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Trabalho Final de Mestrado

Dissertação

Fractional Brownian Motion in Finance

Susana de Matos Neves

Instituto Superior de Economia e Gestão
UNIVERSIDADE TÉCNICA DE LISBOA
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Supervisors:

Prof. Dr. João Guerra

Dr. João Cruz

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Abstract

Some of the statistical properties of the financial data are common to a wide variety of markets: long-range dependence properties, heavy tails, skewness (gain/loss asymmetry), jumps, volatility clustering, etc. The need to seek new models for financial products has increased in recent decades due to the inability of current models to explain some of these facts. One of these models is fractional Brownian motion.

This work aims to give an overview of some studies that were done on the financial applications of fractional Brownian motion, in particular the work of Paolo Guasoni and Patrick Cheridito which shows that if we assume certain restrictions, we can eliminate arbitrage opportunities. Moreover, we also present empirical studies with market data, in order to show how to obtain an estimator for the Hurst index (the fractional Brownian motion parameter). To this end, we used two methods, the Rescaled Range Analysis and the modified Rescaled Range Analysis. This study allows us to discuss the effect of memory on the time series of some market indices.

Keywords: Mathematical Finance, Fractional Brownian motion, Arbitrage, Transaction Costs, Long Memory.

Resumo

Algumas das propriedades estatísticas dos dados financeiros são comuns a uma ampla variedade de mercados: a propriedade de memória longa, as caudas pesadas, assimetria (ganho / perda de assimetria), saltos, agrupamento de volatilidade, etc. A necessidade de procurar novos modelos de produtos financeiros tem aumentado nas últimas décadas devido à incapacidade dos actuais modelos explicarem algumas dessas propriedades estatísticas.

Este trabalho tem como objetivo dar uma visão geral de alguns estudos que foram feitos relativamente à aplicação às finanças do movimento Browniano fracionário, em particular o trabalho de Paolo Guasoni e Cheridito Patrick, que mostram que, se assumirmos certas restrições, podemos eliminar oportunidades de arbitragem. Além disso, também são apresentados estudos empíricos com dados de mercado, com o objectivo de mostrar como se pode obter um estimador para o índice Hurst (o parâmetro do movimento Browniano fracionário). Para este fim, foram utilizados dois métodos, o método *Rescaled Range* e o método modificado do *Rescaled Range*. Este estudo permite-nos discutir o efeito de memória nas séries temporais de alguns índices de mercado.

Palavras Chave: Matemática Financeira, movimento Browniano Fraccionário, Arbitragem, Custos de transacção, Memória Longa.

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Chapter 1

Introduction

Time series with long memory appear in many contexts, for example in financial economics, networks traffic, hydrology, cardiac dynamics, meteorology, etc. The statistical properties of the financial assets return have led many scientists to rethink existing models. One of the alternatives to standard Brownian motion is the fractional Brownian motion. This stochastic process is a generalization of the standard Brownian motion with some of the desired properties for a model that explains well the evolution of financial assets data.

The aim of this work is to study the possible effect of long memory in financial time series using the Hurst parameter of the fractional Brownian motion as a long memory signature. This parameter was initially used to calculate correlations and long memory of natural phenomena (see Hurst [13]) and then reached the financial context (see Mandelbrot and Van Ness [21]). Despite some evidence of long-term dependence in financial data, the using of the fractional Brownian motion in finance began to be questioned, because it could lead to the possibility of making a profit with no risk and no capital investment (arbitrage), as we can see in Rogers [26]. Within the framework of Cheridito [4] and Guasoni [10] work, we discuss how such arbitrage opportunities can be eliminated.

If the Hurst parameter is greater than $\frac{1}{2}$ we have a time series where the increments are positively correlated and exhibits the long memory effect. If the Hurst parameter is equal to $\frac{1}{2}$, the increments are independent and therefore we face the standard Brownian motion. If the Hurst parameter is less than $\frac{1}{2}$ and the series presents the effects of intermittency, the increments are negatively correlated. Intuitively, positively correlated increments means that if the value of the assets tends to increase, then this trend will be maintained. In the case that the increments are negatively correlated, if the tendency of a particular segment is to increase, then the tendency of the following segment will be to decrease and so on (intermittency). This intuition leads us to understand why we can have arbitrage opportunities, because if we can "predict"what will happen then we can explore an arbitrage opportunity.

The fractional Brownian motion has been used to describe the behavior of asset prices and volatilities in stock markets (see Nualart [22]). In [6], Renault and Comte studied a classical extension of the Black and Scholes model for option pricing known as the Hull and White model. The authors specification is that the volatility process is assumed not only to be stochastic, but also to have long-memory features and properties. However, in this text we will only discuss the fractional Brownian motion applied to the asset pricing models.

In order to estimate the Hurst parameter of the fractional Brownian motion, we used two popular methods: the Rescaled Range method and the modified Rescaled Range method. The Rescaled Range method is the classical method of estimation of the Hurst parameter developed by the hidrologist Harold Edwin Hurst (see Hurst [13]). The modified Rescaled Range method was proposed by Lo [19] in order to deal with both heteroskedasticity and short-term memory which are the problems of the Rescaled Range method. However, Willinger, Taqqu and Teverovsky [29] proved that the modified Rescaled Range statistic shows a strong preference for accepting the null hypothesis of no long-range dependence, irrespective of whether long-range dependence is present in data or not.

The structure of this dissertation is as follows: Chapter 2 presents an introduction to fractional Brownian motion, focusing on its definition and its main properties. Chapter 3 presents a brief description of financial concepts which will be used on the next chapters such as trading strategies and arbitrage. Because of the non-semimartingale-property, fractional Brownian motion pricing models admit arbitrage possibilities with continuous trading, so in chapter 4 we make a connection between fractional Brownian motion and arbitrage. We present the main results obtained by Cheridito. This author proves that the arbitrage opportunities will disappear by introducing a minimal period of time between transactions. Moreover, Guasoni proves that they also disappear under proportional transaction costs assumptions. The Hurst parameter has an important role in the theory presented in this text, in Chapter 5 we present two methods for estimating the Hurst parameter, the Rescaled Range method and the modified Rescaled Range method. At the end of the work, in chapter 6, we present conclusions, discuss the controversy about the use of fractional Brownian motion for pricing, and present some topics that deserve further research.

Chapter 2

Fractional Brownian motion: Definition and Basic Properties

The fractional Brownian motion (abbreviated by fBm in the text) is a generalization of Brownian motion which allows the possibility of dependent increments and the presence of long memory. This process has dependent increments, it is not a Markov process and, except for one particular case where the process is reduced to standard Brownian process, is not a semimartingale. The fractional Brownian motion was originally introduced by Kolmogorov in 1940 (see [14]) and the first authors who used the name fractional Brownian motion were Mandelbrot and Van Ness [21]. Let us present the formal definition of fBm.

Definition 2.0.1 *The fractional Brownian motion (fBm) with Hurst index* $H \in (0,1)$ *is a Gaussian process* $B^H = \{B_t^H, t \in \mathbb{R}\}$ *on* $(\Omega, \mathcal{F}, \mathbb{P})$*, having the properties*

1. $B_0^H = 0$,
2. $\mathbb{E}[B_t^H] = 0$, $t \in \mathbb{R}$,
3. $\mathbb{E}[B_t^H B_s^H] = \frac{1}{2}\left(|t|^{2H} + |s|^{2H} - |t-s|^{2H}\right)$, $s, t \in \mathbb{R}$.

Properties 1 *The fBm has the following properties:*

1. *If* $H = \frac{1}{2}$, $B^{\frac{1}{2}}$ *is a standard Brownian motion.*
2. *Mean:* $\mathbb{E}[B_t^H] = 0$. *Variance:* $\mathbb{E}[(B_t^H)^2] = t^{2H}$.
3. *Self-similarity:* $\forall a > 0, \quad a^{-H} B_{at}^H \overset{d}{=} B_t^H$, *where* $\overset{d}{=}$ *means that* $a^{-H} B_{at}^H$ *and* B_t^H *have the same probability distribution.*
4. *Stationary increments:* $\forall s, t \geq 0 \quad B_t^H - B_s^H \overset{d}{=} B_{t-s}^H$.

The proof of these properties can be found in Nualart [22]. We can interpret the property of self-similarity as the fact that changes in the time scale have the same effect as appropriate changes in the space scale.

2.1 Long-range dependence

We now discuss the long-range dependence property.

Definition 2.1.1 *A stationary random sequence $\{X_n\}_{n\in\mathbb{N}}$ exhibits*

- *long-range dependence if the autocovariance function $\rho(n)$ satisfies*

$$\sum_{n=1}^{\infty} \rho(n) = \infty.$$

- *short-range dependence if the autocovariance function $\rho(n)$ satisfies*

$$\sum_{n=1}^{\infty} \rho(n) < \infty.$$

where $\rho(n) = \mathbb{E}[X_0 X_n]$.

Properties 2 *1. If $H > \frac{1}{2}$ then disjoint increments are positively correlated:*

$$\mathbb{E}[(B_t^H - B_s^H)(B_s^H - B_r^H)] > 0$$

and $X_n := B_n^H - B_{n-1}^H$, $n \geq 1$ has long-range dependence.

2. If $H < \frac{1}{2}$ then disjoint increments are negatively correlated:

$$\mathbb{E}[(B_t^H - B_s^H)(B_s^H - B_r^H)] < 0$$

and $X_n := B_n^H - B_{n-1}^H$, $n \geq 1$ exhibits intermittency (or has short-range dependence).

2.2 Regularity

In this subsection, we discuss the path properties of fBm.

Definition 2.2.1 *Let $X, Y : \mathbb{T} \times \Omega \to \mathbb{R}$ be stochastic processes on $(\Omega, \mathrm{F}, \mathbb{P})$. We say that X is a version of Y if for all $t \in \mathbb{T}$*

$$\mathbb{P}[X_t = Y_t] = 1, \quad a.s.$$

Kolmogorov's Criterion 1 *Suppose the process $X : [0, \infty) \times \Omega \to \mathbb{R}$ satisfies the following condition: For all $T > 0$ there exist $\alpha, \beta, C > 0$ such that*

$$\forall 0 \leq t, s \leq T \quad \mathbb{E}[|X_t - X_s|^{\alpha}] \leq C|t - s|^{1+\beta}$$

Then there exists a version of X which is Hölder continuos of order $\gamma \in [0, \frac{\beta}{\alpha})$ a.s.

For a proof see Protter [23].

Properties 3 *1. The fBm admits a version with almost all sample paths Hölder continuous of order strictly less than H: for each such trajectory, there exists a constant c such that*

$$|B^H_t - B^H_s| \leq c|t-s|^{H-\varepsilon}, \quad \forall \varepsilon > 0.$$

2. The sample paths of the fBm are nowhere differentiable a.s.

The proof of these properties can be found in Biagini, Hu, Øksendal and Zhang [30] or Krzywda [17].

