Geometrical definition of a continuous family of time transformations generalizing and including the classic anomalies of the elliptic two-body problem.

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Abstract

This paper is aimed to address the study of techniques focused on the use of a family of anomalies based on a family of geometric transformations that includes the true anomaly f, the eccentric anomaly g and the secondary anomaly f' defined as the polar angle with respect to the secondary focus of the ellipse.

This family is constructed using a natural generalization of the eccentric anomaly. The use of this family allows closed equations for the classical quantities of the two body problem that extends the classic, which are referred to eccentric, true and secondary anomalies.

In this paper we obtain the exact analytical development of the basic quantities of the two body problem in order to be used in the analytical theories of the planetary motion. In addition, this paper includes the study of the minimization of the errors in the numerical integration by an appropriate choice of parameters in our selected family of anomalies for each value of the eccentricity

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1. Introduction

The study of the motion in the solar system is one of strengths of Celestial Mechanics. This issue involves the development of planetary theories and the motion of artificial satellites around the earth. In this paper, we deal with both topics.

To construct a planetary theory two major ways can be considered: the use of a numerical integrator [8], [6] or the use of analytical methods to integrate the problem [2], [22], [23], [26].

The analytical methods are based on the solution of the two body problem (Sun-planet) through a set of orbital elements, for example the third set of Brower and Clemence [1] $(a, e, i, \Omega, \omega, M)$, where $M = M_0 + n(t - t_0)$, n is the mean motion, t_0 is the initial epoch whose value are constant in the unperturbed two body problem and M_o the mean anomaly in the initial epoch t_0. This solution can be considered as a first approximation of the perturbed problem and we can use the Lagrange method of variation of constants to replace the first elements by the osculating ones given by the Lagrange planetary equations [13]

$$\begin{aligned}
\frac{da}{dt} &= \frac{2}{na}\frac{\partial R}{\partial \sigma} \\
\frac{de}{dt} &= -\frac{\sqrt{1-e^2}}{na^2e}\frac{\partial R}{\partial \omega} + \frac{1-e^2}{na^2e}\frac{\partial R}{\partial \sigma} \\
\frac{di}{dt} &= -\frac{1}{na^2\sqrt{1-e^2}\sin i}\frac{\partial R}{\partial \Omega} + \frac{\operatorname{ctg} i}{na^2\sqrt{1-e^2}}\frac{\partial R}{\partial \omega} \\
\frac{d\Omega}{dt} &= \frac{1}{na^2\sqrt{1-e^2}\sin i}\frac{\partial R}{\partial i} \\
\frac{d\omega}{dt} &= \frac{\sqrt{1-e^2}}{na^2e}\frac{\partial R}{\partial e} - \frac{\cos i}{na^2\sqrt{1-e^2}\sin i}\frac{\partial R}{\partial i} \\
\frac{d\sigma}{dt} &= -\frac{2}{na}\frac{\partial R}{\partial a} - \frac{1-e^2}{na^2e}\frac{\partial R}{\partial e}
\end{aligned} \tag{1}$$

σ is a new variable defined by the equation:

$$M = \sigma + \int_{t_0}^{t} n\,dt \tag{2}$$

and it coincides with M_0 in the case of the unperturbed motion. R is the disturbing potential $R = \sum_{k=1}^{N} R_i$ due to the disturbing bodies $i = 1, ..., N$. It is defined as [13]

$$R = \sum_{k=1}^{N} Gm_k \left[\left(\frac{1}{\Delta_k} \right) - \frac{x \cdot x_k + y \cdot y_k + z \cdot z_k}{r_k^3} \right] \tag{3}$$

where $\vec{r} = (x, y, z)$ and $\vec{r}_k = (x_k, y_k, z_k)$ are the heliocentric vector position of the secondary body and the kth disturbing body respectively, Δ_k is the distance between the secondary body and the disturbing body, and m_k the mass of the disturbing body.

In order to integrate the Lagrange planetary equations through analytical methods it is necessary to develop the second member of the Lagrange planetary equations as truncated Fourier series, which is a classical problem in celestial mechanics [26], [9], [1], [3], [12]. The analytical methods provide very long series solution and it is suitable to obtain more compact developments using as temporal variable an appropriate anomaly.

To obtain the expansions according to an anomaly Ψ_i it is necessary to obtain for each planet i the developments of the coordinates and the inverse of the radius in Fourier series of Ψ_i. Then, the integration of the Lagrange planetary equations with respect to the Ψ_i anomalies requires to compute the corresponding Kepler equation $M_i = M_i(\Psi_i)$ [15], [14], [16].

When using numerical integration methods it is more appropriate to consider the equation of motion in the form of the second Newton law. The efficiency of the numerical integrators can be improved through an appropriate change in the temporal variable. In this paper we will study the performance of the previous family of anomalies. To this aim, we select the problem of the motion of an artificial satellite around the Earth. The relative motion of the secondary with respect to the Earth is defined by the second order differential equations

$$\frac{d^2\vec{r}}{dt^2} = -GM\frac{\vec{r}}{r^3} - \vec{\nabla}U - \vec{F} \tag{4}$$

where $\vec{r}$ is the radius vector of the satellite, U the potential from which the perturbative conservative forces are derived and $\vec{F}$ includes the non-conservative forces. To integrate the system (4) it is necessary to known the initial values of the radius vector $\vec{r}_0$ and velocity $\vec{v}_0$.

In order to uniformize the truncation errors when a numerical integrator is used there are three main techniques:

1. The use of a very small stepsize.
2. The use of an adaptative stepsize method.
3. The use of a change in the temporal variable to arrange an appropriate distribution of the points on the orbit so that the points are mostly concentrated in the regions where the speed and curvature are maxima.

