Dynamics of a multipoint variant of Chebyshev-Halley's family

B. Campos

Instituto de Matemáticas y Aplicaciones de Castellón. Universitat Jaume I.Castellón, Spain.

A. Cordero

Instituto de Matemática Multidisciplinar. Universitat Politécnica de Valencia. Spain

Juan R. Torregrosa

Instituto de Matemática Multidisciplinar. Universitat Politécnica de Valencia. Spain

P. Vindel*

Instituto de Matemáticas y Aplicaciones de Castellón. Universitat Jaume I.Castellón, Spain.

Abstract

In this paper, a complex dynamical study of a parametric Chebyshev-Halley type family of iterative methods on quadratic polynomial is presented. The stability of the fixed points is analyzed in terms of the parameter of the family. We also calculate the critical points building their corresponding parameter planes which allow us to analyze the qualitative behaviour of this family. Moreover, we locate some dynamical planes showing different pathological aspects of this family.

Keywords: iterative methods, complex dynamics, Chebyshev-Halley's family, bifurcations.

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1. Introduction

The study of multiple phenomena that commonly appear in various areas of experimental science and technology lead, in a more or less natural way, to formulate prob-

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*Corresponding author
Email addresses:* `campos@uji.es` (B. Campos), `acordero@mat.upv.es` (A. Cordero), `jrtorre@mat.upv.es` (Juan R. Torregrosa), `vindel@uji.es` (P. Vindel)

lems whose mathematical expression is a nonlinear equation or a differential equation or, more often, a system of equations. The problem is that in practice it is very difficult, if not impossible, to find the exact solution of these equations; therefore, it is necessary to resort to numerical approximations by using iterative methods. This means that the output of the method is a sequence of images $\left\{ z_0, \, R\left(z_0\right), \, R^2\left(z_0\right), ..., R^n\left(z_0\right), ...\right\}$ for the initial condition z_0, where R is a function that represents the fixed-point operator of the iterative scheme. Therefore, it can be seen as a discrete dynamical system and we can study it from this point of view.

The historical seed of complex dynamics goes back to Ernst Schröder and Arthur Cayley who, at the end of the nineteenth century, investigated the global dynamics of Newton's method in the complex plane $\mathbb{C}$, applied on polynomials of degree two. They were able to see that there are one neighborhood around each root of the quadratic polynomial where Newton's method converges; in fact, these domains can be extended to two half planes and the boundary straight line between them is precisely the bisectrix. Furthermore, any Newton's map for a quadratic polynomial with two different roots is conformal conjugated to the map z^2 in the Riemann sphere $\hat{\mathbb{C}}$. Nevertheless, Newton's method applied on polynomials of degree greater than two is a more complicated rational function. In this case, the Riemann sphere $\hat{\mathbb{C}}$ is considered as the domain of the rational mapping R associated with the iterative method.

The study of the dynamics of Newton's method has been extended to other one-point iterative schemes used for solving nonlinear equations, with convergence order up to 3 (see, e.g. [1]). In some previous papers, we have considered the dynamical study of Chebyshev-Halley's family [9], the King's family [8], the c-family [5] and finally, the (α, c)-iterative class, which includes Chebyshev–Halley and c-families [6]. A dynamical study of the operators defined by the iterative methods help us to know more widely the regions where these methods have a good behavior.

The natural space for iterating a rational map R is the Riemann sphere $\hat{\mathbb{C}}$. For a given rational map R, the sphere splits into two complementary domains: the Fatou set $\mathcal{F}(R)$ where the family of iterates $\left\{R^n\left(z\right)\right\}_{n\in\mathbb{N}}$ is a normal family, and the Julia set $\mathcal{J}\left(R\right)$ where the family of iterates fails to be a normal family. The Fatou set, when nonempty, is given by the union of, possibly, infinitely many open sets in $\hat{\mathbb{C}}$, usually

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called Fatou components, that is, the Fatou set is composed by the set of points whose orbits tend to an attractor (fixed point, periodic orbit, infinity, ...). On the other hand, it is known that the Julia set is a closed, totally invariant, perfect nonempty set, and coincides with the closure of the set of repelling periodic points. For a deep review on iteration of rational maps see [2].

Given a rational function $R : \hat{\mathbb{C}} \rightarrow \hat{\mathbb{C}}$, where $\hat{\mathbb{C}}$ is the Riemann sphere, the *orbit of a point* $z_0 \in \hat{\mathbb{C}}$ is defined as:

$$\{z_0, R(z_0), R^2(z_0), ..., R^n(z_0), ...\}$$

We are interested in the study of the asymptotic behavior of the orbits depending on the initial condition z_0, that is, we are going to analyze the phase plane of the map R defined by the different iterative methods. To obtain these phase spaces, the first of all is to classify the starting points from the asymptotic behavior of their orbits.

A $z_0 \in \hat{\mathbb{C}}$ is called a *fixed point* if it satisfies: $R(z_0) = z_0$. A *periodic point* z_0 of period $p > 1$ is a point such that $R^p(z_0) = z_0$ and $R^k(z_0) \neq z_0$, $k < p$. A *pre-periodic point* is a point z_0 that is not periodic but there exists a $k > 0$ such that $R^k(z_0)$ is periodic. A *critical point* z_0 is a point where the derivative of rational function vanishes, $R'(z_0) = 0$.

On the other hand, a fixed point z_0 is called *attractor* if $|R'(z_0)| < 1$, *superattractor* if $|R'(z_0)| = 0$, *repulsor* if $|R'(z_0)| > 1$ and *parabolic* if $|R'(z_0)| = 1$.

The basin of attraction of an attractor α is defined as the set of pre-images of any order:

$$\mathcal{A}(\alpha) = \{z_0 \in \hat{\mathbb{C}} : R^n(z_0) \rightarrow \alpha, n \rightarrow \infty\}.$$

As we have said, iterative methods are used for finding roots of a nonlinear equation and, from a dynamical point of view, these roots are fixed points of the operator R associated to the method; we conducted this study in Section 2, with particular emphasis in the study of the region of the parameter plane where the fixed points are attractive (Propositions 1 and 3).