2.3 p-variation and quadratic variation

Consider partitions $\pi = \{0 = t_0 < t_1 < ... < t_n = T\}$ of the interval $[0, T]$. For $p \geq 1$ and $f : [0, T] \to \mathbb{R}$ we denote:

$$v_p(f; \pi) = \sum_{k=1}^{n} |f(t_k) - f(t_{k-1})|^p$$

Definition 2.3.1 *We say that f has finite p-variation if the limit*

$$v^0_p(f) = \lim_{|\pi| \to 0} v_p(f; \pi) \quad \text{exists and is finite.}$$

We say that f has bounded p-variation if

$$v_p(f) = \sup_{\pi} v_p(f; \pi) < \infty.$$

Remark 2.3.2 *The notation $|\pi| \to 0$ means that the mesh size of the partition $|\pi| = \max_i |t_i - t_{i-1}|$ tends to zero.*

Properties 4 *1. For $pH > 1$, $v^0_p(B^H) = 0$ a.s.*

2. For $pH < 1$, $v_p(B^H) = +\infty$ and $v^0_p(B^H)$ does not exists.

The proof of these properties can be found in Krzywda [17].

Definition 2.3.3 *Let $\{\pi_n\}$ be a sequence of partitions of $[0, T]$ such that $|\pi_n| \to 0$. For a stochastic process X_t by the quadratic variation along the sequence $\{\pi_n\}$ we mean*

$$[X, X]_T = \lim_{n \to \infty} \sum_{k=1}^{n} (X_{t_k} - X_{t_{k-1}})^2$$

if the limit exists (in probability).

Properties 5 *1. if $H > \frac{1}{2}$, $[B^H, B^H]_T = 0$*

2. if $H < \frac{1}{2}$, $[B^H, B^H]_T$ does not exists.

2.4 fBm is not a semimartingale

We now discuss the important semimartingale property.

Definition 2.4.1 *By a martingale with respect to the filtration $\{\mathcal{F}_t\}$ we mean a stochastic process X such that:*

1. *X is adapted to $\{\mathcal{F}_t\}$*
2. *$\mathbb{E}[|X_t|] < \infty$*
3. *$\forall s \leq t \quad \mathbb{E}[X_t|\mathcal{F}_s] = X_s$*

Definition 2.4.2 *By a local martingale with respect to the filtration $\{\mathcal{F}_t\}$ we mean an $\{\mathcal{F}_t\}$-adapted stochastic process X for which there exists an increasing sequence of $\{\mathcal{F}_t\}$-stopping times τ_k such that*

1. *$\tau_k \to \infty$ a.s. as $k \to \infty$*
2. *$X_{t\wedge\tau_k}$ is an $\{\mathcal{F}_t\}$-martingale for all k.*

Definition 2.4.3 *$f : \mathbb{R}^+ \to \mathbb{R}$ is a càdlàg function if it is "continue à droite et limitè à gauche"- right continuous with left limits.*

Definition 2.4.4 *By a semimartingale with respect to the filtration $\{\mathcal{F}_t\}$ we mean a stochastic process $X : \Omega \times [0, \infty) \to \mathbb{R}$ that can be decomposed as*

$$X_t = X_0 + M_t + A_t$$

where M is a local martingale and A is a càdlàg adapted process of locally bounded variation. For the case of a continuous semimartingale processes M and A are continuous. Moreover the representation is unique.

Proposition 2.4.5 *The quadratic variation exists for every semimartingales.*

The proof of this property can be found in Revuz and Yor [25].

One can prove that the fBm cannot be a semimartingale, except in the case $H = \frac{1}{2}$, because if

1. $H < \frac{1}{2}$, the quadratic variation is infinite.
2. $H > \frac{1}{2}$, the quadratic variation is zero and the 1-variation (total variation) is infinite.

and a proof can be found in Sottinen [27].

Chapter 3

Concepts of Finance

In this chapter we introduce some basic concepts of finance that will be useful in later chapters.

3.1 Trading Strategies

In this section the time interval is an arbitrary compact interval $[a, b]$. A trading strategy is just a pair $\Theta = (\theta^0, \theta^1)$ of stochastic processes $(\theta_t^0)_{t\in[a,b]}$ and $(\theta_t^1)_{t\in[a,b]}$ taking real values. $\theta_t^0 X_t$ describes the money in the money market account at time t and θ_t^1 the number of stock shares held at time t. Hence, the evolution of the portfolio value of a strategy Θ is given by

$$V_t^\theta := \theta_t^0 X_t + \theta_t^1 Y_t, \quad t \in [a, b].$$

Since we want to use X as numeraire, we require it to be positive. We set

$$\tilde{Y}_t := \frac{Y_t}{X_t} \quad \text{and} \quad \tilde{V}_t^\theta := \frac{V_t^\theta}{X_t}, \quad t \in [a, b].$$

Obviously we need to make some restrictions on a strategy to have some financial sense:

1. Trading strategies should only be based on available information;
2. We assume that at any time $t \in [a, b]$, X_t and Y_t can be observed and no information is lost over time;
3. X and Y have to be progressively measurable with respect to $\mathbb{F}$, where $\mathbb{F} = (\mathcal{F}_t)_{t\in[a,b]}$ is a filtration.

Definition 3.1.1 *Let* $\mathbb{F} = (\mathcal{F}_t)_{t\in[a,b]}$.

1. *The set of simple predictable integrands is given by*

$$S(\mathbb{F}) := \Bigg\{ g_0 1_{\{a\}} + \sum_{j=1}^{n-1} g_j 1_{(\tau_j,\tau_{j+1}]} : a = \tau_1 \leq ... \leq \tau_n = b; \text{all } \tau_j\text{'s are}$$
$$\mathbb{F}\text{-stopping times; } g_0 \text{ is a real, } \mathcal{F}_a\text{-measurable random variable;}$$
$$\text{and the other } g_j\text{'s are real, } \mathcal{F}_{\tau_j}\text{-measurable random variables}\Bigg\}.$$

The class of simple predictable trading strategies is given by

$$\Theta^S(\mathbb{F}) := \{\theta = (\theta^0, \theta^1) : \theta^0, \theta^1 \in S(\mathbb{F})\}.$$

2. *The set of almost simple predictable integrands is given by*

$$aS(\mathbb{F}) := \Bigg\{ g_0 1_{\{a\}} + \sum_{j=1}^{\infty} g_j 1_{(\tau_j,\tau_{j+1}]} : a = \tau_1 \leq ... \leq \tau_n \leq b; \text{all } \tau_j\text{'s are}$$
$$\mathbb{F}\text{-stopping times; } g_0 \text{ is a real, } \mathcal{F}_a\text{-measurable random variable;}$$
$$\text{and the other } g_j\text{'s are real, } \mathcal{F}_{\tau_j}\text{-measurable random variables}$$
$$\mathbb{P}[\exists j \text{ such that } \tau_j = b] = 1\Bigg\}.$$

The class of almost simple predictable trading strategies is given by

$$\Theta^{aS}(\mathbb{F}) := \{\theta = (\theta^0, \theta^1) : \theta^0, \theta^1 \in aS(\mathbb{F})\}.$$

Note that in the above definition $S(\mathbb{F}) \subset aS(\mathbb{F})$ and therefore $\Theta^S(\mathbb{F}) \subset \Theta^{aS}(\mathbb{F})$. Moreover, the dates of transaction could have been fixed in advance, but as in reality the investor does not know beforehand the decision to buy or sell by the date τ_j is taken based on information available at time τ_j. Therefore we require the τ_j's to be $\mathbb{F}$ -stopping times. As the new composition of the portfolio g_j is chosen based on the information available to τ_j makes sense doing the requirement that the variables g_j are $\mathcal{F}_{\tau_j}$-measurable.

Definition 3.1.2 *For $\theta^1 = g_0 1_a + \sum_{j=1}^{\infty} g_j 1_{(\tau_j,\tau_{j+1}]} \in aS(\mathbb{F})$ we define*

$$(\theta^1.Y)_t := \sum_{j=1}^{\infty} g_j (Y_{\tau_{j+1}\wedge t} - Y_{\tau_j \wedge t}), \quad t \in [a, b].$$

Note that this is almost surely a sum of finitely many terms and the process $((\theta^1.Y)_t)_{t\in[a,b]}$ is progressively measurable because $(Y_t)_{t\in[a,b]}$ is.

We are interested in working with strategies that do not receive capital contributions over time and all gains are reinvested. These will be called self-financing strategies, concept formalized in the following definition.

Definition 3.1.3 (Auto-financing strategy) *Let $\theta = (\theta^0, \theta^1) \in \Theta^{aS}(\mathbb{F})$. There exist stopping times $a = \tau_1 \leq \tau_2 \leq ... \leq b$ such that θ^0 and θ^1 can be written as*

$$\theta^0 = f_0 1_{\{a\}} + \sum_{j=1}^{\infty} f_j 1_{(\tau_j, \tau_{j+1}]}, \quad \theta^1 = g_0 1_{\{a\}} + \sum_{j=1}^{\infty} g_j 1_{(\tau_j, \tau_{j+1}]}.$$

We set $\tau_0 = a - 1$ and call Θ self-financing for (X, Y) if for all $j \geq 1$, $k = 1, ..., j$ and $l \geq 0$,

$$1_{\{\tau_{j-k} < \tau_{j-k+1} = \tau_{j+l} < \tau_{j+l+1}\}} \{(f_{j+l} - f_{j-k}) X_{\tau_j} + (g_{j+l} - g_{j-k}) Y_{\tau_j}\} \stackrel{a.s}{=} 0. \qquad (3.1)$$

Furthermore, we set

$$\Theta^S_{sf}(\mathbb{F}) := \{\theta \in \Theta^S(\mathcal{F}) : \theta \text{ is self-financing to } (X, Y)\}$$

$$\Theta^{aS}_{sf}(\mathbb{F}) := \{\theta \in \Theta^{aS}(\mathcal{F}) : \theta \text{ is self-financing to } (X, Y)\}$$

Note: The set used in the definition of self-financing strategy can be rewritten as

$$\{\tau_{j-k} < \tau_{j-k+1} = \tau_{j-k+2} = ... = \tau_j = \tau_{j+1} = ... = \tau_{j+l} < \tau_{j+l+1}\}$$

and then the property described by Eq. 3.1 is independent of the representation of θ.

The proof of the following proposition can be found in Cheridito [4].

Proposition 3.1.4 *Let $\theta = (\theta^0, \theta^1) \in \Theta^{aS}(\mathbb{F})$. Then the following properties are equivalent:*

1. *θ is self-financing for (X, Y)*
2. *$V^\theta_t = V^\theta_a + (\theta^0.X)_t + (\theta^1.Y)_t$ a.s. for all $t \in [a, b]$*
3. *θ is self-financing for $(1, \tilde{Y})_t$*
4. *$\tilde{V}^\theta_t = \tilde{V}^\theta_a + (\theta^1.\tilde{Y})_t$ a.s. for all $t \in [a, b]$.*

The above proposition assures us that if we are working with a self-financing strategy $\theta = (\theta^0, \theta^1)$ we only need to observe the process θ^1 to fully characterize the dynamics of the process of the portfolio value. For θ self-financing we note that

$$\theta^0_t = \tilde{V}^\theta_a + (\theta^1.\tilde{Y})_t - \theta^1_t \tilde{Y}_t, \quad t \in [a, b].$$

3.2 Arbitrage

One of the most important concepts in the construction of mathematical models for the financial market is arbitrage. Informally, the concept of arbitrage is the possibility of making a profit with no risk and no capital investment in a financial market. Let us present the formal definition of 'arbitrage', 'strong arbitrage'and 'free lunch with vanishing risk'.