This paper follows the third technique. Several authors have already studied this question. See for instance, Sundman [24], who introduced a new temporal variable τ related to the time t through $dt = Crd\tau$, Nacozy [21] proposed a new temporal variable $dt = Cr^{3/3}d\tau$, Brumberg [4] proposed the use of the regularized length of arc and Brumberg and Fukushima [5] introduced the elliptic anomaly as temporal variable. Janin [10], [11] and Velez [25] extended this technique defining a new one-parameter family of transformations α called generalized Sundman transformations $dt = Q(r, \alpha)d\tau_\alpha$, where $Q(r, \alpha) = C_\alpha r^\alpha$. The function $Q(r)$ is normally known as partition function. A more complicated family of transformations was introduced by Ferrandiz [7] $Q(r) = r^{2/3}(a_0+a_1 r)^{-1/2}$. López [18] introduces an new family of anomalies, called natural anomalies as $\Psi_\alpha = (1-\alpha)f'+\alpha f$, $\alpha \in [0, 1]$ where f, f' are the true and secondary anomalies it is the angle between the periapsis and the secondary position taking as origin the primary focus F or the secondary focus of the ellipse f' respectively. Analytical and numerical properties of generalized Sundman anomalies and natural have been studied by López et al. [17], [19], [20].

The generalized Sundmand family and the natural anomalies involve several inconveniences: the main quantities of the two-body problem, such as the orbital coordinates (ξ, η), the radius vector and the generalized Kepler equation cannot be written by means of a closed formula except for a small set of values of the parameter α. In general, the coefficients of the necessary developments for the construction of analytical theories of the planetary motion cannot be written using closed formulas, either. Finally, these anomalies have not an easy geometrical interpretation. In this paper we define a new family of anomalies through a geometric family of transformations in order to solve the latter inconveniences. We propose for this family the name of generalized eccentric anomaly.

The rest of this article is organized as follows: In this section the general background has been introduced. In section 2 the properties of generalized eccentric family of anomalies will be described. In addition, we will obtain the differential equations of motion using an arbitrary anomaly from this family. In section 3 the analytical properties of the generalized eccentric anomaly will be studied. This study contains the expansions of eccentric anomaly g, $\sin g$, $\cos g$, Kepler equation $\frac{r}{a}$ and $\frac{a}{r}$ according to the generalized eccentric anomaly. So, an appropriate use of them suffices to obtain the development of the mean anomaly with respect to the generalized eccentric anomaly. It is the Kepler equation. In section 4 a set of numerical examples about the two body problem will be considered. In section 5 the main conclusions and remarks will be exposed.

2. Generalized eccentric anomalies

In this section a new family of anomalies depending on one parameter is defined. We represent in figure 1 the elliptic orbit corresponding to the motion of the two body problem. This ellipse is defined by its major semiaxis $a = \overline{OQ}$ and its eccentricity $e = \frac{c}{a}$, $0 \leq e < 1$ where c is the focal semidistance $c = \frac{\overline{FF'}}{2}$, and the minor semiaxis b is defined as $b = a\sqrt{1-e^2}$. Let O be the center of the ellipse, F the primary focus, F' the secondary focus (also called equality point).

Let us define F_α as the point of coordinates $(\alpha\, e\, a, 0)$, $\alpha \in [-1, 1]$, Q the periapsis and P the position of the secondary in the orbit. The point F_α is the primary focus of an ellipse with the same center and major semiaxe as the orbit and the minor semiaxe $\alpha\, e\, a$ where e is the eccentricity of the orbit and $\alpha \in [-1, 1]$. Notice that in this point $F_\alpha = F'$ if $\alpha = -1$, $F_\alpha = O$ if $\alpha = 0$ and $F_\alpha = F$ if $\alpha = 1$. Let us define the orbital coordinates (ξ, η) referred to the primary focus, and let r and r' be the distance between the secondary P and the primary focus F and the secondary focus F' respectively. The angle g is called eccentric anomaly, the angle f is called true anomaly and for the angle f' we propose the name of secondary true anomaly.

Let P the position of the secondary on the orbit, S the orthogonal projection of P on the major semiaxis and N, R the corresponding points on the ellipses of minor semiaxe $\overline{OB} = a\sqrt{1-\alpha^2 e^2}$ and the major circle of the orbit. In this