The basin of attraction of an attractor needs at least one critical point inside, so, it is important the number of critical points because they are the causative of the instability of numerical methods (Section 3). In Section 3 we also build the parameter planes

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associated to the different free critical points. Finally, in Section 4, we study some methods coming from the parametric family studied in this paper, specially chosen for their stable or unstable behavior, and show the dynamical planes for these values of the parameter.

1.1. A multipoint variant of Chebyshev-Halley's family

In this paper we study the dynamics of a multipoint variant of Chebyshev's method for solving a nonlinear equation $f(z) = 0$. Considering the Newton-like iterative method as a predictor

$$y_n = z_n - \alpha \frac{f(z_n)}{f'(z_n)},$$

Chebyshev-Halley's family is modified by using the second-order derivative on y_n instead of z_n:

$$z_{n+1} = z_n - \frac{f(z_n)}{f'(z_n)} - \frac{1}{2} \frac{f(z_n)^2 f'(z_n) f''(y_n)}{(f'(z_n)^2 - af(z_n) f''(y_n))^2}, \tag{1}$$

If we put $a = 0$ and $\alpha = 0$ we have the Chebyshev's method. In [3], the authors present a new fourth-order variant of Chebyshev method from this family when $\alpha = \frac{1}{3}$ and $a = \frac{1}{2}$. So, we fix the value of $\alpha = \frac{1}{3}$ for developing a dynamical study of this family, depending on one parameter a.

So, the fixed point operator corresponding to the family described in (1) is:

$$O_f(z, a) = z - \frac{f(z)}{f'(z)} - \frac{1}{2} \frac{f(z)^2 f'(z) f''(y)}{(f'(z)^2 - af(z) f''(y))^2}, \tag{2}$$

when

$$y = z - \frac{1}{3} \frac{f(z)}{f'(z)}.$$

In this work, we analyze the dynamics of this operator when it is applied on quadratic polynomials. It is known that any quadratic polynomial can be transformed, by means of an affine map, to $p(z) = z^2 + c$ with no qualitative changes on the dynamics of family (1). Moreover, P. Blanchard [4], by considering the conjugacy map with the following properties:

$$h(\infty) = 1, \qquad h(i\sqrt{c}) = 0, \qquad h(-i\sqrt{c}) = \infty,$$

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proved that, for quadratic polynomials, Newton's operator is always conjugate to the rational map z^2, and $z = 0$ and $z = \infty$ are associated to the roots of the quadratic polynomial $p(z) = z^2 + c$. By the same procedure, after applying this conjugacy map to operator (2) we obtain the rational function:

$$O_p(z,a) = \frac{z^3(-1+2a-z)(2+(3-2a)z+z^2)}{(-1+(2a-1)z)(1+(3-2a)z+2z^2)},\qquad (3)$$

depending on the parameter a. Additionally, it is easy to prove the following result, that will be useful for checking that $z = \infty$ is a fixed point.

Lemma 1. *The operator defined in equation (3) satisfies the following statement:*

$$O_p\left(\frac{1}{z},a\right) = \frac{1}{O_p(z,a)}.$$

2. Study of the fixed points

We study the dynamics of operator $O_p(z,a)$ in terms of parameter a. In this section, we calculate the fixed points of $O_p(z,a)$ analyzing the number and their stability depending on the parameter a.

The fixed points of $O_p(z,a)$ are the roots of the equation $O_p(z,a) = z$. Solving this equation we obtain $z = 0$, $z = \infty$, $z = 1$ (if $a \neq 1$ and $a \neq 3$) and the four roots of the symmetric polynomial equation

$$z^4 + (5-4a)z^3 + 4(2-2a+a^2)z^2 + (5-4a)z + 1 = 0.\qquad (4)$$

These roots are given by:

$$z_{1,2}(a) = \frac{1}{4}\left(4a-5+\sqrt{1-8a}\pm 2\sqrt{\frac{1}{4}\left(4a-5+\sqrt{1-8a}\right)^2-4}\right),\quad (5)$$

$$z_{3,4}(a) = \frac{1}{4}\left(4a-5-\sqrt{1-8a}\pm 2\sqrt{\frac{1}{4}\left(4a-5-\sqrt{1-8a}\right)^2-4}\right).$$

In the following we avoid the dependence of a in the notation of the fixed and critical points, unless necessary.

Due to the symmetry of the polynomial (4), we have that $z_1 = \dfrac{1}{z_2}$ and $z_3 = \dfrac{1}{z_4}$.

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Indeed, the number of fixed points is reduced for some values of a. For example, if $a = 2 \pm i$ then $z_1 = z_2 = 1$, if $a = 0$ then $z_1 = z_2 = -1$. Moreover, $z = -1$ is a pre-periodic point, i. e. $O_p(-1, a) = 1$.

Let us remember that fixed points different from $z = 0, z = \infty$ (that are associated to the roots of the quadratic polynomial) are called *strange fixed points*.

In order to study the stability of the fixed points, we calculate the first derivative of $O_p(z, a)$,

$$O'_p(z, a) = -2z^2 \frac{(1 + (2 - 2a)z + z^2)P(z, a)}{(-1 + (2a - 1)z)^2 (1 + (3 - 2a)z + 2z^2)^2}, \tag{6}$$

where

$$P(z, a) = 6a - 3 + (-12 + 22a - 12a^2)z + (-18 + 32a - 24a^2 + 8a^3)z^2 + (-12 + 22a - 12a^2)z^3 + (6a - 3)z^4. \tag{7}$$

As a fixed point is attractive or repulsive if $|O'_p(z, a)|$ is less than or greater than one, respectively, this function is known as *stability function*.

From (6) we obtain that $z = 0$ and $z = \infty$ are always superattractive fixed points, but the stability of the other strange fixed points changes depending on the values of the parameter a.

Remark 1. *Let us notice that for $a = \frac{1}{2}$ the degree of polynomial (7) decreases, in fact, $P(z, \frac{1}{2}) = -z(4 + 7z + 4z^2)$.*

2.1. Stability of $z = 1$

We begin with the stability of the strange fixed point $z = 1$, when $a \neq 1$ and $a \neq 3$. Then we consider,

$$O'_p(1, a) = \frac{2(a - 2)^2}{(a - 3)(a - 1)}. \tag{8}$$

The stability of this point is shown in the following result.