Definition 3.2.1 *Let ξ be a random variable that takes values in $[0,+\infty]$ such that $\mathbb{P}[\xi > 0] > 0$.*

1. *A strategy θ is an ξ-arbitrage if $\mathbb{P}[\tilde{V}_b^\theta - \tilde{V}_a^\theta = \xi] = 1$.*

   We say that θ is an arbitrage if it is a ξ'-arbitrage for some random variable ξ' taking values in $[0,+\infty]$, with $\mathbb{P}[\xi' > 0] > 0$.

2. *A strategy θ is strong arbitrage if there is $c > 0$ such that $\mathbb{P}[\tilde{V}_b^\theta - \tilde{V}_a^\theta \geq c] = 1$.*

3. *A sequence of strategies $(\theta_n)_{n=1}^{+\infty}$ is ξ-FLVR (Free Lunch with Vanishing Risk) if*

   $$\lim_{n\to+\infty}\left(\tilde{V}_b^{\theta_n} - \tilde{V}_a^{\theta_n}\right) = \xi \quad \text{in probability and,} \lim_{n\to+\infty} \operatorname{ess\,sup}\left(\tilde{V}_b^{\theta_n} - \tilde{V}_a^{\theta_n}\right)^- = 0,$$

   where the essentials supremum of a random variable is defined as

   $$\operatorname{ess\,sup} = \inf\{a \in \mathbb{R} : \mathbb{P}[\{\omega \in \Omega : X(\omega) > a\}] = 0\}.$$

   We say that $(\theta_n)_{n=1}^{+\infty}$ is a FLVR if is a ξ'-FLVR for some random variable ξ' taking values in $[0,+\infty]$ with $\mathbb{P}[\xi' > 0] > 0$.

While we work on the set of possible strategies of definition 3.1.3, we may have arbitrage possibilities even in the standard Black-Scholes (or Samuelson) model (with $H = \frac{1}{2}$). One of these arbitrage possibilities was designated by doubling strategy by Harrison and Pliska [11] and can be ruled out by putting an admissibility condition on the trading strategies.

Definition 3.2.2 (Admissible strategy) *For $c \geq 0$, we call $\theta \in \Theta_{sf}^{aS}(\mathbb{F})$ c-admissible if*

$$\inf_{t\in[a,b]}\left(\tilde{V}_b^\theta - \tilde{V}_a^\theta\right) = \inf_{t\in[a,b]}\left(\theta^1.\tilde{Y}\right)_t \geq -c.$$

We call θ admissible if it is c-admissible for some $c \geq 0$. Furthermore, we set

$$\Theta_{sf,adm}^S(\mathbb{F}) := \{\theta \in \Theta^S(\mathbb{F}) : \theta \text{ is admissible}\}$$

$$\Theta_{sf,adm}^{aS}(\mathbb{F}) := \{\theta \in \Theta^{aS}(\mathbb{F}) : \theta \text{ is admissible}\}.$$

Chapter 4

Fractional Brownian Motion and Arbitrage

Fractional Brownian motion has been used to describe the behaviour of asset prices and volatilies in stock markets. In 1997, Rogers [26] showed that fractional Brownian motion could not be used as a price process for a risky security without introducing arbitrage opportunities.

In the next section we present an example of application to financial mathematics where fBm admits arbitrage opportunities. So we will present the Fractional Black and Scholes model that consists in replacing the standard Brownian motion in the classical Black and Scholes model, which has no memory and is based on the geometric Brownian motion, by the fBm.

4.1 Example of arbitrage using fBm: Fractional Black and Scholes model

In this section we present the example described in [22]. In this model the market stock price of the risky asset is given by

$$S_t = S_0 \exp\left(\mu t + \sigma B_t^H - \frac{\sigma^2}{2} t^{2H}\right) \tag{4.1}$$

where B^H is an fBm with Hurst parameter H, μ is the mean rate of return and $\sigma > 0$ is the volatility. The price of the non-risky assets at time t is given by e^{rt}, where r is the interest rate.

Consider an investor who starts with some initial amount $V_0 \geq 0$ and invests in the assets described above. Let α_t be the number of non-risky assets and let β_t the number of stocks owned by the investor at time t. The pair (α_t, β_t), $t \in [0, T]$ is called a portfolio and we assume that α_t and β_t are stochastic processes. Then the investor's wealth or value of the portfolio at time t is

$$V_t = \alpha_t e^{rt} + \beta_t S_t.$$

We say that the portfolio is self-financing if

$$V_t = V_0 + r\int_0^t \alpha_s e^{rs} ds + \int_0^t \beta_s dS_s. \tag{4.2}$$

The self-financing condition requires the definition of a stochastic integral with respect to the fBm. There are two possibilities: path-wise integrals and Wick-type integrals.

The use of path-wise integrals leads to the existence of arbitrage opportunities, which is one of the main drawbacks of the model (4.1). Let us now present the definition of path-wise integral to the case of $H > \frac{1}{2}$.

Path-Wise Integrals in the case $H > \frac{1}{2}$

Suppose that f, g are Hölder continous functions of orders α and β, with $\alpha + \beta > 1$. Then the Riemann-Stieltjes integral $\int f dg$ exists. If $H > \frac{1}{2}$ and F is regular enough, $\int F(B_s^H) dB_s^H$ exists (in the Riemann-Stieltjes sense). Moreover

$$F(t, B_t^H) = F(0,0) + \int_0^t \frac{\partial F}{\partial t}(s, B_s^H) ds + \int_0^t \frac{\partial F}{\partial x}(s, B_s^H) dB_s^H.$$

Returning to our example consider the case $H > \frac{1}{2}$. Suppose, to simplify, that $\mu = r = 0$. Consider the self-financing portfolio defined by

$$\begin{aligned} \beta_t &= S_t - S_0 \\ \alpha_t &= \int_0^t \beta_s dS_s - \beta_t S_t. \end{aligned}$$

This portfolio satisfies $V_0 = 0$ and $V_t = \int_0^t (S_s - S_0) dS_s = \frac{(S_t - S_0)^2}{2} > 0$, $\forall t > 0$. It is an arbitrage!

If $H = \frac{1}{2}$ we can not conclude the existence of arbitrage opportunities. Because if we apply the Itô formula of the standard Brownian motion to $F(t, S_t) = (S_t - S_0)^2$ we get

$$V_t = \int_0^t (S_r - S_0) dS_r = \frac{(S_t - S_0)^2}{2} - \frac{1}{2}\int_0^t \sigma^2 S_r^2 dr.$$

This can be positive or negative, hence we can not conclude existence of arbitrage.

The existence of arbitrage can be avoided using forward Wick integrals to define the self-financing property 4.2 (see Nualart [22]). But there is much controversy surrounding this type of integral. In [2], Björk and Hult argue that the definition of a self-financing portfolio using the Wick product is quite restrictive and has no economic meaning.

In the three following sections, we present models for application to financial mathematics that do not involve arbitrage possibilities.

4.2 Exclusion of arbitrage in fractional models

In this section we present two different approaches to the exclusion of arbitrage possibilities: the first is based on the inclusion of a minimum time delay between two consecutive transactions and the other is based on the inclusion of transaction costs proportional to the value of the asset.

By the end of this section, we will always consider a market consisting of two assets: a bank account that represents a risk-free asset, and a stock (asset with risk) that does not pay dividends. In our context, all economic activity occurs in a defined time interval $[0, T]$ with $T \in (0, +\infty)$. Short selling is allowed, in other words, a trader who borrowed a particular financial asset may sell it. We suppose further that the interest rates for borrowing and lending are equal and it is possible to buy or sell any fraction of the asset. Finally, we assume also that there is no difference between the purchase price and sale price of assets (the bid-ask spread is zero).

Note that there are two hypotheses that are normally present and are not mentioned in our market definition: the absence of transaction costs and the possibility of transactions at any time $t \in [0, T]$.

Formally we assume that there is a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ where two stochastic processes are defined, $X = (X_t)_{t\in[0,T]}$ and $Y = (Y_t)_{t\in[0,T]}$, which represent, respectively, risk-free assets and risk assets.

Let us study the behavior of the following models:

Fractional Bachelier model:

$$X_t = 1, \quad Y_t = Y_0 + \vartheta(t) B_t^H, \quad t \in [0, T]. \tag{4.3}$$

Fractional Samuelson model (or Fractional Black-Scholes model):

$$X_t = e^{rt} \quad ; \quad Y_t = Y_0 e^{rt+\vartheta(t)+\sigma B_t^H}, \quad t \in [0, T]. \tag{4.4}$$

where $\vartheta \in \mathcal{C}^1[0, T]$, $Y_0, \sigma \in \mathbb{R}$ and $B^H = (B_t^H)_{t\in[0,T]}$ is a fBm.

The first approach, based on the work of Patrick Cheridito [4], presents that there is a restriction on the class of possible strategies, requiring a minimum time (predetermined and as small as you like) between two transactions. The second approach, based on the work of Paolo Guasoni [10], establishes transaction costs proportional to the value of the asset that can be set so low as the real market admits.

4.2.1 Approach by Cheridito

In this section we present the approach by Cheridito [4] to the exclusion of arbitrage in fractional Bachelier model and fractional Black-Scholes model, with $H \in \left(0, \frac{1}{2}\right) \cup \left(\frac{1}{2}, 1\right)$.

Exclusion of Arbitrage

As stated above, one possibility of exclusion of arbitrage strategies involves demanding a minimum time delay between two consecutive transactions. This restriction can be formalized by the construction of the following class of strategies:

Definition 4.2.1 *Let $\mathbb{F} = (\mathcal{F}_t)_{t\in[0,T]}$ be a filtration and $h > 0$. We define*

$$S^h(\mathbb{F}) := \left\{ g_0 I_{\{0\}} + \sum_{j=0}^{n-1} g_j I_{(\tau_j,\tau_{j+1})} \in S(\mathbb{F}) \text{ such that } \forall j, \tau_{j+1} > \tau_j + h \right\}$$

and

$$\Theta^h_{af}(\mathbb{F}) := \left\{ \theta = (\theta^0, \theta^1) \in \Theta^S_{af}(\mathcal{F}) : \theta^0, \theta^1 \in S^h(\mathbb{F}) \right\}.$$

The class of strategies

$$\Pi_C(\mathbb{F}) = \cup_{h>0} \Theta^h_{af}(\mathbb{F}) \tag{4.5}$$

is called class of Cheridito.

Theorem 4.2.2 presents the main result of this section: the absence of arbitrage opportunities in the fractional Bachelier model and fractional Black-Scholes model for strategies in the class of Cheridito (when there is need for a minimum time delay between two transactions).

