MULTILINEAR ISOMETRIES ON FUNCTION ALGEBRAS

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ABSTRACT. Let $A_1, ..., A_k$ be function algebras (or more generally, dense subspaces of uniformly closed function algebras) on locally compact Hausdorff spaces $X_1, ..., X_k$, respectively, and let Z be a locally compact Hausdorff space. A k-linear map $T : A_1 \times ... \times A_k \longrightarrow C_0(Z)$ is called a *multilinear (or k-linear) isometry* if

$$\|T(f_1, ..., f_k)\| = \prod_{i=1}^{k} \|f_i\| \qquad ((f_1, ..., f_k) \in A_1 \times ... \times A_k).$$

Based on a new version of the additive Bishop's Lemma, we provide a weighted composition characterization of such maps. These results generalize the well-known Holsztyński's theorem ([9]) and the bilinear version of this theorem provided in [10] by a different approach.

1. Introduction

Let X be a locally compact Hausdorff space. As usual, $C_0(X)$ (resp. $C(X)$ if X is compact) stands for the Banach algebra of all continuous scalar-valued functions on X which vanish at infinity, endowed with the supremum norm, $\|\cdot\|$. In [9], W. Holsztyński inaugurated a new direction of generalization of the famous Banach-Stone Theorem. Namely, he provided the following non-surjective version: If there exists a (not necessarily onto) linear isometry $T : C(X) \longrightarrow C(Y)$, then T is a weighted composition operator on a subset of Y. More precisely, there are a closed subset Y_0 of Y, a continuous map h from Y_0 onto X and a unimodular continuous function a defined on Y_0 such that $T(f)(y) = a(y)f(h(y))$ for all $y \in Y_0$ and all $f \in C(X)$.

In [10], the authors proved, based on the powerful Stone-Weierstrass Theorem, the following bilinear version of Holsztyński's theorem:

Let $T : C(X) \times C(Y) \longrightarrow C(Z)$ be a bilinear (or 2-linear) isometry. Then there exist a closed subset Z_0 of Z, a surjective continuous mapping $\varphi : Z_0 \longrightarrow X \times Y$ and a unimodular function $a \in C(Z_0)$ such that $T(f, g)(z) = a(z)f(\pi_x(\varphi(z)))g(\pi_y(\varphi(z)))$ for all $z \in Z_0$ and every pair $(f, g) \in C(X) \times C(Y)$, where π_x and π_y are projection maps.

Key words and phrases: function algebra, k-linear isometry, Choquet boundary, additive Bishop's Lemma, peaking function, uniform algebra.

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In this paper we extend this bilinear version of Holsztyński's theorem to a more general context, where Stone-Weierstrass Theorem is not applicable. Namely, let $A_1, ..., A_k$ be function algebras (or more generally, dense subspaces of uniformly closed function algebras) on locally compact Hausdorff spaces $X_1, ..., X_k$, respectively, and let Z be a locally compact Hausdorff space. A k-linear map $T : A_1 \times ... \times A_k \longrightarrow C_0(Z)$ is called a *multilinear (or k-linear) isometry* if

$$\|T(f_1, ..., f_k)\| = \prod_{i=1}^{k} \|f_i\| \qquad ((f_1, ..., f_k) \in A_1 \times ... \times A_k).$$

We provide a complete characterization of such maps as follows: given a k-linear isometry $T : A_1 \times ... \times A_k \longrightarrow C_0(Z)$, there exist a nonempty subset Z_0 of Z, a continuous surjective map $\varphi : Z_0 \longrightarrow Ch(A_1) \times ... \times Ch(A_k)$ and a unimodular continuous function $a : Z_0 \longrightarrow \mathbb{T}$ such that $T(f_1, ..., f_k)(z) = a(z) \prod_{i=1}^{k} f_i(\pi_i(\varphi(z)))$ for all $(f_1, ..., f_k) \in A_1 \times ... \times A_k$ and $z \in Z_0$, where π_i is the ith projection map.

The main tool we use to prove this characterization is a recent stronger version of the additive Bishop's Lemma (see [12] or Lemma 2.2 below). This technique also lets us fix some inaccuracies detected in [6], particularly in the bounds obtained in the proof of [6, Lemma 3.3]. Furthermore, for the sake of completeness and in order to give a unified version of the proofs involved in this topic, the (known) results for 1-linear isometries are also included and proved straightforwardly by using this version of the additive Bishop's Lemma.

