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UNIVERSIDADE TÉCNICA DE LISBOA
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MESTRADO

MATEMÁTICA FINANCEIRA

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DISSERTAÇÃO

OPTIMAL VALUE OF A FIRM INVESTING IN EXOGENEOUS TECHNOLOGY

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Abstract

In this work we study the optimal value for a Firm whose value is function of an exogenous technology level. At any point in time the Firm can invest in a new technology, incurring in an immediate cost and in return it will become able to use that technology yielding profit through a given profit flow function. The technology is modelled by a discrete stochastic process with a time-dependent arrival rate. We study the optimal stopping time that will correspond to the point in time when the firm will invest. We use a dynamic programming approach, finding the Hamilton-Jacobi-Bellman equation whose solution gives us the optimal value of the firm. We particularise for two cases, one with a constant arrival rate and the other with a time-dependent and non-monotonic arrival.

Resumo

Neste trabalho estudamos o valor ótimo para uma Firma cujo valor função depende de um nível de tecnologia exógeno. Em qualquer ponto no tempo a Firma pode investir numa nova tecnologia incorrendo num custo imediato e em retorno passará a utilizar essa nova tecnologia gerando lucros a partir de uma dada função. Estudamos o tempo de paragem ótimo que corresponde ao ponto no tempo em que a empresa investe para obter a valorização ótima. Usamos uma abordagem de programação dinâmica, encontrando a equação de Hamilton-Jacobi-Bellman cuja solução nos dá o valor ótimo da firma. A tecnologia é modelada usando um processo estocástico discreto com uma intensidade dependente do tempo. Particularizamos para dois casos, um em que a intensidade é constante e outro em que é dependente do tempo e não monótona.

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Contents

Chapter 1

Introduction

In this thesis we develop a model for a Firm that wants to optimise its value by investing in a new technology. The Firm value is directly related with the technology by a profit flow function that represents the profit that the Firm earns by using a given technology level. In our model the Firm can invest at any time in a new technology. The technology is assumed to follow a discrete stochastic process with a general arrival rate, that is dependent on the time elapsed since the last innovation.

By investing in a new technology, the Firm incurs an immediate cost (which is assumed to be constant) but its value increases. We derive the Optimal Value for the Firm for two cases, a case where the arrival rate is constant and a case where the arrival rate is time-dependent and non-monotonic.

We use a dynamic programming approach. We start by proving the dynamic programming principle, finding the HJB and finally solving it for the two different cases.

The structure of this thesis is as follows. In Chapter 2 we introduce the concept of Real Options and Technology Investment and review some of the existing literature. In Chapter 3 we construct our initial model, prove the dynamic programming principle and find the Hamilton-Jacobi-Bellman Equation. In Chapter 4 we make a first attempt to solve the model and, upon facing some issues, we develop a new modified problem that is proven to converge to our initial one. We then work with this problem and find the pretended results. In Chapter 5 we conclude with an overview of the results found and an outlook for further studies.

Chapter 2

Real Options and Technology Investment

In this chapter we start by introducing the main concepts of Real Options. We then particularise for the study of Technology Investment and introduce the main motivation for the work presented here. We also include an overview of some of the literature concerning the focus of our work.

2.1 Investment Decisions under The Real Options Approach

Making the decision to invest can be defined as incurring into an immediate cost in the expectation of further rewards [4]. A key word in this definition is "expectation", as it embodies one of the most important factors in an investment decision, the *uncertainty*. That uncertainty is the prime reason for construction of mathematical models to study the investment decisions.

A particular methodology to study investment opportunities is using the "Real Options Approach" or simply Real Options. In this approach, investment decisions are studied using the framework initially developed for financial derivatives, such as the Black-Scholes model for European Call Options. Whereas the study of optimal investment policies is centuries old, the Real Options Approach is relatively new. The term "Real Options" itself was coined in a pioneering article published in 1977 by Stewart C. Myers [9], which was also one of the first applications of Financial Options theory to Corporate Finance.

There are several reasons for using this approach over the NPV[1] methods of traditional Corporate Finance. Most investment projects possess three fundamental characteristics, the *irreversibility* of the decision, the *ability to wait* and postpone the decision and also the *uncertainty* regarding future rewards of the investment. All these characteristics are usually overlooked in traditional NPV methods, and, conversely they are easily modelled with Real Options. In a fundamental survey of Real Options and its applications, published in 1996, Dixit and Pindyck [4] demonstrate for a variety of cases how using the traditional simplified NPV methods over Real Options could lead to drastically different results and consequently investment policies could be simply wrong.

2.2 Technology Investment

One kind of investment that can be studied under the framework of Real Options is the investment by a Firm in Technology Markets which is the focus of work presented here. We now state what is the main motivation for this kind of problems and we review some of the existent work in the field.

2.2.1 Motivation

The increase in productivity of the overall economy in the nineteenth and twentieth century was mostly due to increases in technology [8]. Yet it was not until end of 20th century that the study of optimal technology adoption and diffusion begun. A valid argument is that, until recently, the technological cycles were much longer. Therefore the options for a Firm to invest in technology were very limited in its lifetime.

[1]Net Present Value

In the last decades technological research and development changed considerably, and now we have fast-paced, yet uncertain, improvement of technology crossing a variety of industries. A few examples are the increase in the fuel efficiency in modern passenger vehicles, that resulted from the adoption of innovative technologies in areas such as micro electronics, aerodynamics and material substitution. Also, the boost in productivity in the agricultural sector that stemmed from continuous adoption of new mechanical and biochemical technologies [14]. Finally, a irrefutable example on the fast-paced yet uncertain improvement of technology and its impact on a Firm's productivity is the industry of telecommunications and information technologies, illustrated by Moore's law [5, 14].

As was noted by Kriebel [7] in 1989 50% of new corporate capital expenditures by the biggest companies in the United States was in information and communication technology.

And besides being a major part of the expenditures, the fast development of new technology multiplies the opportunities and options for a Firm to invest. Thus it is crucial to develop a systematic approach to investment in technology.

2.2.2 Related Literature

One of the first publications to study technology investment by a Firm, within the framework of real options, was in 1982 by Baldwin [1]. In his paper, he demonstrated the existence of an "NPV Premium", under certain conditions, that a Firm requires on an optimal technology investment. That premium essentially represents the value of waiting of an option. A few years later, in 1985, McCardle published a paper on another aspect of investing in technology, the uncertainty related not to the technology arrival but to its profitability [8].

A rather extent study of both decision and game-theoretical models on Technology Investment was developed by Huisman [6] in his Ph.D thesis in 2000. Huisman studied a variety of models, where different assumptions where made. He considers two main kind of models,the decision models where a single Firm is considered to be in the market and game-theoretical models where interactions between Firms are considered.

Within the decision theoretical models, Huisman considered several assumptions as well. He studied the optimal choices when the investment cost was constant and when it was variable, along with other variations. More recently, V. Hagspiel [5], developed a model for a changing arrival rate where the rate takes one of two values depending on how much time it elapsed since the last innovation.

In this thesis, a model is built using a dynamic programming approach, following a somewhat similar approach as Huisman, Verena and Farzin [5, 6, 14] we however, use a general intensity rate dependent on the time elapsed since the last innovation.

Chapter 3

Problem Formulation

We want to model a Firm that is considering investing in an exogenous technology. By investing in that technology we mean incurring in an immediate cost and in return it will become able to use that technology yielding profit through a given profit flow function $\tilde{\Pi} : \mathbb{N} \to \mathbb{R}$. In this chapter we introduce the stochastic process that models the technology arrivals, we model the Firm's value, we prove the dynamic programming principle and find the Hamiton-Jacobi-Bellman.

3.1 The Technology Process

We are considering that the most efficient technology level available on the market evolves in a stochastic way, and that it is exogenous to our Firm.

Motivated by the nature of the various technology markets, we will consider that the technology process stays constant for a stochastic amount time and it's changes are discontinuous. Therefore we will be dealing with a discrete stochastic process in continuous time. This stochastic process will be denoted by $\{N_t, t \geq 0\}$, where N_t denotes the number of changes in the technology level until time t.

In addition, we will assume that the improvement in technology is constant.

Therefore, without loss of generality, we can assume that the technology level process

$\{N_t\}$[1] takes values in $\mathbb{N}_0$, with

$$\Pr\{N_s \leq N_t\} = 1 \quad \forall \;\; 0 \leq s \leq t < +\infty, \tag{3.1}$$

$$\lim_{t \to s^+} N_t = N_s. \tag{3.2}$$

We define the time of the arrival of the jth level of technology by T_j, with

$$T_0 = 0 \qquad T_j = \sup\{t : N_t < j\}, \qquad j \in \mathbb{N}, \tag{3.3}$$

and we assume that the sequence $\{T_j, j \in \mathbb{N}\}$ has independent identically distributed increments with distribution,

$$\Pr\{T_n - T_{n-1} \leq t\} = F(t) = 1 - e^{-\int_0^t \lambda(s)ds} \qquad \forall n \in \mathbb{N}, t \geq 0. \tag{3.4}$$

The the density function is,

$$f(t) = \frac{dF(t)}{dt} = \lambda(t)e^{-\int_0^t \lambda(s)ds}, \tag{3.5}$$

where $\lambda : \mathbb{R}_0^+ \to \mathbb{R}_0^+$ is a continuous function. If we introduce the process $\theta_t = t - T_{N_t}$ representing the time elapsed since the arrival of the current most efficient technology, then the two-dimensional process,

$$\{(N_t, t - T_{N_t}) = (N_t, \theta_t), t \geq 0\}. \tag{3.6}$$

is Markovian. Therefore, from now on, once we mention the technology process we will be referring to the Markovian representation of the process defined in (3.6).

