REAL-LINEAR ISOMETRIES AND JOINTLY NORM-ADDITIVE MAPS ON FUNCTION ALGEBRAS

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ABSTRACT. In this paper we describe into real-linear isometries defined between (not necessarily unital) function algebras and show, based on an example, that this type of isometries behave differently from surjective real-linear isometries and classical linear isometries. Next we introduce *jointly norm-additive mappings* and apply our results on real-linear isometries to provide a complete description of these mappings when defined between function algebras which are not necessarily unital or uniformly closed.

1. Introduction

A great deal of work has been done on linear isometries between several spaces. Let X be a locally compact Hausdorff space and let $C_0(X)$ (resp. $C(X)$ if X is compact) denote the Banach space of complex-valued continuous functions defined on X vanishing at infinity, endowed with the supremum norm $\|\cdot\|$. The study of linear isometries (with respect to the supremum norm) is traced back to the classical Banach-Stone theorem which gives the first characterization of surjective linear isometries between $C(X)$-spaces as weighted composition operators ([3, 22]). Several extensions of this theorem have been derived for different settings. In [15, 17], surjective linear isometries between uniform algebras (unital uniformly closed separating subalgebras of $C(X)$-spaces) were described. In another direction, Holsztyński ([11]) considered the non-surjective version of the Banach-Stone theorem and showed that if $T : C(X) \longrightarrow C(Y)$ is a linear isometry (not necessarily onto), then T can be represented as a weighted composition operator on a nonempty subset of Y. Generalizations of this result have been obtained by replacing $C(X)$ by certain subspaces or subalgebras of continuous functions (cf. [1, 18]). We refer the reader to [6] for a survey on the topic.

Another direction of extensions of the Banach-Stone theorem deals with its real-linear version. Given compact Hausdorff spaces, X_1 and X_2, let M_1 be a uniform algebra on X_1 and M_2 be a unital closed separating subspace of $C(X_2)$ such that the Šilov boundaries of M_1 and M_2 are X_1 and X_2, respectively. Ellis proved that if $T : M_1 \longrightarrow M_2$ is a surjective real-linear isometry, then there

2010 *Mathematics Subject Classification.* Primary 46J10; Secondary 47B48, 46J20.

Key words and phrases. real-linear isometry, function algebra, Choquet boundary, uniform algebra, norm-additive maps.

Research of J.J. Font was partially supported by Universitat Jaume I (Projecte P11B2014-35).

exist a clopen subset K of X_2 and a homeomorphism $\varphi : X_2 \longrightarrow X_1$ such that $T(f) = T(1)f \circ \varphi$ on K and $T(f) = T(1)\overline{f \circ \varphi}$ on $X_2 \setminus K$, where $\bar{\cdot}$ denotes the complex conjugate ([5]). This result was extended by Hatori et al in [7] by characterizing surjective real-linear isometries between unital function algebras. Their approach relied heavily on the existence of unit elements in the algebras. In [16], Miura generalized this result to non-unital algebras as follows: If A and B are function algebras on locally compact Hausdorff spaces X and Y, respectively, and $T : A \longrightarrow B$ is a surjective real-linear isometry, then there exist a homeomorphism $\varphi : Ch(B) \longrightarrow Ch(A)$, a continuous function $\omega : Ch(B) \longrightarrow \mathbb{T}$ and a clopen subset K of $Ch(B)$ such that $T(f) = \omega f \circ \varphi$ on K and $T(f) = \omega\overline{f \circ \varphi}$ on $Ch(B) \setminus K$. Following the study in this subject, Hatori and Miura showed that this representation can be extended to the Šilov boundary but not necessarily to the maximal ideal space (see [9, Example 3.2]). Moreover, in [13], real-linear isomeries between certain unital subspaces of continuous functions, and also real-linear isometries between Lipschitz algebras (with respect to a complete norm) were studied. More recently, in [14], the authors characterized surjective real-linear isometries between complex function spaces satisfying certain separating conditions and extended some previous results by a technique based on the extreme points.

In the first part of this paper (Section 3), we provide a Holsztyński-type characterization of the above cited papers and obtain generalizations of some results of [5, 13, 16]. Namely we characterize non-surjective real-linear isometries between (not necessarily unital) function algebras and show, based on an example, that this type of isometries behave differently from surjective real-linear isometries and classical linear isometries. As a consequence of our main result (Theorem 3.8), we are able to give affirmative answers to Question 4 and Question 5 in [8, Section 5] for real-linear isometries.

In [20], Rao, Tonev and Toneva studied maps $T : A \longrightarrow B$ between uniform algebras satisfying $R_\pi(Tf + Tg) = R_\pi(f + g)$ for all $f, g \in A$, where $R_\pi(f)$ (*the peripheral range* of f) is the set of range values of f with maximum modulus, and obtained sufficient conditions for such maps to be algebra isomorphisms. Next, Tonev and Yates ([24]) considered maps preserving additively norm conditions. In particular, they studied maps T which are called *norm-additive* in the sense that $\|Tf + Tg\| = \|f + g\|$ for all $f, g \in A$, and gave conditions under which T is an isometric algebra isomorphism. Moreover, in [7], Hatori, Hirasawa and Miura characterized maps T between unital semisimple commutative Banach algebras satisfying $r(Tf + Tg) = r(f + g)$, where $r(f)$ is the spectral-radius of f. Related results to the norm-additive maps are given between dense subsets of uniformly closed function algebras in [23], which are extended by Miura [16]. Moreover, there are results concerning certain norm-additive type conditions (involving more that one map) between uniform algebras in [21]. For a survey of additive-type preservers, see [8].

In the second part of this paper (Section 4), we introduce a new kind of mappings, which we call *jointly norm-additive*. Namely, given two function algebras $\mathcal{A}$ and $\mathcal{B}$, we say that the mappings $T_1, T_2 : \mathcal{A} \longrightarrow \mathcal{B}$ are jointly norm-additive if $\|T_1(f)+T_2(g)\| = \|f+g\|$ for all $f, g \in \mathcal{A}$. We first apply our results on real-linear isometries to prove the following general result: let P and Q be arbitrary nonempty sets, and let A and B be the uniform closures of two function algebras $\mathcal{A}$ and $\mathcal{B}$ on locally compact Hausdorff spaces X and Y, respectively. We characterize surjections $S_1 : P \longrightarrow \mathcal{A}$, $S_2 : Q \longrightarrow \mathcal{A}$, $T_1 : P \longrightarrow \mathcal{B}$ and $T_2 : Q \longrightarrow \mathcal{B}$ satisfying

$$\|T_1(p) + T_2(q)\| = \|S_1(p) + S_2(q)\| \quad (p \in P, q \in Q).$$

As a corollary, we characterize jointly norm-additive surjections and, as a consequence, we obtain generalizations of several results concerning maps satisfying additively norm conditions mentioned above. In particular, we show that if A and B both have an approximate identity, they are real-algebra isomorphic.