2. PRELIMINARIES

Let X be a locally compact Hausdorff space and let X_∞ be its one point compactification. Let us recall that $C_0(X)$ is the algebra of all continuous scalar-valued functions on X vanishing at infinity. A *function algebra* A on X is a subalgebra of $C_0(X)$ which separates strongly the points of X, i.e. for each $x, x' \in X$ with $x \neq x'$, there exists an $f \in A$ with $f(x) \neq f(x')$ and for each $x \in X$, there exists an $f \in A$ with $f(x) \neq 0$. If X is a compact Hausdorff space, each unital uniformly closed function algebra on X is called a *uniform algebra* on X.

Let A be a function algebra on a locally compact Hausdorff space X. We denote the uniform closure of A by $\overline{A}$. The unique minimal closed subset of X with the property that every function in A assumes its maximum modulus on this set, which exists by [2], is called the *Šilov boundary* for A and is denoted by ∂A. The *Choquet boundary* $Ch(A)$ of A is the set of all $x \in X$ for which δ_x, the evaluation functional at the point x, is an extreme point of the unit ball of the dual space of $(A, \|\cdot\|)$. So it is apparent that $Ch(A) = Ch(\overline{A})$. Besides, note that for a function algebra A, ∂A is the closure of $Ch(A)$ [2, Theorem 1]. A point $x \in X$ is called a *strong boundary point* (or *weak peak point*) for A if for every neighborhood V of x, there exists a function $f \in A$ such that $\|f\| = 1 = |f(x)|$ and $|f| < 1$ on $X \setminus V$. It is known that if A is a uniformly closed function algebra on a locally compact Hausdorff space X, then $Ch(A)$ coincides with the set of all strong boundary points (see [11]). However, according to the example given in [4], this coincidence is not true for all function algebras, although the Choquet boundary always contains the strong boundary points.

A function $f \in A$ is a *peaking function* if $\|f\| = 1$ and for each $x \in X$, either $|f(x)| < 1$ or $f(x) = 1$. If we fix $x_0 \in X$, then $P_A(x_0)$ denotes the set of peaking functions f in A with $f(x_0) = 1$.

Moreover, if A is a subspace of $C_0(X)$, for an element $x \in X$, we set $C_x := \{f \in A : |f(x)| = 1 = \|f\|\}$. Besides, for $g \in A$ we denote the maximum modulus set of g by $M_g := \{x \in X : |g(x)| = \|g\|\}$.

As mentioned in the introduction, the proofs of the technical lemmas preceding our main result are based essentially on extensions of Bishop's Lemma in the context of uniform algebras [3, Theorem 2.4.1], a result which has been generalized in many directions. Next we include the following generalizations (given in [8] and [12] respectively) which we shall use in the next sections.

Lemma 2.1. *Let A be a uniformly closed function algebra on a locally compact Hausdorff space X, $f \in A$ and $x_0 \in Ch(A)$. If $f(x_0) \neq 0$, then there exists a peaking function $h \in P_A(x_0)$ such that $\frac{fh}{f(x_0)} \in P_A(x_0)$.*

Proof. The result can be concluded by the arguments similar to [8, Lemma 2.3], where X is a compact Hausdorff space. □

Lemma 2.2. *Assume that A is a uniformly closed function algebra on a locally compact Hausdorff space X and $f \in A$. Let $x_0 \in Ch(A)$ and arbitrary $r > 1$ (or $r \geq 1$ if $f(x_0) \neq 0$), then there exists a function $h \in r\|f\|P_A(x_0) = \{r\|f\|k : k \in P_A(x_0)\}$ such that*

$$|f(x)| + |h(x)| < |f(x_0)| + |h(x_0)|$$

for every $x \notin M_h$ and $|f(x)| + |h(x)| = |f(x_0)| + |h(x_0)|$ for all $x \in M_h$. Consequently, $\||f| + |h|\|_X = |f(x_0)| + |h(x_0)|$.

Proof. The proof is exactly the same as that of [12, Lemma 1], where X is a compact Hausdorff space. □

Let us remark that Lemma 2.1 is a version of the multiplicative Bishop's Lemma and Lemma 2.2 is the strong version of the additive Bishop's Lemma (see [7] for further details concerning Bishop's Lemma).

3. 1-linear isometries between function algebras

In this section we shall assume that A and B are dense subspaces of uniformly closed function algebras on locally compact Hausdorff spaces X and Y, respectively, and characterize linear (i.e., 1-linear) isometries $T : A \longrightarrow B$. It should be noted that although these results can be deduced from [1], here we provide new shorter proofs based on Lemma 2.2 in order to give a self-contained unified vision of this topic. We refer the reader to [5] for a summary on the study of isometries.

