\le T}|u(t)-u_h(t)|^2_{l^{0,2}(r)}+\int_0^T |u(t)-u_h(t)|^2_{l^{1,2}(r,\rho)}\mathrm{d}t$$
$$\le h^2 N\left(\int_0^T |u(t)|^2_{W^{3,2}(r,\rho)}\mathrm{d}t+|g|^2_{W^{0,2}(r,\rho)}+\int_0^T |f(t)|^2_{W^{-1,2}(r,\rho)}\mathrm{d}t\right).$$

Chapter 5

Discretisation in space and time

5.1 Introduction

In this chapter we combine the results obtained with the time discretisation in abstract spaces in Chapter 3 and with the space discretisation of the PDE in Chapter 4, in order to obtain a rate of convergence for a discretisation in space and time.

In fact, we showed that the discretised problem in Chapter 4 can be cast into the problem in abstract spaces in Chapter 3.

5.2 Numerical approximation in space and time

We reset ourselves in the same general framework established in Section 3.2. We consider, once again, the Cauchy problem for an evolution equation

$$\frac{\mathrm{d}u}{\mathrm{d}t} = A(t)u + f(t) \quad \text{in } [0,T], \qquad u(0) = g, \tag{5.1}$$

with $T \in (0,\infty)$, for every $t \in [0,T]$. $A(t)$ and $\mathrm{d}/\mathrm{d}t$ are linear operators from V to V^*, $f(t) \in V^*$, and $g \in H$.

We make the same assumptions as in Chapter 3, assuming that the operator $L(t)$ is continuous and imposing a coercivity condition, and also some regularity on the free data f and g (see Assumption 3.1).

An existence and uniqueness result for the generalised solution (see Definition 3.1) of problem (5.1) is stated in Theorem 3.2.

The problem we aim to establish for a rate convergence in space and time is the problem considered in Chapter 4

$$Lu - \frac{\partial u}{\partial t} + f = 0 \quad \text{in } Q, \qquad u(0,x) = g(x) \quad \text{in } \mathbb{R}, \tag{5.2}$$

where L is the second-order operator with real coefficients

$$L(t,x) = a(t,x)\frac{\partial^2}{\partial x^2} + b(t,x)\frac{\partial}{\partial x} + c(t,x), \tag{5.3}$$

$Q = [0,T] \times \mathbb{R}$, with $T \in (0,\infty)$, and f and g are given functions. We allow the growth, in the space variables, of the coefficients $a(t,x)$ and $b(t,x)$, and of the free data $f(t,x)$ and $g(x)$.

In Chapter 4, Section 4.3, we proceeded to the space-discretisation of second-order parabolic PDE problem (5.2). We took its version, discrete in space

$$L_h u - u_t + f_h = 0 \ \text{ in } \ Q(h), \quad u(0,x) = g_h(x) \ \text{ in } \ Z_h, \tag{5.4}$$

where $Q(h) = [0,T] \times Z_h$, with $T \in (0,\infty)$, Z_h is a h-grid on $\mathbb{R}$ and L_h the discrete operator

$$L_h(t,x) = a(t,x)\partial^-\partial^+ + b(t,x)\partial^+ + c\,(t,x), \tag{5.5}$$

and set a suitable space-discrete framework, considering the discrete weighted spaces $l^{0,2}(r)$ and $l^{1,2}(r,\rho)$, in order to handle the unbounded data. We then showed that this discrete framework is a particular case of the general framework presented in Section 3.2.

Let us now consider the time-discretisation of the second-order problem (5.4).

Given a non-negative integer n such that $T/n \in (0,1]$, we define the n-grid on $[0,T]$

$$T_n = \{t \in [0,T] : t = jk, \quad j = 0,1,\ldots,n\}, \tag{5.6}$$

where $k := T/n$. We denote $t_j = jk$ for $j = 0,1,\ldots,n$.