2. Preliminaries

Let X be a locally compact Hausdorff space and X_∞ be the one-point compactification of X. By $C_0(X)$ we mean the algebra of all complex-valued continuous functions on X vanishing at infinity. We denote the supremum norm of $f \in C_0(X)$ by $\|f\|$. A *function algebra* A on X means a subalgebra of $C_0(X)$ which strongly separates the points of X, i.e. for each $x, x' \in X$ with $x \neq x'$, there exists a function $f \in A$ with $0 \neq f(x) \neq f(x')$. A *uniformly closed function algebra* on X is a function algebra on X which is a closed subalgebra of $(C_0(X), \|\cdot\|)$. If X is a compact Hausdorff space, a uniformly closed function algebra on X is called a *uniform algebra* on X if it contains the constant functions.

If A is a nonempty subset of $C_0(X)$, a subset E of X is called a *boundary* for A if each function in A attains its maximum modulus within E. We denote the uniform closure of A by $\overline{A}$. Let A be a function algebra on a locally compact Hausdorff space X. The *Šilov boundary* of A, ∂A, is the unique minimal closed boundary for A and it exists by [2]. The *Choquet boundary*, $Ch(A)$, of A is the set of all $x \in X$ for which the evaluation functional δ_x at x is an extreme point of the unit ball of the dual space of $(A, \|.\|)$. So it is apparent that $Ch(A) = Ch(\overline{A})$. Besides, this is a known fact that for a function algebra A, ∂A is the closure of $Ch(A)$ [2, Theorem 1].

A point $x \in X$ is called a *strong boundary point* for A if for every neighborhood V of x, there exists a function $f \in A$ such that $\|f\| = 1 = |f(x)|$ and $|f| < 1$ on $X \setminus V$. It is known that for each uniformly closed function algebra on a locally compact Hausdorff space, the Choquet boundary coincides with the set of all strong boundary points (see [19]). However, according to the example given in [4], it is not true for all function algebras, although the Choquet boundary always contains all strong boundary points.

Now let us include some notions that are used throughout the rest of the paper. Let A be a subspace of $C_0(X)$ (X is a locally compact Hausdorff space). For every $x \in X$, we put $V_x := \{f \in A : f(x) = 1 = \|f\|\}$. For $g \in A$, M_g stands for the maximum modulus set of g, in fact $M_g := \{x \in X : |g(x)| = \|g\|\}$. Moreover, the peripheral range of g is defined by $R_\pi(g) := \{z \in g(X) : |z| = \|g\|\}$.

We finally state the following results concerning the additive and the multiplicative versions of Bishop's lemma adapted to the context of uniformly closed function algebras:

Lemma 2.1. [24, Lemma 1] *Assume that A is a uniformly closed function algebra on a locally compact Hausdorff space X and $f \in A$. Let $x_0 \in Ch(A)$ and arbitrary $r > 1$ (or $r \geq 1$ if $f(x_0) \neq 0$), then there exists a function $h \in A$ with $\|h\| = 1 = h(x_0)$ such that*

$$|f(x)| + r\|f\||h(x)| < |f(x_0)| + r\|f\|$$

for every $x \notin M_h$ and $|f(x)| + r\|f\||h(x)| = |f(x_0)| + r\|f\|$ for all $x \in M_h$. In particular, $\||f| + r\|f\||h|\| = |f(x_0)| + r\|f\|$.

Lemma 2.2. [10, Lemma 2.3] *Let A be a uniformly closed function algebra, $f \in A$ and $x \in Ch(A)$. If $f(x) \neq 0$, then there is a function $h \in A$ with $\|h\| = 1 = h(x)$ such that $M_{fh} = M_h$ and $R_\pi(fh) = \{f(x)\}$.*

3. Real-linear isometries

Let A and B be function algebras on locally compact Hausdorff spaces X and Y, respectively, and let $T : A \longrightarrow B$ be a real-linear isometry.

Notice that we can extend easily $T : A \longrightarrow B$ to a real-linear isometry $T : \overline{A} \longrightarrow \overline{B}$ between the uniform closures. Then in this section, we may assume, without loss of generality, that A and B are uniformly closed function algebras on locally compact Hausdorff spaces X and Y, respectively,

Let us remark that our results are also valid if A and B are dense subspaces of uniformly closed function algebras because real-linear isometries can be extended naturally to the uniform closures of A and B.

In the first part of this section, we present several lemmas to establish a basis for the characterization of real-linear isometries from A into B provided in the second part.

Lemma 3.1. *Let $x \in Ch(A)$ and $\alpha \in \mathbb{T}$. Then the set $\bigcap_{f \in \alpha V_x} M_{T(f)}$ is nonempty.*

Proof. We apply a minor modification of the proof of Lemma 3.1 in [16]. Since for each $f \in \alpha V_x$, $M_{T(f)}$ is a compact set of Y_∞ and $\infty \notin \{y \in Y : |T(f)(y)| = 1\}$, it is enough to show that the family

$\{M_{T(f)} : f \in \alpha V_x\}$ has the finite intersection property. For this, let $f_1, ..., f_n \in \alpha V_x$ and define $f = \frac{1}{n}\sum_{i=1}^{n} f_i$. Then $f \in \alpha V_x$ and so $\|T(f)\| = 1$. Hence there is a point $y \in Y$ with $|T(f)(y)| = 1$. Consequently,

$$1 = |T(f)(y)| \leq \frac{1}{n}\sum_{i=1}^{n} |T(f_i)(y)| \leq \frac{1}{n}\sum_{i=1}^{n} \|T(f_i)\| = 1,$$

which implies that $|T(f_i)(y)| = 1$ for all $i \in \{1, ..., n\}$. This means $y \in \bigcap_{i=1}^{n} M_{T(f_i)}$. Therefore, $\{M_{T(f)} : f \in \alpha V_x\}$ has the finite intersection property and so $\bigcap_{f \in \alpha V_x} M_{T(f)} \neq \emptyset$. □

For each $x \in Ch(A)$ and $\alpha \in \mathbb{T}$, we put $\mathcal{I}_{x,\alpha} := \bigcap_{f \in \alpha V_x} M_{T(f)}$. By Lemma 3.1, $\mathcal{I}_{x,\alpha} \neq \emptyset$.