Theorem 3.1. *Let $T : A \longrightarrow B$ be a linear isometry. Then there exist a nonempty subset Y_0 of Y, a continuous surjective map $\varphi : Y_0 \longrightarrow Ch(A)$, a unimodular continuous function $a : Y_0 \longrightarrow \mathbb{T}$, such that $T(f)(y) = a(y)f(\varphi(y))$ for all $f \in A$ and $y \in Y_0$. Moreover, $a(y) = T(g)(y)$ for any $g \in A$ with $g(\varphi(y)) = 1$.*

First note that we can extend easily $T : A \longrightarrow B$ to a linear isometry $T : \overline{A} \longrightarrow \overline{B}$ between their uniform closures. Besides, notice that the Choquet boundary for a linear subspace of continuous functions on a locally compact Hausdorff space is defined similar to the function algebra case. So since the Choquet boundary of a subspace equals the Choquet boundary of its uniform closure, without loss of generality, we can assume that A and B are uniformly closed function algebras.

Before providing the proof of Theorem 3.1, we need several lemmas.

Lemma 3.2. *Let $x \in Ch(A)$. Then the set $\mathcal{I}_x := \bigcap_{f \in C_x} M_{T(f)}$ is nonempty.*

Proof. The proof is the same as that of [1, Lemma 2.2]. □

Lemma 3.3. *Let $x \in Ch(A)$. If $f \in A$ such that $f(x) = 0$, then $T(f)(y) = 0$ for all $y \in \mathcal{I}_x$.*

Proof. Let $f \in A$ with $f(x) = 0$ and $y \in \mathcal{I}_x$. Suppose, on the contrary, that $T(f)(y) \neq 0$. We may assume, without loss of generality, that $\|f\| = 1$ and $T(f)(y) = \alpha$, where $0 < \alpha \leq 1$. Fix a constant $r > 1$. By Lemma 2.2, there is a peaking function $h \in P_A(x)$ such that $\||f| + r|h|\| = r$. In particular, $\|f + r\bar{\lambda}h\| = r$, where $\lambda = T(h)(y) \in \mathbb{T}$. Hence

$$r = \|f + r\bar{\lambda}h\| = \|T(f + r\bar{\lambda}h)\| \geq |T(f)(y) + r| = \alpha + r,$$

which is a contradiction showing that $T(f)(y) = 0$. □

Lemma 3.4. *If $f \in A$ and $x \in Ch(A)$, then $|T(f)(y)| = |f(x)|$ for all $y \in \mathcal{I}_x$.*

Proof. Let $f \in A$, $x \in Ch(A)$ and $y \in \mathcal{I}_x$. If $f(x) = 0$, then, by the preceding lemma, $T(f)(y) = 0$. Now let us suppose that $f(x) \neq 0$. Since $x \in Ch(A)$, there is a peaking function $h \in C_x$. If we define

$$g(t) := f(t) - f(x)h(t) \quad (t \in X),$$

then $g \in A$ and $g(x) = 0$. So, by Lemma 3.3, $0 = T(g)(y) = T(f)(y) - f(x)T(h)(y)$. Hence $T(f)(y) = f(x)T(h)(y)$. On the other hand, since $y \in \mathcal{I}_x$ and $h \in C_x$, $|T(h)(y)| = 1$. Therefore, $|T(f)(y)| = |f(x)|$. □

Lemma 3.5. *For different points x and x' in $Ch(A)$, $\mathcal{I}_x \cap \mathcal{I}_{x'} = \emptyset$.*

Proof. Choose a peaking function $f \in C_x$ such that $|f(x')| < 1$. Now if $y \in \mathcal{I}_x \cap \mathcal{I}_{x'}$, then from Lemma 3.4, it follows that $|T(f)(y)| = |f(x)| = 1$ and $|T(f)(y)| = |f(x')| < 1$, which is a contradiction. Thereby, $\mathcal{I}_x \cap \mathcal{I}_{x'} = \emptyset$. □

Now we are ready to complete the proof of Theorem 3.1:

Proof. Let $Y_0 := \bigcup_{x \in Ch(A)} \mathcal{I}_x$. Clearly, $Y_0 \neq \emptyset$, by Lemma 3.2. Define the map $\varphi : Y_0 \longrightarrow Ch(A)$ by $\varphi(y) := x$ if $y \in \mathcal{I}_x$. Note that, since for different points x and x' in $Ch(A)$, $\mathcal{I}_x \cap \mathcal{I}_{x'} = \emptyset$, the map φ is well-defined. Furthermore, φ is surjective because $\mathcal{I}_x \neq \emptyset$ for each $x \in Ch(A)$. Meantime, since for all $f \in A$, $|T(f)| = |f \circ \varphi|$ on Y_0 and the set $\{|f| : f \in A\}$ separates the points of X_∞, it is not difficult to check that φ is continuous.

