SEQUENTIALLY COMPACT SUBSETS AND MONOTONE FUNCTIONS: AN APPLICATION TO FUZZY THEORY

JUAN J. FONT AND MANUEL SANCHIS

ABSTRACT. Let $(X, <, \tau_{\mathcal{O}})$ be a first countable compact linearly ordered topological space. If $(Y, \mathscr{D})$ is a uniform sequentially compact linearly ordered space with weight less than the splitting number $\mathfrak{s}$, then we characterize the sequentially compact subsets of the space $\mathcal{M}(X, Y)$ of all monotone functions from X into Y endowed with the topology of the uniform convergence induced by the uniformity $\mathscr{D}$. In particular, our results are applied to identify the compact subsets of $M([0, 1], Y)$ for a wide class of linearly ordered topological spaces, including $Y = \mathbb{R}$. This allows us to provide a characterization of the compact subsets of an extended version of the fuzzy number space (with the supremum metric) where the reals are replaced by certain linearly ordered topological spaces, which corrects some characterizations which appear in the literature.

Since fuzzy analysis is based on the notion of fuzzy number just as much as classical analysis is based on the concept of real number, our results open new possibilities of research in this field.

1. INTRODUCTION

For any linearly ordered set $(X, <)$, let $\tau_{\mathcal{O}}$ be the topology on X that has the collection of all open intervals of $(X, <)$ as a base. The topology $\tau_{\mathcal{O}}$ is called the *open interval topology* of the order $<$ and $(X, <, \tau_{\mathcal{O}})$ is a *linearly ordered topological space* or **LOTS** for short. It is a well-known fact that every **LOTS** is a normal Hausdorff space (indeed, a hereditarily collectionwise normal Hausdorff space). There is a close link between the properties of the order $<$ and the topological properties of a **LOTS**. Recall that $(X, <)$ is *Dedekind complete* if every non-empty subset of X that is bounded above admits a supremum, and $(X, <)$ is said to be a *dense linear order set* if for all $x, y \in X$ with $x < y$ there exists $z \in X$ such that $x < z < y$. Basic results in the theory state that $(X, <, \tau_{\mathcal{O}})$ is connected if, and only if, $(X, <)$ is Dedekind complete and has a dense linear order and that $(X, <, \tau_{\mathcal{O}})$ is compact if, and only if, $(X, <)$ is Dedekind complete and it has a first and a last element.

A function from a linearly ordered set $(X, <)$ into a linearly ordered set $(Y, <)$ is called *nondecreasing* (respectively, *nonincreasing*) if, for all $x, y \in X$, $x < y$ implies $f(x) \leq f(y)$ (respectively, $f(x) \geq f(y)$). A function is *monotone* if it is either nondecreasing or nonincreasing. In particular, a sequence $(x_n)_{n \in \mathbb{N}}$ is called

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Corresponding author: Manuel Sanchis (sanchis@uji.es)

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nondecreasing, nonincreasing or monotone respectively, if it is nondecreasing, nonincreasing or monotone as a function from the natural numbers $\mathbb{N}$ into the set X.

A linearly ordered set $(X, <)$ is said to be *dense* if, for all $x, y \in X$, $x < y$ implies that there exists $z \in D$ such that $x \leq z \leq y$. The order-density of X is defined by

$$d(X) = \min \{|D| \, : \, D \text{ is dense in } X\},$$

where $|D|$ stands for the cardinality of the set D.

One of the major notions in this paper is the concept of *splitting number*. A family S of infinite subsets of $\mathbb{N}$ is said to be *splitting* if, for every infinite subset $A \subseteq \mathbb{N}$, there exists a set $B \in S$ such that $A \cap B$ and $A \setminus B$ are both infinite. The splitting number $\mathfrak{s}$ is defined by

$$\mathfrak{s} = \min \{|S| \, : \, S \text{ is a splitting family}\}.$$

It is readily seen that $\mathfrak{s}$ is greater or equal to the first uncountable cardinal $\aleph_1$ and that it is less than or equal to the continuum $\mathfrak{c}$. It is consistent in ZFC that $\aleph_1 = \mathfrak{s}$ and also that $\mathfrak{c} = \mathfrak{s}$. For more information about the splitting number, the reader can consult [3]. A basic fact in the theory of **LOTS** is that the *weight* of the topological space $(X, <, \tau_{\mathscr{O}})$ coincides with the order-density of $(X, <)$. As usual, $\omega(X)$ will stand for the weight of a topological space (X, τ).