Lemma 3.2. *Let $x \in Ch(A)$, $\alpha \in \mathbb{T}$ and $y \in \mathcal{I}_{x,\alpha}$. Then there exists a unique $\lambda \in \mathbb{T}$ such that $T(\alpha V_x) \subseteq \lambda V_y$.*

Proof. Let $f, g \in V_x$. Then $\alpha f, \alpha g \in \alpha V_x$ and so $|T(\alpha f)(y)| = 1 = |T(\alpha g)(y)|$. It is also clear that $\frac{|T(\alpha f)(y)+T(\alpha g)(y)|}{2} = 1$ because $\frac{\alpha f+\alpha g}{2} \in \alpha V_x$. Hence, from these equations, it follows that $T(\alpha f)(y) = T(\alpha g)(y)$. Now, if we define $\lambda := T(\alpha f)(y)$ for some $f \in V_x$, then the argument implies that $T(\alpha V_x) \subseteq \lambda V_y$. □

Lemma 3.3. *Let $x \in Ch(A)$, $\alpha \in \mathbb{T}$ and $y \in \mathcal{I}_{x,\alpha}$. If $f \in A$ with $f(x) = 0$, then $T(f)(y) = 0$.*

Proof. Let $f \in A$ with $f(x) = 0$. Suppose, on the contrary, that $T(f)(y) \neq 0$. Because of the real-linearity of T, without loss of generality, we may assume that $T(f)(y) = e^{i\theta}$, where $-\pi < \theta \leq \pi$. Fix a real constant $r > 1$ and let $r' = r\|f\|$. By Lemma 2.1, there is a function $h \in V_x$ such that $\||f| + r'|h|\| = r'$. In particular, $\|f + r'\alpha h\| = r'$. Since $T(\alpha h)(y) \in \mathbb{T}$, we may take $T(\alpha h)(y) = e^{i\theta'}$, where $-\pi < \theta' \leq \pi$. According to the values of θ and θ', one of the following cases will happen and as it is seen below all of them lead to contradictions.

- If $\cos(\theta - \theta') > 0$ then

$$r' = \|f + r'\alpha h\| = \|T(f + r'\alpha h)\| \geq |T(f)(y) + r'T(\alpha h)(y)| = |e^{i(\theta-\theta')} + r'| \geq r' + \cos(\theta - \theta').$$

- If $\cos(\theta - \theta') = 0$ then

$$r' = \|f+r'\alpha h\| = \|T(f+r'\alpha h)\| \geq |T(f)(y)+r'T(\alpha h)(y)| = |e^{i(\theta-\theta')}+r'| = |r' \pm i| = \sqrt{r'^2+1} > r'.$$

- If $\cos(\theta - \theta') < 0$ then

$$r' = \|-f+r'\alpha h\| = \|T(-f+r'\alpha h)\| \geq |-T(f)(y)+r'e^{i\theta'}| = |-e^{i(\theta-\theta')}+r'| \geq -\cos(\theta-\theta')+r' > r'.$$

So in all cases we get a contradiction. Thereby, $T(f)(y) = 0$. □

Lemma 3.4. *If $\alpha, \alpha' \in \mathbb{T}$, $x, x' \in Ch(A)$ and $x \neq x'$, then $\mathcal{I}_{x,\alpha} \cap \mathcal{I}_{x',\alpha'} = \emptyset$.*

Proof. Let x and x' be two distinct points in $Ch(A)$, and $\alpha, \alpha' \in \mathbb{T}$. There is a function $g \in A$ such that $g(x) = \alpha$ and $g(x') = 0$. Next, by Lemma 2.2, we can choose a function $h \in V_x$ such that $\frac{gh}{\alpha} \in A$ with $\|\frac{gh}{\alpha}\| = 1 = \frac{gh}{\alpha}(x)$. Then letting $f := gh$, we have $f \in \alpha V_x$ and $f(x') = 0$. Now if $y \in \mathcal{I}_{x,\alpha} \cap \mathcal{I}_{x',\alpha'}$, then $|T(f)(y)| = 1$ since $f \in \alpha V_x$, but on the other hand, from Lemma 3.3, it follows that $T(f)(y) = 0$, which is a contraction. Hence $\mathcal{I}_{x,\alpha} \cap \mathcal{I}_{x',\alpha'} = \emptyset$. $\square$

The sets $\mathcal{I}_{x,\alpha}$, which dates back to Holsztynski [11], are a usual tool in the context of into linear and surjective real-linear isometries on function algebras A (see e.g., [1], [6] and [16]). In all papers dealing with them, a common result is the following: $\mathcal{I}_{x,\alpha} = \mathcal{I}_{x,\alpha'}$ for each $x \in Ch(A)$ and any $\alpha, \alpha' \in \mathbb{T}$. The next simple example shows that this equality is no longer true when we consider non-surjective real-linear isometries:

Example. Define $T : C(\{x\}) \longrightarrow C(\{y_1, y_2\})$ as $T(a+ib)(y_1) := a$ and $T(a+ib)(y_2) := (a+ib)$. It is apparent that T is a non-surjective real-linear isometry for which $\mathcal{I}_{x,1} = \{y_1, y_2\}$ and $\mathcal{I}_{x,i} = \{y_2\}$.

For each $x \in Ch(A)$, assume that $\mathcal{I}_{x,1} \cap \mathcal{I}_{x,i} \neq \emptyset$ and put $\mathcal{I}_x := \mathcal{I}_{x,1} \cap \mathcal{I}_{x,i}$. Let $Y_0 := \{y \in Y : y \in \mathcal{I}_x \text{ for some } x \in Ch(A)\}$. Clearly, $Y_0 \neq \emptyset$. Now we can define a map $\varphi : Y_0 \longrightarrow Ch(A)$ by $\varphi(y) = x$ if $y \in \mathcal{I}_x$ for some $x \in Ch(A)$. Since, by Lemma 3.4, for any distinct points $x, x' \in Ch(A)$ and any scalars $\alpha, \alpha' \in \mathbb{T}$, $\mathcal{I}_{x,\alpha} \cap \mathcal{I}_{x',\alpha'} = \emptyset$, then $\mathcal{I}_x \cap \mathcal{I}_{x'} = \emptyset$ and φ is well-defined. It is clear that φ is surjective. Moreover, let us define a map $\Lambda : Y_0 \times \mathbb{T} \longrightarrow \mathbb{C}$ by $\Lambda(y, \alpha) = \lambda$ such that λ is a unique scalar with $T(\alpha V_{\varphi(y)}) \subseteq \lambda V_y$, by Lemma 3.2. It is apparent that Λ is a well-defined map.

Lemma 3.5. *If $y \in Y_0$, then either $\Lambda(y,i) = i\Lambda(y,1)$ or $\Lambda(y,i) = -i\Lambda(y,1)$.*

Proof. Let $y \in Y_0$, and put $\lambda_i := \Lambda(y,i)$ and $\lambda_1 := \Lambda(y,1)$ for simplicity. For each $f \in V_{\varphi(y)}$ we have

$$\begin{aligned} |\lambda_1 \pm \lambda_i| &= |T(f)(y) \pm T(if)(y)| = |T(f \pm if)(y)| \\ &\leq \|T(f \pm if)\| = \|f \pm if\| \\ &= \|f\||1 \pm i| = \sqrt{2}. \end{aligned}$$

Hence $|\lambda_1 \pm \lambda_i| \leq \sqrt{2}$, and since $|\lambda_1| = |\lambda_i| = 1$, it follows easily that $\lambda_i^2 = -\lambda_1^2$. Consequently, either $\Lambda(y,i) = i\Lambda(y,1)$ or $\Lambda(y,i) = -i\Lambda(y,1)$. $\square$

Since according to the above lemma, $\Lambda(y,i) = \pm i\Lambda(y,1)$ for all $y \in Y_0$, if we set $K = \{y \in Y_0 : \Lambda(y,i) = i\Lambda(y,1)\}$, then $Y_0 \setminus K = \{y \in Y_0 : \Lambda(y,i) = -i\Lambda(y,1)\}$. Now we obtain the following result easily:

