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Attribute implications with unknown information based on weak Heyting algebras

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ABSTRACT

Simplification logic, a logic for attribute implications, was originally defined for Boolean sets. It was extended to distributive fuzzy sets by using a complete dual Heyting algebra. In this paper, we weaken this restriction in the sense that we prove that it is possible to define a simplification logic on fuzzy sets in which the membership value structure is not necessarily distributive. For this purpose, we replace the structure of the complete dual Heyting algebra by the so-called weak complete dual Heyting algebra. We demonstrate the soundness and completeness of this simplification logic, and provide a characterisation of the operations defining weak complete dual Heyting algebras.

1. Introduction

In this paper, our attention is directed towards attribute implications as key elements for describing information enclosed in datasets. The notion of attribute implication comes from the theory of formal concept analysis and is essentially the same as the notion of exact association rule in data mining, functional dependency in database theory, Horn's clause in logic, etc. Attribute implications are pairs of sets of attributes (premise and conclusion) interpreted as "if-then" rules in the sense that "any object that has all the attributes of the premise also has all the attributes of the conclusion". Even for the medium-sized datasets, the number of all implications held uses to be huge, making them difficult to handle. In contrast, an advantage of attribute implications is that all knowledge can be represented by a small part of the implications, and the rest can be inferred using inference mechanisms. One such mechanism is the well-known Armstrong axioms [1]. However, the most efficient and application-oriented alternative is provided by the family of logics called simplification logics. This family includes logics for the classical version of attribute implication and for some extensions of it [2–4]. Also for the fuzzy notion of attribute implication [5], in the framework of fuzzy formal concept analysis [6], Simplification Logic has been extended [7].

All of these logics are based on the so-called simplification paradigm, which, as its name suggests, allows the size of the representation to be reduced without losing knowledge by eliminating redundant information using a "difference" operation. This operator plays the role of a cornerstone to build an inference engine to simplify the sets of implications preserving the meaning (by means of equivalence transformations).

Simplification Logic for classical attribute implications builds its language on the Boolean algebra of the power set, so that the difference of sets can be used to remove redundancy. Extending to the fuzzy framework, no difference operator is available because the fuzzy power set is generally not a Boolean algebra. In [7], we paid attention to sufficient conditions for a suitable

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difference operation to prove the soundness and completeness of the new Simplification logic. The complete dual Heyting algebra was the answer to this question. Our next step was to define a generalised framework to cover different approaches (fuzzy, temporal, etc.) [8]. The so-called Parameterised framework [9] established a guideline for introducing new approaches in the future: first, a suitable difference operator has to be defined for the new truth value structure, consistent with the Simplification paradigm.

This generalised framework was designed to be adaptable to a wide range of situations simply by redefining the difference operator in the appropriate extension. The aim of this paper is to identify and deal with the cases where this extension is not possible. In [10] we present one of these cases. There, following the line initiated in [11], we consider datasets that collect unknown or missing information, and for this purpose we consider a truth value set of three elements, as was done in [12], and later we extend it to consider a fourth element related to the inconsistency. An important result is that we cannot consider a complete dual Heyting algebra, and a weaker structure is needed. It should be noted that various generalisations of Heyting algebras or their duals appear in the literature [13–16], being considered as an important tool for different purposes.

Based on the notion of weak complete dual Heyting algebra introduced in [10], this paper presents a guideline for establishing a comprehensive framework that incorporates unknown information and consistently characterises the formal results.

After presenting the preliminary results and necessary notions used throughout this paper (Section 2), we first introduce the language and present its semantics. In order to deal with unknown information in an abstract sense, a general set is used as the language and a complete lattice is our choice as its semantics (Section 3). Our next step in developing this generalisation of Simplification Logic is to introduce the axiomatic system. As we have explained, we first define a suitable difference operation, and we also prove the soundness of the axiomatic system by characterising the needed properties of the difference operation (Section 4).

One of the key points of the family of Simplification Logics is that they are intended to be executable logics. For this reason, we need to provide equivalence rules to manage the set of implications. The so-called weak complete dual Heyting algebra is introduced as the basic structure to ensure the equivalence issue (Section 4).

The natural next step in this guideline is to prove completeness, which has to be studied in two different situations according to the language classification. In the finite case it is sufficient to have a weak complete dual Heyting algebra. Otherwise we have to impose an additional condition on the algebraic structure: the compactness of the lattice elements (Section 5).

Having introduced the (very general) value set, its semantics and a logic for dealing with it, we formally characterise when the lattice behind the semantics provides a consistent structure to ensure soundness and completeness (Section 6). Finally, we show some conclusions and promising works to continue in this line of research (Section 7).