It will be advantageous to have the previously mentioned stochastic process characterised by its infinitesimal generator.

[1] To ease the notation, we usually omit the index set from the stochastic process. Thus we denote $\{N_t, t \geq 0\}$ simply by $\{N_t\}$

Proposition 1. *Let $\mathcal{X}$ denote the set of all bounded functions $\phi : \mathbb{N}_0 \times \mathbb{R}_0^+ \mapsto \mathbb{R}$ continuously differentiable with respect to the second argument. For the two-dimensional stochastic process defined in* (3.6) *the infinitesimal generator $\mathcal{A}$, is the operator acting on $\mathcal{X}$ is given by*

$$\mathcal{A}\phi(n,t) = \lambda(t)\Big(\phi(n+1,0) - \phi(n,t)\Big) + \frac{\partial}{\partial t}\phi(n,t) \tag{3.7}$$

for $n \in \mathbb{N}$ and $t \in \mathbb{R}_0^+$. ∎

Proof. For a definition of the infinitesimal generator and the proof of the proposition above please refer to the Appendix A.

3.2 Value of The Firm

Let $\tilde{\Pi} : \mathbb{N} \to \mathbb{R}$,with $\tilde{\Pi}(n)$ being the profit flow function for a given technology level n. The value of a Firm that uses the technology at level n and operates forever in this level of technology is the integral of the discounted profit flow over time,

$$R(n) = \int_0^{+\infty} e^{-rt}\tilde{\Pi}(n)\mathrm{dt} = \frac{\tilde{\Pi}(n)}{r} \tag{3.8}$$

where $r > 0$ is a given discount rate, assumed here to be constant in time.

Assuming that the Firm uses a given technology m until the moment ξ, when it switches to a technology level n (and operates in this level forever), then its value will be,

$$R_m(n,\xi) = \int_0^{\xi} e^{-rs}\tilde{\Pi}(m)ds + \int_{\xi}^{+\infty} e^{-rs}\tilde{\Pi}(n)ds - e^{-r\xi}I. \tag{3.9}$$

Here I is the investment cost, assumed to be constant (independent of m, n, ξ). Computing the integral we easily find,

$$R_m(n,\xi) = \frac{\tilde{\Pi}(m)}{r} + \frac{e^{-r\xi}}{r}\Big(\tilde{\Pi}(n) - \tilde{\Pi}(m) - rI\Big). \tag{3.10}$$

Therefore, $R_m(n,\xi)$ is the profit flow that comes from using the technology level m forever plus the discounted difference in the profit flow between using level n and m (taking into account the investment cost, also properly discounted back in time).

3.3 Our Optimization Problem

One of the objectives of the Firm is to derive when it is optimal to switch from its initial technology level m to a new technology level.

Let $\mathcal{T}$ denote the set of all stopping times adapted to the filtration generated by $(N_t,\theta_t)_{t\geq 0}$. If the Firm switches to the best available technology at time $\tau \in \mathcal{T}$, then its value is a random variable hereby denoted by $R_m(N_\tau,\tau)$.

The assumption $\tau \in \mathcal{T}$ means that the investment decision is taken only on the basis of the observation of the process (N_t,θ_t). Since this process is Markovian, only the current value of N_t and θ_t need to be taken into account.

In this framework, the optimal strategy (investment time) when the current state of the technology process is (n,t) is any stopping time $\gamma \in \mathcal{T}$ such that,

$$\mathbb{E}\big[R_m(N_\gamma,\gamma)\big|N_0 = n, \theta_0 = t\big] = \sup_{\tau\in\mathcal{T}} \mathbb{E}\big[R_m(N_\tau,\tau)\big|N_0 = n, \theta_0 = t\big] \tag{3.11}$$

Let us also define,

$$\Pi_m(n) := \tilde{\Pi}(n) - \tilde{\Pi}(m) - rI, \tag{3.12}$$

and,

$$V_m(n,t) := \sup_{\tau \in \mathcal{T}} \mathbb{E}\big[e^{-r\tau}\Pi_m^+(N_\tau)\big|N_0 = n, \theta_0 = t\big] \tag{3.13}$$

where $\Pi_m^+(n) = \max(\Pi_m(n), 0)$. Using (3.10) it is immediate that,[2]

$$\mathbb{E}_{n,t}\big[R_m(N_\tau, \tau)\big] = \frac{\tilde{\Pi}(m)}{r} + \frac{V_m(N_\tau, t)}{r}. \tag{3.14}$$

Therefore, we will consider the following problem.

Problem (The Optimal Investment Strategy)**.** *Find optimal a stopping time* $\tau^* \in \mathcal{T}$ *maximising the functional,*

$$\tau \mapsto \mathbb{E}_{n,t}\big[e^{-r\tau}\Pi_m^+(N_\tau)\big] \qquad \tau \in \mathcal{T} \tag{3.15}$$

which, in view of (3.14)*, is equivalent to find the optimal value of* $\mathbb{E}_{n,t}\big[R_m(N_\tau, \tau)\big]$. ■

To solve this problem we will use a dynamic programming approach. So we start by proving the dynamic programming principle, and after we find the Hamilton-Jacobi-Bellman equation[3]. Finally we solve this equation to find both the optimal time corresponding to the optimal value of the Firm.

3.3.1 Dynamic Programming Principle

We now state and prove the dynamic programming principle.

Theorem 1 (Dynamic Programming Principle)**.** *For any stopping time* $\tilde{\tau} \in \mathcal{T}$ *we have,*

$$V_m(n,t) = \sup_{\tau \in \mathcal{T}} \mathbb{E}_{n,t}\left[e^{-r\tau}\Pi_m^+(N_\tau)\chi_{\tau<\tilde{\tau}} + e^{-r\tilde{\tau}}V_m(N_{\tilde{\tau}}, \theta_{\tilde{\tau}})\chi_{\tau\geq\tilde{\tau}}\right]. \tag{3.16}$$

where χ_A *denotes the indicator function of the proposition* A. ■

[2]From here onwards we shall denote the conditional expectation $\mathbb{E}[X|Y = y, Z = z]$ by $\mathbb{E}_{y,z}[X]$.
[3]HJB equation, for short

Proof. If we define the functional,

$$J_m(n,t,\tau) := \mathbb{E}_{n,t}\Big[e^{-r\tau}\Pi_m(N_\tau)\Big] \tag{3.17}$$

then,

$$V_m(n,t) = \sup_{\tau\in\mathcal{T}} J_m(n,t,\tau). \tag{3.18}$$

We then define

$$\tilde{V}_m(n,t) := \sup_{\tau\in\mathcal{T}} \mathbb{E}_{n,t}\Big[e^{-r\tau}\Pi_m^+(N_\tau)\chi_{\tau<\tilde{\tau}} + e^{-r\tilde{\tau}}V(N_{\tilde{\tau}},\theta_{\tilde{\tau}})\chi_{\tau\geq\tilde{\tau}}\Big] \tag{3.19}$$

in order to prove that $V_m(n,t) = \tilde{V}_m(n,t)$, we prove that

$$\text{(i)}V_m(n,t) \geq \tilde{V}_m(n,t) \quad \text{and}$$
$$\text{(ii)}V_m(n,t) \leq \tilde{V}_m(n,t).$$

(i) Proving that $V_m(n,t) \geq \tilde{V}_m(n,t)$:

For $\epsilon > 0$, we start by choosing the following stopping time,

$$\hat{\tau} = \tau\chi_{\tau<\tilde{\tau}} + \big(\tilde{\tau} + \tau_\epsilon\big)\chi_{\tau\geq\tilde{\tau}}, \tag{3.20}$$

where $\tau_\epsilon := \tau_\epsilon(N_{\tilde{\tau}},\theta_{\tilde{\tau}})$, such that,

$$J_m\big(N_{\tilde{\tau}},\theta_{\tilde{\tau}},\tau_\epsilon\big) \geq V_m(N_{\tilde{\tau}},\theta_{\tilde{\tau}}) - \epsilon. \tag{3.21}$$

Then, it follows that,

$$V(n,t) \geq J(n,t,\hat{\tau}) = \mathbb{E}_{n,t}\Big[e^{-r\hat{\tau}}\Pi_m^+(N_{\hat{\tau}})\Big] \tag{3.22}$$

$$= \mathbb{E}_{n,t}\Big[e^{-r\tau}\Pi_m^+(N_\tau)\chi_{\tau<\tilde{\tau}} + e^{-r(\tilde{\tau}+\tau_\epsilon)}\Pi_m^+(N_{\tilde{\tau}+\tau_\epsilon})\chi_{\tau\geq\tilde{\tau}}\Big] \tag{3.23}$$

$$= \mathbb{E}_{n,t}\Big[e^{-r\tau}\Pi_m^+(N_\tau)\chi_{\tau<\tilde{\tau}} + e^{-r\tilde{\tau}}\mathbb{E}_{N_{\tilde{\tau}},\theta_{\tilde{\tau}}}\big[e^{-r\tau_\epsilon}\Pi_m^+(N_{\tilde{\tau}+\tau_\epsilon})\big]\chi_{\tau\geq\tilde{\tau}}\Big] \tag{3.24}$$

$$= \mathbb{E}_{n,t}\Big[e^{-r\tau}\Pi_m^+(N_\tau)\chi_{\tau<\tilde{\tau}} + e^{-r\tilde{\tau}}J_m(N_{\tilde{\tau}},\theta_{\tilde{\tau}},\tau_\epsilon)\chi_{\tau\geq\tilde{\tau}}\Big] \tag{3.25}$$