A topological space X is said to be *sequentially compact* if every sequence in X has a convergent subsequence. Notice that the convergence of a sequence can be defined in a canonical way in a **LOTS**: a nondecreasing (respectively, nonincreasing) sequence $(x_n)_{n\in\mathbb{N}}$ converges to x_0 if x_0 is the supremum (respectively, the infimum) of the set of all elements of this sequence. In general, a sequence $(x_n)_{n\in\mathbb{N}}$ converges to x_0 if every monotone subsequence converges to x_0. Since every sequence in a linearly ordered set contains a monotone sequence, we have that a **LOTS** $(X, <, \tau_{\mathscr{O}})$ is sequentially compact if and only if every monotone sequence converges. It is a well-known fact that by adding a copy of the open interval $]0, 1[$ between the elements of each jump, every infinite linearly ordered set X can be embedded into a dense linear order $\widehat{X}$ such that $d(\widehat{X}) = d(X)$. Moreover, if X is sequentially compact, then $\widehat{X}$ can be also chosen to be so. Also, convergent sequences in X remain convergent in $\widehat{X}$ with the same limit.

Our interest in the splitting number is motivated by the following interesting result of Fuchino and Plewik:

THEOREM 1.1. ([6, Theorem 7]) *Let $(X, <)$ and $(Y, <)$ be* ***LOTS****. If $(Y, <, \tau_{\mathscr{O}})$ is sequentially compact with weight less than $\mathfrak{s}$, then any sequence of monotone functions from X to Y contains a pointwise convergent subsequence.*

A uniform structure on a set X is a nonempty family $\mathscr{D}$ of pseudometrics on X such that: (i) if $d, e \in \mathscr{D}$, then $d \vee e \in \mathscr{D}$, and (ii) if e is a pseudometric on X and for each $\varepsilon > 0$ there exist $d \in \mathscr{D}$ and $\delta > 0$ such that

$$d(x, y) \leq \delta \quad \text{implies} \quad e(x, y) \leq \varepsilon,$$

for all $x, y \in X$, then $e \in \mathscr{D}$.

By a *Hausdorff uniform structure* on X it is understood a uniform structure $\mathscr{D}$ on X such that for each $x, y \in X$ with $x \neq y$, there exists $d \in \mathscr{D}$ for which $d(x, y) > 0$. In this situation, the pair $(X, \mathscr{D})$ is called a *uniform space.*

A Hausdorff uniform structure $\mathscr{D}$ on X induces a Hausdorff topology $\tau_{\mathscr{D}}$ on X such that for each $x \in X$, the family $\{B_d(x, \varepsilon) : d \in \mathscr{D}, \varepsilon > 0\}$ is a neighborhood base at x, where, as usual, $B_d(x, \varepsilon) = \{y \in X : d(x, y) < \varepsilon\}$ for all $x \in X$ and $\varepsilon > 0$. If X is a topological space, a uniform structure $\mathscr{D}$ on X is said to be *admissible* if the topology induced by $\mathscr{D}$ coincides with the original topology. A uniform space $(X, \mathscr{D})$ enjoys a topological property provided the topological space $(X, \tau_{\mathscr{D}})$ does also. It is well known that a topological space X has an admissible structure if, and only if, X is a Tychonoff space. In particular, this is the case for a **LOTS** $(X, <, \tau_{\mathscr{O}})$.

If $\mathcal{F}$ is a family of functions from a set X into a uniform space $(Y, \mathscr{D})$, then the uniform structure $\mathscr{D}$ induces a uniformity $\mathcal{D}_U$ on $\mathcal{F}$, the so-called *uniformity of the uniform convergence* which is generated by the family of pseudometrics $\{\rho_d : d \in \mathscr{D}\}$ with $\rho_d(f, g) = \sup_{x \in X} d(f(x), g(x))$ for all $f, g \in \mathcal{F}$. It is apparent that a sequence $(f_n)_{n \in \mathbb{N}}$ converges uniformly to a function f if for all $d \in \mathcal{D}$ and all $\varepsilon > 0$, there exists $n_0(d)$ such that $d(f_n(x), f(x)) < \varepsilon$ for all $x \in X$ and all $n \geq n_0(d)$. As usual, a net $(f_i)_{i \in I}$ is said to be a *Cauchy net* if for all $d \in \mathcal{D}$ and all $\varepsilon > 0$, there exists $i_0(d) \in I$ such that $d(f_i(x), f_j(x)) < \varepsilon$ for all $x \in X$ and all $i, j \geq i_0(d)$. The uniform space $(\mathcal{F}, \mathcal{D}_U)$ is called *complete* if every Cauchy net converges. For simplicity's sake, if $(Y, \mathcal{D})$ is a uniform space such that the topology induced by $\mathcal{D}$ makes Y a **LOTS**, then we say that $(Y, \mathcal{D})$ is a **LOTS**. In this case, when we say that $(Y, \mathcal{D})$ is a complete **LOTS**, it means that the uniform space $(Y, \mathcal{D})$ is complete and that X equipped with the topology induced by $\mathcal{D}$ is a **LOTS**, that is, no order notions on completeness are considered.