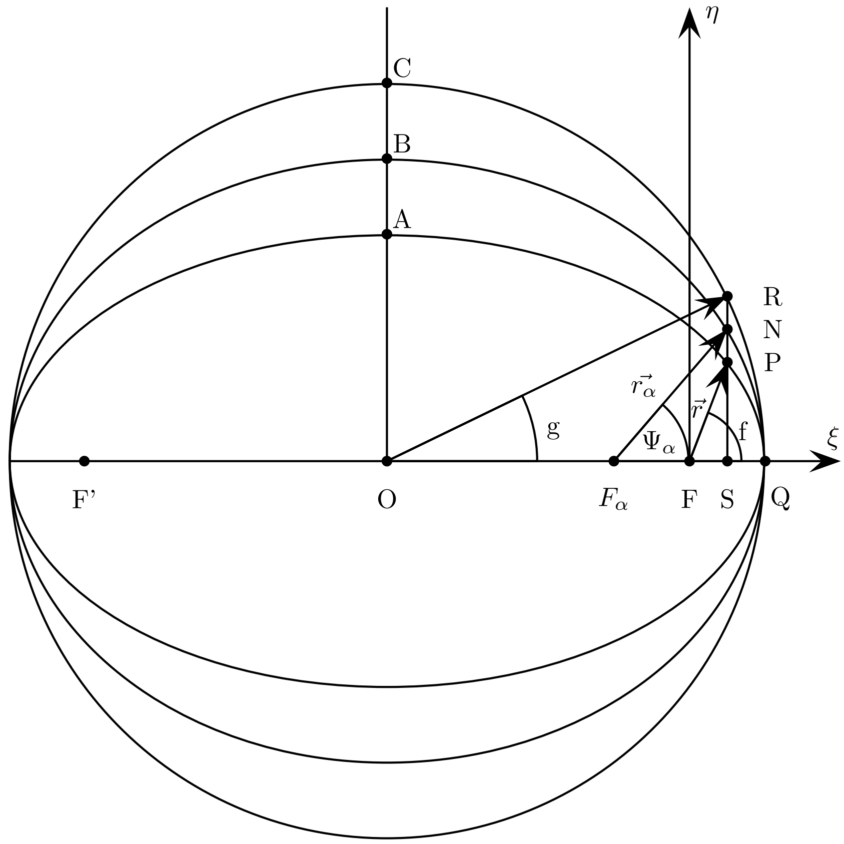


Figure 1: Elliptic motion

family of ellipses we have the relationship:

$$\frac{\overline{SP}}{\overline{OA}} = \frac{\overline{SN}}{\overline{OB}} = \frac{\overline{SR}}{\overline{OC}} \tag{5}$$

The vector radius of point $\overrightarrow{F_\alpha N}$ is defined by its module r_α and the value of the anomaly Ψ_α. The radius r_α is related to Ψ_α through

$$r_\alpha = \frac{a(1-\alpha^2 e^2)}{1+\alpha e\cos\Psi_\alpha}. \tag{6}$$

On the other hand $\overline{OF} = a\,e$, $\overline{OF_\alpha} = \alpha\,a\,e$ and taking into account (5), the coordinates (ξ,η) of the secondary are related to Ψ_α as

$$\xi = r_\alpha\cos\Psi_\alpha - ae(1-\alpha), \quad \eta = \frac{\sqrt{1-e^2}}{\sqrt{1-\alpha^2 e^2}} r_\alpha \sin\Psi_\alpha \tag{7}$$

and so

$$\xi = a\left[\frac{(1-\alpha^2 e^2)\cos\Psi_\alpha}{1+\alpha e\cos\Psi_\alpha} - (1-\alpha)e\right], \quad \eta = a\sqrt{\frac{1-e^2}{1-\alpha^2 e^2}}\frac{(1-\alpha^2 e^2)\sin\Psi_\alpha}{1+\alpha e\cos\Psi_\alpha}. \tag{8}$$

To link g with Ψ_α we consider the classical relations

$$\xi = a(\cos g - e), \quad \eta = a\sqrt{1-e^2}\sin g, \tag{9}$$

and operating we have

$$\cos\Psi_\alpha = \frac{\cos g - \alpha\,e}{1-\alpha\,e\cos g}, \tag{10}$$

and

$$\sin\Psi_\alpha = \frac{\sqrt{1-\alpha^2 e^2}\sin g}{1-\alpha\,e\cos g}, \tag{11}$$

and from (10), (11) it is easy to get

$$\cos g = \frac{\cos\Psi_\alpha + \alpha\,e}{1+\alpha\,e\cos\Psi_\alpha}, \tag{12}$$

and

$$\sin g = \frac{\sqrt{1-\alpha^2 e^2}\sin\Psi_\alpha}{1+\alpha\,e\cos\Psi_\alpha}. \tag{13}$$

To evaluate the orbital velocity according to the anomaly Ψ_α we consider the well-known equations

$$\dot\xi = -\frac{na\sin g}{1-e\cos g} \quad ,\dot\eta = \frac{na\sqrt{1-e^2}\cos g}{1-e\cos g}, \tag{14}$$

and taking into account (12) and (13) we obtain

$$\dot\xi = -\frac{na\sqrt{1-\alpha^2 e^2}\sin\Psi_\alpha}{1-\alpha e^2 + (1-\alpha)e\cos\Psi_\alpha}, \quad \dot\eta = \frac{na\sqrt{1-e^2}(\cos\Psi_\alpha + \alpha e)}{1-\alpha e^2 + (1-\alpha)e\cos\Psi_\alpha}. \tag{15}$$

Finally, replacing (12) in the classical equation $r = a(1 - e\cos g)$, we have for the vector radius

$$r = a\frac{(1-\alpha\, e^2) - e(1-\alpha)\cos\Psi_\alpha}{1+\alpha\, e\cos\Psi_\alpha}, \tag{16}$$

and taking into account (6) we obtain

$$r_\alpha = a(1-\alpha) + \alpha\, r. \tag{17}$$

The anomaly Ψ_α is related with eccentric anomaly by means of:

$$\tan\frac{\Psi_\alpha}{2} = \sqrt{\frac{1+\alpha\, e}{1-\alpha\, e}}\tan\frac{g}{2}. \tag{18}$$

To connect $d\Psi_\alpha$ and dM we proceed derivating (10)

$$-\sin\Psi_\alpha d\Psi_\alpha = -\frac{(1-\alpha^2 e^2)\sin g}{(1-\alpha\, e\cos g)^2}dg, \tag{19}$$

and, taking into account (11) and operating we obtain

$$d\Psi_\alpha = \frac{\sqrt{1-\alpha^2 e^2}dg}{1-\alpha\, e\cos g}, \tag{20}$$

replacing in this equation (16) we get

$$d\Psi_\alpha = \sqrt{1-\alpha^2 e^2}\frac{a}{r_\alpha}dg, \tag{21}$$

and taking into account that $dg = a/r dM$ we obtain

$$dM = \frac{r r_\alpha}{a^2\sqrt{1-\alpha^2 e^2}}d\Psi_\alpha. \tag{22}$$

Which can be written as $dM = Q(r)d\Psi_\alpha$ where $Q(r)$ is the partition function

$$Q(r) = \frac{r\, r_\alpha}{a^2\sqrt{1-\alpha^2 e^2}} = \frac{r(1-\alpha)a + \alpha\, r^2}{a^2\sqrt{1-\alpha^2 e^2}}, \tag{23}$$

notice that the partition function $Q(r)$ is symmetric in its first form.