For all $z \in V$, we have the backward difference quotient in time

$$\Delta^- z(t_{j+1}) = k^{-1}(z(t_{j+1}) - z(t_j)), \quad j = 0,1,\ldots,n-1,$$

and the forward difference quotient in time

$$\Delta^+ z(t_j) = k^{-1}(z(t_{j+1}) - z(t_j)), \quad j = 0,1,\ldots,n-1.$$

Let L_{hk} and f_{hk} be some time-discrete versions of L_h and f_h, respectively. For all $z \in V_h$, denote $L_{hk,j+1}z = L_{hk}(t_{j+1})z$, $f_{hk,j+1} = f_{hk}(t_{j+1})$, $j = 0,1,\ldots,n-1$.

For each $n \geq 1$ fixed, we obtain a time-discrete version of problem (5.4) for the implicit scheme

$$\Delta^- v_{i+1} = L_{hk,i+1}v_{i+1} + f_{hk,i+1} \ \text{ for } \ i = 0,1,\ldots,n-1, \quad v_0 = g_h \tag{5.7}$$

and for the explicit scheme

$$\Delta^+ v_i = L_{hk,i}\, v_i + f_{hk,i} \;\text{ for }\; i = 0, 1, \ldots, n-1, \quad v_0 = g_h, \tag{5.8}$$

where we define $v_j = v(t_j)$, $j = 0, 1, \ldots, n$, a vector in V.

The existence and uniqueness of a solution for both problems was proved in Sections 3.3 and 3.4 under an abstract framework, but that still hold for this particular problem.

Under certain assumptions, we also obtained results concerning the stability of the schemes (see Theorems 3.7 and 3.11), and their convergence (see Theorems 3.8 and 3.12). The rates of convergence in both schemes were obtained in Corollaries 3.9 and 3.13.

We recall Remark 3.7 to note that Assumption 3.5 in Theorems 3.11 and 3.12 for the time-discretisation using a finite-difference explicit scheme is not void.

It remains only to determine the rate of convergence when the discretisation is considered both in space and time. We will prove that the approximation is $(k^\delta + h)$-accurate.

We first establish the result for the case where the implicit scheme is used for the time-discretisation.

Theorem 5.1. *Assume that the hypotheses of Corollary 4.7 and Theorem 3.8 are satisfied. Denote by u the solution of (5.2) in Theorem 4.2, u_h the solution of (5.4) in Theorem 4.4, and $v_{hk,j}$, $j = 0, 1, \ldots, n$, the solution of (5.7) in Theorem 3.3. Then*

$$\max_{0\le j\le n} |v_{hk,j} - u(t_j)|^2_{l^{0,2}(r)} + \sum_{0\le j\le n} |v_{hk,j} - u(t_j)|^2_{l^{1,2}(r,\rho)}\, k$$

$$\le N\left(k^{2\delta} + h^2 \int_0^T |u(t)|^2_{W^{m+3,2}(r,\rho)} \mathrm{d}t\right) + N\left(\sum_{1\le j\le n} \frac{1}{k} \left| L_{hk,j} u_h(t_j) k - \int_{t_{j-1}}^{t_j} L_h(s) u_h(t_j) \mathrm{d}s \right|^2_{l^{0,2}(r)}\right.$$

$$\left. + \sum_{1\le j\le n} \frac{1}{k} \left| f_{hk,j} k - \int_{t_{j-1}}^{t_j} f_h(s) \mathrm{d}s \right|^2_{l^{0,2}(r)} + |g - g_h|^2_{l^{0,2}(r)} + \int_0^T |f(t) - f_h(t)|^2_{l^{0,2}(r)} \mathrm{d}t \right),$$

with N a constant independent of h and k.

Proof. Let us consider separately the two terms in the sum we want to estimate.

For the first term,

$$\begin{aligned} &\max_{0\le j\le n} |v_{hk,j} - u(t_j)|^2_{l^{0,2}(r)} \\ &\le 2 \max_{0\le j\le n} |v_{hk,j} - u_h(t_j)|^2_{l^{0,2}(r)} + 2 \sup_{0\le t\le T} |u(t) - u_h(t)|^2_{l^{0,2}(r)}, \end{aligned} \tag{5.9}$$

and the desired estimate is obtained immediately by using Theorems 4.7 and 3.8.