Historically, **LOTS** and their topological subspaces (the so-called **GO**-spaces) have been valuable sources of counterexamples in topology. Among others, elementary examples include the real line, the space ω_1 of all countable ordinals less than or equal to the first uncountable ordinal, the Sorgenfrey line and the Michael line (i.e., the usual space of real numbers with each irrational isolated). More elaborate examples can be constructed if one is willing to start with more sophisticated linear orders, e.g., those derived from lexicographic products, nonseparable **LOTS** or **GO**-spaces that have countable cellularity (for example, Souslin trees and Souslin lines (see [11])). Moreover, monotone functions have played a role not only in classical analysis but also in many fields of mathematics and its applications. We can find applications in utility theory [1], in the context of equivalent locally uniformly convex norms in function spaces [5], etc.

In this paper, for a wide set of **LOTS**, X and Y, we deal with sequentially compact subsets of the space of all monotone functions from X into Y endowed with the topology of uniform convergence induced by an admissible uniformity on Y. An especially interesting case occurs when X is the unit interval and Y the real line. Our results apply in the space of fuzzy numbers; indeed, in the more general setting of fuzzy subsets on connected **LOTS** with weight less than the splitting number $\mathfrak{s}$. Considering that **LOTS** are widely used in a variety of ways, our results provide a new area for future research and also for further applications of fuzzy theory.

2. Sequential compactness in the space of monotone functions

From now on, we write X instead of $(X, <, \tau_{\mathscr{O}})$. Given two **LOTS** X and Y, if $\mathscr{D}$ is an admissible uniform structure on Y, then $\mathcal{M}(X, Y)$ denotes the set of all monotone functions from X into Y endowed with the topology of uniform convergence induced by $\mathcal{D}$. The aim of this section is to provide a characterization of the sequentially compact subsets of the space $\mathcal{M}(X, Y)$ when X is a first countable compact **LOTS** and Y is a sequentially compact **LOTS** with weight less than $\mathfrak{s}$. As a consequence, we shall obtain a criterion for compactness in $\mathcal{M}(X, Y)$ in several interesting situations, including the useful case $X = [0, 1]$ and $Y = \mathbb{R}$, the real line with its usual topology.

Before stating our result, we need to introduce some notation and fix some details. Let X, Y be two **LOTS** and let $\lambda_0 \in X$. If X is first countable and Y is sequentially compact, given a monotone function $f\colon X \to Y$, we can consider the limit $f(\lambda_0+)$ of f when λ approaches λ_0 from above (right); indeed, we have

$$f(\lambda_0+) = \inf\{f(\lambda_n) : (\lambda_n)_{n\in\mathbb{N}} \text{ is a decreasing sequence converging to } \lambda_0\}.$$

Notice that $f(\lambda_0+)$ does not depend on the choice of the sequence $(\lambda_n)_{n\in\mathbb{N}}$. *Mutatis mutandis*, a statement to the previous one holds for the case of $f(\lambda_0-)$, the limit of f when λ approaches λ_0 from below (left).

DEFINITION 2.1. Let $\{ f_i \}_{i\in I}$ be a family of functions defined from a first countable **LOTS** X into a sequentially compact **LOTS** $(Y, \mathcal{D})$. Given $\lambda_0 \in X$ such that $f_i(\lambda_0+)$ exists for all $i \in I$, the family $\{ f_i \}_{i\in I}$ is said to be *almost-right-equicontinuous at* λ_0 if, for all $d \in \mathcal{D}$ and all $\varepsilon > 0$, there is $\lambda_1 > \lambda_0$ such that $d(f_i(\lambda), f_i(\lambda_0+)) < \varepsilon$ for all $i \in I$ whenever $\lambda \in \,]\lambda_0, \lambda_1[$.

DEFINITION 2.2. Let $\{ f_i \}_{i\in I}$ be a family of functions defined from a first countable **LOTS** X into a sequentially compact **LOTS** $(Y, \mathcal{D})$. Given $\lambda_0 \in X$ such that $f_i(\lambda_0-)$ exists for all $i \in I$, the family $\{ f_i \}_{i\in I}$ is said to be *almost-left-equicontinuous at* λ_0 if, for all $d \in \mathcal{D}$ and all $\varepsilon > 0$, there is $\lambda_1 < \lambda_0$ such that $d(f_i(\lambda), f_i(\lambda_0-)) < \varepsilon$ for all $i \in I$ whenever $\lambda \in \,]\lambda_1, \lambda_0[$.

Notice that, when working with right-continuous (respectively, left-continuous) functions, the usual notions of almost-right-equicontinuity (respectively, almost-left-equicontinuity) and right-equicontinuity (respectively, left-equicontinuity) coincide. If the family $\{ f_i \}_{i\in I}$ is almost-right-equicontinuous (respectively, almost-left-equicontinuous) at λ for all $\lambda \in X$, then we say that $\{ f_i \}_{i\in I}$ is *almost-right-equicontinuous* (respectively, *almost-left-equicontinuous*) on X.