Proposition 1. *For every value $a = x + iy$ of the parameter, the strange fixed point $z = 1$ satisfies the following statements:*

i) $z = 1$ *is an attractor inside curve C defined by:*

$$3y^2 = (-11 + 12x - 3x^2 + 2\sqrt{-11 + 12x - 3x^2}).$$

It is a superattractor for $a = 2$.

ii) $z = 1$ *is a parabolic point for values of the parameter a on curve C, and*

iii) $z = 1$ *is a repulsive fixed point for values of a outside curve C.*

Proof. From equation (8), the stability function of the fixed point $z = 1$ is:

$$\left| O_p'(1, a) \right| = \left| \frac{2(a - 2)^2}{(a - 3)(a - 1)} \right|$$

To know where this point is parabolic, we look for the points where the stability function equals one:

$$\left| O_p'(1, a) \right| = 1,$$

then

$$2 \left| a - 2 \right|^2 = \left| a - 3 \right| \left| a - 1 \right|.$$

By writing $a = x + yi$, we obtain

$$4 \left((x - 2)^2 + y^2 \right)^2 = \left((x - 3)^2 + y^2 \right) \left((x - 1)^2 + y^2 \right),$$

developing on both sides of the equality and simplifying:

$$55 - 104x + 74x^2 - 24x^3 + 3x^4 + (22 - 24x + 6x^2)y^2 + 3y^4 = 0,$$

whose solution is the curve C (Figure 1):

$$3y^2 = (-11 + 12x - 3x^2 + 2\sqrt{-11 + 12x - 3x^2}),$$

where x and y are the real and the imaginary part, respectively, of the parameter a.

As C is a closed curve, it separates the complex plane into two complementary regions, in each of which one of the inequalities is satisfied. From equation (8), it is easy to see that $z = 1$ is a superattractor for $a = 2$; as this value is inside the curve C, the fixed point $z = 1$ must be attractive inside this curve and repulsive outside. ∎

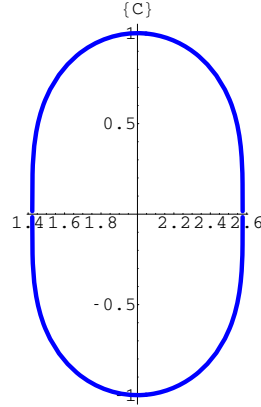As $z = -1$ is a pre-periodic point, its dynamical behaviour is determined by $z = 1$.

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Figure 1: Stability region of $z = 1$ in the parameter plane.

115 *2.2. Stability of the strange fixed points z_i*

In the next result we prove that the stability functions of z_1 and z_2 coincide; therefore, z_1 and z_2 exhibit the same dynamical behaviour. The same occurs for z_3 and z_4.

Proposition 2. *The stability functions of the strange fixed points z_i, $i = 1, 2, 3, 4$ satisfy the following statements:*

$$\left| O'_p(z_1, a) \right| = \left| O'_p(z_2, a) \right|, \qquad \left| O'_p(z_3, a) \right| = \left| O'_p(z_4, a) \right|$$

for any value of the parameter a.

120 **Proof.**

It can be checked that

$$O'_p\left(\frac{1}{z}, a\right) = \frac{(-1 + (2a-1)z)^2 \, (1 + (3 - 2a)z + 2z^2)^2}{z^4 \, (-z + 2a - 1)^2 \, (z^2 + (3 - 2a)z + 2)^2} O'_p(z, a).$$

Then, $O'_p\left(\frac{1}{z}, a\right) = O'_p(z, a)$ when $\dfrac{(-1 + (2a-1)z)^2 \, (1 + (3 - 2a)z + 2z^2)^2}{z^4 \, (-z + 2a - 1)^2 \, (z^2 + (3 - 2a)z + 2)^2} =$ 1. So, we can require

$$(-1 + (2a - 1)z)\,(1 + (3 - 2a)z + 2z^2) = z^2\,(-z + 2a - 1).(z^2 + (3 - 2a)z + 2)$$

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After some algebraic manipulations, we obtain that the values of z that satisfy this request must satisfy the equation:

$$(z - 1)\left(z^4 + (5 - 4a)z^3 + 4(2 - 2a + a^2)z^2 + (5 - 4a)z + 1\right) = 0$$

and this equation is just equation (4), that it is satisfied by the strange fixed points $z_i, i = 1, 2, 3, 4$. As $z_1 = \dfrac{1}{x_2}$ and $z_3 = \dfrac{1}{x_4}$, their corresponding stability functions coincide. ∎

Now, we draw the stability functions of all strange fixed points for real values of parameter a (see Figure 2). In this figure the stability functions are coloured as follows: red colour corresponds to $|O'_p(z_1(a), a)|$, green to $|O'_p(z_3(a), a)|$ and blue to $|O'_p(1, a)|$; black colour is for the unit.
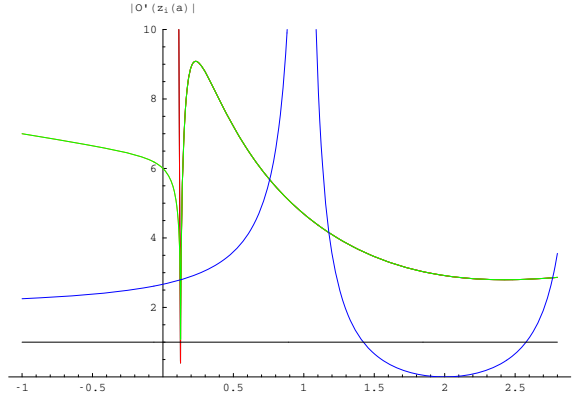
Figure 2: Stability functions of strange fixed points for real values of a.

In Figure 3 we show an enlargement of Figure 2; we can observe that the stability functions of z_1 and z_2 reach values below one whereas that the stability functions of z_3 and z_4 are always above one. This information will be useful for finding the stability regions of these points in the complex plane.