Theorem 4.2.2 *Let $T > 0$, $B^H = (B^H_t)_{t\in[0,T]}$ a fBm with $H \in \left(0, \frac{1}{2}\right) \cup \left(\frac{1}{2}, 1\right)$, $\sigma > 0$ and $\vartheta : [0,T] \to \mathbb{R}$ a measurable function such that $\sup_{t\in[0,T]} |\vartheta(t)| < +\infty$. Consider two cases:*

(i) $\tilde{Y}_t = \vartheta(t) + \sigma B^H_t$, $t \in [0,T]$

(ii) $\tilde{Y}_t = e^{\vartheta(t)+\sigma B^H_t}$, $t \in [0,T]$.

If $\theta^1 = g_0 I_{\{0\}} + \sum_{j=0}^{n-1} g_j I_{(\tau_j,\tau_{j+1}]} \in \Pi_C(\mathbb{F}^{\tilde{Y}})$ and there is any $j \in 1, ..., n-1$ with $\mathbb{P}[g_j \neq 0] > 0$, then in case (i),

$$\mathbb{P}[(\theta^1.\tilde{Y})_T \leq -c] > 0, \quad \forall c \geq 0$$

and in case (ii),

$$\mathbb{P}[(\theta^1.\tilde{Y})_T < 0] > 0.$$

An example of a proof of this theorem can be found in Cheridito [4].

4.2.2 Approach by Guasoni

In the approach proposed by Paolo Guasoni in [10] it is assumed that the market has transaction costs proportional $k \in (0,1)$, in other words, for every transaction of monetary value $\tilde{Y}_t|\Delta\theta^1_t|$ there is a fee of $k\tilde{Y}_t|\Delta\theta^1_t|$ to pay.

Assumption 4.2.3 *We assume that the process $\tilde{Y}_t = (\tilde{Y}_t)_{t\in[0,T]}$ has càdlàg sample paths, is strictly positive almost surely, adapted to the filtration $\mathcal{F}_t$ and continuous.*

In this section we use strategies that are more general than the simple strategies in definition 3.1.3. We will use strategies that are locally of bounded variation. The reason for this is that, as strategy θ^1 of infinite variation in some interval would lead to immediate ruin of the agent, is sufficient to consider only strategies that are locally of bounded variation.

Definition 4.2.4 (Variation) *For a given function $f : [0, T] \to \mathbb{R}$ the variation of f in $[0, t]$ is defined as*

$$V(f)_t = \sup_{P \in \mathcal{P}} \sum_{i=0}^{nP-1} |f(S_{i+1}) - f(S_i)|$$

where the supreme is taken on the set of all partitions of the interval $[0, t]$.
We say that f is locally of bounded variation if $V(f)_t < +\infty$ for all $t \leq T$.

Analyzing strategies with bounded variation is quite reasonable from an economic standpoint, since they include strategies of definition 3.1.3. These strategies has the economic meaning of buy-and-hold.

Based on the concept of self-financing strategies in markets without transaction costs (Proposition 3.1.4) we propose a definition for the process of evolution of the value of a portfolio with strategy θ^1 of bounded variation, which is approximated by almost simple strategies.

Definition 4.2.5 *The process governing the value of the portfolio strategy with $\theta^1 = g_0 I_{\{0\}} + \sum_{j=1}^{n-1} g_j I_{(\tau_j, \tau_{j-1}]} \in S(\mathbb{F})$ is defined as*

$$\tilde{V}_t^\theta = \sum_{\tau_j < t} g_j(\tilde{Y}_{\tau_{j+1}} - \tilde{Y}_{\tau_j}) - k \sum_{\tau_j < t} \tilde{Y}_{\tau_j} |g_{j+1} - g_j| - k\tilde{Y}_t |g_t|.$$

As we can approximate a strategy θ^1 of bounded variation by almost simple strategies, the value of the portfolio becomes

$$\tilde{V}_t^\theta = (\theta^1 . \tilde{Y}_t) - k \int_{[0,t]} \tilde{Y}_s d|D\theta^1|_s - k\tilde{Y}_t |\theta_t^1|,$$

where $D\theta^1$ is the derivative in the sense of distributions of θ^1 and $|D\theta^1|$ the total variation of this measure. We denote by $|D\theta^1|_t$ the measure $|D\theta^1|$ applied to the set $[0, t]$, i.e $|D\theta^1|_t = |D\theta^1|([0, t])$.

The above definition should be interpreted as follows: the first term takes into account capital gains, the second term accounts for the costs indicated in the various transactions and the third represents the potential costs of liquidation of the portfolio.

Lemma 4.2.6 *Let $\tilde{Y}, \tilde{Y}^* : [0, +\infty) \to (0, +\infty)$ be processes with càdlàg trajectories such that*

$$|\tilde{Y}_t - \tilde{Y}_t^*| < k\tilde{Y}_t, \quad \forall t \in [0, +\infty).$$

If $\theta^1 : [0, +\infty) \to \mathbb{R}$ *is a left-continuous function of bounded variation, then*

$$V_t^\theta \leq (\theta^1.\tilde{Y}^*)_t = \int_{[0,t]} \theta_s^1 d\tilde{Y}_s^*, \quad \forall t \in [0, +\infty)$$

and equality holds if and only if $\theta_s^1 = 0, \forall s \leq t$.

Note: Under the assumptions of the lemma above, a strategy θ generates a lower payoff when applied to the process $\tilde{Y}$ in a market with transaction costs than when connected to $\tilde{Y}^*$ without transaction costs. Thus, the no-arbitrage condition applied to $\tilde{Y}^*$ can be immediately extended to $\tilde{Y}$.

Proposition 4.2.7 *Let* $\tilde{Y} = (\tilde{Y}_t)_{t\in[0,T]}$ *be a stochastic process that satisfies assumption 4.2.3 and* $k, T > 0$. *If for any stopping time* τ *such that* $\mathbb{P}[\tau < T] > 0$ *we have*

$$\mathbb{P}\left[\sup_{t\in[\tau,T]} \left|\frac{\tilde{Y}_\tau}{\tilde{Y}_t} - 1\right| < k, \quad \tau < t\right] > 0,$$

then $\tilde{Y}$ *is free of arbitrage with transaction costs* k *in the interval* $[0, T]$.

Making an analysis of the previous proposition, we can make a brief discussion of the assumptions. Obviously, to get an arbitrage at time τ we need to start trading. As we are dealing with markets with transaction costs, the cost incurred in this transaction must be recovered at some future time. If the asset does not vary enough, the agent would not be able to recover the initial costs. Therefore, if there is a possibility of arbitrary small variations in the price of the asset in every moment, then the risk associated with a drop in the price of the asset can not be eliminated and it becomes impossible to carry out an arbitrage strategy.

The proposition 4.2.7 leads us to believe that for certain processes, there are no arbitrage opportunities even with transaction costs k arbitrarily small and under a time horizon T arbitrarily large. We will see that this is the case when we are working with processes defined by Guasoni (2006) in [10] as 'sticky'.

Definition 4.2.8 (Sticky Process) *A stochastic process progressively measurable* $\tilde{Y} = (\tilde{Y}_t)_{t\in[0,+\infty)}$ *is called sticky on the filtration* $\{\mathcal{F}_t\}_{t\in[0,+\infty)}$ *if, for every* $\varepsilon, T > 0$ *and all stopping times* τ *such that* $\mathbb{P}[\tau < T] > 0$ *we have*

$$\mathbb{P}\left[\sup_{t\in[\tau,T]} |\tilde{Y}_\tau - \tilde{Y}_t| < \varepsilon, \quad \tau < t\right] > 0.$$

Proposition 4.2.9 *Let* $\tilde{Y} = (\tilde{Y}_t)_{t\in[0,T]}$ *be a process that satisfies hypothesis 4.2.3. If* $Z = (\log \tilde{Y}_t)_{t\in[0,T]}$ *is sticky, then* $\tilde{Y}$ *is free of arbitrage with transaction costs* k *on the interval* $[0, T]$, $\forall k, T > 0$.

Finally, we present a theorem of exclusion of arbitrage for the fractional Black-Scholes model when we introduce proportional transaction costs arbitrarily small. The proof can be found in [10].

Theorem 4.2.10 *Let $Z = (Z_t)_{t\in[0,T]}$ be a stochastic process such that $Z_t = \vartheta_t + \sigma B_t^H$ with $\sigma > 0$ and $\vartheta : \mathbb{R}^+ \to \mathbb{R}$ is a continuous function. Then Z is sticky and therefore for all $k, T > 0$, the process $\tilde{Y} = (\tilde{Y}_t)_{t\in[0,T]}$ with $\tilde{Y}_t = e^{Z_t}$ is free of arbitrage with transaction costs k in the interval $[0, T]$.*

Chapter 5

Estimation of the Hurst Parameter

In this chapter we present two methods of estimation of the parameter H of the fBm, in order to provide methods for evaluating the long memory of time series. The main motivation for the study of fractional Brownian motion is to use their flexible covariance structure to capture the covariance structure of the data. This can only be done through the estimation of the Hurst parameter H. We present two of the most used ones in the following text, the Rescaled Range analysis and the modified Rescaled Range analysis.

5.1 The Rescaled Range Analysis

The Rescaled Range method is the oldest one of the Hurst parameter estimation methods and was developed by hydrologist Harold Edwin Hurst [13] while working as an engineer in Egypt. Hurst spent his life studying the Nile River and the problems associated with storage of water. The Nile River is known to have characteristics of long memory, i.e. long periods of drought are followed by long periods of flooding. Hurst invented a new statistical method - The Rescaled Range Analysis (R/S). The method was later applied to financial time series by Mandelbrot [20]. We provide a detailed description of the method together with discussion of its weaknesses in the following sections. Let us introduce the procedure of the R/S method (see for example [8] or [15]).

5.1.1 Procedure

1. Transform the original price series $(P_0, P_1, ..., P_T)$ into a series of returns $(r_0, r_1, ..., r_T)$, where $r_i = \frac{P_i - P_{i-1}}{P_{i-1}}$, for $i = 1, 2, .., T$.

2. Divide the time period T into N adjacent sub-periods of length k while $N * k = T$. Each sub-period is to be labeled as I_n with $n = 1, 2, ..., N$. Moreover, each element in I_n is labeled $r_{v,n}$ with $v = 1, 2, ..., k$.

3. For each sub-period, calculate the average value as $\bar{r}_n = \frac{1}{k} \sum_{i=1}^{k} r_{i,n}$.

4. Create new series of accumulated deviations from the arithmetic mean values for each sub-period as $X_{v,n} = \sum_{i=1}^{v} r_{i,n} - \bar{r}_n$, $v = 1, ..., k$.

5. Calculate the range defined as a difference between maximum and minimum value of $X_{v,n}$ for each sub-period as $R_{I_n} = \max(X_{1,n}, ..., X_{k,n}) - \min(X_{1,n}, ..., X_{k,n})$.

6. Calculate the sample standard deviation series $S_{I_n} = \sqrt{\frac{1}{k}\sum_{i=1}^{k}(X_{i,n} - \bar{X}_{i,n})^2}$.

