Invariant Sylow subgroups and solvability of finite groups

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Abstract. Let A and G be finite groups of relatively prime orders and assume that A acts on G via automorphisms. We study how certain conditions on G imply its solvability when we assume the existence of a unique A-invariant Sylow p-subgroup for p equal to 2 or 3.

1. Introduction

The number of Sylow p-subgroups for a prime p is restricted arithmetically by the properties of a finite group G and reciprocally. An immediate consequence of the Feit-Thompson Theorem, on the solvability of odd order groups, is that every finite group G that has a normal Sylow 2-subgroup, i.e., with $\nu_2(G) = 1$, is necessarily solvable. Also, by using the well-known property (see [6] for a proof) that the only non-abelian simple finite groups whose order is not divisible by 3 are the Suzuki simple groups, $\mathrm{Sz}(q)$, with $q = 2^r$ and $r > 1$ odd, one can easily prove (by induction on the order) that if a group G satisfies $\nu_3(G) = 1$ and has no composition factor isomorphic to the simple group $\mathrm{Sz}(q)$, then G is solvable too.

We suppose that the group G is acted on by an automorphism group A and investigate what information on the number of A-invariant Sylow subgroups of G can imply the solvability of G. This is especially relevant within the coprime action scenario, that is, when $(|A|, |G|) = 1$, which becomes quite a usual situation in Finite Group Theory. In this case, it turns out that G always has A-invariant Sylow p-subgroups for every prime p (dividing the order of G) and there exists exactly one A-invariant Sylow p-subgroup P if and only if P is normalized by the fixed point subgroup, $C = \mathbf{C}_G(A)$. Precisely, under

the assumption of the existence of exactly one A-invariant Sylow p-subgroup in G for certain primes p, we obtain certain conditions on the structure of G that characterize the solvability.

Theorem. *Suppose that a finite group A acts coprimely on a finite group G.*

  a) *Assume that G has exactly one A-invariant Sylow 2-subgroup. Then G is solvable if and only if G has no composition factor isomorphic to $\mathrm{PSL}(2, 3^r)$, with $r \geq 5$.*
  b) *Assume that G has exactly one A-invariant Sylow 3-subgroup. Then G is solvable if and only if G has no composition factor isomorphic to either $\mathrm{PSL}(2, 2^r)$, with $r \geq 5$, or $\mathrm{Sz}(q)$ with $q = 2^r$ and $r > 1$ an odd integer.*

If the action is not coprime, then the result is simply not true. It is easy to find examples of simple groups which are acted on by other groups in such a way that they only have one A-invariant Sylow p-subgroup for $p = 2$ or 3. For instance, let $G = \mathrm{Alt}(5)$ and let $A = \mathbf{N}_G(P)$, with P a Sylow p-subgroup of G, where $p = 2$ or 3. If we consider the action of A on G by conjugation, it turns out that G has exactly one A-invariant Sylow p-subgroup, for $p = 2$ or 3, respectively.

The proof of our main result employs the Classification of the Finite Simple Groups (CFSG), and we use [2] to compute the normalizers of the Sylow subgroups of certain simple groups.

2. Preliminaries

We start with some background on coprime action as well as some useful results. Nevertheless, we refer for the non-familiarized reader to Chapter 8 of [4] for instance, which compiles the main properties of coprime action in detail. Under the coprime action hypothesis, a group A acting on a group G, we define $\nu_p^A(G)$ to be the number of A-invariant Sylow p-subgroups of G for each prime p. We recall that $\nu_p^A(G) = 1$ if and only if there exists an A-invariant Sylow subgroup of G which is normalized by $\mathbf{C}_G(A)$. In order to establish certain divisibility properties of these A-invariant Sylow numbers we need the following.

Lemma 2.1. *Suppose that A is a finite group acting coprimely on a finite group G, and let H be an A-invariant subgroup of G. Let $C = \mathbf{C}_G(A)$. Then $|C : C \cap H|$ divides $|G : H|$.*

Proof. This is Lemma 2.1 of [5]. □

Lemma 2.2. *Suppose that A is a finite group acting coprimely on a finite group G and let $C = \mathbf{C}_G(A)$. Then, for every prime p,*

  a) *$\nu_p^A(G) = |C : \mathbf{N}_C(P)|$ for every A-invariant Sylow p-subgroup P of G.*
  b) *$\nu_p(C)$ divides $\nu_p^A(G)$ and $\nu_p^A(G)$ divides $\nu_p(G)$.*

c) *if N is an A-invariant normal subgroup of G, then $\nu_p^A(N)$ and $\nu_p^A(G/N)$
divide $\nu_p^A(G)$.*