Now we define the function $a : Y_0 \longrightarrow \mathbb{T}$. For this purpose, let $y \in Y_0$. Then take $f \in A$ with $f(\varphi(y)) = 1$ and define $a(y) := T(f)(y)$. Note that the definition is independent of the choice of f because if $f, f' \in A$ and $f(\varphi(y)) = 1 = f'(\varphi(y))$, then $f - f' \in A$ with $(f - f')(\varphi(y)) = 0$. Hence, by Lemma 3.3, we conclude that $T(f - f')(y) = 0$ and so $T(f)(y) = T(f')(y)$. Moreover, by Lemma 3.4, it is evident that $|a(y)| = 1$.

Next, we give the representation of T. Let $f \in A$ and $y \in Y_0$. The function $g := f - f(\varphi(y))k$, where k is a function in $P_A(\varphi(y))$, belongs to A and $g(\varphi(y)) = 0$. So by Lemma 3.3, $T(f)(y) = f(\varphi(y))T(k)(y)$, i.e., $T(f)(y) = a(y)f(\varphi(y))$.

We finally show the continuity of a. Let $y_0 \in Y_0$ and choose $f \in A$ such that $f(\varphi(y_0)) \neq 0$. If we define $W := \{x \in Ch(A) : f(x) \neq 0\}$, then $\varphi^{-1}(W)$ is a neighborhood of y_0. Moreover, $a(y) = \frac{T(f)(y)}{(f \circ \varphi)(y)}$ holds for all $y \in \varphi^{-1}(W)$. Now from the continuity of $\frac{T(f)}{f \circ \varphi}$ on $\varphi^{-1}(W)$, it follows that a is also continuous at y_0. □

Remark 3.6. (i) Notice that φ sends $Ch(T(A))$ onto $Ch(A)$. In fact, $T : A \longrightarrow T(A)$ is a bijective isometry, then the adjoint of T, $T^* : T(A)^* \longrightarrow A^*$ is a bijective isometry. Therefore, $ext(T(A)^*_1)$ is sent onto $ext(A^*_1)$, where $T(A)^*_1$ and A^*_1 are the unit ball of $T(A)^*$ and A^*, respectively. Thus, by Lemma 3.4, it follows easily that $\varphi(Ch(T(A))) \subseteq Ch(A)$. Next repeating the same arguments for T^{-1} and noting that $(T^{-1})^* = (T^*)^{-1}$, finally we conclude that $\varphi(Ch(T(A))) = Ch(A)$. In particular, if T is surjective, then φ is a homeomorphism of $Ch(B)$ onto $Ch(A)$.

(ii) We note that if a map $T : A \longrightarrow C_0(Y)$ is defined by $T(f) = af \circ \varphi$ on Y_0, where $Y_0 \subseteq Y$ is a boundary for $T(A)$, a is a unimodular continuous function on Y_0, and $\varphi : Y_0 \longrightarrow Ch(A)$ is a surjective map, then T is a linear isometry.

4. k-LINEAR ISOMETRIES BETWEEN FUNCTION ALGEBRAS

Let $A_1, ..., A_k$ be dense subspaces of uniformly closed function algebras on locally compact Hausdorff spaces $X_1, ..., X_k$, respectively, and let Z be a locally compact Hausdorff space. We recall that a k-linear map $T : A_1 \times ... \times A_k \longrightarrow C_0(Z)$ is called a multilinear (or k-linear) isometry if

$$\|T(f_1, ..., f_k)\| = \prod_{i=1}^{k} \|f_i\| \qquad ((f_1, ..., f_k) \in A_1 \times ... \times A_k).$$

In this section we shall deepen in these maps. First note that it is not difficult to extend $T : A_1 \times ... \times A_k \longrightarrow C_0(Z)$ to a k-linear isometry $T : \overline{A_1} \times ... \times \overline{A_k} \longrightarrow C_0(Z)$, where $\overline{A_i}$ is the uniform closure of A_i $(i = 1, ..., k)$. So, as before, without loss of generality, we can assume each A_i $(i = 1, ..., k)$ is a uniformly closed function algebra.

Let us recall that for an element $x_i \in X_i$, we set $C_{x_i} := \{f \in A_i : |f(x_i)| = 1 = \|f\|\}$. Moreover, for $g \in C_0(Z)$, $M_g := \{z \in Z : |g(z)| = \|g\|\}$ stands for the maximum modulus set of g.

Lemma 4.1. *Let $(x_1, ..., x_k) \in Ch(A_1) \times ... \times Ch(A_k)$. The set*

$$\mathcal{I}_{x_1,...,x_k} := \{z \in Z : z \in M_{T(f_1,...,f_k)} \text{ for all } (f_1, ..., f_k) \in C_{x_1} \times ... \times C_{x_k}\}$$

is nonempty.