7. Calculate the rescaled range series $(R/S)_{I_n} = \frac{R_{I_n}}{S_{I_n}}$

8. We repeat the process for each sub-period of length k and get the average rescaled range as $(R/S)_k = \frac{1}{N}\sum_{n=1}^{N}(R/S)_{I_n}$ as shown in the figure 5.1.

9. The length k is increased and the whole process is repeated.

10. The Hurst parameter is estimated by fitting the power law

$$\left(\frac{R}{S}\right)_k = Ck^H \tag{5.1}$$

to the data. We may use linar regression to find the slope H for the log plot of R/S against the log plot of k, i.e.

$$\log(R/S)_k \approx H\log(k) + R,$$

where R is independent of H.

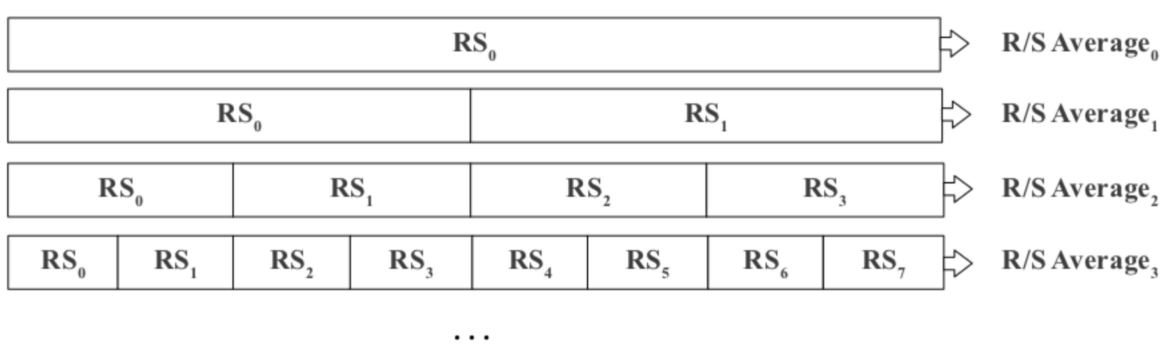


Figure 5.1: Sample Blocks

5.1.2 Notes

We use the length k equal to the power of a set integer value b and a maximum power $pmax$ so that we get sub-periods of length $k = b, b^2, ..., b^{pmax}$ and $b^{pmax} \leq T$. As assumption we use a minimum scale of at least 10 observations and a maximum scale of a half of the time series length, as proposed by several authors (see [15] for further details). In the step 10 we may use logarithm with basis equal to b, i.e.

$$\log_b(R/S)_k \approx H \log_b(k) + R.$$

5.2 Modified Rescaled Range Analysis

As the R/S analysis is known for a long time, it has been a subject to a lot of testing and criticism. The method is mostly criticized for its problematic use for heteroskedastic time series (due to use of sample standard deviation together with a filtration of a constant trend makes R/S analysis sensitive to non-stationarities) and for the series with short-term memory (see Lo [19]). To deal with short-term dependence in the time series, modified rescaled range (M-R/S) is a popular technique proposed by Lo.

5.2.1 Procedure

The procedure of this method differs of the R/S method only in the definition of the standard deviation (S_{I_n}) which deals with both the heteroskedasticity and the short-range dependence. The new equation is defined with a use of auto-covariance function γ of the selected sub-interval I_n up to the lag q as follows

$$S_q^M(I_n) = \sqrt{S_{I_n}^2 + 2\sum_{j=1}^{q} \gamma_j \left(1 - \frac{j}{1+q}\right)}.$$

Thus, R/S turns into a special case of M-R/S with $q = 0$. There are two estimators of optimal lag suggested in the literature. The first one proposed by Lo [19] is the more complicated and still the most used one. The optimal lag is based on the first-order autocorrelation coefficient $\hat{\rho}(1)$:

$$q^* = \left(\frac{3k}{2}\right)^{\frac{1}{3}} \left(\frac{2\hat{\rho}(1)}{1-(\hat{\rho}(1))^2}\right)^{\frac{2}{3}}.$$

The second one by Chin [5] is based on the length of the sub-interval only and sets the optimal lag as

$$q^* = 4\left(\frac{k}{100}\right)^{\frac{2}{9}}.$$

Note that the optimal lag q^* is recalculated for each length of specific sub-period k. We will use these two estimators and compare them.

Following Lo [19], in order to calculate the modified Rescaled Range statistic, $V_q(T)$, instead of considering multiple lags, we only focus on lag $k = T$, the length of the series. Then, the definition of the modified R/S - statistic is the following:

$$V_q(T) = T^{-1/2} \frac{R(T)}{S_q^M(T)}.$$

Lo uses the interval $[0.809, 1.862]$ as the 95% acceptance region for testing the null hypothesis

$$H_0 = \{\text{no long-range dependence, i.e., } H = 0.5\}$$

against

$$H_1 = \{\text{there is long-range dependence, i.e., } 0.5 < H < 1\}.$$

Therefore, at a level of significance of 5%, the null hypothesis is rejected if $V_q(T)$ does not belong to the interval $[0.809, 1.862]$.

5.3 Data

We will analyze the behavior of the stock market returns using daily data of four indices: a Japanese index, the Nikkei; a German index, the DAX; a US index, the Dow Jones; and a UK index, the FTSE. These data were extracted from the internet site `finance.yahoo.com`. The data contains open, high, low, close and volume daily indices. The range of each index is: from 4 January 1984 to 18 December 2009 for Nikkei index, from 26 November 1990 to 18 December 2009 for DAX index, from 1 October 1928 to 18 December 2009 for Dow Jones index and from 2 April 1984 to 18 December 2009 for FTSE index. The daily records are described in business time in the sense that the time units correspond to business days. Returns were computed as $r_t = \frac{P_t - P_{t-1}}{P_{t-1}}$, with P_t being the close index at time t, as shown in figure 5.2.

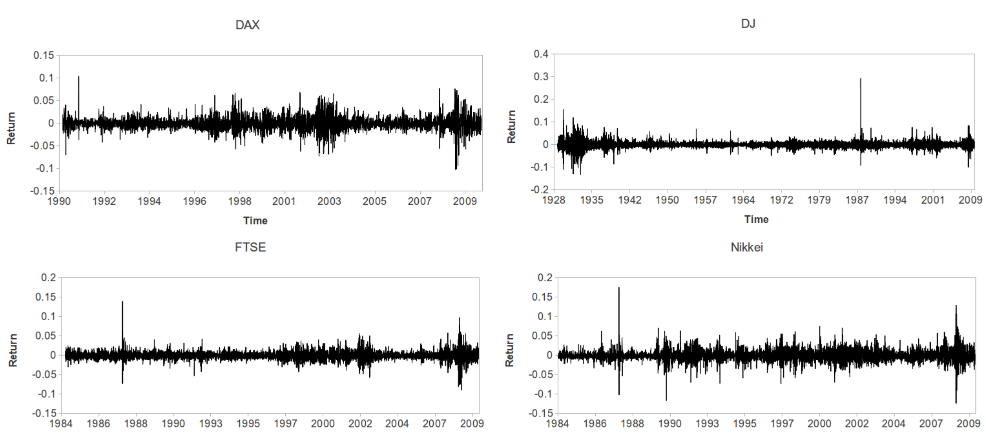


Figure 5.2: Returns of the indices.

To see how the indices vary during the day, we import the data from finance.yahoo.com. We consider data (returns) in periods of one minute.

5.4 Empirical Results

In this section we will present the results of each analysis. First we estimate the Hurst parameter of each series. After that, we estimate the Hurst parameter of the daily series with intervals of one minute. Finally, we present a brief study of the Hurst parameter, e.g. we calculate the Hurst parameter variations for different values of the sliding and the window sizes. The Hurst parameters are obtained by the two methods described above.

5.4.1 Results of Historical Data

Before we show the results of R/S and M-R/S methods of returns for the complete period, we will discuss what basis we consider in order to choose the length k in each method. The Hurst parameter is estimated by calculating the average rescaled range over multiple regions of the data. The figure 5.3 shows the evolution of the Hurst parameter considering different values for the basis, assuming the R/S method. The powers of the basis will be the size k.

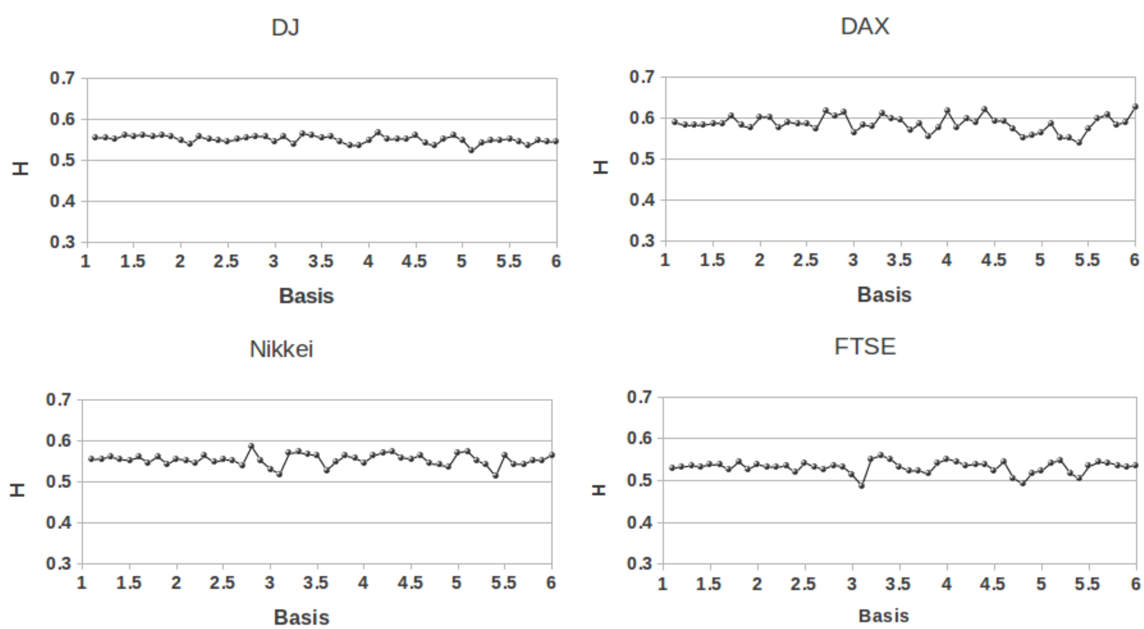


Figure 5.3: Evolution of the Hurst parameter considering different basis.

We can observe that the Hurst parameter does not remain constant for the various basis. However, we cannot detect a trend in the variation of H with the basis size. So, we can calculate the Hurst parameter considering the basis 2 and considering powers of 2 for the values of k. This is the procedure we follow in order to estimate H. The table 5.1 gives the results of R/S and M-R/S methods of returns for the complete period.