3. Analytical developments

In order to integrate the Lagrange planetary equations using analytical or semi-analytical methods it is necessary to develop their second member as Fourier series according to the selected anomalies for each couple of planets. To this aim, it is necessary to obtain the expansions with respect to the selected anomaly of the two-body problem quantities g, $\sin g$, $\cos g$, r/a, a/r, and M. In the rest of this section we write Ψ instead of Ψ_α to ease the notation.

To obtain the development of g according with Ψ we consider in the first place the equation (18). From this equation we have

$$\tan\frac{g}{2} = \sqrt{\frac{1-\alpha\, e}{1+\alpha\, e}}\tan\frac{\Psi}{2}. \tag{24}$$

let us define q by

$$\sqrt{\frac{1-\alpha\, e}{1+\alpha\, e}} = \frac{1+q}{1-q}, \tag{25}$$

we have [26]

$$g = \Psi + \sum_{k=1}^{\infty}\frac{2q^k}{k}\sin k\Psi, \tag{26}$$

where $q = -\alpha\, e/(1+\sqrt{1-\alpha^2 e^2})$, and so:

$$g = \Psi + \sum_{k=1}^{\infty}\frac{(-1)^k}{k}\frac{2\alpha^k e^k}{(1+\sqrt{1-\alpha^2 e^2})^k}\sin k\Psi, \tag{27}$$

To obtain the expansions of $\sin g$, we employ equation (13). Let us define a new complex variable $z = exp(\sqrt{-1}\Psi)$, the values of $\sin g$ and $\cos g$ are related to the new variable by

$$\sin g = \frac{\sqrt{1-\alpha^2 e^2}}{\sqrt{-1}}\frac{z^2-1}{\alpha\, e\, z^2 + 2z + \alpha\, e} = \frac{\sqrt{1-\alpha^2 e^2}}{\sqrt{-1}}\frac{z^2-1}{\alpha\, e(z-z_1)(z-z_2)}, \tag{28}$$

where $z_1 = -\alpha\, e/(1+\sqrt{1-\alpha^2 e^2})$, $z_2 = -(1+\sqrt{1-\alpha^2 e^2})/(\alpha\, e)$ are the roots of the equation $\alpha\, e\, z^2 + 2z + \alpha\, e = 0$. For $|\alpha\, e| < 1$, $\exists r_1, r_2 > 0$ such that $0 < |z_1| < r_1 < 1 < r_2 < |z_2|$ and the we can expand (28) as Laurent series $\sin g = \sum\limits_{k=-\infty}^{\infty} a_k z^k$ in the ring $\mathcal{H} = \{z \in\ r_1 \leq |z| \leq r_2\}$. Notice that the circumference of radius one is contained in the ring. On the other hand, we replace z by z^{-1} in equation (28) and we obtain the development of $-\sin g$, and then $a_k = -a_{-k}$, $k = 0, 1, \ldots$. Let γ be the circumference in the complex plane centered in the origin and radius one; the coefficient a_{-k} is given by

$$a_{-k} = \frac{1}{2\pi\sqrt{-1}}\int_{\gamma}\frac{\sqrt{1-\alpha^2 e^2}}{\sqrt{-1}}\frac{z^2-1}{\alpha\, e(z-z_1)(z-z_2)}z^{k-1}dz, \tag{29}$$

let us define

$$G_k(z) = \frac{\sqrt{1-\alpha^2 e^2}}{\sqrt{-1}}\frac{z^2-1}{\alpha\, e(z-z_2)}z^{k-1}, \tag{30}$$

this function is holomorphic in a disk containing the circumference γ and so, the value of a_{-n} is given by $a_{-k} = G_k(z_1)$.

$$a_k = -a_{-k} = \frac{(-1)^k}{2\sqrt{-1}}\left(\frac{\alpha^2 e^2}{(1+\sqrt{1-\alpha^2 e^2})^2} - 1\right)\frac{\alpha^{k-1}e^{k-1}}{(1+\sqrt{1-\alpha^2 e^2})^{k-1}}, \tag{31}$$

and thus

$$\sin g = \sum_{k=1}^{\infty} 2(-1)^k \left(\frac{\alpha^2 e^2}{(1+\sqrt{1-\alpha^2 e^2})^2} - 1 \right) \frac{\alpha^{k-1} e^{k-1}}{(1+\sqrt{1-\alpha^2 e^2})^{k-1}} \sin k\Psi. \quad (32)$$

To obtain $\cos g$ we can proceed similarly, replacing $z = exp(\sqrt{-1}\Psi)$ in (12) we obtain

$$\cos g = \frac{z^2+1+2\alpha\, ez}{\alpha\, ez^2+2z+\alpha\, e} = \frac{z^2+1+2\alpha\, ez}{\alpha\, e(z-z_1)(z-z_2)}, \quad (33)$$