For the second term, we have

$$\begin{aligned}&\sum_{j=0}^{n}|v_{hk,j}-u(t_j)|^2_{l^{1,2}(r,\rho)}\,k\\&\leq 2\sum_{j=0}^{n}|v_{hk,j}-u_h(t_j)|^2_{l^{1,2}(r,\rho)}\,k+2\sum_{j=0}^{n}|u_h(t_j)-u(t_j)|^2_{l^{1,2}(r,\rho)}\,k.\end{aligned} \tag{5.10}$$

Let us determine an estimate for the second term in (5.10).

Denote $|\ \ |_{l^{1,2}(r,\rho)} := |\ \ |_1$. We have that

$$\begin{aligned}&\left|\sum_{j=0}^{n}|u_h(t_j)-u(t_j)|_1^2\,k-\int_0^T|u_h(t)-u(t)|_1^2\,\mathrm{d}t\right|\\&=|u_h(t_0)-u(t_0)|_1^2k+\left|\sum_{j=0}^{n-1}\int_{t_j}^{t_{j+1}}\left(|u_h(t_{j+1})-u(t_{j+1})|_1^2-|u_h(s)-u(s)|_1^2\right)\mathrm{d}s\right|.\end{aligned} \tag{5.11}$$

For the integral in (5.11), using Cauchy's inequality and Assumption 3.3, we have

$$\begin{aligned}&\int_{t_j}^{t_{j+1}}\left(|u_h(t_{j+1})-u(t_{j+1})|_1^2-|u_h(s)-u(s)|_1^2\right)\mathrm{d}s\\&\leq 2|u_h(t_{j+1})-u(t_{j+1})|_1\int_{t_j}^{t_{j+1}}(|u_h(t_{j+1})-u(t_{j+1})|_1-|u_h(s)-u(s)|_1)\,\mathrm{d}s\\&\leq\lambda k|u_h(t_{j+1})-u(t_{j+1})|_1^2+\frac{1}{\lambda k}\left(\int_{t_j}^{t_{j+1}}(|u_h(t_{j+1})-u(t_{j+1})|_1-|u_h(s)-u(s)|_1)\,\mathrm{d}s\right)^2\\&\leq\lambda k|u_h(t_{j+1})-u(t_{j+1})|_1^2+\frac{k}{\lambda}Nk^{2\delta},\end{aligned} \tag{5.12}$$

with $\lambda > 0$.

From (5.11) and (5.12),

$$\left|\sum_{j=0}^{n} |u_h(t_j) - u(t_j)|_1^2 \, k - \int_0^T |u_h(t) - u(t)|_1^2 \, \mathrm{d}t\right|$$
$$\leq |u_h(t_0) - u(t_0)|_1^2 k + \lambda \sum_{j=0}^{n-1} |u_h(t_{j+1}) - u(t_{j+1})|_1^2 k + Nk^{2\delta}$$
$$= (1-\lambda)|u_h(t_0) - u(t_0)|_1^2 k + \lambda \sum_{j=0}^{n} |u_h(t_{j+1}) - u(t_{j+1})|_1^2 k + Nk^{2\delta}$$
$$\leq \lambda \sum_{j=0}^{n} |u_h(t_{j+1}) - u(t_{j+1})|_1^2 k + Nk^{2\delta}$$

and, for $0 < \lambda < 1$ we finally obtain

$$\sum_{j=0}^{n} |u_h(t_j) - u(t_j)|_1^2 \, k \leq N \int_0^T |u_h(t) - u(t)|_1^2 \, \mathrm{d}t + Nk^{2\delta}. \tag{5.13}$$

From (5.10) and (5.13), the desired estimate is obtained immediately owing to Corollary 4.7. The result is proved. □

Next result follows immediately from Theorem 5.1.