	R/S	M-R/S			
Indices		Lo	V_q	Chin	V_q
Nikkei	0.5548	0.5516	1.5311	0.5296	1.5420
DAX	0.6006	0.5931	1.4999	0.5693	1.527
DJ	0.5469	0.5416	1.512	0.5262	1.5081
FTSE	0.5381	0.5334	1.3615	0.5008	1.3991

Table 5.1: Results of Hurst parameter for each method and the M-R/S statistic.

In this table we can see that all indices have Hurst parameter greater than 0.5 with both methods. For the Nikkei, DJ and FTSE the Hurst parameter is close to 0.5 and therefore we can not conclude that the series of returns of these indices exhibit long-range dependence (or it is present but in a very weak form). However, for the DAX index, the Hurst parameter estimated is close to 0.6 which seems to indicate the presence of long memory. The highest values of the Hurst parameter are observed by the R/S method. The M-R/S has values closer to the values of R/S when we use the Lo estimator. The M-R/S statistic is always within the range $[0.809, 1.862]$ which means that we do not reject the null hypothesis, i.e., at a level of significance of 5%, these series do not exhibit long-range dependence. However, in [29], the authors discuss the consistency of Lo's statistic and argue that the M-R/S statistic shows a strong preference for accepting the null hypothesis of no long-range dependence even if long-range dependence is present in data. In [28], the authors show that as the truncation lag q increases, the test statistic V_q has a strong trend toward accepting the null hypothesis, even in situations of purely long-range dependent data.

5.4.2 Results of Recent Data

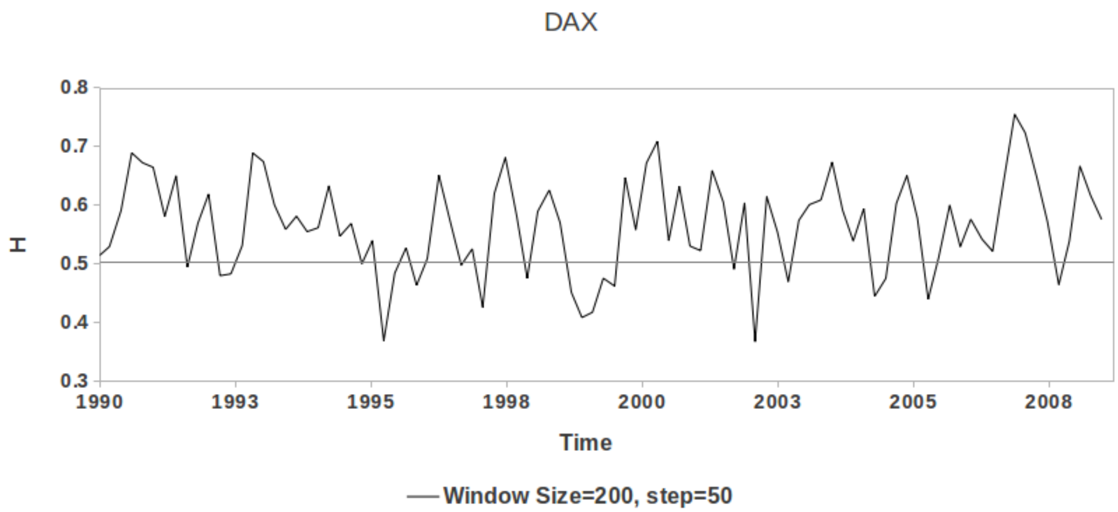
In this section we will observe the evolution of the Hurst parameter throughout the day in intervals of 1 minute. We have considered five stock market indices as DAX, FTSE, Nikkei, NASDAQ and Treasury Yield 30 Years, extracted from the internet site `yahoo.finance.com`. The days considered to calculate the Hurst parameter were from April 17 until May 15 2012. We considered powers of two for size the sample in each method.

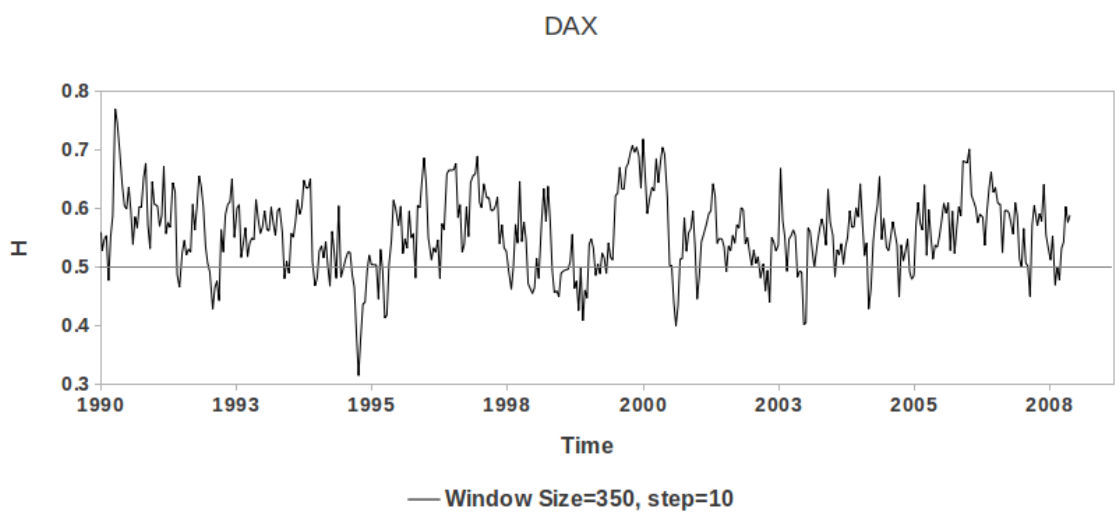
Figure 5.4: Evolution of the Hurst parameter throughout the day in intervals of 1 minute.

We can observe that the two methods (R/S and M-R/S) show the same trend but the values of H estimated by the R/S method are almost always higher. We also observed that using the approach proposed by Chin for computing M-R/S, the H values are always lower than when using the method proposed by Lo. The evolution of H seems to be very erratic and a possible explanation can be found in the effects of the market microstructure which can be more important at short time scales (see Cont [7]).

5.4.3 Time variation of the Hurst parameter

The variations of the Hurst parameter suggest us to estimate the time evolution for subsample of the returns time series, and this task was executed by applying R/S for subsamples corresponding to overlapped sliding windows. In this section we will study the Hurst parameter as a function of time. To this end, we use the historical data described above and calculate the Hurst parameter considering a window and stepping through the data until the end. We now present the results for each time series.


DAX
0.8
0.7
0.6
0.5
0.4
0.3
H
1990
1993
1995
1998
2000
2003
2005
2008
Time
Window Size=200, step=50


DAX
0.8
0.7
0.6
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0.4
0.3
H
1990
1993
1995
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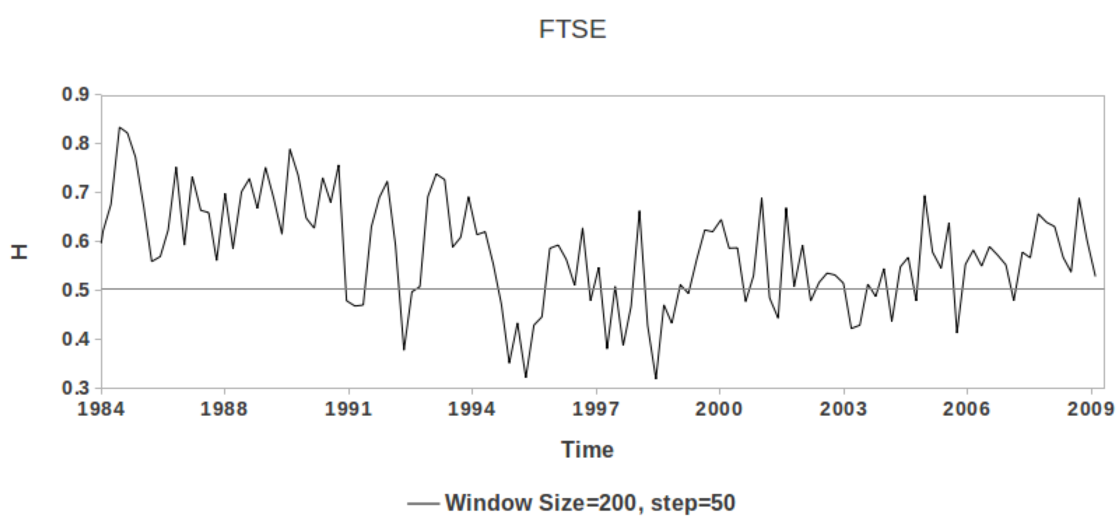
2000
2003
2005
2008
Time
Window Size=350, step=10