Lemma 3.6. *Let $y \in Y_0$ and $\alpha \in \mathbb{T}$. Then*

$$\Lambda(y,\alpha) = \begin{cases} \alpha\Lambda(y,1) & y \in K, \\ \overline{\alpha}\Lambda(y,1) & y \in Y_0 \setminus K. \end{cases}$$

Proof. Let $\alpha = a+ib$, where $a, b \in \mathbb{R}$. Take also $\lambda_i := \Lambda(y,i)$, $\lambda_1 := \Lambda(y,1)$ and $\lambda_\alpha := \Lambda(y,\alpha)$. Since we have $T(\alpha V_{\varphi(y)}) \subseteq \lambda_\alpha V_y$, then, for a given $f \in V_{\varphi(y)}$, $T(\alpha f)(y) = \lambda_\alpha$. Hence $\lambda_\alpha = T(\alpha f)(y) = T(af + ibf)(y) = aT(f)(y) + bT(if)(y)$, and so, from Lemma 3.5,

$$\lambda_\alpha = \begin{cases} a\lambda_1 + b\lambda_i = a\lambda_1 + ib\lambda_1 = (a+ib)\lambda_1 = \alpha\lambda_1 & y \in K, \\ a\lambda_1 + b\lambda_i = a\lambda_1 - ib\lambda_1 = (a-ib)\lambda_1 = \overline{\alpha}\lambda_1 & y \in Y_0 \setminus K. \end{cases}$$

Therefore,

$$\Lambda(y,\alpha) = \begin{cases} \alpha\Lambda(y,1) & y \in K, \\ \overline{\alpha}\Lambda(y,1) & y \in Y_0 \setminus K. \end{cases}$$

□

Remark 3.7. We define the map $\omega : Y_0 \longrightarrow \mathbb{T}$ by $\omega(y) = \Lambda(y,1)$ for all $y \in Y_0$. Hence if $y \in Y_0$, then, by the above lemma, we have

$$T(\alpha f)(y) = \begin{cases} \alpha\omega(y) & y \in K, \\ \overline{\alpha}\omega(y) & y \in Y_0 \setminus K, \end{cases}$$

for all $\alpha \in \mathbb{T}$ and $f \in V_{\varphi(y)}$.

Now we are ready to prove the main result of this section:

Theorem 3.8. *Let $T : A \longrightarrow B$ be a real-linear isometry and assume that $\mathcal{I}_{x,1} \cap \mathcal{I}_{x,i} \neq \emptyset$ for each $x \in Ch(A)$. Then there exist a nonempty subset Y_0 of Y, a continuous surjective map $\varphi : Y_0 \longrightarrow Ch(A)$, a unimodular continuous function $\omega : Y_0 \longrightarrow \mathbb{T}$ and a clopen subset K of Y_0 such that for all $f \in A$ and $y \in Y_0$,*

$$T(f)(y) = \omega(y) \begin{cases} f(\varphi(y)) & y \in K, \\ \overline{f(\varphi(y))} & y \in Y_0 \setminus K. \end{cases}$$

Moreover, $\omega(y) = T(g)(y)$ for any $g \in A$ with $g(\varphi(y)) = 1$.

Proof. Let $\varphi : Y_0 \longrightarrow Ch(A)$ and $\omega : Y_0 \longrightarrow \mathbb{T}$ be the maps defined after Lemma 3.4. Assume that $f \in A$ and $y \in Y_0$. If $f(\varphi(y)) = 0$, then from Lemma 3.3, $T(f)(y) = 0$. Now suppose that $f(\varphi(y)) \neq 0$. Choose a function h in $V_{\varphi(y)}$. If we define $g := f - f(\varphi(y))h$, then $g \in A$ and $g(\varphi(y)) = 0$, so, again from Lemma 3.3, $T(f)(y) = T(f(\varphi(y))h)(y)$. If we set $\alpha := \frac{f(\varphi(y))}{|f(\varphi(y))|}$, then

$|\alpha| = 1$ and $T(f)(y) = T(\alpha|f(\varphi(y))|h)(y) = |f(\varphi(y))|T(\alpha h)(y)$. Therefore, according to Remark 3.7, we observe that

$$T(f)(y) = |f(\varphi(y))|T(\alpha h)(y) = \begin{cases} |f(\varphi(y))|\alpha\omega(y) & y \in K, \\ |f(\varphi(y))|\overline{\alpha}\omega(y) & y \in Y_0 \setminus K. \end{cases}$$

Then

$$T(f)(y) = \begin{cases} f(\varphi(y))\omega(y) & y \in K, \\ \overline{f(\varphi(y))}\omega(y) & y \in Y_0 \setminus K. \end{cases}$$

We now claim that φ is continuous. Contrary to what we claim, assume that there is a net, (y_i), in Y_0 converging to $y_0 \in Y_0$ but $\varphi(y_i)$ does not approach $\varphi(y_0)$. Then by passing to a subnet if necessary, we may suppose that $\varphi(y_i)$ converges to $x_0 \in X_\infty$ with $x_0 \neq \varphi(y_0)$. From the derived representation of T, it follows that for each $f \in A$, $|T(f)| = |f \circ \varphi|$ on Y_0, which implies that

$$|f(\varphi(y_0))| = |T(f)(y_0)| = \lim |T(f)(y_i)| = \lim |f(\varphi(y_i))| = |f(x_0)|.$$

Thus $|f(\varphi(y_0))| = |f(x_0)|$, but it is impossible because we can choose a function $f \in V_{\varphi(y_0)}$ such that $|f(x_0)| < 1$. Hence, φ is a continuous map.

Next we prove that K is a clopen subset of Y_0. Indeed, it is shown that

$$K = \bigcap_{f \in A} \{y \in Y_0 : T(if)(y) = iT(f)(y)\} \text{ and } Y_0 \setminus K = \bigcap_{f \in A} \{y \in Y_0 : T(if)(y) = -iT(f)(y)\}.$$

We only check the first equality since the second one can be concluded similarly. By the above representation of T, it is clear that $K \subseteq \bigcap_{f \in A} \{y \in Y_0 : T(if)(y) = iT(f)(y)\}$. Conversely, let $y \in Y_0$ such that $T(if)(y) = iT(f)(y)$ for all $f \in A$. If $y \in Y_0 \setminus K$, then, according to the obtained representation of T we deduce that, for any function f in $V_{\varphi(y)}$, we have $T(if)(y) = -i\omega(y)$ while $T(f)(y) = \omega(y)$, which is a contradiction. This contradiction yields $y \in K$. Therefore,

$$K = \bigcap_{f \in A} \{y \in Y_0 : T(if)(y) = iT(f)(y)\}.$$

Now, these equations ensure us that K and $Y_0 \setminus K$ are closed in Y_0, which is to say that K is a clopen subset of Y_0.