PROPOSITION 2.3. *Let $\{ f_n \}_{n\in\mathbb{N}}$ be a sequence of functions defined from a first countable **LOTS** X into a sequentially compact **LOTS** $(Y, \mathcal{D})$ which is almost-right-equicontinuous at a point $\lambda_0 \in X$. If $\{ f_n \}_{n\in\mathbb{N}}$ converges pointwise to a function f on X and $f(\lambda_0+)$ exists, then $\{ f_n(\lambda_0+) \}_{n\in\mathbb{N}}$ converges to $f(\lambda_0+)$.*

Proof. Let $d \in \mathcal{D}$ and $\varepsilon > 0$. By hypothesis, $f(\lambda_0+)$ exists and, since $\{ f_n \}_{n\in\mathbb{N}}$ is almost-right-equicontinuous at λ_0, we know that $f_n(\lambda_0+)$ also exists and that there is $\lambda \in X$ such that

$$d(f_n(\lambda), f_n(\lambda_0+)) < \varepsilon \quad \text{for all } n \in \mathbb{N}$$

and

$$d(f(\lambda), f(\lambda_0+)) < \varepsilon.$$

Moreover, since $\{ f_n \}_{n\in\mathbb{N}}$ converges pointwise to f on X, there is $n_0(\lambda) \in \mathbb{N}$ such that, for all $n \geq n_0(\lambda)$, we have

$$d(f_n(\lambda), f(\lambda)) < \varepsilon.$$

Then, if $n \geq n_0(\lambda)$, we obtain

$$d(f_n(\lambda_0+), f(\lambda_0+)) \leq d(f_n(\lambda_0+), f_n(\lambda)) + d(f_n(\lambda), f(\lambda)) + d(f(\lambda), f(\lambda_0+)) < 3\varepsilon,$$

which completes the proof. □

An argument similar to the one used in the previous proposition allows us to obtain

PROPOSITION 2.4. *Let $\{ f_n \}_{n\in\mathbb{N}}$ be a sequence of functions defined from a first countable **LOTS** X into a sequentially compact **LOTS** $(Y, \mathcal{D})$ which is almost-left-equicontinuous at a point $\lambda_0 \in X$. If $\{ f_n \}_{n\in\mathbb{N}}$ converges pointwise to a function f on X and $f(\lambda_0-)$ exists, then $\{ f_n(\lambda_0-) \}_{n\in\mathbb{N}}$ converges to $f(\lambda_0-)$.*

We are now ready to prove our promised characterization of sequentially compact subsets of the space $(\mathcal{M}(X, Y), \mathcal{D}_U)$. Notice that a compact **LOTS** X can be written as a closed interval $[m, M]$ where m and M are, respectively, the first and the last element of X.

THEOREM 2.5. *If $X = [m, M]$ is a first countable compact **LOTS** and $(Y, \mathcal{D})$ is a sequentially compact **LOTS** with weight less than $\mathfrak{s}$, then a subset S of the space $(\mathcal{M}(X, Y), \mathcal{D}_U)$ is sequentially compact if, and only if, it is almost-left-equicontinuous on $]m, M]$ and almost-right-equicontinuous on $[m, M[$.*

Proof. Sufficiency Assume that S is almost-left-equicontinuous on $]m, M]$ and almost-right-equicontinuous on $[m, M[$ and consider a sequence $\{ f_n \}_{n\in\mathbb{N}} \subset S$. Assume, with no loss of generality, that every f_n is nondecreasing. We shall show that $\{ f_n \}_{n\in\mathbb{N}}$ has a subsequence which converges uniformly. Indeed, by Theorem 1.1, we can assume that $\{ f_n \}_{n\in\mathbb{N}}$ converges pointwise to a function, say f, on X. Since the pointwise limit of a sequence of nondecreasing functions is a nondecreasing function, we infer that f belongs to $\mathcal{M}(X, Y)$.

We shall next prove that $f_n \to f$ uniformly on X. If we assume, contrary to what we claim, that the convergence is not uniform, then we can choose $d \in \mathcal{D}$,

$\varepsilon > 0$, an infinite sequence of natural numbers $n_1 < n_2 < n_3 < \dots$ and a sequence $\{ \lambda_{n_k} \}_{k\in\mathbb{N}} \subset X$ such that

$$d(f_{n_k}(\lambda_{n_k}), f(\lambda_{n_k}) \geq 3\varepsilon.$$

Let us suppose, with no loss of generality, that the sequence $\{ \lambda_{n_k} \}_{k\in\mathbb{N}}$ converges to a point $\lambda_0 \in X$. We shall consider two cases.