We can also obtain the stability functions of these strange points in a three dimensional picture (Figure 4). The horizontal plane is the complex plane where the parameter a varies and the vertical axis corresponds to the stability functions of strange points, i.e., $\left|O'_p(z_1(a), a)\right|$ and $\left|O'_p(1, a)\right|$. Moreover, we have also drawn here the curves
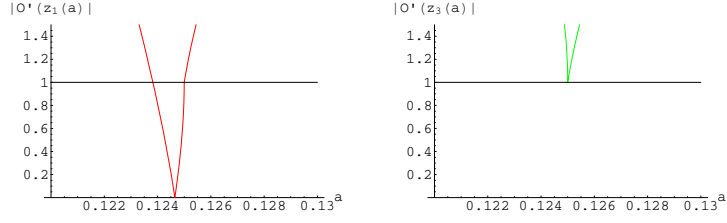
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Figure 3: Detail of the stability functions of strange fixed points z_i for real values of a.
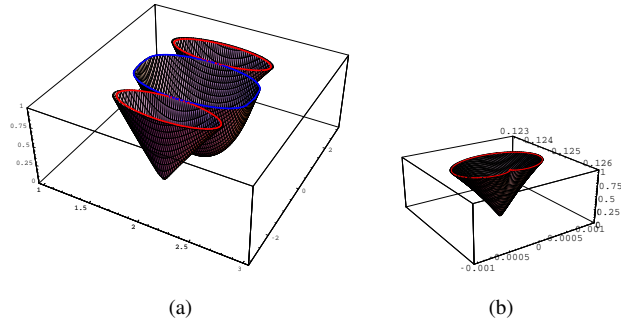
obtained in Propositions 1 and 3.

(a)   (b)

Figure 4: Stability functions of the strange fixed points, z_1, z_2 and $z = 1$ and the curves C, C_1, C_2 and C_3 in the complex plane.

From Proposition 2, we know that the stability regions of the fixed points z_1 and z_2 in the parameter plane are the same and these regions are defined by $\left|O'_p(z_1(a), a)\right| < 1$. The curves $\left|O'_p(z_1(a), a)\right| = 1$ are difficult to find, so we look for algebraic curves that approximate them according Figure 4. The results we obtain are shown in the following proposition.

Proposition 3. *The fixed points $z_1(a)$ and $z_2(a)$ are attractors for $a = x + iy \in D$ such that $D = D_1 \cup D_2 \cup D_3$, where D_1 is the disk delimited by the circumference C_1:*
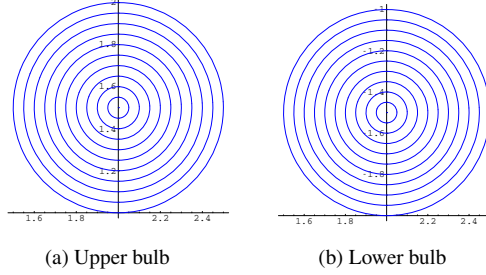
$$(x - 2)^2 + (y - 1.5)^2 = \frac{1}{4}, \tag{9}$$

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(a) Upper bulb   (b) Lower bulb

Figure 5: Fiber of circles inside the bulbs

D_2 *is the disk delimited by the circumference* C_2*:*

$$(x - 2)^2 + (y + 1.5)^2 = \frac{1}{4}, \tag{10}$$

and D_3 *is the region delimited by the cardioid* C_3*:*

$$
\begin{aligned}
x(t) &= \frac{1}{8} - \frac{1}{4}\frac{6.9}{6000} + \frac{6.9}{6000}\left(\frac{1}{2}\cos t - \frac{1}{4}\cos 2t\right), \\
y(t) &= \frac{6.9}{6000}\left(\frac{1}{2}\sin t - \frac{1}{4}\sin 2t\right),\ t \in [0, 2\pi].
\end{aligned}
$$

Proof.

We know that $z_1 = z_2 = 1$ for $a = 2 \pm i$; so, we look for two bulbs which

are tangent to the curve C at these points. As we see in the following, z_1 and z_2 are attractive inside these bulbs. In fact, $a = 2 \pm i$ are bifurcations points where $z = 1$ changes from attractive to repulsive and the strange points z_1 and z_2 change from repulsive to attractive. We can parametrize a bundle of circles inside each bulb (Figure 5); the equations of the outer circles are:

$$
\begin{aligned}
C_1 &: \quad (x - 2)^2 + (y - 1.5)^2 = \frac{1}{4}, \\
C_2 &: \quad (x - 2)^2 + (y + 1.5)^2 = \frac{1}{4}.
\end{aligned}
$$

We deduce that the strange fixed points z_1 and z_2 are attractive inside these bulbs by drawing the stability function $\left|O'_p(z_1(a), a)\right|$ applied on the different circles previously defined and checking that the stability function $\left|O'_p(z_1(a), a)\right|$ on the points belonging to the bundle of circles have values lower than one (see Figure 6). In both
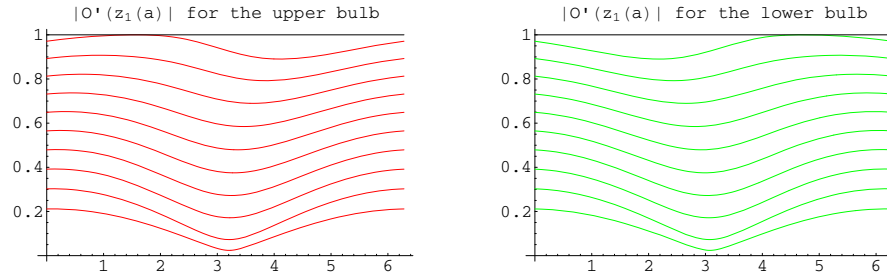
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Figure 6: Stability functions for $z_1(a)$ for values of a belonging to the bundle of circles inside upper and lower bulbs.

plots, the upper coloured curves correspond to $\left|O'_p(z_1,a)\right|$ applied on the outer circumferences C_1 and C_2. The black line is the unit. Moreover, the upper curves have value 1 when $a = 2 \pm i$, respectively.

Let us observe that both stability functions seem to be symmetric.
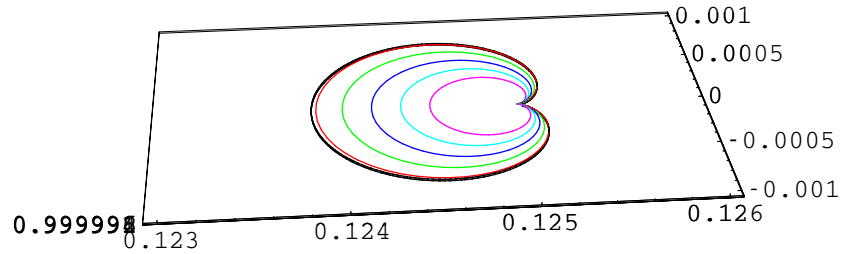

Figure 7: Bundle of cardioids inside the little bulb.