We finally show the continuity of ω. Fix $y_0 \in Y_0$ and choose a function $f \in A$ such that $f(\varphi(y_0)) \neq 0$. If we consider $W := \{x \in Ch(A) : f(x) \neq 0\}$, then $\varphi^{-1}(W)$ is a neighborhood of y_0. Hence

$$\omega(y) = \begin{cases} \frac{T(f)(y)}{(f \circ \varphi)(y)} & y \in \varphi^{-1}(W) \cap K, \\ \frac{T(f)(y)}{\overline{(f \circ \varphi)(y)}} & y \in \varphi^{-1}(W) \cap (Y_0 \setminus K). \end{cases}$$

Now from the continuity of $\frac{T(f)}{f \circ \varphi}$ and $\frac{T(f)}{\overline{f \circ \varphi}}$ on $\varphi^{-1}(W) \cap Y_0$ and the openness of K and $Y_0 \setminus K$, it follows that ω is continuous at y_0. $\square$

Remark 3.9. 1) Notice that in Theorem 3.8, the set Y_0 can be neither open nor closed in Y.

2) Since for each $f \in A$, $|T(f)| = |f \circ \varphi|$ on Y_0, $\varphi(Y_0) = Ch(A)$ and $Ch(A)$ is a boundary for A, then it is concluded easily that Y_0 is a boundary for $T(A)$. In particular, if $f, g \in A$, the equation $Tf = Tg$ on Y_0 ensures that $f = g$. We also remark that if $Ch(A)$ is compact, then Y_0 is a closed boundary for $T(A)$.

3) If $T(A)$ separates the points of Y in the sense that for each distinct points $y, y' \in Y$, there exists $g \in T(A)$ such that $|g(y)| \neq |g(y')|$, then clearly Y_0 is heomeomorphic to $Ch(A)$.

The following result, which is a consequence of Theorem 3.8, gives affirmative answers to Question 4 and Question 5 in [8, Section 5] for real-linear isometries.

Corollary 3.10. *(i) If $T : A \longrightarrow B$ is a real-linear isometry and $T(ih_0) = iT(h_0)$ for some $h_0 \in A$ with $h_0 \neq 0$ on $Ch(A)$, then $K = Y_0$ and T is a weighted composition operator on Y_0, where K and Y_0 are given by Theorem 3.8.*

(ii) Under the hypotheses of Theorem 3.8, if furthermore, $R_\pi(T(h)) = R_\pi(h)$ for all functions $h \in A$, then $T(f) = f \circ \varphi$ for each $f \in A$ on Y_0.

Proof. (i) First notice that since $T(ih_0) = iT(h_0)$ and T is a real-linear map, then we have $T(\alpha h_0) = \alpha T(h_0)$ for all $\alpha \in \mathbb{T}$. Suppose that $x \in Ch(A)$, $y \in \mathcal{I}_{x,1}$ and $f \in V_x$. Lemma 3.3 implies that $T(\frac{ih_0}{h_0(x)} - if)(y) = 0$ and $T(\frac{h_0}{h_0(x)} - f)(y) = 0$. Then $T(if)(y) = \frac{i}{h_0(x)}T(h_0)(y)$ and $T(f)(y) = \frac{1}{h_0(x)}T(h_0)(y)$, which easily lead to this fact that $T(if)(y) = iT(f)(y)$. In particular, it is concluded that $\mathcal{I}_{x,1} = \mathcal{I}_{x,i}$. Therefore, from Theorem 3.8, there exist a nonempty subset $Y_0 = \bigcup_{x \in Ch(A)} \mathcal{I}_{x,1}$ of Y, a continuous surjective map $\varphi : Y_0 \longrightarrow Ch(A)$, a unimodular continuous function $\omega : Y_0 \longrightarrow \mathbb{T}$ and a clopen subset K of Y_0 such that for all $f \in A$ and $y \in Y_0$,

$$T(f)(y) = \omega(y) \begin{cases} f(\varphi(y)) & y \in K, \\ \overline{f(\varphi(y))} & y \in Y_0 \setminus K. \end{cases}$$

Now since $T(ih_0) = iT(h_0)$, it is easy to see $K = Y_0$ and get the result.

(ii) It is clear. ☐

We now consider the onto case and obtain the main theorem in [16].

Corollary 3.11. *Let $T : A \longrightarrow B$ be a surjective real-linear isometry. Then there exist a homeomorphism φ from $Ch(B)$ onto $Ch(A)$, a unimodular continuous function $\omega : Ch(B) \longrightarrow \mathbb{T}$ and a clopen subset K of $Ch(B)$ such that for each $f \in A$ and $y \in Ch(B)$,*

$$T(f)(y) = \omega(y) \begin{cases} f(\varphi(y)) & y \in K, \\ \overline{f(\varphi(y))} & y \in Ch(B) \setminus K. \end{cases}$$

Proof. Let $x \in Ch(A)$.

the first option:

Take $z \in \mathcal{I}_{x,1}$. Clearly,

$$N := \bigcap_{f \in V_x} \{z' \in Y : T(f)(z') = T(f)(z)\} \subseteq \mathcal{I}_{x,1},$$

which implies that $\mathcal{I}_{x,1} \cap Ch(B) \neq \emptyset$ since the intersection of the p-set N and the Choquet boundary is nonempty (see e.g., [12, Lemma 3.2] or [16, Proposition 2.1]).

2nd option:

Since the intersection of each p-set and the Choquet boundary is nonempty, then we may conclude that $\mathcal{I}_{x,1} \cap Ch(B) \neq \emptyset$ (see e.g., the arguments in [12, Page 86] or [16, Lemma 3.2]).

Let $y \in \mathcal{I}_{x,1} \cap Ch(B)$. By Lemma 3.2, there exists $\lambda \in \mathbb{T}$ such that $T(V_x) \subseteq \lambda V_y$. Similarly, for this y, there exists a unique $\alpha \in \mathbb{T}$ and $x' \in X$ such that $T^{-1}(\lambda V_y) \subseteq \alpha V_{x'}$. Hence

$$V_x \subseteq T^{-1}(\lambda V_y) \subseteq \alpha V_{x'},$$

which easily yields $x = x'$ and $\alpha = 1$. Therefore, $T(V_x) = \lambda V_y$. Similarly, there exist $\lambda' \in \mathbb{T}$ and $y' \in Ch(B)$ such that $T(iV_x) = \lambda' V_{y'}$. We claim that $y = y'$. Otherwise, choose $F \in B$ such that $F(y) = 1 = \|F\|$ and $|F(y')| < 0.1$. If we set $E = \{z \in Y : |F(z)| \geq 0.1\}$, then y' belong to the open set $Y \setminus E$. We take $F' \in B$ with $F'(y') = 1 = \|F'\|$ and $|F'| < 0.1$ on E. Now letting $f_1, f_2 \in V_x$ with $T(f_1) = \lambda F$ and $T(if_2) = \lambda' F'$, we have

$$1.1 \geq \|\lambda F + \lambda' F'\| = \|T(f_1) + T(if_2)\| = \|T(f_1 + if_2)\| = \|f_1 + if_2\| \geq \sqrt{2},$$

which is a contradiction. This argument shows that $y = y'$. In particular, we deduce that $\mathcal{I}_{x,1} \cap \mathcal{I}_{x,i} \neq \emptyset$. Hence from Theorem 3.8, there exist a nonempty subset Y_0 of Y, a continuous surjective map $\varphi : Y_0 \longrightarrow Ch(A)$, a unimodular continuous function $\omega : Y_0 \longrightarrow \mathbb{T}$ and a clopen subset K of Y_0 such that for all $f \in A$, $T(f) = \omega f \circ \varphi$ on K, and $T(f) = \omega \overline{f \circ \varphi}$ on $Y_0 \setminus K$.