$$\geq \mathbb{E}_{n,t}\Big[e^{-r\tau}\Pi_m^+(N_\tau)\chi_{\tau<\tilde{\tau}} + e^{-r\tilde{\tau}}\Big(V_m(N_{\tilde{\tau}},\theta_{\tilde{\tau}}) - \epsilon\Big)\chi_{\tau\geq\tilde{\tau}}\Big] \tag{3.26}$$

$$\geq \underbrace{\mathbb{E}_{n,t}\Big[e^{-r\tau}\Pi_m^+(N_\tau)\chi_{\tau<\tilde{\tau}} + e^{-r\tilde{\tau}}V_m(N_{\tilde{\tau}},\theta_{\tilde{\tau}})\chi_{\tau\geq\tilde{\tau}}\Big]}_{=\tilde{V}_m(n,t)} - \epsilon\underbrace{\mathbb{E}_{n,t}\Big[e^{-r\tilde{\tau}}\chi_{\tau\geq\tilde{\tau}}\Big]}_{\leq 1} \tag{3.27}$$

$$\geq \tilde{V}_m(n,t) - \epsilon \Leftrightarrow \boxed{V_m(n,t) \geq \tilde{V}_m(n,t) - \epsilon} \tag{3.28}$$

where in (3.24) and (3.25) we use the definition of $J_m(n,t,\tau)$, and in (3.26) we use (3.20). Making $\epsilon \to 0$ we obtain (i).

(ii) Proving that $V_m(n,t) \leq \tilde{V}_m(n,t)$:

We note that from the definition of $V_m(n,t)$ as a supremum, it follows that for any stopping time ν,

$$J_m(N_{\tilde{\tau}},\theta_{\tilde{\tau}},\nu) \leq V_m(N_{\tilde{\tau}},\theta_{\tilde{\tau}}). \tag{3.29}$$

Directly from the definition of $V_m(n,t)$ and using (3.29)

$$\tilde{V}_m(n,t) = \sup_{\tau\in\mathcal{T}}\mathbb{E}_{n,t}\Big[e^{-r\tau}\Pi_m^+(N_\tau)\chi_{\tau<\tilde{\tau}} + e^{-r\tilde{\tau}}V_m(N_{\tilde{\tau}},\theta_{\tilde{\tau}})\chi_{\tau\geq\tilde{\tau}}\Big] \geq \tag{3.30}$$

$$\geq \sup_{\tau\in\mathcal{T}}\mathbb{E}_{n,t}\Big[e^{-r\tau}\Pi_m^+(N_\tau)\chi_{\tau<\tilde{\tau}} + e^{-r\tilde{\tau}}J_m(N_{\tilde{\tau}},\theta_{\tilde{\tau}},\tau-\tilde{\tau})\chi_{\tau\geq\tilde{\tau}}\Big] \tag{3.31}$$

We note that due to the Tower property we have,

$$\mathbb{E}_{n,t}\Big[\mathbb{E}_{N_{\tilde{\tau}},\theta_{\tilde{\tau}}}\big[e^{-r(\tau-\tilde{\tau})}\Pi_m^+(N_\tau)\big]\Big] = \mathbb{E}_{n,t}\big[e^{-r(\tau-\tilde{\tau})}\Pi_m^+(N_\tau)\big] \tag{3.32}$$

Now simply using the definition of $J_m(n,t,\tau)$ and (3.32), it follows that,

$$\tilde{V}_m(n,t) \geq \sup_{\tau\in\mathcal{T}} \mathbb{E}_{n,t}\Big[e^{-r\tau}\Pi_m^+(N_\tau)\chi_{\tau<\tilde{\tau}} + e^{-r\tilde{\tau}}\mathbb{E}_{N_{\tilde{\tau}},\theta_{\tilde{\tau}}}\big[e^{-r(\tau-\tilde{\tau})}\Pi_m^+(N_\tau)\big]\chi_{\tau\geq\tilde{\tau}}\Big] = \tag{3.33}$$

$$= \sup_{\tau\in\mathcal{T}} \mathbb{E}_{n,t}\Big[e^{-r\tau}\Pi_m^+(N_\tau)\chi_{\tau<\tilde{\tau}} + e^{-r\tilde{\tau}}e^{-r(\tau-\tilde{\tau})}\Pi_m^+(N_\tau)\chi_{\tau\geq\tilde{\tau}}\Big] \tag{3.34}$$

$$= \sup_{\tau\in\mathcal{T}} \mathbb{E}_{n,t}\Big[e^{-r\tau}\Pi_m^+(N_\tau)\Big] = V_m(n,t) \tag{3.35}$$

$$\Leftrightarrow \boxed{\tilde{V}_m(n,t) \geq V_m(n,t)} \tag{3.36}$$

Therefore (3.28) and (3.36) imply that that $\tilde{V}_m(n,t) = V_m(n,t)$ and thus:

$$V_m(n,t) = \sup_{\tau\in\mathcal{T}} \mathbb{E}_{n,t}\Bigg[e^{-r\tau}\Pi_m^+(N_\tau)\chi_{\tau<\tilde{\tau}} + e^{-r\tilde{\tau}}V_m(N_{\tilde{\tau}},\theta_{\tilde{\tau}})\chi_{\tau\geq\tilde{\tau}}\Bigg]. \tag{3.37}$$

□

3.3.2 HJB Equation

Let us now state and prove the Hamilton-Jacobi-Bellman equation for our problem.

Proposition 2. *If $V_m(n,t) \in \mathcal{X}$ then,*

$$\max(\Pi_m^+(n) - V_m(n,t), \mathcal{A}V_m(n,t) - rV_m(n,t)) = 0. \tag{3.38}$$

for $m, n \in \mathbb{N}_0$ and $t \geq 0$. ■

Proof. From the definition of $V_m(n,t)$ as a supremum,

$$V_m(n,t) \geq E_{n,t}[e^{-r\tau}\Pi^+(N_\tau)], \qquad \tau \in \mathcal{T}. \tag{3.39}$$

For $\tau = 0$ this inequality reduces to,

$$\boxed{\Pi_m^+(n) - V_m(n,t) \leq 0.} \tag{3.40}$$

Using the dynamic programming principle with $\tilde{\tau} \equiv h \to 0^+$ then,

$$0 = -V_m(n,t) + \sup_{\tau \in \mathcal{T}} \mathbb{E}_{n,t}\Big[e^{-r\tau}\Pi_m^+(N_\tau)\chi_{\tau<h} + e^{-rh}V_m(N_h,\theta_h)\chi_{\tau\geq h}\Big]. \tag{3.41}$$

By the definition of $V_m(n,t)$,

$$V_m(n,t) \geq \sup_{\tau} J(n,\tau,\tau \vee h) = e^{-rh}\mathbb{E}_{n,t}\big[V(N_h,\theta_h)\big] \tag{3.42}$$

i.e.,

$$0 \geq -V_m(n,t) + e^{-rh}\mathbb{E}_{n,t}[V_m(N_h,\theta_h)]. \tag{3.43}$$

We can expand the exponential function in the equation above and note that,

$$\left(\sum_{i=2}^{+\infty}\frac{(-rh)^i}{i!}\right)\mathbb{E}_{n,t}[V_m(N_h,\theta_h)] \to 0 \quad \text{when} \quad h \to 0. \tag{3.44}$$

Therefore, neglecting all terms of order higher than h, (3.43) becomes

$$0 \geq -V_m(n,t) + (1-rh)\mathbb{E}_{n,t}[V_m(N_h,\theta_h)]$$

dividing all terms by h we find,

$$0 \geq \frac{\mathbb{E}_{n,t}[V_m(N_h,\theta_h)] - V_m(n,t)}{h} - r\mathbb{E}_{n,t}[V_m(N_h,\theta_h)] \tag{3.45}$$

$$= \mathcal{A}V_m(n,t) + \frac{o(h)}{h} - r(V_m(n,t) + \mathcal{A}V_m(n,t)h + o(h)). \tag{3.46}$$

Letting $h \to 0$ we end up with

$$\mathcal{A}V_m(n,t) - rV_m(n,t) \leq 0. \tag{3.47}$$

All is left to prove is that at least one of the statements has to be zero. Let us now assume that we have both,

$$0 > -V_m(n,t) + \Pi_m^+(n), \tag{3.48}$$

$$\text{and} \qquad 0 > -V_m(n,t) + \frac{1}{r}\mathcal{A}V_m(n,t). \tag{3.49}$$

Let,

$$\hat{\tau}_h = \inf\{s : N_s > n \vee s \geq h\} \tag{3.50}$$

We use the functional under the dynamic programming principle with $\hat{\tau}$,

$$\mathbb{E}_{n,t}\left[e^{-r\tau}\Pi^+(N_\tau)\chi_{\tau<\hat{\tau}} + e^{-r\hat{\tau}}V(N_{\hat{\tau}},\theta_{\hat{\tau}})\chi_{\tau\geq\hat{\tau}}\right] = \tag{3.51}$$

$$=\mathbb{E}_{n,t}\left[e^{-r\tau}(V(n,t+\tau)-\delta)\chi_{\tau<\hat{\tau}} + e^{-r\hat{\tau}}V(N_{\hat{\tau}},\theta_{\hat{\tau}})\chi_{\tau\geq\hat{\tau}}\right] = \tag{3.52}$$