Similarly, we deduce from Figure 3 the existence of a small region in the complex plane where z_1 and z_2 are also attractive (see Figure 4b; to prove that D_3 is inside this region we perform a bundle of cardioids inside D_3 (Figure 7). The parametrization of

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the outer cardioid C_3 is:

$$x(t) = \frac{1}{8} - \frac{1}{4}\frac{6.9}{6000} + \frac{6.9}{6000}\left(\frac{1}{2}\cos t - \frac{1}{4}\cos 2t\right)$$
$$y(t) = \frac{6.9}{6000}\left(\frac{1}{2}\sin t - \frac{1}{4}\sin 2t\right),\ t \in [0, 2\pi].$$

As before, we apply the stability function $\left|O_p'(z_1, a)\right|$ on the points belonging to the bundle and we obtain values lower than one (Figure 8). The red curve correspond to $\left|O_p'(z_1, a)\right|$ applied on C_3. The black line is the unit.
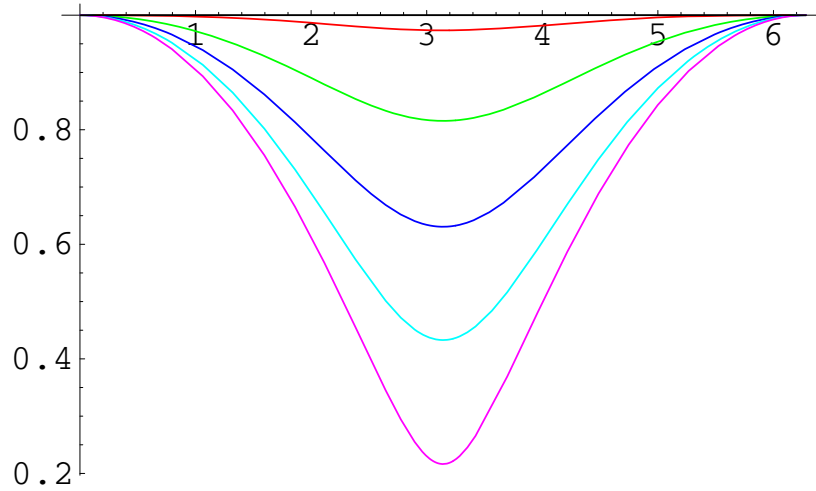

Figure 8: Stability functions for z_1 for values of a belonging to the bundle of cardioids.

■

On the other hand, we have checked numerically that the only point where $\left|O_p'(z_3, a)\right| = 1$ is for $a = \frac{1}{8}$. So, from Figures 2 and 3 we can deduce that the other strange fixed points z_3 and z_4 are always repulsive.

3. Study of the critical points and parameter planes

As we have pointed at the beginning, the critical points of $O_p(z, a)$ are the roots of $O_p'(z, a) = 0$. From equation (6), we know that these roots are $z = 0$, $z = \infty$,

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the solutions of $1 + (2 - 2a)z + z^2 = 0$ and the solutions of $P(z, a) = 0$, where $P(z, a) = 0$ is described in (7).

The roots of $1 + (2 - 2a)z + z^2 = 0$ are:

$$c_{\pm} = a - 1 \pm \sqrt{a^2 - 2a},$$

satisfying $c_+ = \dfrac{1}{c_-}$. The roots of the symmetric fourth degree polynomial $P(z, a)$ are given by:

$$
\begin{aligned}
c_{1,2} &= \frac{1}{2}\left(x_+ \pm \sqrt{x_+^2 - 4} \right), \\
c_{3,4} &= \frac{1}{2}\left(x_- \pm \sqrt{x_-^2 - 4} \right),
\end{aligned}
\tag{11}
$$

where

$$x_{\pm} = \frac{6 - 11a + 6a^2 \pm a\sqrt{1 + 36a - 12a^2}}{3(2a - 1)}.$$

Let us remark that, due to the symmetry of $P(z, a)$, $c_1 = \dfrac{1}{c_2}$ and $c_3 = \dfrac{1}{c_4}$.

Then, there are six critical points (called *free critical points*), different from the roots of the polynomial, but the parameter planes of inverse critical points coincide, i.e. there are only three independent free critical points. This number decreases in the following cases:

- If $a = 0$ then, the only free critical point is -1.

- If $a = 1$ or $a = 3$ then, $c_1 = c_2 = 1$.

- If $a = 2$ then, $c_+ = c_- = c_1 = c_2 = 1$.

- If $a = \frac{1}{6}\left(9 \pm 2\sqrt{21}\right)$ then, $c_1 = c_3$ and $c_2 = c_4$. There are four free critical points.

The dynamical behaviour of operator $O_p(z, a)$ depends on the values of the parameter a. The parameter plane is obtained by iterating one critical (free) point; each point of the parameter plane is associated with a complex value of a, i.e., with an element of the family. To build this parameter plane we use the algorithms designed in [7], with MatLab software. The following figures are made by using these algorithms, by using a mesh of 2000×2000 points, a maximum of 200 iterations and a tolerance of 10^{-3}.
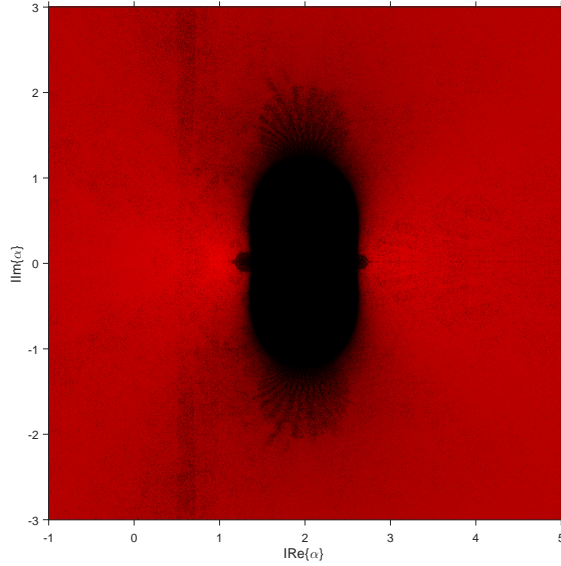
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Figure 9: Parameter plane for the critical point c_+

Red colour in Figures 9, 10 and 11 means that the critical point is into the basins of attraction of $z = 0$ or $z = \infty$, whereas that black colour indicates that the critical point generates its own dynamics.