Next we show that φ is injective. Let $y_1, y_2 \in Y_0$ such that $\varphi(y_1) = \varphi(y_2)$. Then, from the representation of T obtained in Theorem 3.8, we have

$$|T(f)(y_1)| = |f(\varphi(y_1))| = |f(\varphi(y_2))| = |T(f)(y_2)| \quad (f \in A),$$

which, by the surjectivity of T, implies that $y_1 = y_2$ because we can choose $g \in B$ such that $g(y_1) = 1$ and $g(y_2) = 0$. Therefore φ is injective. In particular, the argument also shows that for each $x \in Ch(A)$, the set $\mathcal{I}_x$ is a singleton.

We next prove that $Y_0 = Ch(B)$. Take $y_0 \in Y_0$ and $x_0 = \varphi(y_0)$. Since $\mathcal{I}_x \cap Ch(B) \neq \emptyset$ and $\mathcal{I}_x$ is a singleton, then we conclude that $y_0 \in Ch(B)$. Hence $Y_0 \subseteq Ch(B)$.

Conversely, let $y_0 \in Ch(B)$. Consider T^{-1} (the inverse of T), which is a real-linear isometry from B onto A. Similar to the above arguments for T^{-1}, there exists a continuous surjective map ψ from the nonempty subset $X_0 = \bigcup_{y \in Ch(B)} \mathcal{I}_y$ of X onto $Ch(B)$, where $\mathcal{I}_y = \bigcap_{g \in (V_y \cup iV_y)} M_{T^{-1}(g)}$. Set $x_0 \in X_0$ such that $y_0 = \psi(x_0)$. Then, for each $g \in B$, $|T^{-1}(g)(x_0)| = |g(y_0)|$. In other words, $|f(x_0)| = |T(f)(y_0)|$ for each $f \in A$, which shows that $y_0 \in \mathcal{I}_{x_0} \subseteq Y_0$ and $\varphi(y_0) = x_0$. Thereby, $Y_0 = Ch(B)$. Furthermore, note that $\varphi(\psi(x_0)) = x_0$ and $\psi(\varphi(y_0)) = y_0$. A similar argument for the surjective real-linear isometry T^{-1} shows that $X_0 = Ch(A)$, and in fact, ψ is the inverse of φ. This means that φ is a homeomorphism from $Ch(B)$ onto $Ch(A)$. $\square$

If in the above corollary we assume that X and Y are compact Hausdorff spaces, then $\omega = T(1)$. Moreover, $T_1(f) := \overline{T(1)}T(f)$ defines a real-algebra isomorphism from A onto B, and φ can be extended to a homeomorphism from the maximal ideal space M_B of B onto the maximal ideal space M_A of A (see [9]). It should be noted that the example given in [9] shows that, in general, A and B need not be real-algebra isomorphic and moreover, M_A is not necessarily homeomorphic to M_B.

Corollary 3.12. *Let X and Y be compact metric spaces and let $T : \mathrm{Lip(X)} \longrightarrow \mathrm{Lip(Y)}$ be a surjective real-linear isometry. Then there exist a continuous function $\omega : Y \longrightarrow \mathbb{T}$, a bi-Lipschitz homeomorphism $\varphi : Y \longrightarrow X$ and a clopen subset K of Y such that*

$$T(f)(y) = \omega(y) \begin{cases} f(\varphi(y)) & y \in K, \\ \overline{f(\varphi(y))} & y \in Y \setminus K, \end{cases}$$

for all $y \in Y$ and $f \in \mathrm{Lip(X)}$. A similar result is valid for the pointed Lipschitz algebras.

Proof. Since each point in the underling spaces is a strong boundary point for Lipschitz algebras, by Corollary 3.11, there exist a homeomorphism $\varphi : Y \longrightarrow X$ and a clopen subset K of Y such that

$$T(f)(y) = T(1)(y) \begin{cases} f(\varphi(y)) & y \in K, \\ \overline{f(\varphi(y))} & y \in Y \setminus K, \end{cases}$$

for all $y \in Y$ and $f \in \mathrm{Lip(X)}$. In particular, the above representation shows that $\||Tf| + |Tg|\| = \||f| + |g|\|$ holds for all $f, g \in \mathrm{Lip(X)}$, then [12, Corollary 3.7] implies that φ is a bi-Lipschitz homeomorphism.

We would like to remark that it can be checked that the Closed Graph theorem is true for real-linear maps. So we can also apply the common method to show that φ is bi-Lipschitz. Consider $\mathrm{Lip(X)}$ with the complete norm

$$\|f\|_L = \|f\| + L(f) \quad (f \in \mathrm{Lip(X)}),$$

where $L(f)$ is the Lipschitz constant of f. Indeed, $T : (\mathrm{Lip(X)}, \|\cdot\|_{\mathrm{L}}) \longrightarrow (\mathrm{Lip(Y)}, \|\cdot\|_{\mathrm{L}})$ is continuous by the Closed Graph theorem. Then there exists $t > 0$ such that for every $f \in \mathrm{Lip}(X)$,

$\|T(f)\|_L \leq t\|f\|_L$. Let y, y' be two distinct elements in Y. Define $f_0(z) = d(\varphi(y), z)$ for all $z \in X$. Obviously, $f_0 \in \mathrm{Lip(X)}$ with $\|f_0\|_L \leq k$, where $k = 1+\mathrm{diam}(X)$. Since T is continuous, $\|T(f_0)\|_L \leq tk$ and, in particular, the Lipschitz constant $L(T(f_0)) \leq tk$.

Since $f_0(\varphi(y)) = 0$ then $T(f_0)(y) = 0$. Moreover, $T(f_0)(y') = T(1)(y')d(\varphi(y), \varphi(y'))$. Hence

$$\frac{d(\varphi(y), \varphi(y'))}{d(y, y')} = \frac{|T(f_0)(y) - T(f_0)(y')|}{d(y, y')} \leq tk.$$

Therefore, $\sup\limits_{\substack{y,y' \in Y \\ y \neq y'}} \frac{d(\varphi(y),\varphi(y'))}{d(y,y')} \leq tk$; that is, φ satisfies the Lipschitz condition on Y. Similarly φ^{-1} is a Lipschitz function on X.