$$=\mathbb{E}_{n,t}[\left(e^{-r\hat{\tau}}V(n,t+\tau)+(e^{-r\tau}-e^{-r\hat{\tau}})V(n,t+\tau)-e^{-r\tau}\delta\right)\chi_{\tau<\hat{\tau}}+ \tag{3.53}$$

$$e^{-r\hat{\tau}}V(N_{\hat{\tau}},\theta_{\hat{\tau}})\chi_{\tau\geq\hat{\tau}}] \leq$$

$$\leq\mathbb{E}_{n,t}[e^{-r\hat{\tau}}V(N_{\hat{\tau}},\theta_{\hat{\tau}})] + \mathbb{E}_{n,t}[((1-e^{-rh})V(n,t+\tau)-e^{-rh}\delta)\chi_{\tau<\hat{\tau}}] \tag{3.54}$$

$$\leq\mathbb{E}_{n,t}[e^{-r\hat{\tau}}V(N_{\hat{\tau}},\theta_{\hat{\tau}})] = \tag{3.55}$$

$$=\mathbb{E}_{n,t}[e^{-rh}V(N_h,\theta_h)\chi_{N_h=n} + e^{-r(T_1-t)}V(n+1,0)\chi_{N_h>n}] \tag{3.56}$$

$$=\mathbb{E}_{n,t}[e^{-rh}V(N_h,\theta_h)] + \mathbb{E}_{n,t}[(e^{-r(T_1-t)}V(n+1,0)-e^{-rh}V(N_h,\theta_h)\chi_{N_h>n}] \tag{3.57}$$

$$=\mathbb{E}_{n,t}[e^{-rh}V(N_h,\theta_h)] + o(h) \tag{3.58}$$

$$=V_m(n,t)+(\mathcal{A}V_m(n,t)-rV_m(n,t))h+o(h) < V_m(n,t) \qquad \text{for } h \text{ small enough} \tag{3.59}$$

Since,

$$V_m(n,t) = \sup_{\tau\in\mathcal{T}}\mathbb{E}_{n,t}\left[e^{-r\tau}\Pi_m^+(N_\tau)\chi_{\tau<h} + e^{-rh}V_m(N_h,\theta_h)\chi_{\tau\geq h}\right] \tag{3.60}$$

From Eq. (3.51)-(3.59)

$$V_m(n,t) < V_m(n,t) \tag{3.61}$$

which is clearly a contradiction. So, at least one of the inequalities (3.48 and 3.49) has to be a non-strict inequality.

Therefore, we have proven that,

$$\max\left(\Pi_m(n)-V_m(n,t),\mathcal{A}V_m(n,t)-rV_m(n,t)\right) = 0. \tag{3.62}$$

□

Reading the HJB equation above essentially tells us that the optimal value function $V(n,t)$ has to be either equal or greater than $\Pi^{+}(n)$ and if it is strictly greater than $\Pi^{+}(n)$ then $V(n,t)$ is the solution of

$$\lambda(t)\Big(v(n+1,0)-v(n,t)\Big)+v'(n,t)-rv(n,t)\Big)=0. \tag{3.63}$$

Chapter 4

Results

In this chapter we start by showing that the HJB derived in the previous chapter reduces to a free boundary condition problem. We then proceed to solve a modified version of the problem where boundary conditions are imposed.

Furthermore, we particularise the general solution of the Hamilton Jacobi Bellman equation for two different cases, one with a constant intensity $\lambda(t) = \lambda$, and the other with intensity $\lambda(t) = \frac{t^2}{1-t^3}$.

4.1 A First Approach to Solving the HJB Equation

Let us recall the previously obtained HJB equation

$$\max\left(\Pi_m^+(n) - V_m(n,t), \mathcal{A}V_m(n,t) - rV(n,t)\right) = 0. \tag{4.1}$$

We see that the possible solution is,

$$V_m(n,t) = \Pi_m^+(n). \tag{4.2}$$

which implies, since $V_m(n,t)$ is constant in t,

$$0 \leq \big(r + \lambda(x)\big)V_m(n,t) - \lambda(x)V_m(n+1,0) \tag{4.3}$$

$$\Leftrightarrow \Pi^+(n) \leq \frac{\lambda(x)}{r+\lambda(x)}V_m(n+1,0) \tag{4.4}$$

Regarding the second term of eq.(4.1), we have the following:

$$\begin{aligned}
\mathcal{A}V_m(n,t) - rV_m(n,t) \leq 0 \Leftrightarrow \\
\lambda(t)\bigg(V_m(n+1,0) - V_m(n,t)\bigg) + \frac{\partial V_m(n,t)}{\partial t} - rV_m(n,t) \leq 0 \Leftrightarrow \\
\lambda(t)V_m(n+1,0) + \frac{\partial}{\partial t}V_m(n,t) - (r+\lambda(t))V_m(n,t) \leq 0 \Leftrightarrow \\
\frac{\partial}{\partial t}V_m(n,t) \leq \Big(r+\lambda(t)\Big)V_m(n,t) - \lambda(t)V_m(n+1,0)
\end{aligned}$$

Therefore all the solutions of the following differential equation

$$\frac{\partial}{\partial t}V_m(n,t) = \big(r+\lambda(t)\big)V_m(n,t) - \lambda(t)V_m(n+1,0) \tag{4.5}$$

will also be a solution for the HJB as long as

$$V_m(n,t) \geq \Pi_m(n)^+. \tag{4.6}$$

Therefore, eq.(4.1) could have an infinity number of solutions, given accordingly to the following general form:

$$V_m(n,t) = e^{\int_0^t \lambda(s)+r\mathrm{ds}}\left(c_1 + V_m(n+1,0)\int_0^t \lambda(s)e^{-\int_1^s r+\lambda(y)\mathrm{dy}}\mathrm{d}z\right), \tag{4.7}$$

where c_1 and c_2 depend on boundary conditions, inexistent in this case. Thus, in view of the infinite number of solutions of eq.(4.1) we need to propose an alternative method to solve, in an unique way, eq.(4.1).

4.2 Modified Problems

To overcome the lack of boundary conditions for eq.(4.1) we will consider the following modified problems.

For $k \in \mathbb{N}$, define the following function,

$$\Pi_m^k(n) := \begin{cases} \Pi_m^+(k) & n \geq k \\ \Pi_m^+(n) & n < k. \end{cases}$$

Thus $\Pi_m^k(n)$ denote the profit flow such that it stays constant once the efficiency level reaches level k. For this new profit function, we define the following optimal value function,

$$V_m^k(n,t) := \sup_{\tau \in \mathcal{T}} \mathbb{E}_{n,t}[e^{-r\tau}\Pi_m^k(N_\tau)]. \tag{4.8}$$

One immediately sees that, for $n > k$

$$\tau^* = \arg\max_{\tau \in \mathcal{T}} \mathbb{E}_{n,t}[e^{-r\tau}\Pi_m^k(N_\tau)] = \begin{cases} \infty & \text{if} \quad \Pi_m(k) \leq 0 \\ 0 & \text{if} \quad \Pi_m(k) > 0 \end{cases} \tag{4.9}$$

and,

$$V_m^k(n,t) = \Pi_m^+(k) \qquad \text{for } n \geq k \tag{4.10}$$

Equation (4.9) essentially means that upon achieving the evolution threshold k, we either invest immediately ($\tau^* = 0$) or we don't invest ever at all ($\tau^* = \infty$) (a now or never decision).

Furthermore, we also define the following intensity rate,

$$\lambda^T(t) := \begin{cases} 0 & \text{for} \quad t \geq T \\ \lambda(t) & \text{for} \quad t < T, \end{cases}$$

which corresponds to the case where no further innovation will ever come forth if the time elapsed since the last innovation exceeds T. We define $\{(N_t^T, \theta_t^T), t \geq 0\}$ the process analogous to (N_t, θ_t) but with intensity $\{\lambda^T(t), t \geq 0\}$ instead of $\{\lambda(t), t \geq 0\}$.

Again, this modified problem will yield a new optimal value function,

$$V_m^{k,T}(n,t) := \sup_{\tau \in \mathcal{T}} \mathbb{E}_{n,t}\big[e^{-r\tau}\Pi_m^k(N_\tau^T)\big]. \tag{4.11}$$

In a similar way as with V^k, we have that for $t > T$,

$$\tau^* = \arg\max_{\tau \in \mathcal{T}} \mathbb{E}_{n,t}[e^{-r\tau}\Pi_m^k(N_\tau^T)] \Rightarrow \begin{cases} \infty & \text{if } \Pi_m(k) \leq 0 \\ 0 & \text{if } \Pi_m(k) > 0 \end{cases} \tag{4.12}$$

with,

$$V_m^{k,T}(n,t) = \Pi_m^+(k) \qquad \text{for } t \geq T \tag{4.13}$$

which is also interpreted as a *now or never* investment decision. This is quite understandable since in this region ($t > T$) we won't have any more innovations so that waiting is merely losing in terms of cost of opportunity.

The following relationship is quite obvious,

$$V_m^{k,T}(n,t) \leq V_m^k(n,t) \leq V_m(n,t). \tag{4.14}$$

The following Proposition will enable us to study our initial problem as a limit case of the latter two.