$\cos g$ is holomorphic in the ring $\mathcal{H}$ and so, it can be developed as Laurent series as $\cos g = \sum\limits_{k=-\infty}^{\infty} b_k z^k$ and due to symmetry $a_k = a_{-k}$. For $k = 0$ we have

$$\begin{aligned} b_0 &= \frac{1}{2\pi\sqrt{-1}} \int_\gamma \frac{z^2+1+2\alpha\, ez}{(\alpha\, ez^2+2z+\alpha\, e)z} dz = \frac{1}{2\pi\sqrt{-1}} \int_\gamma \frac{z^2+1+2\alpha\, ez}{\alpha\, e(z-z_1)(z-z_2)z} dz = \\ &= \frac{1}{2\pi\sqrt{-1}} \int_\gamma \frac{z^2+1+2\alpha\, ez}{\alpha\, e(z-z_2)(z-z_1)} z_2 dz - \frac{1}{2\pi\sqrt{-1}} \int_\gamma \frac{z^2+1+2\alpha\, ez}{\alpha\, e(z-z_2)z} z_2 dz = \\ &= \frac{z_1^2+1+2\alpha\, e\, z_1}{\alpha\, e(z_1-z_2)} z_2 + \frac{1}{\alpha\, e} = \frac{\alpha\, e}{1+\sqrt{1-\alpha^2\, e^2}} \end{aligned} \quad (34)$$

and for $k \neq 0$ we have

$$b_k = b_{-k} = \frac{1}{2\pi\sqrt{-1}} \int_\gamma \frac{z^2+1+2\alpha\, ez}{\alpha\, ez^2+2z+\alpha\, e} z^{k-1} dz = \frac{1}{2\pi\sqrt{-1}} \int_\gamma \frac{z^2+1+2\alpha\, ez}{\alpha\, e(z-z_1)(z-z_2)} z^{k-1} dz, \quad (35)$$

and so

$$b_k = \frac{z_1^2+1+2\alpha\, e\, z_1}{\alpha\, e(z_1-z_2)} z_1^{k-1} = (-1)^{k-1} \frac{\sqrt{1-\alpha^2 e^2}}{(1+\sqrt{1-\alpha^2 e^2})^k} \alpha^{k-1} e^{k-1} \quad (36)$$

consequently

$$\cos g = \frac{\alpha\, e}{1+\sqrt{1-\alpha^2\, e^2}} + \sum_{k=1}^{\infty} 2(-1)^{k-1} \frac{\sqrt{1-\alpha^2 e^2}}{(1+\sqrt{1-\alpha^2 e^2})^k} \alpha^{k-1} e^{k-1} \cos k\Psi, \quad (37)$$

Finally, after replacing (27) and (32) in the Kepler equation $M = g - e\sin g$ it is easy to obtain its the development for the anomaly Ψ. Taking into account (37), we obtain the expansion of $r/a = 1 - e\cos g$ as Fourier series according to Ψ.

To develop of a/r we consider (16) and from this equation we have

$$\frac{a}{r} = \frac{1+\alpha\, e\cos\Psi}{(1-\alpha\, e^2)(1-\beta\cos\Psi)}, \quad (38)$$

where $\beta = e(1-\alpha)/(1-\alpha\, e^2)$. It is easy to demonstrate that $0 \leq \beta < 1$ if $e \in [0,1[$ and $\alpha \in [-1,1]$. Replacing $\cos\Psi = (z+z^{-1})/2$ in (38) we obtain

$$\frac{a}{r} = \frac{2z+\alpha\, e\, z^2+\alpha\, e}{(1-\alpha\, e)(2z-\beta z^2-\beta)} = -\frac{2z+\alpha\, e\, z^2+\alpha\, e}{(1-\alpha\, e)\beta(z-z_3)(z-z_4)}, \quad (39)$$

where $z_3 = (1 - \sqrt{1-\beta^2})/\beta$, and $z_4 = (1 + \sqrt{1-\beta^2})/\beta$. Note that $0 \leq z_3 < 1 < z_4$ and so a/r is a holomorphic function in a ring centered in the origin that contains the circumference γ. Following a method similar to the one used to expand $\cos g$ we obtain

$$\frac{a}{r} = c_0 + \sum_{k=1}^{\infty} c_k \cos k\,\Psi, \tag{40}$$

where

$$c_0 = -\frac{\alpha}{1-\alpha} + \frac{\beta + \alpha\, e}{(1-\alpha)e\sqrt{1-\beta^2}}, \tag{41}$$

and

$$c_k = \frac{2(\alpha\, e + \beta)}{(1-\alpha)e\sqrt{1-\beta^2}} \frac{\beta^k}{(1+\sqrt{1-\beta^2})^k}, \tag{42}$$

it is easy to demonstrate that $\lim_{\alpha\to 1} c_0 = 1/(1-e^2)$, $\lim_{\alpha\to 1} c_1 = e/(1-e^2)$ and $\lim_{\alpha\to 1} c_k = 0$ for $k > 1$.

These results are necessary to expand the second member of planetary Lagrange equations. It is worth to remark that the coefficients of these developments are given in closed form.

4. Numerical examples

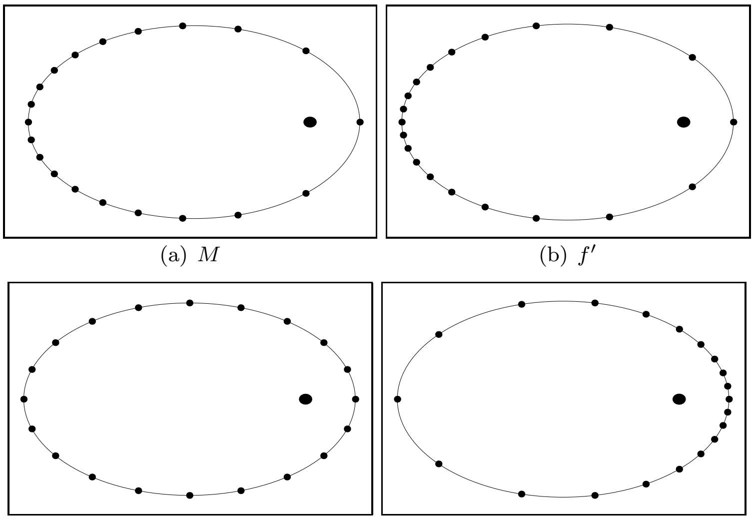
In general, the perturbative forces are small, for this reason it is convenient to test the numerical methods applying them to the well-known two body problem, referred to the orbital coordinate system $(x, y, 0)$, in order to select an appropriate new temporal variable with the aim of minimizing the distribution of the truncation errors on the orbit. Let us define a generic family Ψ_α of anomalies depending on a parameter α as $dt = Q_\alpha(r) d\Psi_\alpha$, for each α we have