Case 1. There exists an infinite subsequence of $\{ \lambda_{n_k} \}_{k\in\mathbb{N}}$ whose elements are less than λ_0. For the sake of simplicity, we shall keep denoting this subsequence by $\{ \lambda_{n_k} \}_{k\in\mathbb{N}}$. Now, the definition of $f(\lambda_0-)$, the fact that $\{ f_{n_k} \}_{k\in\mathbb{N}}$ is a left-equicontinuous sequence at λ_0 that converges pointwise to f and Proposition 2.4 permits us to choose $k_0 \in \mathbb{N}$ such that

$$d(f_{n_k}(\lambda_{n_k}), f_{n_k}(\lambda_0-)) < \varepsilon, \quad d(f_{n_k}(\lambda_0-), f(\lambda_0-)) < \varepsilon, \quad d(f(\lambda_{n_k}), f(\lambda_0-)) < \varepsilon$$

for all $k \geq k_0$. Thus,

$$d(f_{n_k}(\lambda_{n_k}), f(\lambda_{n_k})) < 3\varepsilon$$

whenever $k \geq k_0$, which contradicts our assumption above.

Case 2. There exists an infinite subsequence of $\{ \lambda_{n_k} \}_{k\in\mathbb{N}}$ whose elements are greater than λ_0. As above, for simplicity, we shall denote this subsequence again by $\{ \lambda_{n_k} \}_{k\in\mathbb{N}}$.

Now, the definition of $f(\lambda_0+)$ and the fact that $\{ f_{n_k} \}_{k\in\mathbb{N}}$ is a almost-right-equicontinuous sequence at λ_0 tell us that there exists $k_0 \in \mathbb{N}$ such that

$$d(f_{n_k}(\lambda_{n_k}), f_{n_k}(\lambda_0+)) < \varepsilon, \quad d(f(\lambda_{n_k}), f(\lambda_0+)) < \varepsilon$$

for all $k \geq k_0$. Moreover, by Proposition 2.3, we can choose such k_0 satisfying the additional condition

$$d(f_{n_k}(\lambda_0+), f(\lambda_0+)) < \varepsilon$$

for all $k \geq k_0$. Therefore

$$d(f_{n_k}(\lambda_{n_k}), f(\lambda_{n_k})) < 3\varepsilon,$$

which provides the promised contradiction.

Therefore, $f_n \to f$ uniformly on X. Thus, S is sequentially compact.

Necessity. Suppose that S is not almost-right-equicontinuous at a point $\lambda_0 \in]m, M]$. Then, we can asssume, without loss of generality, that there exists $d \in \mathcal{D}$, $\varepsilon > 0$, a decreasing sequence $\{ \lambda_n \}_{n\in\mathbb{N}}$ converging to the right to λ_0 and a sequence $\{ f_n \}_{n\in\mathbb{N}} \subset M$ such that

$$d(f_n(\lambda_n), f_n(\lambda_0+)) \geq 3\varepsilon \quad \text{for all } n \in \mathbb{N}. \tag{1}$$

Since S is compact, there is a subsequence $\{ f_{n_r} \}_{r\in\mathbb{N}}$ converging uniformly to a function f. Then, taking also into account Proposition 2.3, there exists $r_0 \in \mathbb{N}$ such that, for all $r \geq r_0$,

$$\begin{aligned} d(f_{n_r}(\lambda_{n_r}), f_{n_r}(\lambda_0+)) \leq & d(f_{n_r}(\lambda_{n_r}), f(\lambda_{n_r})) \\ & + d(f(\lambda_{n_r}), f(\lambda_0+)) + d(f(\lambda_0+), f_{n_r}(\lambda_0+)) \\ < & \varepsilon + \varepsilon + \varepsilon = 3\varepsilon, \end{aligned}$$

a contradiction with (1). Thus S is almost-right-equicontinuous on $]m, M]$. An argument resembling the previous one shows that S is almost-left-equicontinuous on $[m, M[$. This completes the proof. □

An interesting particular case in applications arises when the uniform space $(Y, \mathcal{D})$ is complete. In this context, the previous result can be applied in order to characterize sequentially compact subsets of the space $(\mathcal{M}(X, Y), \mathcal{D}_U)$ when $(Y, \mathcal{D})$ is not necessarily sequentially compact.

To this end, we consider the notion of *uniform boundedness* of a family of functions as follows: Let X, Y be two **LOTS**. A family S of functions from X into Y is said to be *uniformly bounded* if there exist $x, y \in Y$ such that $x \leq f(z) \leq y$ for all $z \in X$ and all $f \in S$, i.e., $f(X)$ is included in the closed interval $[x, y]$ for all $f \in S$. Recall that a subset B of a space X is called *functionally bounded* (in X) if the restriction to B of every real-valued function on X is bounded. It is a well-known fact that closed functionally bounded subsets of a complete uniform space are compact. We feel free to use this property without special mention. Now we need a lemma which we include for the sake of completeness.