Proof. The proof is a modification of the proof of [6, Lemma 3.1]. Since for each $(f_1, ..., f_k) \in C_{x_1} \times ... \times C_{x_k}$, the maximum modulus set of $T(f_1, ..., f_k)$, $M_{T(f_1,...,f_k)}$, is a compact subset of Z_∞, so it is enough to check that the family $\{M_{T(f_1,...,f_k)} : (f_1, ..., f_k) \in C_{x_1} \times ... \times C_{x_k}\}$ has the finite intersection property. For this, let $(f_1^1, ..., f_k^1), ..., (f_1^n, ..., f_k^n)$ be members in $C_{x_1} \times ... \times C_{x_k}$. Define

$$f_i := \frac{1}{n} \sum_{j=1}^{n} \frac{1}{f_i^j(x_i)} f_i^j, \quad i \in \{1, ..., k\}.$$

Clearly, $(f_1, ..., f_k) \in C_{x_1} \times ... \times C_{x_k}$. Hence $\|T(f_1, ..., f_k)\| = \|f_1\|...\|f_k\| = 1$. Then there is a point $z_0 \in Z$ such that

$$1 = |T(f_1, ..., f_k)(z_0)| = \frac{1}{n^k} \left| \sum_{1 \le i_1,...,i_k \le n} \frac{1}{f_1^{i_1}(x_1)} \cdots \frac{1}{f_k^{i_k}(x_k)} T(f_1^{i_1}, ..., f_k^{i_k})(z_0) \right|.$$

Since for each $1 \le i_1, ..., i_k \le n$, $f_1^{i_1} \in C_{x_1}$, ..., $f_k^{i_k} \in C_{x_k}$ and $\|T(f_1^{i_1}, ..., f_k^{i_k})\| = 1$, we conclude that $|T(f_1^{i_1}, ..., f_k^{i_k})(z_0)| = 1$. In particular, $z_0 \in \bigcap_{i=1}^{n} M_{T(f_1^i,...,f_k^i)}$. Therefore $\bigcap_{i=1}^{n} M_{T(f_1^i,...,f_k^i)} \neq \emptyset$, as was to be proved. □

Lemma 4.2. *Fix $i \in \{1, ..., k\}$ and let $(x_1, ..., x_k) \in Ch(A_1) \times ... \times Ch(A_k)$. If $f = (f_1, ..., f_k) \in C_{x_1} \times ... \times C_{x_{i-1}} \times A_i \times C_{x_{i+1}} \times ... \times C_{x_k}$ such that $f_i(x_i) = 0$ and $z \in \mathcal{I}_{x_1,...,x_k}$ then $T(f)(z) = 0$.*

Proof. For simplicity, we can take $i = 1$. Let $f = (f_1, ..., f_k) \in A_1 \times C_{x_2} \times ... \times C_{x_k}$ such that $f_1(x_1) = 0$ and suppose that there exists $z_0 \in \mathcal{I}_{x_1,...,x_k}$ such that $T(f)(z_0) \neq 0$. We can assume, without loss of generality, that $\|f_1\| = 1$ and $T(f)(z_0) = \alpha$, where $0 < \alpha \leq 1$. Fix a constant $r > 1$. By Lemma 2.2, there is a peaking function $h_1 \in A_1$ such that $h_1(x_1) = 1$ and $\||f_1| + r|h_1|\| = r$. In particular, $\|f_1 + r\bar{\lambda}h_1\| = r$, where $\lambda = T(h_1, f_2, ..., f_k)(z_0) \in \mathbb{T}$. Then we have

$$r = \|f_1 + r\bar{\lambda}h_1\|\|f_2\|...\|f_k\| = \|T(f_1 + r\bar{\lambda}h_1, f_2, ..., f_k)\|,$$

while

$$T(f_1 + r\bar{\lambda}h_1, f_2, ..., f_k)(z_0) = T(f_1, f_2, ..., f_k)(z_0) + r\bar{\lambda}T(h_1, f_2, ..., f_k)(z_0) = \alpha + r,$$

a contradiction which yields $T(f)(z) = 0$ for all $z \in \mathcal{I}_{x_1,...,x_k}$. □

Lemma 4.3. *Let $(x_1, ..., x_k) \in Ch(A_1) \times ... \times Ch(A_k)$ and $z \in \mathcal{I}_{x_1,...,x_k}$. Let also I and J be two disjoint sets with $I \neq \emptyset$ and $I \cup J = \{1, ..., k\}$. Assume that for each $j \in J$, $h_j \in C_{x_j}$ and for each $i \in I$, $f_i \in A_i$ with $f_i(x_i) = 0$, then $T(F_1, ..., F_k)(z) = 0$, where $F_t = f_t$ if $t \in I$ and $F_t = h_t$ if $t \in J$.*