$$\frac{d}{dt} = n\frac{d}{dM} = n\frac{d}{d\Psi_\alpha}\frac{d\Psi_\alpha}{dM} = \frac{n}{Q_\alpha(r)}\frac{1}{d\Psi_\alpha} \tag{43}$$

so

$$\begin{aligned}
\frac{dx}{d\Psi_\alpha} &= \frac{Q_\alpha(r)}{n} v_x, \quad & \frac{dv_x}{d\Psi_\alpha} &= -\frac{Q_\alpha(r)}{n}\left[GM\frac{x}{r^3} + \frac{\partial V}{\partial x} - F_x\right] \\
\frac{dy}{d\Psi_\alpha} &= \frac{Q_\alpha(r)}{n} v_y, \quad & \frac{dv_y}{d\Psi_\alpha} &= -\frac{Q_\alpha(r)}{n}\left[GM\frac{y}{r^3} + \frac{\partial V}{\partial y} - F_y\right]
\end{aligned} \tag{44}$$

In order to test the performance of this method we use a fictitious artificial satellite with the same elements than HEOS II used by Brumberg [4] ($a = 118363.47 Km$, $e = 0.942572319$, $i = 28^o.16096$, $\Omega = 185^o.07554$, $\omega = 270^o.07151$, $M_0 = 0^o$), except for its eccentricity, that is changed in order to study the optimum value of α depending on the value of the eccentricity e. In figure 2 we show a sample of twenty points for Ψ_α with homogeneous distribution over the orbit.



(a) M (b) f' (c) g (d) f

Figure 2: Points distribution for M, f', g, f

Table 1 shows the minimum of the errors for this fictitious satellite with the same elements $(a, e, i, \Omega, \omega, M)$ than HEOSII and several values of parameter $\alpha = -1.00, -.95, \ldots, 0, \ldots, 1$. This table has been carried out using a classic fourth order Runge Kutta integrator with 10000 uniform steps. In this table we see the dependence of the errors in position and velocity with respect to the value of the parameter α.

Table 2 shows the minimum of the errors in position (in Km) and velocity (in Km/s) for this fictitious satellite with the same a) than HEOSII , $\Omega = \omega = i = M_0 = 0$, and different values of eccentricity ($e = 0.0, 0.05, \ldots, 0.95$). This table has been carried out using a classic fourth order Runge Kutta integrator with 1000 uniform steps. In this table, we see that the value of α where the errors in position reach their minimum depends on the eccentricity. Notice that the values contained in table 2 improve the ones obtained using Sundman generalized anomalies [19] and the natural anomalies [18].

The value of α that minimize the error position can be fit by means of a fifth order polynomial:

$$\alpha(e) = 0.554 + 0.326x - 0.609x^2 + 1.196x^3 - 1.204x^4 + 0.755x^5. \quad (45)$$

Figure 3 shows the difference between the calculated values and the least square fit.

Figures 4,5,6,7 show the local integration errors, in position and velocity, for a satellite with $a = 118363.47Km$ and $e = 0.8$ for the values of $\alpha =$

Table 1: Errors in Heos II position in Km and velocity in Km/s for several values of α.

α	$\lvert\Delta\vec{r}\rvert$	$\lvert\Delta\vec{v}\rvert$	α	$\lvert\triangle\vec{r}\rvert$	$\lvert\triangle\vec{v}\rvert$
M	9.536e+00	7.709e-03	0.00	1.120e-05	9.076e-09
-1.00	2.597e+00	2.099e-03	0.05	8.025e-06	6.506e-09
-0.95	2.916e-01	2.357e-04	0.10	5.751e-06	4.664e-09
-0.90	7.246e-02	5.857e-05	0.15	4.111e-06	3.336e-09
-0.85	2.564e-02	2.073e-05	0.20	2.916e-06	2.367e-09
-0.80	1.108e-02	8.960e-06	0.25	2.057e-06	1.672e-09
-0.75	5.449e-03	4.405e-06	0.30	1.433e-06	1.165e-09
-0.70	2.929e-03	2.368e-06	0.35	9.928e-07	8.085e-10
-0.65	1.680e-03	1.358e-06	0.40	6.742e-07	5.498e-10
-0.60	1.012e-03	8.182e-07	0.45	4.546e-07	3.714e-10
-0.55	6.334e-04	5.122e-07	0.50	2.934e-07	2.404e-10
-0.50	4.087e-04	3.305e-07	0.55	1.883e-07	1.547e-10
-0.45	2.702e-04	2.185e-07	0.60	1.170e-07	9.659e-11
-0.40	1.822e-04	1.474e-07	0.65	7.030e-08	5.843e-11
-0.35	1.248e-04	1.009e-07	0.70	3.766e-08	3.169e-11
-0.30	8.661e-05	7.008e-08	0.75	1.996e-08	1.705e-11
-0.25	6.073e-05	4.914e-08	0.80	8.703e-09	7.807e-12
-0.20	4.290e-05	3.473e-08	0.85	3.362e-09	3.265e-12
-0.15	3.052e-05	2.471e-08	0.90	9.436e-10	1.255e-12
-0.10	2.179e-05	1.765e-08	0.95	1.928e-10	2.923e-13
-0.05	1.562e-05	1.265e-08	1.00	9.146e-10	2.947e-13