Proposition 3. *For the optimal value functions* $V_m^{k,T}(\cdot,\cdot)$*,* $V_m^k(\cdot,\cdot)$*,* $V_m(\cdot,\cdot)$ *defined as above, the following holds*

$$\lim_{T+\infty} V_m^{k,T}(n,t) = V_m^k(n,t) \qquad \forall (n,t) \in \mathbb{N}\times\mathbb{R}_0^+. \tag{4.15}$$

$$\lim_{k\to+\infty} V_m^k(n,t) = V_m(n,t) \qquad \forall (n,t) \in \mathbb{N}\times\mathbb{R}_0^+. \tag{4.16}$$

■

Proof. It comes directly from $\forall \tau \in \mathcal{T}$,

$$\mathbb{E}_{n,t}\Big[e^{-r\tau}\Pi_m^k(N_\tau^T)\Big] \nearrow \mathbb{E}_{n,t}\Big[e^{-r\tau}\Pi_m^k(N_\tau)\Big] \nearrow \mathbb{E}_{n,t}\Big[e^{-r\tau}\Pi_m^+(N_\tau)\Big]. \tag{4.17}$$

□

Now we are in position to state the HJB equation for this combination of problems (with frozen profit function in level k and frozen intensity rate at T):

$$\max\Big(\Pi_m^+(n) - V_m^{k,T}(n,t), \lambda\big(V_m^{k,T}(n+1,0) - V_m^{k,T}(n,t)\big) + \frac{\partial}{\partial t}V_m^{k,T}(n,t) - rV_m^{k,T}(n,t)\Big) = 0 \tag{4.18}$$

with boundary conditions,

$$V_m^{k,T}(n,t) = \Pi_m^+(n) \qquad \forall t \geq T \quad \wedge \quad \forall n < k \tag{4.19}$$

$$V_m^{k,t}(n,t) = \Pi_m^+(k) \qquad \forall n \geq k. \tag{4.20}$$

We see that we can solve the initial problem of finding $V_m(n,t)$ by solving the modified problems and then taking limits, i.e. $V_m(n,t) = \lim_{k\to+\infty}\big(\lim_{T\to+\infty} V_m^{k,T}(n,t)\big)$

4.3 Case 1: Constant λ

Let us now particularise in the simplest case possible, the one where $\lambda(\cdot)$ is constant,i.e. $\lambda(t) = \lambda$.

The HJB for this case is then eq. (4.18) with $\lambda(t) = \lambda$,

$$\max\Big(\Pi_m^+(n) - V_m^{k,T}(n,t), \lambda\big(V_m^{k,T}(n+1,0) - V_m^{k,T}(n,t)\big) + \frac{\partial}{\partial t}V_m^{k,T}(n,t) - rV_m^{k,T}(n,t)\Big) = 0. \tag{4.21}$$

Proposition 4 (HJB Solution for a constant λ)**.** *The solution of the eq.* (4.21) *is,*

$$V_m^{k,T}(n,t) = \begin{cases} \Pi_m^+(n) & \textit{if} \quad \frac{\lambda}{r+\lambda}V_m^{k,T}(n+1,0) < \Pi_m^+(n) \\ e^{(r+\lambda)(t-T)}\left(\Pi_m^+(n) - \frac{\lambda}{r+\lambda}V_m^{k,T}(n+1,0)\right) + \frac{\lambda}{r+\lambda}V_m^{k,T}(n+1,0) & \textit{if} \quad \frac{\lambda}{r+\lambda}V_m^{k,T}(n+1,0) \geq \Pi_m^+(n) \end{cases} \tag{4.22}$$

yielding, with $k, T \to \infty$,

$$V_m^{\infty,\infty}(n,t) = \sup_{i\geq 0}\left\{\left(\frac{\lambda}{\lambda+r}\right)^i \Pi^+(n+i)\right\}. \tag{4.23}$$

■

Proof. To simplify the notation, from here onwards and when no confusion arises, we shall drop the subscript $_m$ from $V_m^{k,T}(\cdot)$ and $\Pi^+(n)$ and the superscript ^{k,T} from $V_m^{k,T}(\cdot)$.

We start by noting that one of two things must happen,

$$\text{either} \quad \frac{\lambda}{r+\lambda}V(n+1,0) \leq \Pi^+(n) \tag{4.24}$$

$$\text{or} \quad \frac{\lambda}{r+\lambda}V(n+1,0) > \Pi^+(n). \tag{4.25}$$

We will first prove that if (4.24) holds then the unique solution of the HJB equation is $V(n,t) = \Pi^+(n)$. Since at the boundary T, $V(n,T) = \Pi^+(n)$, we look for the last moment where $V(t,n)$ is different from $\Pi^+(n)$, let then T_1 be such moment, with:

$$T_1 = \begin{cases} \sup\{t : V(n,t) > \Pi^+(n)\} & \text{if} \quad \exists t \in [0,T] : V(n,t) > \Pi^+(n) \\ 0 & \text{if} \quad \forall t \in [0,T] : V(n,t) \leq \Pi^+(n) \end{cases} \tag{4.26}$$

If $T_1 > 0$ then there must be a point

$$\hat{t} \in]0, T_1[\quad \text{such that} \quad \left.\frac{\partial V(n,t)}{\partial t}\right|_{t=\hat{t}} < 0 \quad \text{and} \tag{4.27}$$

by the HJB,

$$\frac{\partial}{\partial t} V(n,\hat{t}) = (r+\lambda)V(n,\hat{t}) - \lambda V(n+1,0) \tag{4.28}$$

$$\geq (r+\lambda)\Pi^+(n) - \lambda V(n+1,0) \geq 0. \tag{4.29}$$

This yields $0 > 0$ and therefore we conclude that $T_1 = 0$.

On the other hand, if (4.25) holds, we will prove that the unique solution to the HJB is the solution to the following ODE,

$$\frac{\partial}{\partial t} V(n,t) = (\lambda + r)V(n,t) - \lambda V(n+1,0) \qquad V(n,T) = \Pi^+(n). \tag{4.30}$$

We now assume that $T_1 < T$, therefore,

$$\left.\frac{\partial}{\partial t} V(n,t)\right|_{t=\rho} = 0 \qquad a.e. \quad \rho \in]T_1, T[. \tag{4.31}$$

From the HJB it comes directly, [1]

$$\lambda V(n+1,0) - (r+\lambda)\Pi^+(n) \leq 0 \qquad \forall \rho \in]T_1, T[. \tag{4.32}$$

[1] Since $\left.\frac{\partial}{\partial t} V(n,t)\right|_{t=\rho} = 0$ and $V(n,\rho) = \Pi^+(n), \qquad \forall \rho \in]T_1, T[$.

This contradicts our assumption (4.25) and implies $T_1 = T$. Therefore,

$$\frac{\lambda}{r+\lambda}V(n+1,0) > \Pi^+(n) \tag{4.33}$$

$$\Rightarrow \frac{\partial}{\partial t}V(n,t) = (\lambda+r)V(n,t) - \lambda V(n+1,0), \quad \text{with} \quad V(n,T) = \Pi^+(n). \tag{4.34}$$

Solving the ODE (4.34) we obtain Eq.(4.22).

Let time $T \to \infty$ we conclude finally that,

$$\begin{aligned} V(n,t) &= \begin{cases} \Pi^+(n) & \text{if} \quad \frac{\lambda}{r+\lambda}V(n+1,0) < \Pi^+(n) \\ \frac{\lambda}{r+\lambda}V(n+1,0) & \text{if} \quad \frac{\lambda}{r+\lambda}V(n+1,0) \geq \Pi^+(n) \end{cases} \\ &= \max\left\{\Pi^+(n), \frac{\lambda}{\lambda+r}V(n+1,0)\right\} \end{aligned} \tag{4.35}$$

Equation (4.35) can be solved recursively using the boundary condition,

$$V(k,t) = \Pi^+(k).$$

Therefore the following recursive procedure holds,

$$n = k-1: \qquad V(k-1,t) = \max\left\{\Pi^+(k-1), \frac{\lambda}{\lambda+r}\Pi^+(k)\right\}$$

$$\begin{aligned} n = k-2: \qquad V(k-2,t) &= \max\left\{\Pi^+(k-2), \frac{\lambda}{\lambda+r}\max\left\{\Pi^+(k-1), \frac{\lambda}{\lambda+r}\Pi^+(k)\right\}\right\} \\ &= \max\left\{\Pi^+(k-2), \frac{\lambda}{\lambda+r}\Pi^+(k-1), \left(\frac{\lambda}{\lambda+r}\right)^2\Pi^+(k)\right\} \end{aligned}$$

$$\begin{aligned} n = k-p: \qquad V(k-p,t) &= \max\left\{\Pi^+(k-p), \cdots, \left(\frac{\lambda}{\lambda+r}\right)^p\Pi^+(k)\right\} \\ &= \max_{j=0,\cdots,p}\left\{\left(\frac{\lambda}{\lambda+r}\right)^j\Pi^+(k-p+j)\right\} \end{aligned}$$

i.e,

$$V(n,t) = \max_{j=0,\cdots,k-n} \left\{ \left(\frac{\lambda}{\lambda + r} \right)^j \Pi^+(n+j) \right\}$$

Finally, letting $k \to \infty$, we end up with equation (4.23)

$$V(n,t) = \sup_{i \geq 0} \left\{ \left(\frac{\lambda}{\lambda + r} \right)^i \Pi^+(n+i) \right\}$$