LEMMA 2.6. *Let B be a subset of a **LOTS** X. If every sequence $(\lambda_n)_{n\in\mathbb{N}}$ in B has a subsequence converging to a point $\lambda_0 \in X$, then $\mathrm{cl}_X B$ is sequentially compact.*

Proof. It is straightforward that, if f is a continuous real-valued function on X, then the restriction of f to B is bounded. Now the result follows from [10, Theorem 2.2]. □

THEOREM 2.7. *If $X = [m, M]$ is a first countable compact **LOTS** and $(Y, \mathcal{D})$ is a complete **LOTS** with weight less than $\mathfrak{s}$, then a subset S of the space $(\mathcal{M}(X, Y), \mathcal{D}_U)$ is sequentially compact if, and only if, the following hold:*

(i) S is uniformly bounded.
(ii) S is almost-left-equicontinuous on $]m, M]$ and almost-right-equicontinuous on $[m, M[$

Proof. Suppose that S satisfies conditions (i) and (ii). By condition (i) we can consider the set S as a family of functions from X to a compact **LOTS** space B with $\omega(B) \leq \omega(Y) < \mathfrak{s}$. Now B can be embedded in a compact **LOTS** Z endowed with a dense linear order and such that $\omega(Z) = \omega(B)$ (see, for instance, [6, Lemma 2]).

Let us denote by $\mathcal{V}$ the unique admissible uniform structure on Z ([7, 15H]). Then Theorem 2.5 tells us that S is a sequentially compact subset of $\mathcal{M}(X, Z, \mathcal{V}_U)$ and, *a posteriori*, it is a sequentially compact subset of $\mathcal{M}(X, B, \mathcal{V}_U|_B)$. The result now follows from the fact that, by compactness, $\mathcal{V}|_B = \mathcal{D}|_B$.

Conversely, define $H = \bigcup\{f(X) : f \in S\}$. We shall show that $\mathrm{cl}_Y H$ is sequentially compact. By Theorem 2.5 and Lemma 2.6, it suffices to show that every increasing sequence $(\nu_n)_{n\in\mathbb{N}}$ in H has a convergent subsequence (in Y). For this purpose, consider two sequences $(f_n)_{n\in\mathbb{N}} \subset S$ and $(\lambda_n)_{n\in\mathbb{N}} \subset X$ such that $f_n(\lambda_n) = \nu_n$ for all $n \in \mathbb{N}$. Since S is sequentially compact and X is first countable and compact, we can suppose, with no loss of generality, that $(f_n)_{n\in\mathbb{N}}$ converges uniformly to a function $f \in S$ and $(\lambda_n)_{n\in\mathbb{N}}$ converges to $\lambda_0 \in X$. Then given $\varepsilon > 0$ and $d \in \mathcal{D}$, there exists $n_0 \in \mathbb{N}$ such that, for all $n \geq n_0$, we have

$$\begin{aligned} d(f_n(\lambda_n), f(\lambda_0-)) \leq & d(f_n(\lambda_n), f(\lambda_n)) + d(f(\lambda_n), f(\lambda_0-)) \\ \leq & \varepsilon/2 + \varepsilon/2 \leq \varepsilon, \end{aligned}$$

which proves that $(f_n(\lambda_n))_{n\in\mathbb{N}}$ converges to $f(\lambda_0-)$. Thus, $\mathrm{cl}_Y H$ is sequentially compact (and consequently, functionally bounded). Since $(Y, \mathcal{D})$ is complete, $\mathrm{cl}_Y H$ is compact. Hence there exist $x, y \in Y$ such that $\mathrm{cl}_Y H \subset [x, y]$, that is, condition (i) holds. To see condition (ii), it suffices to notice that, as in the above argument, the interval $[x, y]$ can be embedded in a compact **LOTS** Z equipped with a dense linear order and such that $\omega([x, y]) = \omega(Z)$. Then Theorem 2.5 applies. □

Compact spaces need not be sequentially compact (this is the case, for instance, of $\beta\mathbb{N}$, the Stone-Čech compactification of the natural numbers endowed with the discrete topology). However, as we have used implicitly in the previous result, it is a matter of fact that every compact linearly ordered topological space is sequentially compact. The converse can fail to be true: ω_1, the linearly ordered topological space of all countable ordinals less than or equal to the first uncountable ordinal, is sequentially compact but not compact. In spite of this, under several set-theorical or topological properties, sequentially compactness implies compactness. Among them, it seems interesting to mention paracompactness (which includes metrizability), to be a pure space in the sense of Arhangelskiĭ (in particular, spaces with a (quasi)-G_δ-diagonal), etc... (an interesting survey on this topic can be found in [11]). In this case, an argument similar to the one used in Theorem 2.7 allows us to obtain the following result. Let $\mathcal{P}$ denote the set of all properties P such that *sequentially compact* + $P \Longrightarrow$ *compact.*