Proof. Let us suppose, contrary to what we claim, that there exists $z_0 \in \mathcal{I}_{x_1,...,x_k}$ such that $T(F_1, ..., F_k)(z_0) \neq 0$. Without loss of generality, we may assume that $\|f_i\| = 1$ for each $i \in I$ and $T(F_1, ..., F_k)(z_0) = \alpha$ with $0 < \alpha \leq 1$. Fix a constant $r > 1$. For each $i \in I$, we can choose, by Lemma 2.2, a peaking function $h_i \in C_{x_i}$ such that $\||f_i| + r|h_i|\| = r$. In particular, for each $i \in I$ we have $\|f_i + r\bar{\lambda}h_i\| = r$, where $\lambda = T(h_1, ..., h_k)(z_0) \in \mathbb{T}$.

Let us first suppose that $I = \{1, 2\}$. Hence, by Lemma 4.2, we can conclude that

$$\begin{aligned} T(f_1 + r\bar{\lambda}h_1, f_2 + rh_2, h_3, ..., h_k)(z_0) &= T(f_1, f_2, h_3, ..., h_k)(z_0) + r\bar{\lambda}T(h_1, f_2, h_3, ..., h_k)(z_0) \\ &\quad + rT(f_1, h_2, h_3, ..., h_k)(z_0) + r^2\bar{\lambda}T(h_1, ..., h_k)(z_0) = \alpha + r^2 \\ &> r^2 = \|f_1 + r\bar{\lambda}h_1\|\|f_2 + rh_2\|\|h_3\|...\|h_k\| \\ &= \|T(f_1 + r\bar{\lambda}h_1, f_2 + rh_2, h_3, ...h_k)\|, \end{aligned}$$

a contradiction which implies that the result is true when $I = \{1, 2\}$. Similarly, this result is held for all the cases where $card(I) = 2$.

Now we can continue by induction: noting to the above explanation, let us assume that the result is true for $card(I) = l - 1$ and $3 \leq l \leq k$. We shall show that the result is held if $card(I) = l$. We suppose that $card(I) = l$ and $I = \{x_1, ..., x_l\}$, without loss of generality. If $l < k$, then we get

$$\begin{aligned}
r^l &= \|f_1 + r\bar{\lambda}h_1\|\|f_2 + rh_2\|...\|f_l + rh_l\|\|h_{l+1}\|...\|h_k\| \\
&= \|T(f_1 + r\bar{\lambda}h_1, f_2 + rh_2, ..., f_l + rh_l, h_{l+1}, ..., h_k)\| \\
&\geq |T(f_1 + r\bar{\lambda}h_1, f_2 + rh_2, ..., f_l + rh_l, h_{l+1}, ..., h_k)(z_0)| \\
&= |T(f_1, ..., f_l, h_{l+1}, ..., h_k)(z_0) + r^l\bar{\lambda}T(h_1, ..., h_k)(z_0)| = \alpha + r^l,
\end{aligned}$$

which is impossible. Therefore, $T(f_1, ..., f_l, h_{l+1}, ..., h_k)(z) = 0$ for all $z \in \mathcal{I}_{x_1,...,x_k}$. Now if $l = k$, then $I = \{x_1, ..., x_k\}$ and

$$\begin{aligned}
r^k &= \|f_1 + r\bar{\lambda}h_1\|\|f_2 + rh_2\|...\|f_k + rh_k\| = \|T(f_1 + r\bar{\lambda}h_1, f_2 + rh_2, ..., f_k + rh_k)\| \\
&\geq |T(f_1 + r\bar{\lambda}h_1, f_2 + rh_2, ..., f_k + rh_k)(z_0)| \\
&= |T(f_1, ..., f_k)(z_0) + r^k\bar{\lambda}T(h_1, ..., h_k)(z_0)| = \alpha + r^k,
\end{aligned}$$

which is a contradiction showing that $T(f_1, ..., f_k)(z) = 0$ for all $z \in \mathcal{I}_{x_1,...,x_k}$. □

Lemma 4.4. *Let $(x_1, ..., x_k)$ and $(x'_1, ..., x'_k)$ be distinct points in $Ch(A_1) \times ... \times Ch(A_k)$. Then $\mathcal{I}_{x_1,...,x_k} \cap \mathcal{I}_{x'_1,...,x'_k} = \emptyset$.*