Proof. a) Let P be an A-invariant Sylow p-subgroup of G. Then a) follows
from the fact that the A-invariant Sylow subgroups of G are all C-conjugate,
so $\nu_p^A(G)$ is exactly the number of distinct C-conjugates of P, and this is
exactly equal to $|C : \mathbf{N}_C(P)|$.

b) If P is an A-invariant Sylow subgroup of G, then by coprime action
properties $P \cap C$ is a Sylow p-subgroup of C and obviously, $\mathbf{N}_C(P) \subseteq \mathbf{N}_C(P \cap C)$. Hence $\nu_p(C) = |C : \mathbf{N}_C(P \cap C)|$ divides $\nu_p^A(G) = |C : \mathbf{N}_C(P)|$. On the
other hand, by applying Lemma 2.1 to the A-invariant subgroup $\mathbf{N}_G(P)$, we
get that $\nu_p^A(G) = |C : \mathbf{N}_C(P)|$ divides $\nu_p(G) = |G : \mathbf{N}_G(P)|$.

c) If P is an A-invariant Sylow p-subgroup of G, then PN/N is an
A-invariant Sylow subgroup of G/N and by a) and 8.2.2 of [4] we have

$$\nu_p^A(G/N) = |\mathbf{C}_{G/N}(A) : \mathbf{N}_{\mathbf{C}_{G/N}(A)}(PN/N)| =$$
$$= |CN/N : \mathbf{N}_{CN/N}(PN/N)| = |CN/N : \mathbf{N}_C(P)N/N|,$$

which certainly divides $\nu_p^A(G) = |C : \mathbf{N}_C(P)|$.

On the other hand, observe that $P \cap N$ is an A-invariant Sylow p-
subgroup of N and then

$$\nu_p^A(N) = |C \cap N : \mathbf{N}_{C \cap N}(P \cap N)| = |(C \cap N)\mathbf{N}_C(P \cap N) : \mathbf{N}_C(P \cap N)|.$$

Now, by the Frattini argument, we write $G = N\mathbf{N}_G(P)$, so by 8.2.11 of [4],
we have $C = (C \cap N)\mathbf{N}_C(P)$. As $\mathbf{N}_C(P) \subseteq \mathbf{N}_C(P \cap N)$, in particular we get
$C = (C \cap N)\mathbf{N}_C(P \cap N)$. Therefore, $\nu_p^A(N)$ divides $|C : \mathbf{N}_C(P)| = \nu_p^A(G)$,
as claimed. $\qquad\square$

The following result establishes some properties of the action of a group
on a direct product and of the behavior of the invariant Sylow subgroups.
Note that the coprime hypothesis is not needed in the first two properties
but it is required in the last one.

Lemma 2.3. *Suppose that a finite group A acts on a finite group G which
allows a direct decomposition $G = H_1 \times \ldots \times H_n$, that is A-invariant under
A, i.e., $H_i^a \in \{H_1, \ldots, H_n\}$ for all $a \in A$ and all $i \in \{1, \ldots, n\}$. Assume
further that A acts transitively on $\{H_1, \ldots, H_n\}$. Let $H \in \{H_1, \ldots, H_n\}$, let
$B = \mathbf{N}_A(H)$, and let S be a transversal for the cosets of B in A. Then*

a) *$\mathbf{C}_G(A) = \{\prod_{s \in S} e^s \mid e \in \mathbf{C}_H(B)\}$.*
b) *The projection π of $\mathbf{C}_G(A)$ on H is a monomorphism. In particular,
$\mathbf{C}_G(A) \cong \mathbf{C}_H(B)$.*
c) *Suppose that $(|A|, |G|) = 1$. For every prime p, we have $\nu_p^A(G) = \nu_p^B(H)$.*

Proof. a) This is exactly 8.1.6.a of [4].

b) Let $c_1, c_2 \in \mathbf{C}_G(A)$. By a) we can write $c_1 = \prod_{s \in S} e_1^s$ and $c_2 = \prod_{s \in S} e_2^s$ with $e_1, e_2 \in \mathbf{C}_H(B)$, and notice that such factorizations are unique.
Then

$$\pi(c_1 c_2) = \pi(\prod_{s \in S} e_1^s \prod_{s \in S} e_2^s) = \pi(\prod_{s \in S} (e_1 e_2)^s) = e_1 e_2 = \pi(c_1)\pi(c_2),$$

so π is an homomorphism from $\mathbf{C}_G(A)$ into $\mathbf{C}_H(B)$. Moreover, if $e \in \mathbf{C}_H(B)$, then $c = \prod_{s \in S} e^s$ is a preimage of e. If $\pi(c) = 1$, then clearly $c = 1$, so π is bijective.