FTSE
0.9
0.8
0.7
0.6
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0.4
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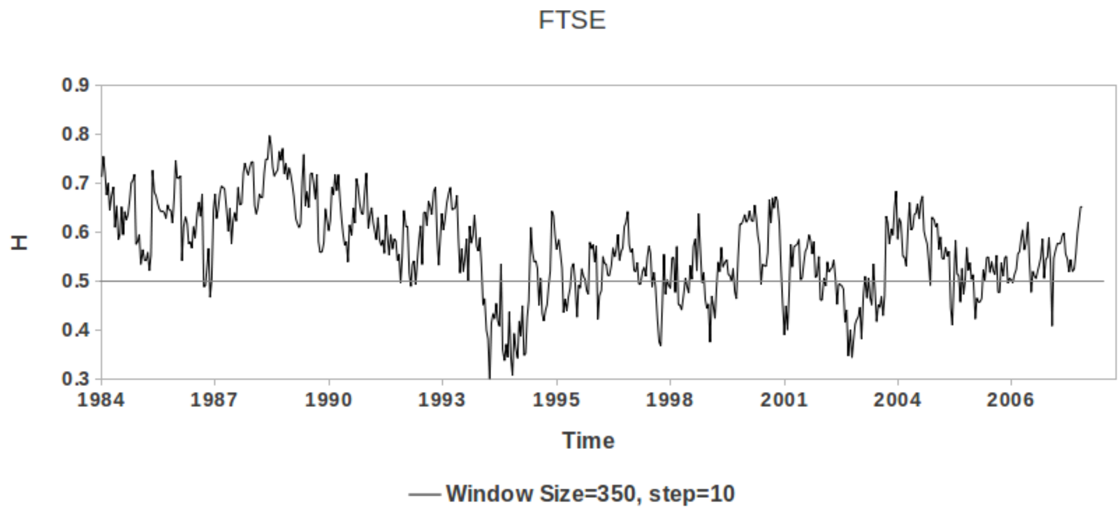
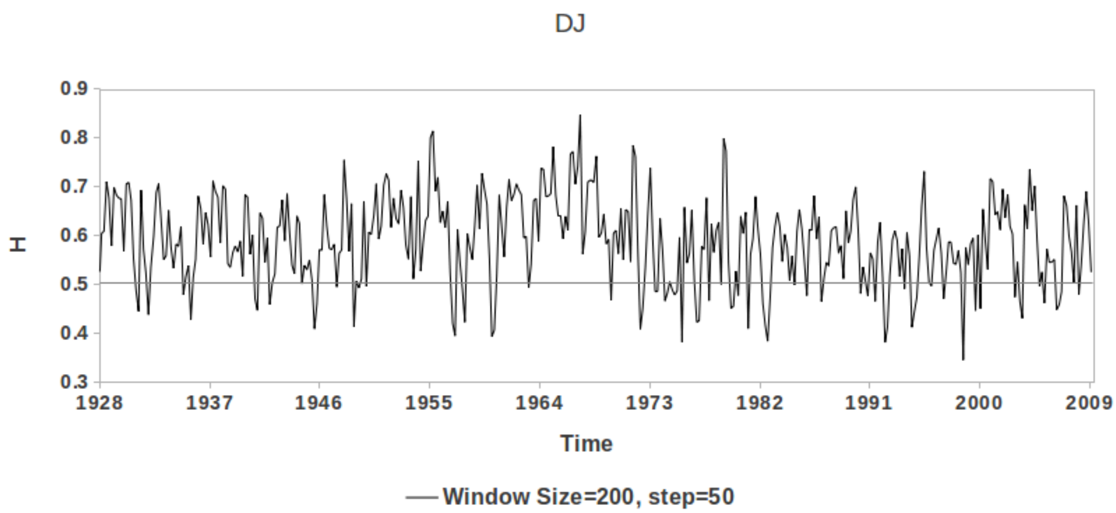
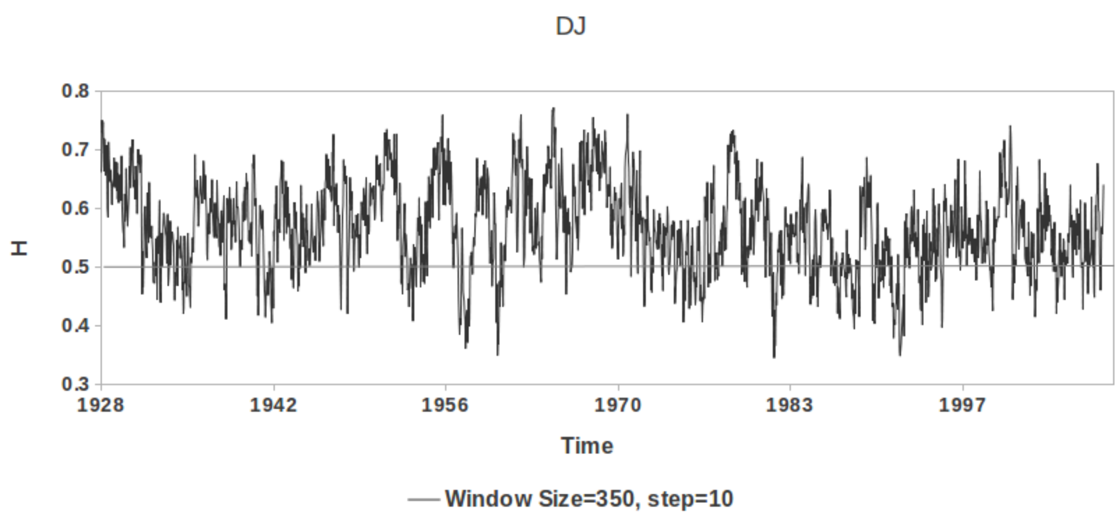
H
1984
1988
1991
1994
1997
2000
2003
2006
2009
Time
Window Size=200, step=50


FTSE
0.9
0.8
0.7
0.6
0.5
0.4
0.3
H
1984
1987
1990
1993
1995
1998
2001
2004
2006
Time
Window Size=350, step=10


DJ
0.9
0.8
0.7
0.6
0.5
0.4
0.3
H
1928
1937
1946
1955
1964
1973
1982
1991
2000
2009
Time
Window Size=200, step=50


DJ
0.8
0.7
0.6
0.5
0.4
0.3
H
1928
1942
1956
1970
1983
1997
Time
Window Size=350, step=10

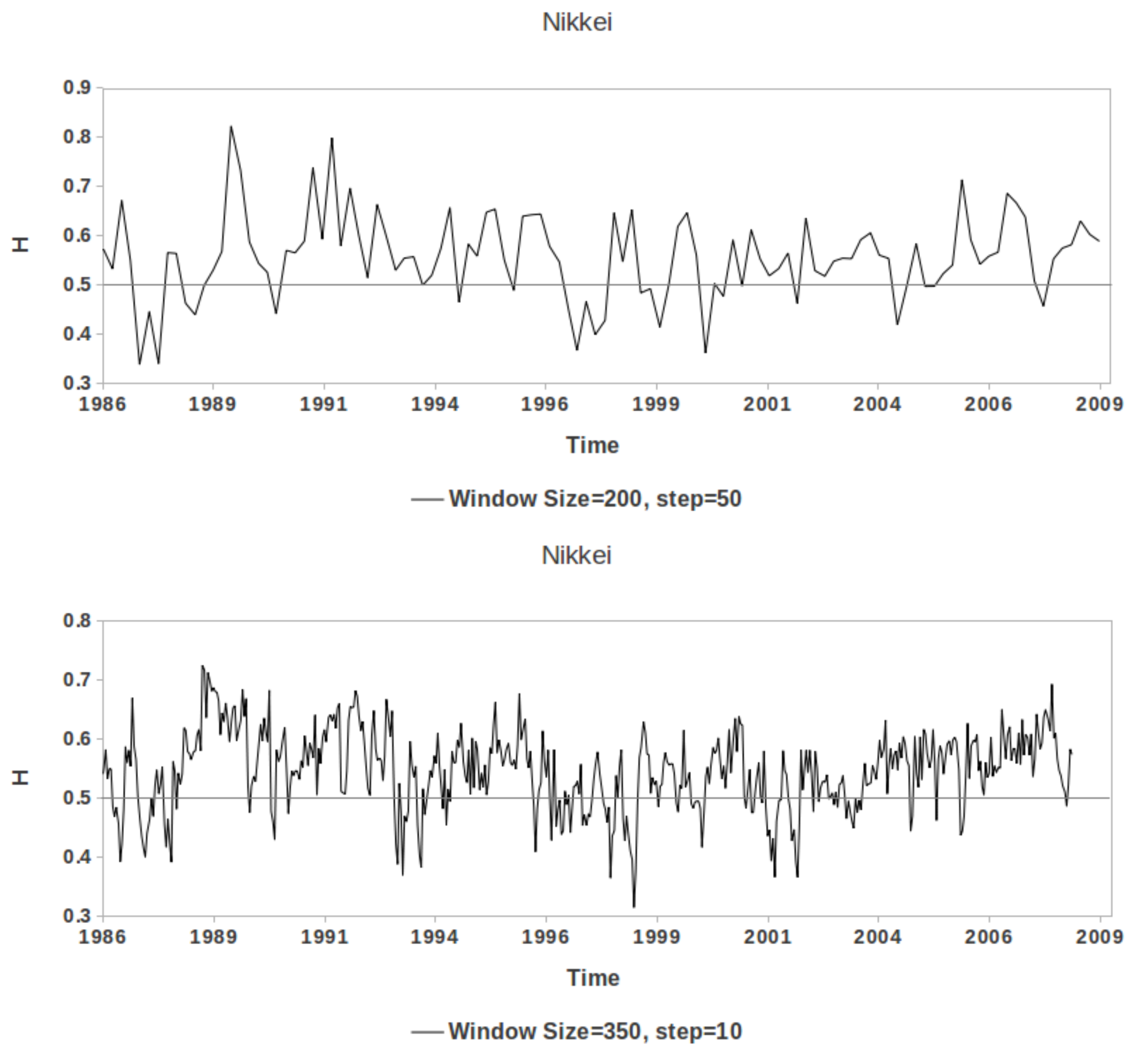


Figure 5.5: Evolution of the Hurst parameter as a function of time.

The Hurst parameter displays an erratic dynamics and exhibits some periods of persistent and anti-persistent behavior. The DAX index shows various periods of high Hurst parameter ($H > 0.6$) denoting periods of persistence effects (eg, between 1990 and 1992, 1999 and 2001, 2007 and 2008, etc.) and also some periods of anti-persistent effects ($H < 0.5$) (eg, 1992, 1995, 1997, etc.). The same effects are observed in the other indices, having periods of persistent and anti-persistent effects.

Looking at the graphs, the one which seems to have a change of trend over time is the FTSE index. It is possible to observe a change in trend around the year 1994. To confirm this observation, we adjusted a linear regression before 1994 and after 1994 as shown in figure 5.6.

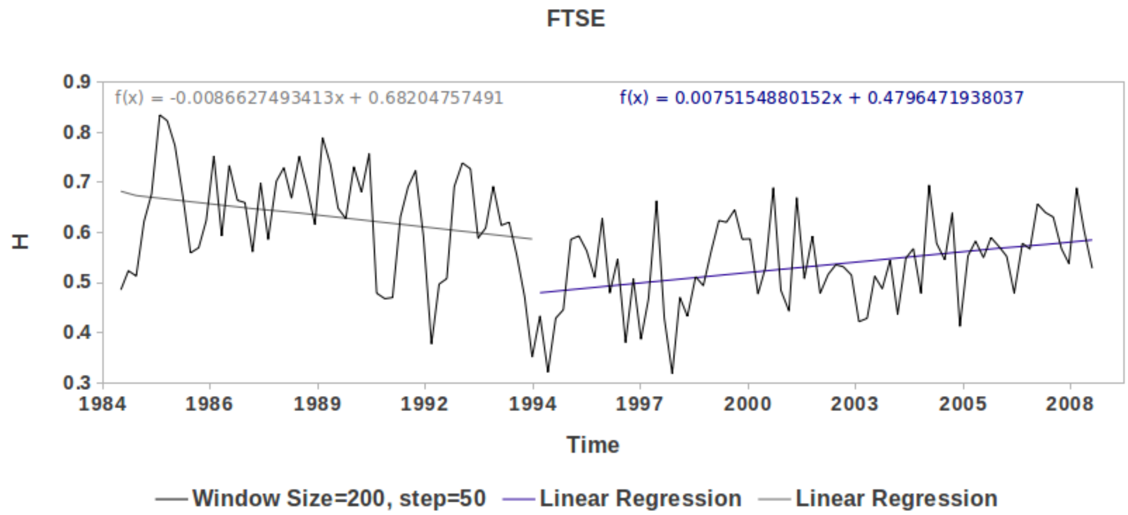


Figure 5.6: Trend of FTSE index before 1994 and after 1994.

Before 1994 the trend is negative, which could mean that the market were becoming more efficient with time (H tends to decline with time, approaching to $H = 0.5$) but after 1994 the trend is positive, the market is becoming more persistent (greater long-memory effects).