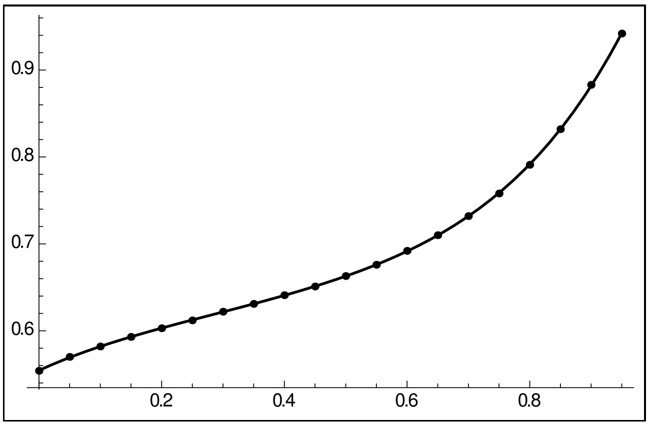


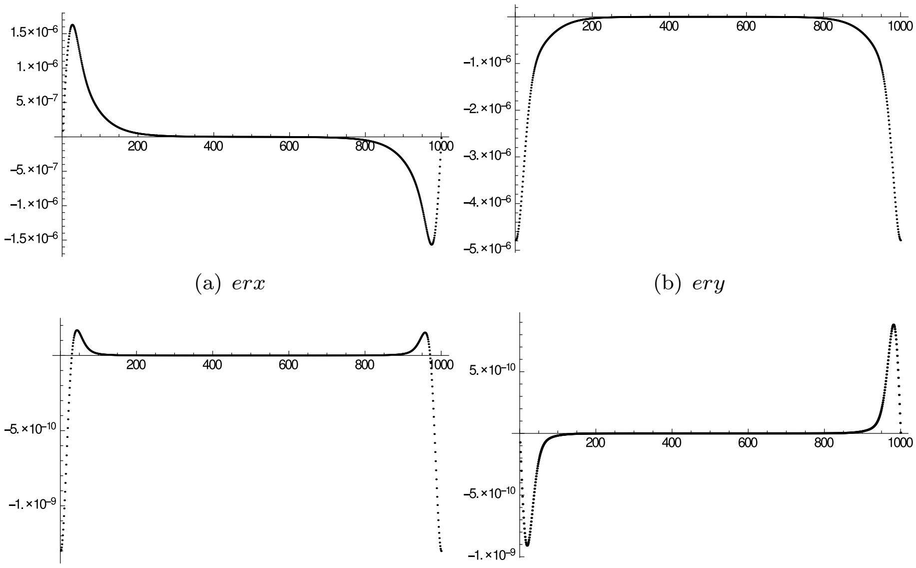
Figure 3: Dependence of α on e

0, 0.5 , 0.75, 1.0. These errors have been obtained by comparison of the values obtained integrating one step the differential equations (44) with the initial conditions given by (8) and (15) for each $\Psi_\alpha = i * h$ where $h = 2\pi/1000$, $i =$

Table 2: Optimal α for e

e	α	$\lvert\triangle\vec{r}\rvert$	$\lvert\triangle\vec{v}\rvert$	e	α	$\lvert\triangle\vec{r}\rvert$	$\lvert\triangle\vec{v}\rvert$
0.00	0.554	3.73e-07	2.90e-12	0.50	0.663	1.71e-07	5.92e-10
0.05	0.570	3.71e-07	6.15e-11	0.55	0.676	1.55e-07	6.23e-10
0.10	0.582	3.60e-07	1.23e-10	0.60	0.692	1.26e-07	6.45e-10
0.15	0.593	3.46e-07	1.87e-10	0.65	0.710	9.95e-08	6.35e-10
0.20	0.603	3.29e-07	2.51e-10	0.70	0.732	1.05e-07	5.89e-10
0.25	0.612	3.15e-07	3.14e-10	0.75	0.758	9.28e-08	4.61e-10
0.30	0.622	2.92e-07	3.80e-10	0.80	0.791	1.06e-07	2.21e-10
0.35	0.631	2.57e-07	4.39e-10	0.85	0.832	1.68e-07	2.79e-10
0.40	0.641	2.33e-07	4.98e-10	0.90	0.883	3.27e-07	1.43e-09
0.45	0.651	2.08e-07	5.46e-10	0.95	0.942	1.03e-06	5.49e-09

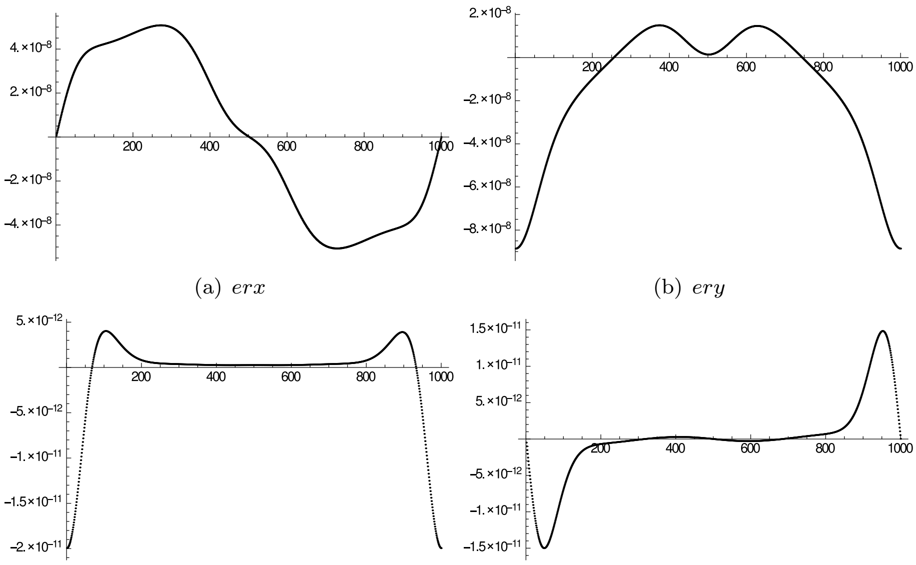
$0, \ldots, 999$ with the exact values obtained from (8) and (15) for $\Psi_\alpha = (i+1) * h$.