Given a compact metric space X with distinguished base point e_X, the pointed Lipschitz algebra on X is a function algebra on the locally compact Hausdorff space $X \setminus \{e_X\}$ and every point in $X \setminus \{e_X\}$ is a strong boundary point. Then a similar argument can be applied to get the result for the pointed Lipschitz algebras. □

4. Jointly norm-additive mappings

Below we characterize jointly norm-additive maps (see Introduction) when defined between function algebras (not necessarily unital or uniformly closed) and obtain an extension of the main theorem by Shindo [21, Theorem 1]. We note that the maps here are not necessarily linear.

Theorem 4.1. *Let P and Q be arbitrary nonempty sets, and let A and B be the uniform closures of two function algebras $\mathcal{A}$ and $\mathcal{B}$ on locally compact Hausdorff spaces X and Y, respectively. Let $S_1 : P \longrightarrow \mathcal{A}$, $S_2 : Q \longrightarrow \mathcal{A}$, $T_1 : P \longrightarrow \mathcal{B}$ and $T_2 : Q \longrightarrow \mathcal{B}$ be surjections satisfying*

$$\|T_1(p) + T_2(q)\| = \|S_1(p) + S_2(q)\| \quad (p \in P, q \in Q).$$

Then there exist a homeomorphism φ from $Ch(B)$ onto $Ch(A)$, a unimodular continuous function $\omega : Ch(B) \longrightarrow \mathbb{T}$ and a clopen subset K of $Ch(B)$ such that for each $p \in P$, $q \in Q$ and $y \in Ch(B)$,

$$T_1(p)(y) - T_1(p_0)(y) = \omega(y) \begin{cases} S_1(p)(\varphi(y)) & y \in K, \\ \overline{S_1(p)(\varphi(y))} & y \in Ch(B) \setminus K, \end{cases}$$

and

$$T_2(q)(y) - T_2(q_0)(y) = \omega(y) \begin{cases} S_2(q)(\varphi(y)) & y \in K, \\ \overline{S_2(q)(\varphi(y))} & y \in Ch(B) \setminus K, \end{cases}$$

where $p_0 \in P$ and $q_0 \in Q$ are elements with $S_1(p_0) = S_2(q_0) = 0$. Moreover, if $p \in P$ and $q \in Q$ such that $S_1(p) = S_2(q)$, then $T_1(p) - T_1(p_0) = T_2(q) - T_2(q_0)$.

Proof. We first show that

$$\|T_1(p) - T_1(p')\| = \|S_1(p) - S_1(p')\| \text{ and } \|T_2(q) - T_2(q')\| = \|S_2(q) - S_2(q')\| \tag{4.1}$$

for all $p, p' \in P$ and $q, q' \in Q$. We only need to show the first equation since the conditions are symmetric for T_2, S_2 in comparison with T_1, S_1. From our assumptions, we may conclude easily that $T_2(q) = -T_1(p)$ if and only if $S_2(q) = -S_1(p)$ for $p \in P$ and $q \in Q$. For any $p \in P$, let $q_p \in Q$ with $T_2(q_p) = -T_1(p)$. Then if $p, p' \in P$ we have

$$\|T_1(p) - T_1(p')\| = \|T_1(p) + T_2(q_{p'})\| = \|S_1(p) + S_2(q_{p'})\| = \|S_1(p) - S_1(p')\|.$$

Hence $\|T_1(p) - T_1(p')\| = \|S_1(p) - S_1(p')\|$ for all $p, p' \in P$.

Let us define two maps $\mathcal{T}_1$ and $\mathcal{T}_2$ on $\mathcal{A}$ as follows:

$$\mathcal{T}_1(S_1(p)) := T_1(p) \text{ and } \mathcal{T}_2(S_2(q)) := T_2(q) \quad (p \in P, q \in Q).$$

According to (4.1), $\mathcal{T}_1$ and $\mathcal{T}_2$ are well-defined surjections from $\mathcal{A}$ onto $\mathcal{B}$ such that

$$\|\mathcal{T}_1(f) - \mathcal{T}_1(g)\| = \|f - g\| \text{ and } \|\mathcal{T}_2(f) - \mathcal{T}_2(g)\| = \|f - g\|$$

for all $f, g \in \mathcal{A}$. Then the Mazur-Ulam theorem (see [25]) implies that the maps $\mathcal{T}_1 - \mathcal{T}_1(0)$ and $\mathcal{T}_2 - \mathcal{T}_2(0)$ are real-linear. So $\mathcal{T}_1 - \mathcal{T}_1(0)$ and $\mathcal{T}_2 - \mathcal{T}_2(0)$ can be extended naturally to real-linear isometries between the uniform closures of $\mathcal{A}$ and $\mathcal{B}$, A and B, respectively. We denote the extensions by $\mathcal{T}_1 - \mathcal{T}_1(0)$ and $\mathcal{T}_2 - \mathcal{T}_2(0)$.

Let $f \in \mathcal{A}$, then $\mathcal{T}_1(f) = -\mathcal{T}_2(-f)$ since, letting $p \in P$ and $q \in Q$ such that $S_1(p) = -S_2(q) = f$, we have

$$\|\mathcal{T}_1(f) + \mathcal{T}_2(-f)\| = \|T_1(p) + T_2(q)\| = \|S_1(p) + S_2(q)\| = 0,$$

which yields the desired conclusion. Now from the real-linearity of $\mathcal{T}_2 - \mathcal{T}_2(0)$ it follows that

$$\mathcal{T}_1(f) - \mathcal{T}_1(0) = -\mathcal{T}_2(-f) + \mathcal{T}_2(0) = -(\mathcal{T}_2(-f) - \mathcal{T}_2(0)) = \mathcal{T}_2(f) - \mathcal{T}_2(0),$$

thus $\mathcal{T}_1(f) - \mathcal{T}_1(0) = \mathcal{T}_2(f) - \mathcal{T}_2(0)$. Now if $f \in A$, then there exists a sequence $\{f_n\}$ in $\mathcal{A}$ converging uniformly to f. From the above we conclude that

$$\begin{aligned} \mathcal{T}_1(f) - \mathcal{T}_1(0) &= \lim_n \mathcal{T}_1(f_n) - \mathcal{T}_1(0) = \lim_n \mathcal{T}_2(f_n) - \mathcal{T}_2(0) \\ &= \mathcal{T}_2(f) - \mathcal{T}_2(0). \end{aligned}$$

Therefore, $\mathcal{T}_1(f) - \mathcal{T}_1(0) = \mathcal{T}_2(f) - \mathcal{T}_2(0)$ for all $f \in A$. Applying Corollary 3.11 to the real-linear isometry $\mathcal{T}_1 - \mathcal{T}_1(0)$ from A onto B, there exist a homeomorphism φ from $Ch(B)$ onto $Ch(A)$, a unimodular continuous function $\omega : Ch(B) \longrightarrow \mathbb{T}$ and a clopen subset K of $Ch(B)$ such that for each $f \in A$,

$$\mathcal{T}_1(f) - \mathcal{T}_1(0) = \mathcal{T}_2(f) - \mathcal{T}_2(0) = \omega \begin{cases} f \circ \varphi & \text{on } K, \\ \overline{f \circ \varphi} & \text{on } Ch(B) \setminus K. \end{cases}$$

Then for each $p \in P$, $q \in Q$ and $y \in Ch(B)$,

$$T_1(p)(y) - T_1(p_0)(y) = \omega(y) \begin{cases} S_1(p)(\varphi(y)) & y \in K, \\ \overline{S_1(p)(\varphi(y))} & y \in Ch(B) \setminus K, \end{cases}$$

and

$$T_2(p)(y) - T_2(q_0)(y) = \omega(y) \begin{cases} S_2(p)(\varphi(y)) & y \in K, \\ \overline{S_2(p)(\varphi(y))} & y \in Ch(B) \setminus K, \end{cases}$$

where $p_0 \in P$ and $q_0 \in Q$ satisfy $S_1(p_0) = S_2(q_0) = 0$.