THEOREM 2.8. *If* $X = [m, M]$ *is a first countable compact* ***LOTS*** *and* $(Y, \mathcal{D})$ *is a* ***LOTS*** *which has property* $P \in \mathcal{P}$ *and with weight less than* $\mathfrak{s}$*, then a subset* S *of the space* $(\mathcal{M}(X, Y), \mathcal{D}_U)$ *is sequentially compact if, and only if, the following hold:*

(i) S *is uniformly bounded.*
(ii) S *is almost-left-equicontinuous on* $]m, M]$ *and almost-right-equicontinuous on* $[m, M[$.

In the case that the space $(Y, \mathcal{D})$ is metrizable (that is, the uniform structure $\mathcal{D}$ has a countable base), the uniform space $(\mathcal{M}(X, Y), \mathcal{D}_U)$ is also metrizable. Since compactness and sequential compactness are equivalent in the realm of metric spaces, we have the following result which includes the case $X = [0, 1]$ and $Y = \mathbb{R}$.

THEOREM 2.9. *If $X = [m, M]$ is a first countable compact **LOTS** and $(Y, \mathcal{D})$ is a metrizable **LOTS** with weight less than $\mathfrak{s}$, then a subset S of the space $(\mathcal{M}(X, Y), \mathcal{D}_U)$ is compact if, and only if, the following hold:*

(i) S is uniformly bounded.

(ii) S is almost-left-equicontinuous on $]m, M]$ and almost-right-equicontinuous on $[m, M[$.

3. An Application to Fuzzy Theory

Let $F(Y)$ denote the family of all fuzzy subsets on a connected **LOTS** Y. For $u \in F(Y)$ and $\lambda \in [0, 1]$, the λ-level set of u is defined by

$$[u]^\lambda := \{\, x \in Y \,:\, u(\lambda) \geq \lambda \,\}, \quad \lambda \in]0, 1], \quad [u]^0 := \mathrm{cl}_Y \{\, \lambda \in Y \,:\, u(\lambda) > 0 \,\}.$$

Let $\mathbb{K}_c(Y)$ be the set of elements u of $F(Y)$ satisfying the following properties:

(1) u is normal, i.e., there exists $\lambda_0 \in Y$ with $u(\lambda_0) = 1$;
(2) u is convex, i.e., for all $x, y \in Y$, $u(z) \geq \min\{u(x), u(y)\}$ for all $x \leq z \leq y$;
(3) u is upper-semicontinuous;
(4) $[u]^0$ is a compact set in Y.

Notice that if $u \in \mathbb{K}_c(Y)$, then the λ-level set $[u]^\lambda$ of u is a compact interval for each $\lambda \in [0, 1]$. We denote $[u]^\lambda = [u^-(\lambda), u^+(\lambda)]$. Every element $y \in Y$ can be considered an element of $\mathbb{K}_c(Y)$ since y can be identified with the element of $\mathbb{K}_c(Y)$ $\tilde{y}$ defined as

$$\tilde{y}(t) := \begin{cases} 1 & \text{if } t = y, \\ 0 & \text{if } t \neq y. \end{cases}$$

The typical case of a $\mathbb{K}_c(Y)$ is when Y is the space of the real numbers equipped with its usual topology. In this set-up we face the so-called fuzzy numbers which were introduced by Dubois and Prade ([2]) to provide formalized tools to deal with non-precise quantities. It is worth mentioning that fuzzy analysis is based on the notion of fuzzy number just as much as classical analysis is based on the concept of real number. It has significant applications in fuzzy optimization, fuzzy decision making, etc. (see, for instance, [9], [12], [13]). It is also worth noting that, with the development of the theory and applications of fuzzy numbers, these are becoming increasingly important.

Goetschel and Voxman proposed an equivalent representation of such numbers in a topological vector space setting, which eased the development of the theory and applications of fuzzy numbers (see [8]). In a straightforward way, the proof of Goetschel–Voxman's representation theorem can be adapted in order to obtain:

THEOREM 3.1. *Let $u \in \mathbb{K}_c(Y)$ and $[u]^\lambda = [u^-(\lambda), u^+(\lambda)]$, $\lambda \in [0, 1]$. Then the pair of functions $u^-(\lambda)$ and $u^+(\lambda)$ has the following properties:*

(i) $u^-(\lambda)$ is a bounded left-continuous nondecreasing function on $]0, 1]$;

(ii) $u^+(\lambda)$ is a bounded left-continuous nonincreasing function on $]0, 1]$;

(iii) $u^-(\lambda)$ and $u^+(\lambda)$ are right-continuous at $\lambda = 0$;

(iv) $u^-(1) \leq u^+(1)$.