195   The critical points verify $c_+ = \dfrac{1}{c_-}$, $c_1 = \dfrac{1}{c_2}$ and $c_3 = \dfrac{1}{c_4}$. As $O_p\left(\frac{1}{z}, a\right) = \dfrac{1}{O_p(z, a)}$ (Lemma 1) the conjugated critical points exhibit the same dynamics; so, we consider only three independent critical points and we draw the parameter plane corresponding to each of them (Figures 9, 10 and 11).

We can also ensure that the critical point c_1 is in the basin of $z = 1$ or z_1 for those

200   values of the parameter for which $z = 1$ or z_1 are attractive, by overlaying their basins of attraction (Figure 4) with the parameter plane of c_1 (Figure 10). This is illustrated in Figure 12 .

4. Dynamical planes

A classical result establishes that there is at least one critical point associated with

205   each invariant Fatou component. As $z = 0$ and $z = \infty$ are both superattractive fixed
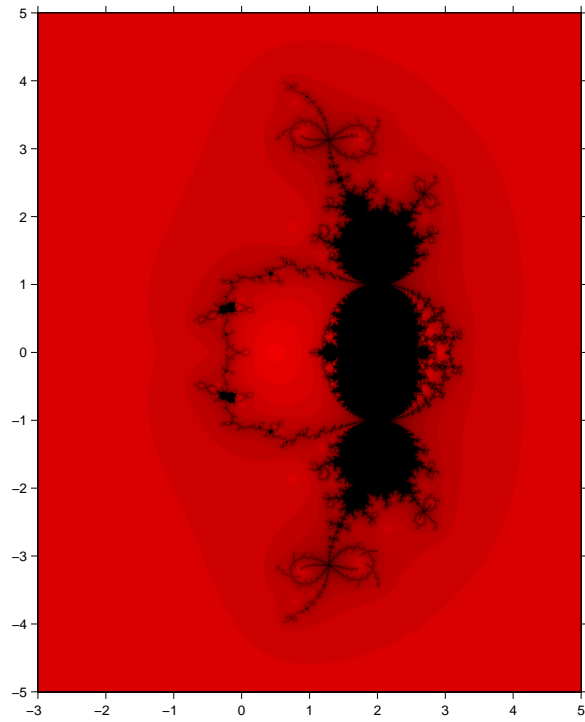
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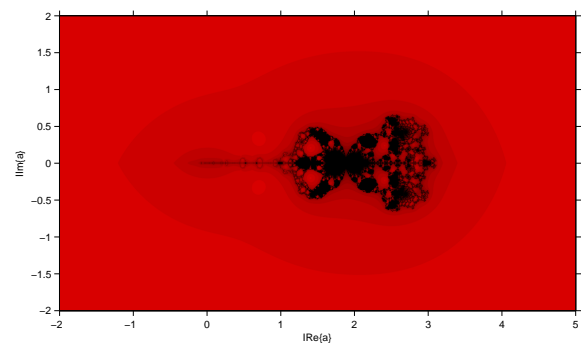Figure 10: Parameter plane for the critical point c_1.

Figure 11: Parameter plane for the critical point c_3.

points, they also are critical points and give rise to their respective Fatou components.
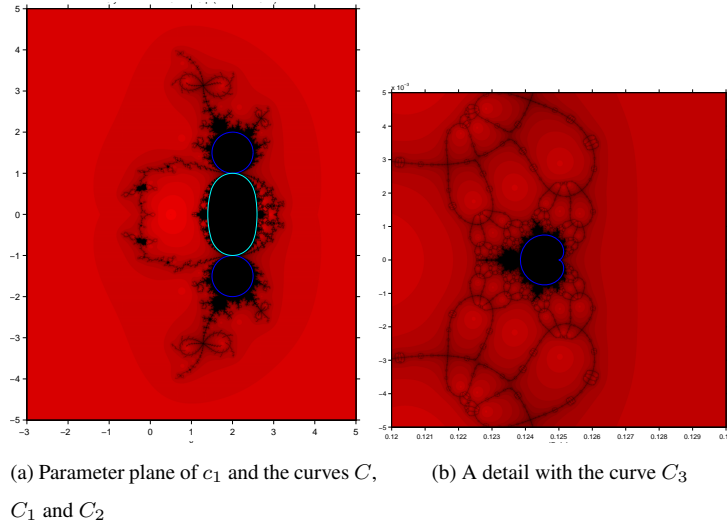The other Fatou components need at least one free critical point.

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(a) Parameter plane of c_1 and the curves C, (b) A detail with the curve C_3
C_1 and C_2

Figure 12: The basins of attraction of $z = 1, z_1$ and z_2 in the parameter plane of c_1.

Therefore, the number of free critical points for a given value of the parameter determines how rich is the dynamics of the rational function. However, what is interesting from a dynamical point of view, it is not from the point of view of stability of the numerical method. In this section, we consider those methods with a small number of critical points and show the dynamical planes for given values of the parameter. The dynamical planes are built by using the algorithms designed in [7], with MatLab software, by using a mesh of 800×800 points, a maximum of 80 iterations and a tolerance of 10^{-3}.

In the following figures we use different colours for the different basins of attraction: blue colour corresponds to the basin of attraction of $z = \infty$, orange colour is for $z = 0$, black colour indicates the existence of attractive periodic orbits and the other colours correspond to basins of attraction of strange fixed points.