Moreover, if $p \in P$ and $q \in Q$ such that $S_1(p) = S_2(q)$, then we have

$$T_1(p) - T_1(p_0) = \mathcal{T}_1(S_1(p)) - \mathcal{T}_1(0) = \mathcal{T}_2(S_2(q)) - \mathcal{T}_2(0) = T_2(q) - T_2(q_0),$$

therefore $T_1(p) - T_1(p_0) = T_2(q) - T_2(q_0)$. $\square$

Composing the following and Corollary 3.10, we can give generalizations of [16, Corollary], [21, Corollary 5], [24, Theorems 13, 16, 20] and [23, Theorems 3.6, 4.1, and their corollaries] under weaker conditions between function algebras (not necessarily unital or uniformly closed).

Corollary 4.2. *(i) Let A and B be the uniform closures of two function algebras $\mathcal{A}$ and $\mathcal{B}$, respectively. Let also $T_1, T_2 : \mathcal{A} \longrightarrow \mathcal{B}$ be jointly norm-additive surjections, i.e., $\|T_1(f) + T_2(g)\| = \|f + g\|$ for all $f, g \in \mathcal{A}$. Then for each $f \in \mathcal{A}$,*

$$T_1(f) - T_1(0) = T_2(f) - T_2(0) = \omega \begin{cases} f \circ \varphi & on\, K, \\ \overline{f \circ \varphi} & on\, Ch(B) \setminus K, \end{cases}$$

where ω, φ and K are given by the above theorem.

(ii) If a surjection $T : \mathcal{A} \longrightarrow \mathcal{B}$ satisfies $\|T(f) + T(g)\| = \|f + g\|$ for all $f, g \in \mathcal{A}$, then

$$T(f) = \omega \begin{cases} f \circ \varphi & on\, K, \\ \overline{f \circ \varphi} & on\, Ch(B) \setminus K, \end{cases}$$

where ω, φ and K are as in (i).

(iii) If surjections $T_1, T_2 : \mathcal{A} \longrightarrow \mathcal{B}$ satisfy $\|T_1(f) - T_2(g)\| = \|f - g\|$ for all $f, g \in \mathcal{A}$, then $T_1(f) = T_2(f)$ and

$$T_1(f) - T_1(0) = \omega \begin{cases} f \circ \varphi & on\, K, \\ \overline{f \circ \varphi} & on\, Ch(B) \setminus K, \end{cases}$$

where ω, φ and K are as in (i).

Proof. (i) If we let $P = Q = \mathcal{A}$ and $S_1 = S_2 = id : \mathcal{A} \longrightarrow \mathcal{A}$, then, from Theorem 4.1, we conclude that $T_j - T_j(0)$ is a real-linear isometry, and there exist a homeomorphism φ from $Ch(B)$ onto

$Ch(A)$, a unimodular continuous function $\omega : Ch(B) \longrightarrow \mathbb{T}$ and a clopen subset K of $Ch(B)$ such that for each $f \in \mathcal{A}$,

$$T_j(f) - T_j(0) = \omega \begin{cases} f \circ \varphi & \text{on } K, \\ \overline{f \circ \varphi} & \text{on } Ch(B) \setminus K, \end{cases}$$

where $j \in \{1, 2\}$.

(ii) From the assumption, it is apparent that $T(0) = 0$. So (ii) is a direct consequence of (i).

(iii) It is clear that $T_1(f) = T_2(f)$ for each $f \in \mathcal{A}$. Now apply (i) for the maps T_1 and T_2', where T_2' is defined by $T_2'(g) = -T_2(-g)$ for all $g \in \mathcal{A}$, to get the result. □

Corollary 4.3. *Under the hypotheses of Theorem 4.1, if furthermore, A and B both have an approximate identity, respectively, then there exist a real-algebra isomorphism $\mathcal{T}$ from A onto B, a homeomorphism $\phi : M_A \longrightarrow M_B$ and a clopen subset $\mathcal{K}$ of M_B such that for each $f \in A$,*

$$\widehat{\mathcal{T}(f)} = \begin{cases} \hat{f} \circ \phi & \textit{on}\, \mathcal{K}, \\ \overline{\hat{f} \circ \phi} & \textit{on}\, M_B \setminus \mathcal{K}. \end{cases}$$

Moreover, $\mathcal{K} \cap Ch(B) = K$ and $\phi = \varphi$ on $Ch(B)$.

Proof. According to the proof of Theorem 4.1, there exists a real-linear isometry $\mathcal{T}_1 - \mathcal{T}_1(0)$ from A onto B. Moreover,

$$T_1(f) - \mathcal{T}_1(0) = \omega \begin{cases} f \circ \varphi & \text{on } K, \\ \overline{f \circ \varphi} & \text{on } Ch(B) \setminus K, \end{cases}$$

where ω, K and φ are from the previous theorem. Since both A and B have an approximate identity, then, from [9, Corollary 3.4], it follows B is the algebra which consists of the extensions of the continuous functions $\frac{T_1(f) - \mathcal{T}_1(0)}{\omega}|_{Ch(B)}$, where $f \in A$ and also the operator

$$\mathcal{T}(f) = \begin{cases} f \circ \varphi & \text{on } K, \\ \overline{f \circ \varphi} & \text{on } Ch(B) \setminus K, \end{cases}$$

defines a real-algebra isomorphism from A onto B. In fact, φ can be extended to a homeomorphism ϕ from M_A onto M_B, and there exists a clopen subset $\mathcal{K}$ of M_B such that $\mathcal{K} \cap Ch(B) = K$ and for each $f \in A$, $\widehat{\mathcal{T}(f)} = \hat{f} \circ \phi$ on $\mathcal{K}$ and $\widehat{\mathcal{T}(f)} = \overline{\hat{f} \circ \phi}$ on $M_B \setminus \mathcal{K}$. □

The above corollary may be considered as an extension of [21, Corollary 4].

The example given in [9] (Example 3.2) shows that, in general, the above conclusion is not valid and B need not be real-algebra isomorphism if both A and B do not have an approximate identity.

Finally we mention that the results are valid if $\mathcal{A}$ and $\mathcal{B}$ are replaced by dense subspaces of uniformly closed function algebras A and B, by following the proofs.

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