Proof. Contrary to what we claim, assume that there exists $z_0 \in \mathcal{I}_{x_1,...,x_k} \cap \mathcal{I}_{x'_1,...,x'_k}$. Since $(x_1, ..., x_k)$ and $(x'_1, ..., x'_k)$ are distinct, the set $L = \{i : 1 \leq i \leq k, x_i \neq x'_i\}$ is nonempty. For each $i \in L$, we can choose a function $g_i \in A_i$ such that $g_i(x_i) = 1$ and $g_i(x'_i) = 0$, and then, by Lemma 2.1, a peaking function $h_i \in P_{A_i}(x_i)$ such that $g_ih_i \in P_{A_i}(x_i)$. Now if we let $f_i = g_ih_i$ for every $i \in L$, then $f_i \in C_{x_i}$ with $f_i(x_i) = 1$ and $f_i(x'_i) = 0$. Moreover, for each $j \in \{1, ..., k\} \setminus L$, we can also choose a peaking function $f_j \in C_{x_j}$. On one side, since $(f_1, ..., f_k) \in C_{x_1} \times ... \times C_{x_k}$, $|T(f_1, ..., f_k)(z_0)| = 1$. On the other side, by Lemma 4.3, $T(f_1, ..., f_k)(z_0) = 0$, which is impossible. Therefore, $\mathcal{I}_{x_1,...,x_k} \cap \mathcal{I}_{x'_1,...,x'_k} = \emptyset$. □

Theorem 4.5. *Suppose that $T : A_1 \times ... \times A_k \longrightarrow C_0(Z)$ is a k-linear isometry. Then there exist a nonempty subset Z_0 of Z, a continuous surjective map $\varphi : Z_0 \longrightarrow Ch(A_1) \times ... \times Ch(A_k)$ and a unimodular continuous function $a : Z_0 \longrightarrow \mathbb{T}$ such that $T(f_1, ..., f_k)(z) = a(z) \prod_{i=1}^{k} f_i(\pi_i(\varphi(z)))$ for all $(f_1, ..., f_k) \in A_1 \times ... \times A_k$ and $z \in Z_0$, where π_i is the ith projection map.*

Proof. Let $Z_0 := \{z \in \mathcal{I}_{x_1,...,x_k} : (x_1, ..., x_k) \in Ch(A_1) \times ... \times Ch(A_k)\}$ which is a nonempty set, by Lemma 4.1. Fix $(x_1, ..., x_k) \in Ch(A_1) \times ... \times Ch(A_k)$ and $h_i \in C_{x_i}$ with $h_i(x_i) = 1$ for each i, $i = 1, ..., k$. Then for each i, $i = 1, ..., k$, we can define an isometry as follows:

$$\begin{cases} T_i : A_i \longrightarrow C_0(Z) \\ T_i(f) = T(h_1, ..., h_{i-1}, f, h_{i+1}, ..., h_k). \end{cases}$$

According to Theorem 3.1, there exist a subset Z_i of Z, a continuous surjective map $\varphi_i : Z_i \longrightarrow Ch(A_i)$ such that

$$T_i(f_i)(z) = T(h_1, ..., h_k)(z) f_i(\varphi_i(z)), \qquad (f_i \in A_i, z \in Z_i).$$

Namely, $Z_i \supseteq \bigcup_{x'_i \in Ch(A_i)} \mathcal{I}_{x_1,...,x'_i,...,x_k}$ and if $z \in \mathcal{I}_{x_1,...,x'_i,...,x_k}$, then $\varphi_i(z) = x'_i$.

Let $(f_1, ..., f_k) \in A_1 \times ... \times A_k$. Now for a given $z \in \mathcal{I}_{x_1,...,x_k}$, by Lemma 4.3 and using the above reasonings, we conclude that

$$\begin{aligned}
0 &= T(f_1 - f_1(x_1)h_1, f_2 - f_2(x_2)h_2, h_3, ..., h_k)(z) \\
&= T(f_1, f_2, h_3, ..., h_k)(z) - f_1(x_1)T(h_1, f_2, h_3, ..., h_k)(z) \\
&- f_2(x_2)T(f_1, h_2, h_3, ..., h_k)(z) + f_1(x_1)f_2(x_2)T(h_1, ..., h_k)(z) \\
&= T(f_1, f_2, h_3, ..., h_k)(z) - f_1(x_1)T_2(f_2)(z) - f_2(x_2)T_1(f_1)(z) + f_1(x_1)f_2(x_2)T(h_1, ..., h_k)(z) \\
&= T(f_1, f_2, h_3, ..., h_k)(z) - f_1(x_1)T(h_1, ..., h_k)(z)f_2(x_2) \\
&- f_2(x_2)T(h_1, ..., h_k)(z)f_1(x_1) + f_1(x_1)f_2(x_2)T(h_1, ..., h_k)(z) \\
&= T(f_1, f_2, h_3, ..., h_k)(z) - f_1(x_1)f_2(x_2)T(h_1, ..., h_k)(z).
\end{aligned}$$