Chapter 6

Conclusions

From the data obtained in the previous section we can say that the evidence of long memory effects on the financial returns time series is not clear, since the Hurst parameter is usually greater than 0.5 but in many cases is very close to 0.5. The long memory effects, if they exist, are not strong, since H is between 0.5 and 0.6. However, in the DAX index case, by the R/S analysis, $H = 0.6006$, which is a value compatible with the presence of long-range dependence. Recently, R/S analysis has been shown to overestimate H when compared to other methods (see Krištoufek [15]), as we also saw. Note that the estimates can also be influenced by the choice of minimum and maximum scale. Looking at the modified R/S statistic proposed by Lo in chapter 5, leads us to conclude that the time series used have no long-range dependence effect, because the calculated statistics belongs to the interval [0809, 1862] which means that we do not reject the null hypothesis (no long-range dependence). However, in the article by Willinger, Taqqu and Teverovsky [29], the authors study the Lo's statistical reliability and conclude that there is a strong preference for accepting the null hypothesis of no long-range dependence even if long-range dependence is present in date. The authors conclude that an acceptance of the null hypothesis of no long-range dependence based on the modified R/S statistic should never be viewed as the "final word", mainly because of the serious difficulties that the V_q has in identifying "genuine"long-range dependence. Instead, an acceptance of the null hypothesis based on the test-statistic V_q should always be accompanied and supported by further analysis of the data.

The fact that we can not reject, at least for some time series, that there is long memory in these series implies that the fBm may be useful in such cases as a model for the derivative product pricing, provided that we use models and/or restrictions adequate to exclude arbitrage opportunities, e.g. the model of Cheridito (with restriction of introducing a minimal amount of time between transactions) and the model with proportional transaction costs of Guasoni, which were discussed in chapter 4 of this thesis.

Other way to exclude arbitrage opportunities has been studied by several authors. In [30], the authors proposed a fractional integral called Wick integral and in [22], the author shows that using this integral, there is no arbitrage opportunities. However, there is much controversy around this integral. In [2], Björk and Hult argue that this type of integral has no economic meaning, because the definition of the self-financing trading strategies and/or the definition of the value of a portfolio used in for example [12], does not have a reasonable economic interpretation, and thus that the results in these papers are not economically meaningful. So, we can use this integral mathematically but we don't know what this means economically.

Proposals for future research:

- Analysis of the stochastic behavior of $H(t)$.
- The study of possible applications to finance and the discussion of arbitrage problems for the mixed fractional Brownian motion $W(t) + B^H(t)$ (see Cheridito [3]), where W is the standard Brownian motion. The fractional Brownian motion implies arbitrage opportunities in finance. Does the mixed fractional Brownian motion also implies arbitrage opportunities?
- Stochastic volatility models where we model the volatility process by fractional Brownian motion (see Comte and Renault [6]).
- Study the various types of fractional integrals and their applicability to financial markets, for example the Wick integral and the Path-wise integral. In order to study the Wick integral it is essential to consider the criticisms made by Björk and Hult [2].

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Appendix A

C++ code

In this appendix we present the code developed for the practical part of this work.

A.1 Rescaled Range method

```
#include ⟨cstdlib⟩
#include ⟨iostream⟩
#include ⟨fstream⟩
#include ⟨cstdio⟩
#include ⟨vector⟩
#include ⟨list⟩
#include ⟨math.h⟩

using namespace std;

    double calc_media( vector ⟨double⟩ ∗v, size_t N)
{
    double sum = 0.0;
    for (size_t i = 0; i < N; i++) {
    sum = sum + v->at(i);
    }
    double mean = sum / N;
    return mean;
}

void le_ficheiro(vector ⟨double⟩ ∗data_final, vector⟨string⟩ ∗ano){
    std::ifstream ifile;
    long size = pathconf(".", _PC_PATH_MAX);
    char∗ direc;
    direc=(char∗) malloc((size_t)size);
```

```
    getcwd(direc,(size_t)size);
    char file[size];
    vector⟨string⟩ date;
    vector⟨double⟩ Open, max, min, final, volume;
    string fich;
    cout « "Nome do ficheiro de input: ";getline(cin,fich);
    sprintf(file,"%s/%s",direc,fich.c_str());
    cout « fich« endl;
    cout « file « endl;
    ifile.open(file,ifstream::in);
    if(ifile.fail()) {
        cout « "Falhou abertura do ficheiro " « fich « endl;
        system("PAUSE");return ;
    }
    else
        cout « "Ficheiro "« fich « " aberto!"« endl;
    for(int i=0; ifile.good(); i++){
        char data[10];
        double pmax=0., open=0., pmin=0., pfinal=0., vol=0.;
        ifile » data » open » pmax » pmin » pfinal » vol;
        final.push_back(pfinal);
        date.push_back(data);
    }
    for(size_t i=0; i<final.size()-2;i++)
        double f1,f0;
        f0=final[i];
        f1=final[i+1];
        data_final->push_back(log(f1)-log(f0));
        ano->push_back(date[i]);
    }
    ifile.close();
    return ;
}

/∗
    Calculate the rescaled range for a single region of data.
∗/

double calc_RS_( vector ⟨double⟩ ∗v, const size_t boxSize )
{
    double RS = 0.0;
    if (boxSize > 0) {
        double min;
```

```
        double max;
        double runningSum;
        double runningSumSqr;
        double mean = calc_media( v, boxSize );
        min = 0.0;
        max = 0.0;
        runningSum = 0.0;
        runningSumSqr = 0.0;
        for (size_t i = 0; i < boxSize; i++) {
            double devFromMean = v->at(i) - mean;
            runningSum = runningSum + devFromMean;
            runningSumSqr = runningSumSqr + (devFromMean * devFromMean);
            if (runningSum < min)
                min = runningSum;
            if (runningSum > max)
                max = runningSum;
        }
        double variance = runningSumSqr / static_cast<double>(boxSize);
        double stdDev = sqrt( variance );
        double range = max - min;
        RS = range / stdDev;
    }
    return RS;
}

/*
    Calculate the R/S Average for the R/S values calculated on a set of "boxes" (regions)
of size boxSize.
    v the data set used to estimate the Hurst exponent
    N the size of the data set
    boxSize is the k size mentioned on chapter 5. */

double calc_RS_ave( vector<double> *v, const size_t N, size_t boxSize )
{
    double RSAve=0.0;
    size_t i;
    size_t numBoxes = N / boxSize;
    if (numBoxes > 0) {
        double RS, RSSum=0.0;
        for (i = 0; i+boxSize <= N; i = i + boxSize)
            vector <double> *boxStart= new vector<double>;
            for (size_t j=i; j < i + boxSize; j++)
                boxStart->push_back(v->at(j));
```

```
            RS = calc_RS_( boxStart, boxSize );
            RSSum = RSSum + RS;
            delete boxStart;
        }
        RSAve = RSSum / static_cast⟨double⟩( numBoxes );
    }
    return RSAve;
}

double Hurst (vector ∠double⟩ ∗xx, vector ∠double⟩ ∗yy){
    double meanX=0.0, meanY=0.0;
    meanX= calc_media(xx, xx->size()-1);
    meanY=calc_media(yy, yy->size()-1);
    double difXX=0.0, difXY=0.0;
    for(size_t i=0; i<= xx->size()-1;i++){
        difXY = difXY + (xx->at(i)-meanX)∗(yy->at(i)-meanY);
        difXX = difXX + (xx->at(i)-meanX)∗(xx->at(i)-meanX);
    }
    double H = difXY / difXX;
    return H;
}

int main() {
    vector ⟨double⟩ data_final;
    vector⟨string⟩ ano;
    double RS_Ave, H, base = 2;
    le_ficheiro(&data_final, &ano);
    size_t j=0;
    vector ⟨double⟩ ∗x= new vector⟨double⟩;
    vector ⟨double⟩ ∗y= new vector⟨double⟩;
    vector ⟨double⟩ ∗RSAve= new vector⟨double⟩;
    while(pow(base,j)<=10){
        j++;
    }
    for(size_t i=j; pow(base,i) < data_final.size()/2 ;i++ ){
        RS_Ave=calc_RS_ave(&data_final, data_final.size(), pow(base,i));
        RSAve->push_back(RS_Ave);
        x->push_back(i);
        y->push_back(log(RSAve->at(i-j))/log(base));
    }
    H = Hurst(x, y);
    cout « H « " " « x->size() « endl;
    delete x;
```

```
    delete y;
    return 0;
}
```

A.2 Modified Rescaled Range method

In this section we only present the functions code that are different from the previous section, because this methods only differs in the definition of standard deviation.

```
double covariance(vector⟨double⟩ ∗v, size_t N, int k) {
    double sum=0;
    double cov=0;
    for(size_t i=0; i<N-k;i++){
        sum=sum+(v->at(i)-calc_media(v,N))*(v->at(i+k)-calc_media(v,N));
    }
    cov=(double)sum/(N-k);
    return cov;
}

double q (vector⟨double⟩ ∗v, size_t N) {
    double corr=covariance(v,N,1)/covariance(v,N,0);
    //cout « corr « endl;
    //double q= (double) pow((double)(3*N)/2,1.0/3.0)*pow(pow((2*corr/(1-(corr*corr))),2),1.0/3.0);
//Lo(1991)
    double q=4*pow(N/100,2/9); // Chin (2008)
    //cout « q « endl;
    return q;
}

double calc_RS_( vector ⟨double⟩ ∗v, const size_t boxSize )
{
    double RS = 0.0;
    if (boxSize > 0) {
        double min;
        double max;
        double runningSum;
        double runningSumSqr;
        double mean = calc_media( v, boxSize );
        min = 0.0;
        max = 0.0;
        runningSum = 0.0;
        runningSumSqr = 0.0;
```

```
        for (size_t i = 0; i < boxSize; i++) {
            double devFromMean = v->at(i) - mean;
            runningSum = runningSum + devFromMean;
            runningSumSqr = runningSumSqr + (devFromMean * devFromMean);
            if (runningSum < min)
                min = runningSum;
            if (runningSum > max)
                max = runningSum;
        }
        double variance = runningSumSqr / static_cast<double>(boxSize);
        double est = q(v,boxSize);
        double range = max - min;
        double S = 0.0, sum=0.0;
        for(int i=1; i<=est;i++)
            sum = sum + (1-(i/(est+1)))*covariance(v,boxSize,i);
        S=sqrt( variance + 2*sum);
        RS = range / S;
    }
    return RS;
}

/*
    Calculate the M-R/S statistic.
*/

double Estatistica (vector<double> *v, size_t N)
{
    double Vq;
    if (N > 0) {
        double min;
        double max;
        double runningSum;
        double runningSumSqr;
        double mean = calc_media( v, N );
        min = 0.0;
        max = 0.0;
        runningSum = 0.0;
        runningSumSqr = 0.0;
        for (size_t i = 0; i < N; i++) {
            double devFromMean = v->at(i) - mean;
            runningSum = runningSum + devFromMean;
            runningSumSqr = runningSumSqr + (devFromMean * devFromMean);
            if (runningSum < min)
```

```
            min = runningSum;
        if (runningSum > max)
            max = runningSum;
    }
    double variance = runningSumSqr / static_cast⟨double⟩(N);
    double est = q(v,N);
    double range = max - min;
    double S = 0.0, sum=0.0;
    for(int i=1; i<=est;i++)
        sum = sum + (1-(i/(est+1)))*covariance(v,N,i);
    S=sqrt( variance + 2*sum);
    Vq = (1/sqrt(N))*(range/S);
  }
  return Vq;
}
```