Corollary 5.2. *Assume that the hypotheses of Theorem 5.1 are satisfied, and denote by u the solution of (5.2) in Theorem 4.2, u_h the solution of (5.4) in Theorem 4.4, and $v_{hk,j}$, $j = 0, 1, \ldots, n$, the solution of (5.7) in Theorem 3.3. If there exists a constant N independent of h and k such that*

$$\left|L_{hk,j} u_h(t_j) - \frac{1}{k}\int_{t_{j-1}}^{t_j} L_h(s) u_h(t_j) \mathrm{d}s\right|_{l^{0,2}(r)}^2 + \left|f_{hk,j} - \frac{1}{k}\int_{t_{j-1}}^{t_j} f_h(s) \mathrm{d}s\right|_{l^{0,2}(r)}^2 \leq Nk^{2\delta},$$

for $j = 1, 2, \ldots, n$, and

$$|g - g_h|_{l^{0,2}(r)}^2 + \int_0^T |f(t) - f_h(t)|_{l^{0,2}(r)}^2 \, \mathrm{d}t \leq Nh^2 \left(|g|_{W^{m,2}(r,\rho)}^2 + \int_0^T |f(t)|_{W^{m-1,2}(r,\rho)}^2 \mathrm{d}t \right)$$

then

$$\max_{0 \leq j \leq n} |v_{hk,j} - u(t_j)|_{l^{0,2}(r)}^2 + \sum_{0 \leq j \leq n} |v_{hk,j} - u(t_j)|_{l^{1,2}(r,\rho)}^2 \, k$$
$$\leq Nk^{2\delta} + Nh^2 \left(\int_0^T |u(t)|_{W^{m+3,2}(r,\rho)}^2 \mathrm{d}t + |g|_{W^{m,2}(r,\rho)}^2 + \int_0^T |f(t)|_{W^{m-1,2}(r,\rho)}^2 \mathrm{d}t \right).$$

Now, we determine the rate of convergence, in the case where the explicit scheme is used for the discretisation in time. The proof is the same as for Theorem 5.1.

Theorem 5.3. *Let the hypotheses of Corollary 4.7 and Theorem 3.12 be satisfied. Denote by u the solution of (5.2) in Theorem 4.2, u_h the solution of (5.4) in Theorem 4.4, and $v_{hk,j}$, $j = 0, 1, \ldots, n$, the solution of (5.8). Then*

$$\max_{0\le j\le n} |v_{hk,j} - u(t_j)|^2_{l^{0,2}(r)} + \sum_{0\le j\le n} |v_{hk,j} - u(t_j)|^2_{l^{1,2}(r,\rho)}\, k$$

$$\le N\left(k^{2\delta} + h^2\int_0^T |u(t)|^2_{W^{m+3,2}(r,\rho)}\mathrm{d}t\right) + N\left(\sum_{j=0}^{n-1}\frac{1}{k}\left|L_{hk,j}u_h(t_j)k - \int_{t_j}^{t_{j+1}} L_h(s)u_h(t_j)\mathrm{d}s\right|^2_{l^{0,2}(r)}\right.$$

$$\left. + \sum_{j=0}^{n-1}\frac{1}{k}\left|f_{hk,j}k - \int_{t_j}^{t_{j+1}} f_h(s)\mathrm{d}s\right|^2_{l^{0,2}(r)} + |g - g_h|^2_{l^{0,2}(r)} + \int_0^T |f(t) - f_h(t)|^2_{l^{0,2}(r)}\,\mathrm{d}t\right)$$

with N a constant independent of h and k.

Finally, we state a corollary as immediate consequence of Theorem 5.3.