- If $a = 0$, then
$$O_p(z, 0) = \frac{z^3(z+2)}{2z+1},$$
the fixed points are $0, \infty, 1$ and $\frac{-3\pm\sqrt{5}}{2}$. From
$$O_p'(z, 0) = \frac{6z^2(z+1)^2}{(2z+1)^2},$$

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we obtain that the critical points are $0, \infty$ and -1; but, as we pointed out in the first section, -1 is a pre-image of 1, then its dynamics is given by the dynamical behaviour of $z = 1$. As $z = 1$ is repulsive for this value of the parameter (Proposition 1) $z = -1$ is in the Julia set. As the other strange points satisfy that $|O'_p \left(\frac{-3 \pm \sqrt{5}}{2}, 0 \right)| > 1$, they are also repellors. So, in this case, the only attractive fixed points are $z = 0$ and $z = \infty$, that corresponds to the roots of the polynomial. The dynamical plane has only two Fatou components: the basins of attraction of $z = 0$ and $z = \infty$ (see Figure 13).

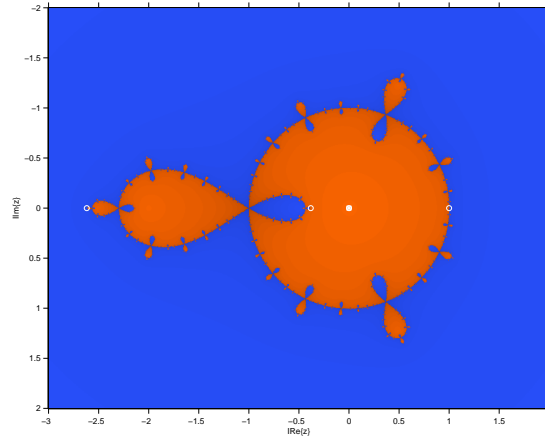Figure 13: Dynamical plane for $a = 0$

- When $a = 1$, then
$$O_p \left(z, 1 \right) = -z^3 \frac{z^2 + z + 2}{2z^2 + z + 1}.$$

In this case, the fixed points are $0, \infty, z_1, z_2, z_3$ and z_4.

In order to establish their stability, we calculate the derivative of the fixed point operator
$$O'_p \left(z, 1 \right) = \frac{-2z^2 \left(3z^4 + 4z^3 + 6z^2 + 4z + 3 \right)}{\left(2z^2 + z + 1 \right)^2}.$$

It leads to the free critical points: $c_\pm = \pm i$ and $c_{3,4} = -\frac{2}{3} \pm \frac{1}{3} i \sqrt{5}$. It can be checked that $O_p \left(i, 1 \right) = O_p \left(-i, 1 \right) = 1$ and $O_p \left(1, 1 \right) = -1$. Then $\{-1, 1\}$ is

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a periodic orbit of period two and $\pm i$ are pre-images of 1. Since $\left|O'_p(1,1)\right| > 1$, this periodic orbit is repulsive and these critical points are in the Julia set. The two free critical points c_3, c_4 generate their own dynamics: we can see the existence of two nearby periodic orbits of period 6 in Figures 14a and 14b.
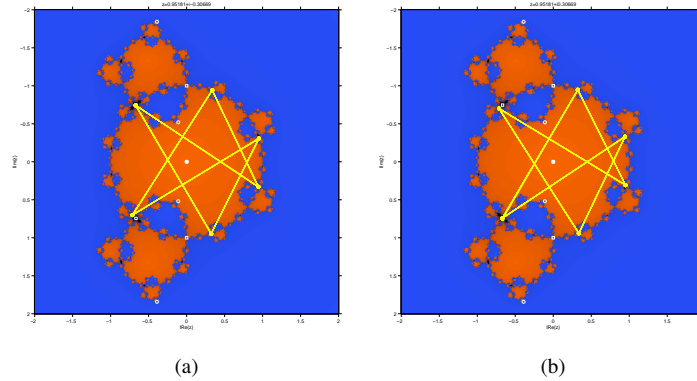

(a) (b)

Figure 14: Dynamical plane for $a = 1$ and periodic orbits of period 6

235 • If $a = 2$ then,

$$O_p(z, 2) = -\frac{z^3 (z - 3) (z^2 - z + 2)}{(3z - 1) (2z^2 - z + 1)}.$$

The fixed points are $0, \infty, 1, z_1, z_2, z_3$ and z_4. Moreover:

$$O'_p(z, 2) = -\frac{2z^2 (z - 1)^4 (9z^2 + 2z + 9)}{(3z - 1)^2 (-z + 2z^2 + 1)^2}$$

gives $c_1 = c_2 = c_+ = c_- = 1$ and $c_{3,4} = -\frac{1}{9}\left(1 \pm 4i\sqrt{5}\right)$ as free critical points. As $z = 1$ is a fixed and critical point, it has its own basin of attraction. We can observe these three basins of attraction of attractive fixed points in the dynamical plane (Figure 15).

240 The other two more free critical points are in the basins of attraction of two period orbits depicted in Figures 15a and 15b.

• If $a = 3$ then,
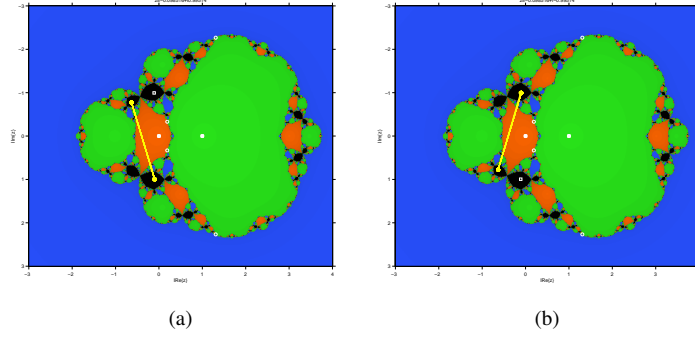
$$O_p(z, 3) = -\frac{z^3 (z - 2) (z - 5)}{10z^2 - 7z + 1}.$$

(a)   (b)

Figure 15: Dynamical plane for $a = 2$

The fixed points are 0, ∞, z_1, z_2, z_3 and z_4. On the other hand,

$$O'_p(z, 3) = -6 \frac{z^2 \left(-4z + z^2 + 1\right) \left(-8z + 5z^2 + 5\right)}{(5z - 1)^2 (2z - 1)^2}$$

gives four critical points $c_\pm = 2 \pm \sqrt{3}$ and $c_{3,4} = \frac{4}{5} \pm \frac{3}{5} i$.