□

4.4 Case 2: $\{\lambda(t) = \frac{t^2}{1+t^3}, t \geq 0\}$

Proposition 5. *The solution of the HJB of the modified problem, with* $\lambda(t) = \frac{t^2}{1+t^3}$*, for a given* n *is*

$$V^{k,T}(n,t) = \max\left(\Pi^+(n), v(n,t)\right)$$

with,

$$v(n,t) = (t^3+1)^{1/3} e^{rt} \left(\frac{e^{-rt_2}\Pi^+(n)}{t_2^{1/3}} + V(n+1,0) \int_t^{t_2} \frac{e^{-rx} x^2}{(1+x^3)^{4/3}} dx \right) \tag{4.36}$$

and

$$t_2 = \sup\{t \in [0,T] : p(t) > 0\} \tag{4.37}$$

where,

$$p(t) = -(t^2 + rt^3 + r)\Pi^+(n) + t^2 V(n+1,0) \tag{4.38}$$

■

Proof. We recall our HJB Equation,

$$\max\Big(\Pi^+(n) - V(n,t), \lambda(t)\big(V(n+1,0) - V(n,t)\big) + \frac{\partial}{\partial t}V(n,t) - rV(n,t)\Big) = 0. \tag{4.39}$$

If we have $V(n,t)$ constant, then we must have,[2]

$$\lambda(t)\big(V(n+1,0) - V(n,t)\big) - rV(n,t) \leq 0 \Leftrightarrow \tag{4.40}$$

$$\Leftrightarrow -(t^2 + rt^3 + r)\Pi^+(n) + t^2V(n+1,0) \leq 0 \tag{4.41}$$

For a given n the expression above is a third degree polynomial of t which we'll denote by $p(t)$. Therefore, the expression above becomes,

$$p(t) = -(t^2 + rt^3 + r)\Pi^+(n) + t^2V(n+1,0) \leq 0 \tag{4.42}$$

We will now study $p(t)$ to find the region of values for which t yields a $p(t)$ satisfying the inequality above. We first consider the case when $\Pi^+(n) = 0$, then one sees that,

$$p(t) \leq 0 \Leftrightarrow \begin{cases} t \in [0, +\infty) & \text{if} \quad V(n+1,0) = 0 \\ t = 0 & \text{if} \quad V(n+1,0) > 0. \end{cases} \tag{4.43}$$

We now consider the remaining cases where $\Pi^+(n) > 0$ and $V(n+1,0) \geq 0$. We immediately see that

$$\lim_{t\to+\infty} p(t) = -\infty \qquad \lim_{t\to-\infty} p(t) = +\infty. \tag{4.44}$$

And taking the derivative,

$$p'(t) = -(2t + r3t^2)\Pi^+(n) + 2tV(n+1,0) \tag{4.45}$$

[2]Recall that $V(n,T) = \Pi^+(n)$ therefore if $V(n,t)$ is constant it must cannot be different from $\Pi^+(n)$

we'll find the critical points to be

$$p'(t) = 0 \Rightarrow \begin{cases} t = 0 \vee t = \frac{2}{3r}\frac{V(n+1,0)-\Pi^+(n)}{\Pi^+(n)} < 0 & \text{if} \quad V(n+1,0) < \Pi^+(n) \\ t = 0 & \text{if} \quad V(n+1,0) = \Pi^+(n) \\ t = 0 \vee t = \frac{2}{3r}\frac{V(n+1,0)-\Pi^+(n)}{\Pi^+(n)} > 0 & \text{if} \quad V(n+1,0) > \Pi^+(n). \end{cases} \tag{4.46}$$

Since at the origin,

$$p(0) = -r\Pi^+(n) < 0 \tag{4.47}$$

Combining Eqs.(4.44) and (4.46) we'll find that the regions for which t satisfies Eq.(4.42) are[3],

$$p(t) \leq 0 \Leftrightarrow \begin{cases} t \in [0, +\infty) & \text{if} \quad V(n+1,0) \leq \Pi^+(n) \\ t \in \mathcal{Y} & \text{if} \quad V(n+1,0) > \Pi^+(n). \end{cases} \tag{4.48}$$

We now proceed to define the set $\mathcal{Y}$,

$$\mathcal{Y} = \begin{cases} [0, +\infty) & \text{if} \quad p(t^*) \leq 0 \\ [0, t_1] \cup [t_2, +\infty] & \text{if} \quad p(t^*) > 0 \end{cases} \tag{4.49}$$

where $t^* = \frac{2}{3r}\frac{V(n+1,0)-\Pi^+(n)}{\Pi^+(n)}$ and t_1, t_2 are the unique zeros of $p(t)$ in $]0, t^*[,]t^*, +\infty[$ respectively. And by evaluating $p(t^*)$, we'll find that

$$\mathcal{Y} = \begin{cases} [0, +\infty) & \text{if} \quad \frac{\Delta^3}{\Pi^+(n)^3} \leq \frac{27r^3}{4} \\ [0, t_1] \cup [t_2, +\infty] & \text{if} \quad \frac{\Delta^3}{\Pi^+(n)^3} > \frac{27r^3}{4} \end{cases} \tag{4.50}$$

where $\Delta = V(n+1,0) - \Pi^+(n)$.

[3] We note that we're only considering positive values of t as it's not very reasonable to consider negative time.

Summarising, we have that,

$$p(t) \leq 0 \Leftrightarrow \begin{cases} t \in [0, +\infty) & \text{if} \quad \Delta \leq 0 \\ t \in [0, t_1] \cup [t_2, +\infty] & \text{if} \quad \Pi^+(n) > 0 \quad \frac{\Delta^3}{\Pi^+(n)^3} > \frac{27r^3}{4}. \\ t = 0 & \text{if} \quad \Pi^+(n) = 0 \quad V(n+1, 0) > 0. \end{cases} \tag{4.51}$$

Since, $t_2 = \sup\{t \in [0, T] : p(t) > 0\}$ from (4.52) we find that,

$$\begin{cases} t_2 = -\infty & \text{if} \quad \Delta \leq 0 \\ t_2 \in (0, T) & \text{if} \quad \Pi^+(n) > 0 \quad \frac{\Delta^3}{\Pi^+(n)^3} > \frac{27r^3}{4} \text{ and } T \text{ large enough} \\ t_2 = T & \text{if} \quad \Pi^+(n) = 0 \quad V(n+1, 0) > 0. \end{cases} \tag{4.52}$$

And now we have a unique solution for the differential equation,

$$\begin{cases} \lambda(t)\big(v(n+1, 0) - v(n, t)\big) + \frac{\partial}{\partial t} v(n, t) - rv(n, t) = 0 \\ v(n, t_2) = \Pi^+(n) \end{cases} \tag{4.53}$$

$$\Rightarrow v(n, t) = (t^3 + 1)^{1/3} e^{rt} \left(\frac{e^{-rt_2} \Pi^+(n)}{t_2^{1/3}} + V(n+1, 0) \int_t^{t_2} \frac{e^{-rx} x^2}{(1 + x^3)^{4/3}} \mathrm{dx} \right) \tag{4.54}$$

So, the solution of the HJB equation is,

$$V^{k,T}(n, t) = \max\big(\Pi^+(n), v(n, t)\big)$$

with,

$$v(n, t) = (t^3 + 1)^{1/3} e^{rt} \left(\frac{e^{-rt_2} \Pi^+(n)}{t_2^{1/3}} + V(n+1, 0) \int_t^{t_2} \frac{e^{-rx} x^2}{(1 + x^3)^{4/3}} \mathrm{dx} \right) \tag{4.55}$$

□

We can now particularise for $n = k - 1$, since in that case we know the value for

$V(n+1,0)$ we become with a completely known solution,

$$V^{k,T}(k-1,t) = \max\big(\Pi^+(k-1), v(k-1,t)\big)$$

with,

$$v(k-1,t) = (t^3+1)^{1/3} e^{rt}\left(\frac{e^{-rt_2}\Pi^+(k-1)}{t_2^{1/3}} + \Pi^+(k)\int_t^{t_2} \frac{e^{-rx}x^2}{(1+x^3)^{4/3}}\mathrm{dx}\right) \tag{4.56}$$

Note that since we've found $V^{k,T}(k-1,t)$ we can find $V^{k,T}(k-1,0)$ and therefore also fully characterise $V^{k,T}(k-2,t)$.

We can follow this recursive fashion to find for $V^{k,T}(n,t)$ where $n = k-p$.

Chapter 5

Discussion

Following the dynamic programming approach and studying the modified problem we find that the optimal value for the Firm, is given by

$$V_m^{\infty,\infty}(n,t) = \sup_{i\geq 0}\left\{\left(\frac{\lambda}{\lambda+r}\right)^i \Pi^+(n+i)\right\}. \tag{5.1}$$

This result is analogous to the one found in previous work [6]. A key aspect of the result above is that the optimal value is not dependent on time, and the stopping time will always be when the jump occurs, that is, if τ^* is the optimal time to invest then $\theta_{\tau^*} = 0$.

As for the second case, where $\lambda(t) = \frac{t^2}{1+t^3}$, we found for $n = k-1$, the value is given by,

$$V^{k,T}(k-1,t) = \max\left(\Pi^+(k-1), v(k-1,t)\right)$$

with,

$$v(k-1,t) = (t^3+1)^{1/3} e^{rt}\left(\frac{e^{-rt_2}\Pi^+(k-1)}{t_2^{1/3}} + \Pi^+(k)\int_t^{t_2} \frac{e^{-rx}x^2}{(1+x^3)^{4/3}}\mathrm{dx}\right) \tag{5.2}$$

This solution is clearly dependent on time, and studying the signal of $\frac{\partial V(k-1,t)}{\partial t}$ one finds that we can have a non-monotonic value function and having the “value of waiting” fluctuate in time even without having any jump in the technology level. Moreover, we can have a value function whose value is $V(k-1,t) = \Pi^{+}(k-1)$ until a given time, say t_p and after that $V(k-1,t) > \Pi^{+}(k-1)$. In that situation, we have that until t_p one should invest immediately but after that time one should wait. This is highly relevant as it proves that if although it would be optimal to invest until t_p we've waited beyond t_p then we should not invest anymore. This last result different from the first case, where if we've achieved an optimal technology level, we should invest independently of the time elapsed since the last jump.