(a) *erx* (b) *ery*

(c) *ervx* (d) *ervy*

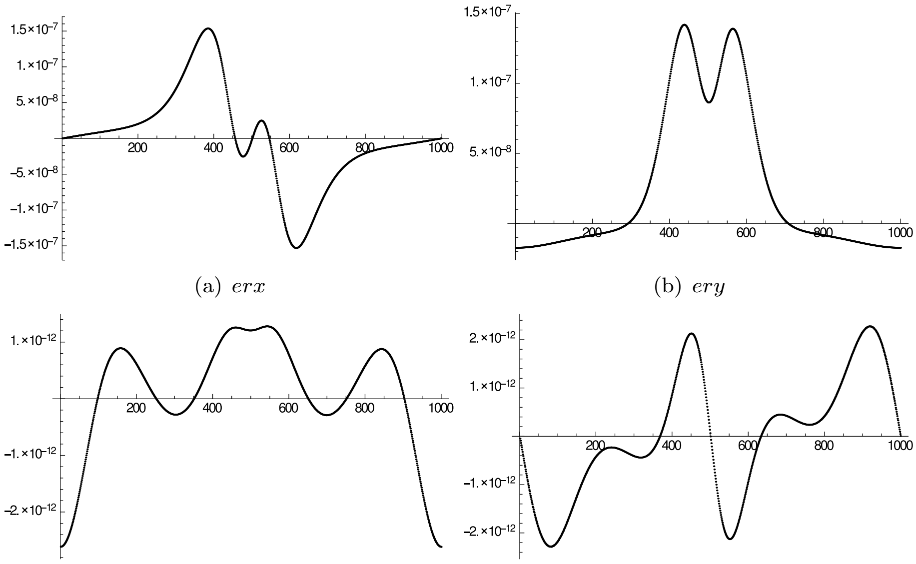
Figure 4: Local integration errors distribution $e = 0.8$, $\alpha = -0.5$



(a) *erx* (b) *ery*

(c) *ervx* (d) *ervy*

Figure 5: Local integration errors distribution $e = 0.8$, $\alpha = 0.5$



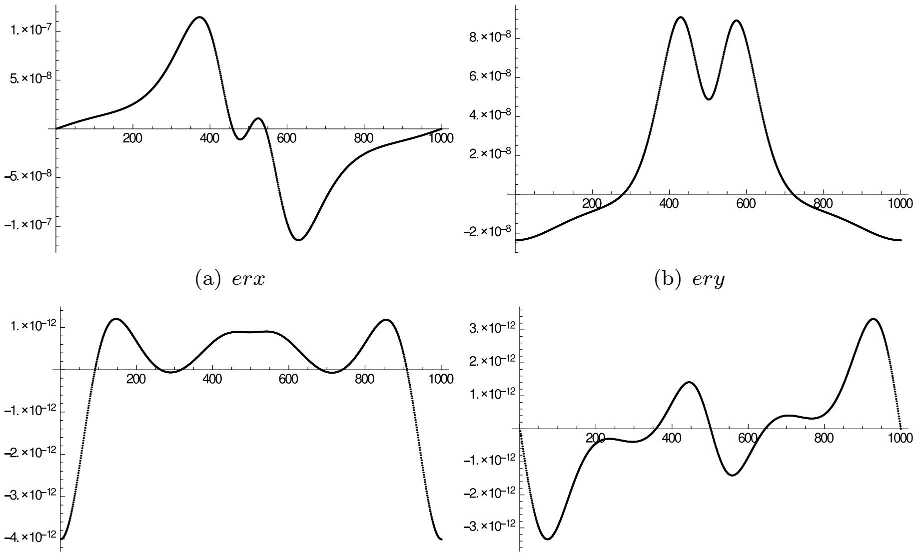
(a) *erx* (b) *ery*

(c) *ervx* (d) *ervy*

Figure 6: Local integration errors distribution $e = 0.8$, $\alpha = 0.85$

5. Concluding Remarks

In this paper a new one-parametric family of anomalies, named as generalized eccentric anomalies, has been defined.



(a) erx (b) ery

(c) $ervx$ (d) $ervy$

Figure 7: Distribution of the local integration errors $e = 0.8$, $\alpha = 0.791$

The generalized eccentric family of anomalies includes the eccentric anomaly, the true anomaly and the secondary anomaly. This family can be considered as a family of ellipses with the same major semiaxis a minor semiaxis $a\sqrt{1-\alpha^2 e^2}$ and the focus in $F_\alpha = \alpha\, e\, a$ position.

It is very important to emphasize that the main quantities of the two body problem (position, velocity, vector radius, sinus and cosinus of the eccentric anomaly) obtained in the section 2, can be written in a closed form using the described family of anomalies. To study the optimal value of α in order to increase the accuracy of the numerical methods, a set of numerical experiments on the unperturbed two-body problem have been carried out.

An important property of this family of anomalies is its performance in the construction of analytical theories because the coefficients of the developments as Fourier series of the main quantities of two-body problem obtained in section 3, also can be written in closed form.

It is also remarkable that the family of generalized eccentric anomalies can be used in order to improve the integration errors in the numerical methods. In this sense a numerical study of the optimal value of α in order to increase the accuracy of the numerical methods, using a set of numerical experiments on two body problem has been carried out leading us to the conclusion that optimal value of α to minimize the global integration errors in a revolution increases with the eccentricity.

6. Acknowledgements

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