Corollary 5.4. *Assume that the hypotheses of Theorem 5.3 are satisfied, and denote by u the solution of (5.2) in Theorem 4.2, u_h the solution of (5.4) in Theorem 4.4, and $v_{hk,j}$, $j = 0, 1, \ldots, n$, the solution of (5.8). If there exists a constant N independent of h and k such that*

$$\left|L_{hk,j}u_h(t_j) - \frac{1}{k}\int_{t_j}^{t_{j+1}} L_h(s)u_h(t_j)\mathrm{d}s\right|^2_{l^{0,2}(r)} + \left|f_{hk,j} - \frac{1}{k}\int_{t_j}^{t_{j+1}} f_h(s)\mathrm{d}s\right|^2_{l^{0,2}(r)} \le Nk^{2\delta},$$

for $j = 0, 1, \ldots, n-1$ and

$$|g - g_h|^2_{l^{0,2}(r)} + \int_0^T |f(t) - f_h(t)|^2_{l^{0,2}(r)}\,\mathrm{d}t \le Nh^2\left(|g|^2_{W^{m,2}(r,\rho)} + \int_0^T |f(t)|^2_{W^{m-1,2}(r,\rho)}\mathrm{d}t\right)$$

then

$$\max_{0\le j\le n} |v_{hk,j} - u(t_j)|^2_{l^{0,2}(r)} + \sum_{0\le j\le n} |v_{hk,j} - u(t_j)|^2_{l^{1,2}(r,\rho)}\, k$$

$$\le Nk^{2\delta} + Nh^2\left(\int_0^T |u(t)|^2_{W^{m+3,2}(r,\rho)}\mathrm{d}t + |g|^2_{W^{m,2}(r,\rho)} + \int_0^T |f(t)|^2_{W^{m-1,2}(r,\rho)}\mathrm{d}t\right).$$

Chapter 6

Conclusions and further research

Many financial problems induce a deep study in mathematics and are described by partial and stochastic differential equations.

The celebrated Black-Scholes equation is a model for option pricing that has been widely used. However, even though for the basic Black- Scholes equation there is an explicit solution, that is not the case, for instance, when coefficients (drift and volatility) are not constant or when multi-dimensionality is considered. In these models, the numerical approximation of the stochastic equation or of the corresponding PDE is compulsory.

In this work, we used finite-differences methods to discretise a PDE problem obtained from the stochastic modelling of a vanilla option of European type under the framework of a general version of Black-Scholes model, where the asset appreciation rate and the volatility are taken time and space-dependent.

Making use of both the implicit and the explicit finite-difference schemes in time, we considered the PDE solvability in the framework of the variational approach and proceeded to the discretisation of a linear parabolic equation in abstract spaces, obtaining the convergence of the schemes under a smoothness condition weaker than the usual Hölder-continuity.

We also used finite-differences methods to approximate in space the weak solution of a Cauchy problem for the one dimension case. The same approach for the multidimensional PDEs has been done previously (see [20, 21]). The same order of accuracy was obtained under regularity assumptions weaker than those required in [20, 21] for the corresponding convergence result.

The discretisation in space and time of a general linear evolution equation in abstract spaces, led us to estimate the rate of convergence of the numerical approximation.

We considered also a particular model with the introduction of costs in the financial transactions, that led us to a non-linear problem with a Black-Scholes type equation and stated results for the existence, uniqueness and localisation of the solution using topological methods.

In fact, functional analysis and topological methods play an important role in the study of non-linear differential problems. Most of the studied cases in the literature concern linear problems but the interest for non-linear versions of the Black-Scholes equation has been increasing rapidly, with results concerning analytical and numerical treatment. See for instance papers from Pablo Amster *et al.* ([2]), Matthias Ehrhardt *et al.* ([3]), Valeri Zakamouline ([59]) and Daniel Ševčovič ([51]) for non-linear Black-Scholes type equation, modelling European and American options.

Another possibility is to use the finite element method, more suitable when solving the equations in general domains and closer to the weak form of the equations, making the mathematical and numerical analysis of the discrete problem profit from functional analysis results valid for the continuous problem.

We are also interested in the investigation of the PDE solvability in the framework of the variational approach and, without the assumption of a nondegeneracy condition, discretise a linear parabolic equation in abstract spaces (see Gyöngy [27]).

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