Conversely, if a pair of functions $\alpha(\lambda)$ and $\beta(\lambda)$ from $[0,1]$ into Y satisfy the above conditions (i)-(iv), then there exists a unique $u \in \mathbb{K}_c(Y)$ such that $[u]^\lambda = [\alpha(\lambda), \beta(\lambda)]$ for each $\lambda \in [0,1]$.

The previous results allow us to consider different topologies on $\mathbb{K}_c(Y)$ defined by means of different types of convergence of families of functions. From now on, if $(Y, \mathcal{D})$ is a uniform **LOTS**, then we endow $\mathbb{K}_c(Y)$ with the topology of the uniform convergence $\mathcal{D}_U$ induced by $\mathcal{D}$, that is, a net $(u_\alpha)_{\alpha \in I} \subset \mathbb{K}_c(Y)$ converges to $u \in \mathbb{K}_c(Y)$ if the net $(u_\alpha^-)_{\alpha \in I}$ uniformly converges to u^- and the net $(u_\alpha^+)_{\alpha \in I}$ uniformly converges to u^+. When $Y = \mathbb{R}$, the topology of uniform convergence is induced by the supremum metric defined by using the Hausdorff distance on the hyperspace all nonempty compact intervals. Moreover, it is apparent that the functions u^- and u^+ which appear in Goetschel–Voxman's theorem belong to $\mathcal{M}([0,1], Y)$. Thus, the outcomes of the previous section permit us to characterize the sequentially compact subsets of $(\mathbb{K}_c(Y), \mathcal{D}_U)$.

THEOREM 3.2. *Let $(Y, \mathcal{D})$ be a connected complete **LOTS**. If the weight of $(Y, \mathcal{D})$ is less than $\mathfrak{s}$, then a subset S of the space $(\mathbb{K}_c(Y), \mathcal{D}_U)$ is sequentially compact if, and only if, the following hold:*

(i) S is uniformly bounded.
(ii) $\{ u^+ : u \in S \}$ and $\{ u^- : u \in S \}$ are left-equicontinuous on $]m, M]$ and almost-right-equicontinuous on $[m, M[$.

Notice that Theorem 2.8 tells us that *completeness* can be replaced by a property $P \in \mathcal{P}$.

If the uniform space $(Y, \mathcal{D})$ is metrizable, Theorem 2.9 applies in order to obtain

THEOREM 3.3. *If $(Y, \mathcal{D})$ is metrizable with weight less than $\mathfrak{s}$, then a subset S of the space $(\mathbb{K}_c(Y), \mathcal{D}_U)$ is compact if, and only if, the following hold:*

(i) S is uniformly bounded.
(ii) $\{ u^+ : u \in S \}$ and $\{ u^- : u \in S \}$ are left-equicontinuous on $]m, M]$ and almost-right-equicontinuous on $[m, M[$.

REMARK 3.4. Condition (i) in the previous theorems is equivalent to: there exist $x, y \in Y$ such that $[u^-(m), u^+(m)] \subset [x, y]$ for all $u \in S$.

REMARK 3.5. In the theory of fuzzy numbers, a subset $M \subset \mathbb{K}_c(\mathbb{R})$ is called *support bounded* if there exists a constant $L > 0$ such that $|u^-(0)| \leq L$ and $|u^+(0)| \leq L$ for all $u \in M$. Notice that this notion is equivalent to our notion of uniform boundedness.

REMARK 3.6. Fang and Xue obtained a characterization of compact subsets of the space $\mathbb{K}_c(\mathbb{R})$ in [4] by means of a stimulating approach: the characterization was obtained by using Goetschel-Voxman's representation theorem. The interest was twofold: they use one of the most helpful tools in the theory of fuzzy numbers (Goetschel-Voxman's representation theorem) and, consequently, only intrinsic properties are used: it is not necessary to pass to *external* structures as, for instance, Banach spaces, hyperspaces, etc. Moreover, it suffices to work only with two well-known basic notions: monotonic functions and uniform convergence. Unfortunately, Fang and Xue's proof has a gap: actually, it is straightforward to see that their result implies that the functions u^+ and u^- must be continuous. Our Theorem 3.3 gives a correct version of Fang and Xue's theorem in its own context: Goetschel-Voxman's representation theorem.

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Current address: (Both authors) Departament de Matemàtiques, and Institut Universitari de Matemàtiques i Aplicacions de Castelló (IMAC), Universitat Jaume I, Campus del Riu Sec. s/n, 12071 Castelló (Spain)

E-mail address: `font@uji.es (Juan J. Font); sanchis@uji.es (Manuel Sanchis)`