c) Let p be a prime, and let P be an A-invariant Sylow p-subgroup of G. For every $s \in S$, it is clear that $P \cap H^s$ is a Sylow p-subgroup of H^s and that $P = \prod_{s \in S}(P \cap H^s)$. Then

$$\mathbf{N}_G(P) = \prod_{s \in S} \mathbf{N}_{H^s}(P \cap H^s).$$

Now, if $c \in \mathbf{N}_{\mathbf{C}_G(A)}(P)$, then by a) we can write $c = \prod_{s \in S} e^s$, with $e \in \mathbf{C}_H(B) \cap \mathbf{N}_G(P \cap H)$. Thus, when the projection π of b) is restricted to $\mathbf{N}_{\mathbf{C}_G(A)}(P)$, we get a monomorphism into H. In fact, as in b), we deduce that

$$\mathbf{N}_G(P) \cap \mathbf{C}_G(A) \cong \mathbf{N}_H(P \cap H) \cap \mathbf{C}_H(B).$$

Therefore, by applying b) and Lemma 2.2.a, we obtain

$$\nu_p^A(G) = |\mathbf{C}_G(A) : \mathbf{N}_{\mathbf{C}_G(A)}(P)| = |\mathbf{C}_H(B) : \mathbf{N}_{\mathbf{C}_H(B)}(P \cap H)| = \nu_p^B(H),$$

so the lemma is proved. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \square$

3. Proof

Proof of the Theorem. Suppose that G has exactly one A-invariant Sylow 2-subgroup and that G is solvable. It is certainly true that G cannot possess any composition factor isomorphic to the non-abelian simple group $\mathrm{PSL}(2, 3^r)$, with $r \geq 5$. In fact, further information can be deduced. Since by hypothesis the action of A is coprime, G cannot have as a composition factor any such simple group for those integers r such that $\mathrm{PSL}(2, 3^r)$ admits nontrivial coprime action. The same happens for proving the direct sense of part b). One only has to consider those r for which $\mathrm{PSL}(2, 2^r)$ and $\mathrm{Sz}(2^r)$ allow a nontrivial and coprime automorphism group.

Conversely, in order to prove the "if" part in a) and b), we argue by induction on $|GA|$, where GA denotes the semidirect product of G by A. We assume that p is 2 or 3. Suppose that N is a proper A-invariant normal subgroup of G. As $\nu_p^A(G) = 1$, by Lemma 2.2 b) we have $\nu_p^A(N) = \nu_p^A(G/N) = 1$. Moreover, N and G/N have no composition factors isomorphic to the corresponding simple groups of the statements of a) and b), so by the inductive hypothesis, N and G/N are solvable. Thus, G is solvable too and the proof is finished.

Consequently, we can assume that G has no proper A-invariant normal subgroup. In this case, we know that G is either elementary abelian, and the proof is finished, or $G = H_1 \times \cdots \times H_t$, where the subgroups H_i are isomorphic non-abelian simple groups. We will show that this leads to a contradiction. Notice that A acts transitively on the set $\{H_1, \ldots, H_t\}$. Let $H = H_1$, and let $B = \mathbf{N}_H(A)$. We are in the situation of Lemma 2.3, and then there exists an isomorphism $\pi : \mathbf{C}_G(A) \longrightarrow \mathbf{C}_H(B)$, given by the projection of $\mathbf{C}_G(A)$ into H. If $t > 1$, then by Lemma 2.3.c, we get $\nu_p^B(H) = 1$, and of course, H cannot

have any composition factor isomorphic to one of the simple groups that appear in the statement. Thus, by induction, H is solvable, a contradiction.

As a result, in the following we can assume that $t = 1$, that is, G is non-abelian simple. Also, we can assume that A acts faithfully on G. If not, we consider $\bar{A} := A/\mathbf{C}_A(G)$, where $\mathbf{C}_A(G)$ is the kernel of the action of A on G. As the set of A-invariant Sylow p-subgroups of G coincides with the set of $\bar{A}$-invariant Sylow p-subgroups of G, the cardinality of both sets coincide, so by induction we get that G is solvable, a contradiction.