One could use the framework develop in this thesis and explore a number of other variations such as other time-dependent arrival rates or a time-dependent discount-rate. It would also be of interest to study the same problems with specific profit flow functions $\tilde{\Pi}$, and eventually do numerical applications with empirical data.

Appendix A

Infinitesimal Generator

In this appendix we introduce the definition infinitesimal generator and then we find the infinitesimal generator for the process that models the arrival of technology $\{(N_t, \theta_t), t \geq 0\}$.

A.1 Definition

Definition 1 (Infinitesimal Generator)**.** *Let $\{X_t, t \geq 0\}$ be a stochastic process with values in $\mathbb{R}^n$. The infinitesimal generator of X_t is the operator $\mathcal{A}$, defined for suitable functions $\phi : \mathbb{R}^n \to \mathbb{R}$, such that*

$$\mathcal{A}\phi(x) = \lim_{h \to 0} \frac{\mathbb{E}\big[\phi(X_{t+h})|X_t = x\big] - \phi(x)}{h} \tag{A.1}$$

A.2 The Infinitesimal Generator for (N_t, θ_t)

Let us a recall the stochastic process that we're considering,

$$\{(N_t, t - T_{N_t}) = (N_t, \theta_t), t \geq 0\}. \tag{A.2}$$

where $\{N_t\}$ is the technology level at time t and $T_{N(t)}$ denote the arrival time of the last technology innovation occurred until time t. Moreover, we have that the distribution and density function, respectively, of the (independent) increments of $T_{N(t)}$, $t > 0$

$$F(t) = 1 - e^{-\int_0^t \lambda(s)ds} \qquad f(t) = \frac{dF(t)}{dt} = \lambda(t)e^{-\int_0^t \lambda(s)ds}, \tag{A.3}$$

where $\lambda : \mathbb{R}_0^+ \to \mathbb{R}_0^+$ is continuous.

Proposition 6. *For the two-dimensional stochastic process defined in* (A.2) *the infinitesimal $\mathcal{A}$, defined for suitable functions $\phi : \mathbb{R}_0^+ \times \mathbb{N}_0 \to \mathbb{R}$, is given by*

$$\mathcal{A}\phi(n,t) = \lambda(t)\Big(\phi(n+1,0) - \phi(n,t)\Big) + \frac{\partial}{\partial t}\phi(n,t) \tag{A.4}$$

for $n \in \mathbb{N}$ and $t \in \mathbb{R}_0^+$. ■

Proof. Directly from the definition of $\mathcal{A}$ we have,

$$\mathcal{A}\phi(n,\tau) = \lim_{h\to 0} \frac{\mathbb{E}[\phi(N_h,\theta_h)|N_0 = n, \theta_0 = \tau] - \phi(n,\tau)}{h} \tag{A.5}$$

where $\phi : \mathbb{N} \times \mathbb{R}_0^+ \to \mathbb{R}$ is a suitable test function and $(n,\tau) \in \mathbb{N} \times \mathbb{R}$.

Before we proceed any further it is useful to introduce some notation. We introduce a random variable to denote for the increments of T_i,

$$X_i := T_i - T_{i-1} \geq 0 \tag{A.6}$$

since the increments will be independent random variables, it follows,

$$f_{X_1,\cdots,X_k}(x_1,\cdots,x_k) = \prod_{i=1}^{k} f(x_i) \tag{A.7}$$

where, as we mentioned before,

$$f(x) = \lambda(x)e^{-\int_0^x \lambda(s)ds} \qquad F(x) = 1 - e^{-\int_0^x \lambda(s)ds}.$$

denote the density and distribution function of X_i, respectively. Assume that τ is given and consider the following sets,

(A.8)
$$A_0 = \{x_1 : x_1 \geq q\}, \qquad h + \tau =: q$$

(A.9)
$$A_1 = \{(x_1, x_2) : q - x_2 < x_1 \leq q,\ x_1 \geq \tau,\ x_2 \geq 0\},$$

(A.10)
$$A_2 = \{(x_1, x_2, x_3) : q - x_2 - x_3 < x_1 \leq q - x_2\ ,\ x_1 \geq \tau,\ x_2 \geq 0,\ x_3 \geq 0\},$$

and for $k > 2$,

(A.11)
$$A_k = \{(x_1, \cdots, x_{k+1}) : q - \sum_{i=2}^{k-1} x_i < x_1 \leq q - x_2\ ,\ x_1 \geq \tau,\ x_2 \geq 0, \cdots,\ x_{k+1} \geq 0\}$$

Finally,

(A.12)
$$f_{\{N_h = n+k | N_0 = n\}}(x_1, \cdots, x_k)$$

denotes the density function of the inter arrival times of k events, given that k and only k arrival took place between the time interval of length h.

We can now make use of the notation to expand the expectation in $A.5$

$$\mathbb{E}[\phi(N_h, \theta_h)|N_0 = n, \theta_0 = \tau] =$$

(A.13)

$$= \sum_{k=0}^{+\infty} \int_0^{+\infty} f_{\{N_h = n+k | N_0 = n\}}(x_1, \cdots, x_k) \Big[\phi\Big(n+k, q - \sum_{i=1}^{k-1} x_i\Big)\Big] \mathrm{d}x_0 \cdots \mathrm{d}x_{k+1}$$
$$= \sum_{k=0}^{+\infty} \int_{A_k} f_{X_1,\cdots,X_k}(x_1, \cdots, x_k) \Big[\phi\Big(n+k, q - \sum_{i=1}^{k-1} x_i\Big)\Big] \mathrm{d}x_1 \cdots \mathrm{d}x_{k+1} \Big(1 - \int_0^{\tau} f_{X_0}(x_0)\mathrm{d}x_0\Big)^{-1}$$
$$= \sum_{k=0}^{+\infty} \int_{A_k} f_{X_1,\cdots,X_k}(x_1, \cdots, x_k) \Big[\phi\Big(n+k, q - \sum_{i=1}^{k-1} x_i\Big)\Big] \mathrm{d}x_1 \cdots \mathrm{d}x_{k+1} \Big(1 - F_X(\tau)\Big)^{-1}.$$

It is clear that (A.13) we are summing the probability of having k possible jumps between τ and $h + \tau = q$. Therefore, we might as well compute the terms individually, starting from $k = 0$.

For $k = 0$

$$\begin{aligned}
\frac{\int_{A_0} f_{X_1}(x_1)\phi(n,q)\mathrm{dx}_1}{1 - F_X(\tau)} &= \frac{\int_q^{+\infty} \lambda(x_1) e^{-\int_0^{x_1}\lambda(s)ds}\phi(n,q)\mathrm{d}x_1}{1 - F_X(\tau)} \\
&= \frac{\phi(n,q)(1 - F_X(q))}{1 - F_X(\tau)} = \frac{\phi(n,q)e^{-\int_0^q \lambda(s)ds}}{e^{\int_0^\tau \lambda(s)ds}} \\
&= \phi(n,q)e^{-\int_\tau^q \lambda(s)ds} = \phi(n,q)\sum_{i=0}^{\infty} \frac{(-\int_\tau^q \lambda(s)ds)^i}{i!} \\
&= \phi(n,q)\Big(1 - \int_\tau^q \lambda(s)ds + \sum_{i=2}^{\infty} \frac{(-\int_\tau^{\tau+h}\lambda(s)ds)^i}{i!}\Big)
\end{aligned}$$

Note that we have,

$$\lim_{h\to 0} \frac{\int_\tau^q \lambda(s)ds}{h} = \lim_{h\to 0} \frac{\lambda(q)}{1} = \lambda(\tau)$$

and,

$$\int_{\tau}^{q} \lambda(s)ds \leq \|\lambda(s)\|_{\infty} h \Rightarrow \frac{\Big(-\int_{\tau}^{q}\lambda(s)ds\Big)^{i}}{h} \leq \big(\|\lambda(s)\|_{\infty}\big)^{i} h^{i-1} \sim \mathcal{O}(h^{i-1})$$

So,

$$\lim_{h\to 0} \frac{\int_{A_0} f_{X_1}(x_1)\phi(n,q)\mathrm{d}x_1}{h\,(1-F_X(\tau))} = \phi(n,q)\Big(1-\lambda(\tau)+\mathcal{O}(h)\Big) \tag{A.14}$$

For $k = 1$

We start by solving the following integral,

$$\begin{aligned}
&\int_{A_1} f_{X_1,X_2}(x_1,x_2)\phi(n+1,q-x_1)\mathrm{d}x_2\mathrm{d}x_1 = \\
=&\int_{A_1} \lambda(x_1)e^{\int_0^{x_1}\lambda(s)ds}\lambda(x_2)e^{\int_0^{x_2}\lambda(s)ds}\phi(n+1,q-x_1)\mathrm{d}x_2\mathrm{d}x_1 \\
=&\int_{\tau}^{q}\int_{q-x_1}^{+\infty} \lambda(x_1)e^{\int_0^{x_1}\lambda(s)ds}\lambda(x_2)e^{\int_0^{x_2}\lambda(s)ds}\phi(n+1,q-x_1)\mathrm{d}x_2\mathrm{d}x_1 \\
=&\underbrace{\int_{\tau}^{q} \lambda(x_1)e^{-\int_0^{x_1}\lambda(s)ds}\phi(n+1,q-x_1)\mathrm{d}x_1}_{\text{A}} - \\
&-\underbrace{\int_{\tau}^{q} \lambda(x_1)e^{-\int_0^{x_1}\lambda(s)ds}F_X(q-x_1)\phi(n+1,q-x_1)\mathrm{d}x_1}_{\text{B}}
\end{aligned}$$