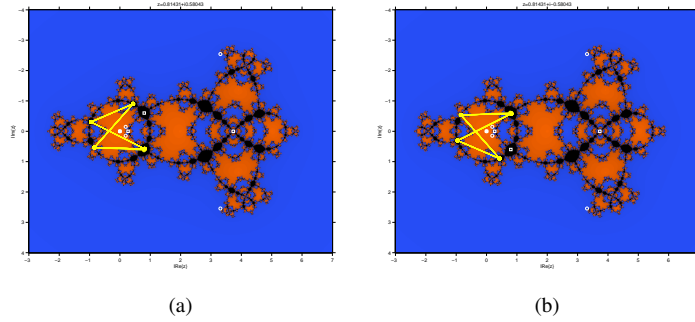These critical points are in the basins of attraction of two 4-periodic orbits (Figure 16a and 16b).

(a)   (b)

Figure 16: Dynamical plane for $a = 3$

- If $a = \frac{1}{6} \left(9 \pm 2\sqrt{21}\right)$ then $c_1 = c_3$ and $c_2 = c_4$. Due to $c_1 = \frac{1}{c_2}$ there are two independent free critical points: c_+ and c_1. The associated operators are:

$$O_p\left(z, \frac{1}{6}\left(9 + 2\sqrt{21}\right)\right) = \frac{z^3 \left(6 + 2\sqrt{21} - 3z\right)\left(-6 + 2\sqrt{21}z - 3z^2\right)}{\left(-3 + 2\left(3 + \sqrt{21}\right)z\right)\left(3 - 2\sqrt{21}z + 6z^2\right)},$$

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$$O_p\left(z, \frac{1}{6}\left(9 - 2\sqrt{21}\right)\right) = \frac{z^3\left(-6 + 2\sqrt{21} + 3z\right)\left(6 + 2\sqrt{21}z + 3z^2\right)}{\left(3 + 2\left(-3 + \sqrt{21}\right)z\right)\left(3 + 2\sqrt{21}z + 6z^2\right)}.$$

The fixed points are 0, ∞, 1, z_1, z_2, z_3 and z_4, but it can be checked that all the strange fixed points are repulsive. In the dynamical planes of Figures 17a and 17b we observe the non existence of attractive periodic orbits; then, there are only two basins of attraction corresponding to 0 and ∞.
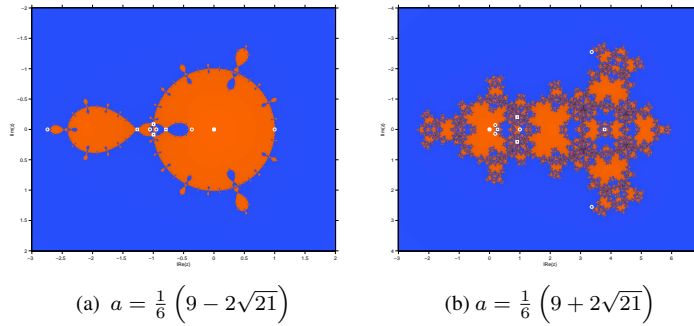

(a) $a = \frac{1}{6}\left(9 - 2\sqrt{21}\right)$　　　　(b) $a = \frac{1}{6}\left(9 + 2\sqrt{21}\right)$

Figure 17: Dynamical planes for $a = \frac{1}{6}\left(9 \pm 2\sqrt{21}\right)$

- Finally, we study the dynamical plane for $a = \frac{1}{2}$, where the numerical method is of order four. For this value:

$$O_p\left(z, \frac{1}{2}\right) = z^4\frac{2 + 2z + z^2}{1 + 2z + 2z^2},$$

whose fixed points are 0, 1 and ∞ and the four strange z_i, $i = 1, 2, 3, 4$. All the strange fixed points are repulsive; so, they are located on the Julia set.

On the other hand,

$$O_p'\left(z, \frac{1}{2}\right) = z^3\frac{(1 + z + z^2)(4 + 7z + 4z^2)}{(1 + 2z + 2z^2)^2}$$

that gives four free critical points, that are in the Julia set. The dynamical plane is given in Figure 18.

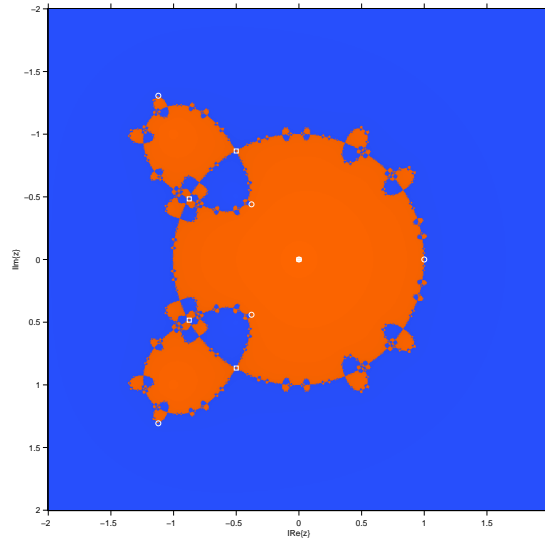This dynamical plane shows that the only attractive regions are those corresponding to $z = 0$ and $z = \infty$.

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Figure 18: Dynamical plane for $a = \frac{1}{2}$

5. Final Remarks

In this paper, a complex dynamical study of a parametric Chebyshev-Halley type family of iterative methods, on quadratic polynomial, is presented. Once the associated rational operator has been found and its symmetric property has been proved, the fixed and critical points have been obtained. The relevance of this kind of analysis is showed in the dynamical richness of the family: several fixed and critical points, different from the roots of the polynomials, appear showing a particular behaviour, that can be stable or unstable depending on the value of the parameter. In order to better understanding these facts, we have got the associated parameter planes to each independent free critical point. They have showed us which are the loci of bifurcation, that is, the values of the parameter where the numerical stability of the methods changes. Some dynamical planes show us different pathological aspects, such us attracting periodic orbits of several periods, basins of attraction of strange fixed points that do not correspond to the solution of the problem, as well as perfectly stable basins. The last ones are the most interesting elements of the family, under the numerical point of view, in terms of

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stability and reliability.

Acknowledgments

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