The alternating groups and the 26 sporadic simple groups do not admit nontrivial coprime action, as one can check by looking up the corresponding outer automorphism groups in [2]. Then we can assume, by using the CFSG, that G is a simple group of Lie type, defined over some finite field F. Furthermore, if we replace A by some conjugate in $\text{Aut}(G)$, the group A can be assumed to be induced by a group of automorphisms of F. Now, if $|A| = r$, it follows that $|F| = q^r$ for some prime power q. Following the notation of [1], we can write $G = G(q^r)$, the group of Lie type "G", and then $C := \mathbf{C}_G(A) = G(q)$, where $G(q)$ is the Lie group of the same type as G, but is defined over the field of q elements. We prove that when G is a simple group of Lie type, except the cases described in the statement, then it has more than one A-invariant Sylow subgroup for $p = 2$ or 3. In fact, all the above groups of Lie type are simple in almost all the cases, except in exactly eight cases, and consequently, they satisfy $\nu_2(C) > 1$ and $\nu_3(C) > 1$, if we also exclude the Suzuki group $^2B_2(2^r)$. This group must be ruled out because it is the only non-abelian simple group satisfying $\nu_3(G) = 1$, as 3 does not divide its order (see [6]). Then, by Lemma 2.2.b, for all these groups we would have $\nu_p^A(G) > 1$, for $p = 2$ and 3, which is a contradiction.

We analyze the cases in which C is one of the eight non-simple groups mentioned above, which are the following: $A_1(2) \cong \text{Sym}(3)$; $A_1(3) \cong \text{Alt}(4)$; $^2A_2(2^2) \cong \text{PSU}(3, 2^2)$, which has of order 72; $^2B_2(2) \cong \text{Sz}(2)$, the Frobenius group of order 20; $B_2(2) \cong \text{Sym}(6)$; $G_2(2)$ which has order 12096 and the derived subgroup is isomorphic to $\text{PSU}_3(3^2)$; $^2G_2(3)$ of order 1512 and its derived subgroup is isomorphic to $\text{PSL}_2(2^3)$; $^2F_4(2)$ whose derived subgroup is simple of order $2^{11}.3^2.5^2.13$. All of them satisfy that $\nu_2(C) > 1$ except $\text{Alt}(4)$. Therefore, we obtain $\nu_2^A(G) > 1$, a contradiction, except at most in the case $G = A_1(3^r) \cong \text{PSL}(2, 3^r)$. Moreover, in this latter case, when $r < 5$ then G does not admit nontrivial coprime action, so $\nu_2^A(G) = \nu_2(G) > 1$. Hence, we conclude that $G = A_1(3^r)$, with $r \geq 5$. As we are assuming in the hypotheses that this possibility cannot occur, we obtain the final contradiction for $p = 2$ and a) is proved. Analogously, all of the above eight groups satisfy that $\nu_3(C) > 1$ except $\text{Sym}(3)$ and $\text{Sz}(2)$. Therefore, $\nu_3^A(G) > 1$, which contradicts our assumptions, except at most when $G = A_1(2^r)$ for some integer $r > 1$ or $G = \text{Sz}(q^r)$ with $r > 1$ odd. Moreover, if $G = A_1(2^r)$ and $r < 5$, then G does not admit nontrivial coprime action, so $\nu_3^A(G) = \nu_3(G) > 1$, a contradiction. Thus, we can assume $r \geq 5$ in the case $G = A_1(2^r)$. Since these possibilities

are excluded in the hypotheses, we achieve the final contradiction and part b) is proved.

Remark 3.1. The condition on the composition factors in the Theorem cannot be much improved. In fact, let $\mathrm{GF}(3^r)$ be the field with 3^r elements, which is the underlying field of the group $G = \mathrm{PSL}(2, 3^r)$. Let us consider A to be the Galois group of the extension $\mathrm{GF}(3^r)/\mathrm{GF}(3)$, which is a cyclic group of order r and induces a group action on G. Then $C := \mathbf{C}_G(A) \cong \mathrm{PSL}(2,3)$. Since $|G| = 3^r(3^r - 1)(3^r + 1)/2$, if we take r such that $(r, |G|) = 1$ (so the action is coprime), it easily follows that $|G|_2 = 4$. Therefore, a Sylow 2-subgroup P of C is also an A-invariant Sylow 2-subgroup of G. Then $\nu_2^A(G) = |C : \mathbf{N}_C(P)| = \nu_2(C) = 1$, that is, G has exactly one A-invariant Sylow 2-subgroup.

Analogously, let $G = \mathrm{PSL}(2, 2^r)$ and let A be the Galois group of the field extension $\mathrm{GF}(2^r)/\mathrm{GF}(2)$, which induces an action on G. We have $|G| = 2^r(2^r - 1)(2^r + 1)$, and if we take r such that $(r, |G|) = 1$, then in particular we have $(r, 6) = 1$ and this implies that $|G|_3 = 3$. Hence, the Sylow 3-subgroup P of $C \cong \mathrm{PSL}(2,2)$ is an A-invariant Sylow 3-subgroup of G. By applying Lemma 2.2.a, we have $\nu_3^A(G) = |C : \mathbf{N}_C(P)| = \nu_3(C) = 1$, that is, G has exactly one A-invariant Sylow 3-subgroup.

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