Let us work on A and B separately,

$$\begin{aligned}
\lim_{h\to 0}\frac{A}{h} =& \lim_{h\to 0}\frac{\int_{\tau}^{q}\lambda(x_1)e^{-\int_0^{x_1}\lambda(s)ds}\phi(n+1,q-x_1)\mathrm{d}x_1}{h} \\
=& \lim_{h\to 0}\frac{\lambda(q)e^{-\int_0^{q}\lambda(s)ds}\phi(n+1,q-q)}{1} \\
=& \lambda(\tau)e^{-\int_0^{\tau}\lambda(s)ds}\phi(n+1,0)
\end{aligned} \tag{A.15}$$

where in (A.15) we used L'Hôpital's rule. Similarly

$$\begin{aligned}\lim_{h\to 0}\frac{B}{h} =&\lim_{h\to 0}\frac{\int_\tau^q \lambda(x_1)e^{-\int_0^{x_1}\lambda(s)ds}F(q-x_1)\phi(n,q-x_1)\mathrm{d}x_1}{h}\\ =&\lim_{h\to 0}\frac{\lambda(q)e^{-\int_0^{q}\lambda(s)ds}F(q-q)\phi(n,q-q)}{1}\\ =&\lambda(\tau)e^{-\int_0^{\tau}\lambda(s)ds}F(0)\phi(n,0)=0\end{aligned}$$

therefore,

$$\begin{aligned}\lim_{h\to 0}\frac{A+B}{h} =&\lim_{h\to 0}\frac{\int_{A_1} f_{X_1,X_2}(x_1,x_2)\phi(n+1,q-x_1)\mathrm{d}x_2\mathrm{d}x_1}{h\,(1-F(\tau))}\\ =&\big(\lambda(\tau)e^{-\int_0^{\tau}\lambda(s)ds}\phi(n+1,0)+\lambda(\tau)e^{-\int_0^{\tau}\lambda(s)ds}F(0)\phi(n+1,0)\big)\cdot e^{\int_0^{\tau}\lambda(s)ds}\\ =&\lambda(\tau)\phi(n,0)+\lambda(\tau)\underbrace{F(0)}_{=0}\phi(n,0)=\lambda(\tau)\phi(n+1,0)\end{aligned}$$

As for $k=2$

$$\begin{aligned}&\int_{A_2} f_{X_1,X_2,X_3}(x_1,x_2,x_3)\phi(n+2,q-x_1-x_2)\mathrm{d}x_3\mathrm{d}x_2\mathrm{d}x_1=\\ =&\int_{A_2}\prod_{i=1}^{3}\lambda(x_i)e^{-\int_0^{x_i}\lambda(u)du}\phi(n+2,q-x_1-x_2)\mathrm{d}x_1\mathrm{d}x_2\mathrm{d}x_3=\\ =&\int_\tau^q\int_0^{q-x_1}\int_{q-x_1-x_2}^{+\infty}\prod_{i=1}^{3}\lambda(x_i)e^{-\int_0^{x_i}\lambda(u)du}\phi(n+2,q-x_1-x_2)\mathrm{d}x_1\mathrm{d}x_2\mathrm{d}x_3=\\ =&\int_\tau^q\int_0^{q-x_1}\prod_{i=1}^{2}\lambda(x_i)e^{-\int_0^{x_i}\lambda(u)du}\phi(n+2,q-x_1-x_2)\big(1-F(q-x_1-x_2)\big)\mathrm{d}x_2\mathrm{d}x_1\end{aligned}$$

if we define,

$$\Gamma(x_1,\cdots,x_k):=\prod_{i=1}^{k}\lambda(x_i)e^{-\int_0^{x_i}\lambda(u)du}\phi(n+k,q-\sum_{i=1}^{k}x_i)$$

we see that,

$$\lim_{h\to 0} \frac{\int_\tau^q \int_0^{q-x_1} \Gamma(x_1,x_2)\big(1-F(q-x_1-x_2)\big)\mathrm{d}x_2\mathrm{d}x_1}{h} =$$
$$= \lim_{h\to 0} \frac{\int_0^{q-\tau-h} \Gamma(q,x_2)\big(1-F(q-q-x_2)\big)\mathrm{d}x_2}{1} = 0$$

yielding,

$$\lim_{h\to 0} \frac{\int_{A_2} f_{X_1,X_2,X_3}\phi(n+2,q-x_1-x_2)\mathrm{d}x_3\mathrm{d}x_2\mathrm{d}x_1}{h(1-F_X(\tau))} =$$
$$= \lim_{h\to 0} \frac{\int_\tau^q \int_0^{q-x_1} \Gamma(x_1,x_2)\big(1-F(q-x_1-x_2)\big)\mathrm{d}x_2\mathrm{d}x_1}{h(1-F_X(\tau))} = 0$$

For $k > 2$

$$\int_{A_k} f_{X_1,\cdots,X_{k+1}}(x_1,\cdots,x_n)\phi\Big(n+k,q-\sum_{i=1}^{k} x_i\Big)\mathrm{d}x_{k+1}\cdots\mathrm{d}x_1 =$$
$$= \int_{A_k} \prod_{i=1}^{k+1} \lambda(x_i)e^{-\int_0^{x_i}\lambda(u)du}\phi\Big(n+k,q-\sum_{i=1}^{k} x_i\Big)\mathrm{d}x_{k+1}\cdots\mathrm{d}x_1 =$$
$$= \int_\tau^q \int_0^{q-x_1}\cdots\int_0^{q-\sum_{i=0}^{k-1}x_i}\int_{q-\sum_{i=1}^{k}x_i}^{+\infty} \prod_{i=1}^{k+1} \lambda(x_i)e^{-\int_0^{x_i}\lambda(u)du}\phi\Big(n+k,q-\sum_{i=1}^{k} x_i\Big)\mathrm{d}x_{k+1}\cdots\mathrm{d}x_1 =$$
$$= \int_\tau^q \int_0^{q-x_1}\cdots\int_0^{q-\sum_{i=0}^{k-1}x_i} \prod_{i=1}^{k} \lambda(x_i)e^{-\int_0^{x_i}\lambda(u)du}\phi\Big(n+k,q-\sum_{1}^{k} x_i\Big)\cdot$$
$$\Big(1-F\Big(q-\sum_{i=0}^{k} x_i\Big)\Big)\mathrm{d}x_k\cdots\mathrm{d}x_1 =$$
$$= \int_\tau^q \cdots \int_0^{q-\sum_{i=1}^{k-1}x_i} \Gamma(x_1,\cdots,x_k)\Big(1-F\Big(q-\sum_{i=0}^{k} x_i\Big)\Big)\mathrm{d}x_k\cdots\mathrm{d}x_1$$

Again we see that,

$$\lim_{h\to 0} \frac{\int_\tau^q \int_0^{q-x_1} \cdots \int_0^{q-\sum_{i=1}^{k-1} x_i} \Gamma(x_1, \cdots, x_k)\Big(1 - F\Big(q - \sum_{i=0}^{k} x_i\Big)\Big)\mathrm{d}x_k \cdots \mathrm{d}x_1}{h} =$$

$$= \lim_{h\to 0} \frac{\int_0^{\tau-h-\tau-h} \cdots \int_0^{q-\sum_{i=1}^{k-1} x_i} \Gamma(q, \cdots, x_k)\Big(1 - F\Big(q - \sum_{i=2}^{k} x_i\Big)\Big)\mathrm{d}x_k \cdots \mathrm{d}x_2}{1} = 0$$

so,

$$\lim_{h\to 0} \frac{\int_{A_k} f_{X_1,\cdots,X_{k+1}} \phi\Big(n+1, q - \sum_{i=1}^{k} x_i\Big)\mathrm{d}x_{k+1} \cdots \mathrm{d}x_1}{h(1 - F_X(\tau))} = 0$$

We can now fully compute the expectation value (A.5) and, consequently, find the infinitesimal generator,

$$\begin{aligned}
\mathcal{A}\phi(n,\tau) &= \lim_{h\to 0} \frac{\mathbb{E}[\phi(N_h, \theta_h) | N_0 = n, \theta_0 = \tau] - \phi(n,\tau)}{h} \\
&= \lim_{h\to 0} \frac{\phi(n,q)\Bigg(1 - \int_\tau^q \lambda(s)ds + \sum_{i=2}^{\infty} \frac{(-\int_\tau^{\tau+h} \lambda(s)ds)^i}{i!}\Bigg) - \phi(n,\tau)}{h} + \lambda(\tau)\phi(n+1,0) \\
&= \lim_{h\to 0} \frac{\phi(n,\tau+h) - \phi(n,\tau)}{h} + \lim_{h\to 0} \frac{\phi(n,\tau+h)\big(-\int_\tau^q \lambda(s)ds + \sum_{i=2}^{\infty} \frac{(-\int_\tau^{\tau+h} \lambda(s)ds)^i}{i!}\big)}{h} + \\
&+ \lambda(\tau)\phi(n+1,0) \\
&= \frac{\partial}{\partial\tau}\phi(n,\tau) - \lambda(\tau)\phi(n,\tau) + \lambda(\tau)\phi(n+1,0) \\
&= \lambda(\tau)\Big(\phi(n+1,0) - \phi(n,\tau)\Big) + \frac{\partial}{\partial\tau}\phi(n,\tau)
\end{aligned}$$

□

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