Thus $T(f_1, f_2, h_3, ..., h_k)(z) = T(h_1, ..., h_k)(z)f_1(x_1)f_2(x_2)$. By continuing this process and applying Lemma 4.3, finally we see that

$$\begin{aligned}
0 &= T(f_1 - f_1(x_1)h_1, ..., f_k - f_k(x_k)h_k)(z) \\
&= T(f_1, ..., f_k)(z) - T(h_1, ..., h_k)(z)f_1(x_1)...f_k(x_k),
\end{aligned}$$

thereby, $T(f_1, ..., f_k)(z) = T(h_1, ..., h_k)(z)f_1(x_1)...f_k(x_k)$.

Now we define the map $\varphi : Z_0 \longrightarrow Ch(A_1) \times ... \times Ch(A_k)$ by $\varphi(z) := (x_1, ..., x_k)$ if $z \in \mathcal{I}_{x_1,...,x_k}$. Since for distinct points $(x_1, ..., x_k)$ and $(x'_1, ..., x'_k)$ in $Ch(A_1) \times ... \times Ch(A_k)$, Lemma 4.4 yields $\mathcal{I}_{x_1,...,x_k} \cap \mathcal{I}_{x'_1,...,x'_k} = \emptyset$, so the map φ is well-defined. Moreover, we can define the unimodular function $a : Z_0 \longrightarrow \mathbb{T}$ such that if $z \in Z_0$ then $a(z) := T(h_1, ..., h_k)(z)$, where $h_i \in P_{A_i}(\pi_i(\varphi(z)))$. Lemma 4.3 implies that the definition of $a(z)$ is independent of the choice of $h_1, ..., h_k$. Besides, from the above argument, it follows that if $z \in Z_0$ with $\varphi(z) = (x_1, ..., x_k)$ and $(f_1, ..., f_k) \in A_1 \times ... \times A_k$ then

$$T(f_1, ..., f_k)(z) = a(z) \prod_{i=1}^{k} f_i(x_i) = a(z) \prod_{i=1}^{k} f_i(\pi_i(\varphi(z))).$$

Next we prove that φ is continuous. Suppose that $z_0 \in Z_0$, $\varphi(z_0) = (x_1, ..., x_k)$ and $U_1 \times ... \times U_k$ is a neighborhood of $(x_1, ..., x_k)$ in $Ch(A_1) \times ... \times Ch(A_k)$. For each i, $i = 1, ..., k$, there is a neighborhood U'_i of x_i in X_i with $U_i = U'_i \cap Ch(A_i)$. Choose a peaking function $f_i \in C_{x_i}$ such that $|f_i| < \frac{1}{2}$ on

$X_i \setminus U_i'$ $(i = 1, ..., k)$. Then $|T(f_1, ..., f_k)(z_0)| = 1$. Set

$$V := \{z \in Z_0 : |T(f_1, ..., f_k)(z)| > \frac{1}{2}\}.$$

Clearly V is a neighborhood of z_0 such that $\varphi(V) \subseteq U_1 \times ... \times U_k$ because if $z \in V$ and $\varphi(z) = (x_1', ..., x_k')$, then

$$\frac{1}{2} < |T(f_1, ..., f_k)(z)| = \prod_{i=1}^{k} |f_i(x_i')| \leq |f_i(x_i')| \quad (i = 1, ..., k).$$

Hence $x_i' \in U_i$ and so $(x_1', ..., x_k') \in U_1 \times ... \times U_k$.

To complete the proof, it suffices to check the continuity of a. Let $z_0 \in Z_0$. Then $z_0 \in \mathcal{I}_{x_1,...,x_k}$ for a unique $(x_1, ..., x_k)$ in $Ch(A_1) \times ... \times Ch(A_k)$. For each i, $i = 1, ..., k$, choose a peaking function $f_i \in P_{A_i}(x_i)$ and take

$$U_i := \{x \in Ch(A_i) : f_i(x) \neq 0\}.$$

Then $U = U_1 \times ... \times U_k$ is a neighborhood of $(x_1, ..., x_k)$ in $Ch(A_1) \times ... \times Ch(A_k)$ and consequently $\varphi^{-1}(U)$ is a neighborhood of z_0. We have

$$a(z) = \frac{T(f_1, ..., f_k)(z)}{\prod_{i=1}^{k} f_i(\pi_i(\varphi(z)))} \quad (z \in \varphi^{-1}(U)).$$

So from the continuity of the function $\frac{T(f_1,...,f_k)}{\prod_{i=1}^{k} f_i \circ \pi_i \circ \varphi}$ on $\varphi^{-1}(U)$, we conclude that a is continuous at z_0. □

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