2. Preliminary definitions and results

2.1. Complete lattices and dual Heyting algebras

This subsection briefly introduces some of the mathematical backgrounds about ordered structures utilised in the paper. For more details, we recommend [17,18].

Definition 1. An ordered set $(L, \leqslant)$ is a complete lattice if every subset $X \subseteq L$ has both a supremum (denoted by $\bigvee X$) and an infimum (denoted by $\bigwedge X$).

It follows that every complete lattice is bounded, with a maximum element (denoted by $\top$) and a minimum element (denoted by $\bot$). Given $x, y \in L$, as usual, $x \vee y$ denotes $\bigvee\{x, y\}$ and $x \wedge y$ denotes $\bigwedge\{x, y\}$.

The comparability relationship, which we define below, will be relevant for the results presented in this paper.

Definition 2. Let $(L, \leqslant)$ be a complete lattice and $x, y \in L$. We say that x and y are *comparable*, denoted by $x \nparallel y$, if $x \leqslant y$ or $y \leqslant x$, whereas, if this condition is not fulfilled, we say that x and y are *incomparable* and denote it by $x \parallel y$.

The notions of reducibility and irreducibility will also be relevant throughout this paper.

Definition 3. An element $x \in L$ is said to be $\vee$-*irreducible* when, for all $y, z \in L$, $x = y \vee z$ implies that $y = x$ or $z = x$; otherwise, we say that x is $\vee$-*reducible*. The notions of $\wedge$-*irreducible* and $\wedge$-*reducible* are introduced in the same way.

On the other hand, the relevance of the concept of the closure operator, and its counterpart closure system, in mathematics, logic and computer science is indisputable.

Definition 4. Let $(L, \leqslant)$ be a complete lattice.

- A mapping $\mathsf{c} : L \to L$ is a *closure operator* if it is isotone (i.e. $x \leqslant y$ implies $\mathsf{c}(x) \leqslant \mathsf{c}(y)$ for all $x, y \in L$), extensive (i.e. $x \leqslant \mathsf{c}(x)$ for all $x \in L$), and idempotent (i.e. $\mathsf{c}(\mathsf{c}(x)) = \mathsf{c}(x)$ for all $x \in L$).
- A subset $S \subseteq L$ is a *closure system* if $\bigwedge X \in S$ for all $X \subseteq S$.

The mentioned closely relationship between both notions is the following: if c is a closure operator, the set $\mathrm{c}(L) = \{\mathrm{c}(x) \mid x \in L\} = \{x \in L \mid \mathrm{c}(x) = x\}$ is a closure system; conversely, if S is a closure system, the mapping $\mathrm{c}: L \to L$ defined as $\mathrm{c}(x) = \bigwedge\{s \in S \mid x \leqslant s\}$ is a closure operator. In addition, both correspondences between closure operators and closure systems are bijective and each one is the inverse mapping of the other one.

As we mentioned in the introduction, a difference operation is considered a key notion in our development and, in particular, for the definition of the notions of complete dual Heyting algebra and its (weak) extension.

Definition 5. A tuple $(L, \leqslant, \smallsetminus)$ is said to be a *complete dual Heyting algebra* (briefly, *cdHa*) if $(L, \leqslant)$ is a complete lattice and $\smallsetminus: L \times L \to L$, named *difference* operation, satisfies the following adjoint property:

$$a \leqslant b \vee c \quad \text{if and only if} \quad a \smallsetminus b \leqslant c, \quad \text{for all } a, b, c \in L. \tag{1}$$

It is well known that, for a complete lattice $(L, \leqslant)$, the following condition is both necessary and sufficient for the presence of an operation $\smallsetminus$ such that $(L, \leqslant, \smallsetminus)$ forms a cdHa:

$$\min\{x \mid a \leqslant b \vee x\} \text{ exists}, \quad \text{for all } a, b \in L. \tag{2}$$

In fact, Condition (1) is equivalent to:

$$a \smallsetminus b = \min\{x \mid a \leqslant b \vee x\}, \quad \text{for all } a, b \in L. \tag{3}$$

Other characterisations of cdHas can be found in [14]. In addition, in [7], it was also proved that the following property holds in any cdHa:

$$a \vee ((a \vee b) \smallsetminus c) = a \vee (b \smallsetminus c), \quad \text{for all } a, b, c \in L. \tag{4}$$

Remark 1. It is well known that every cdHa $(L, \leqslant, \smallsetminus)$ satisfies that $(L, \leqslant)$ is distributive. In fact, in the finite case, $(L, \leqslant)$ is distributive if and only if $(L, \leqslant, \smallsetminus)$ is a cdHa where $a \smallsetminus b = \bigwedge\{x \mid a \leqslant b \vee x\}$. This construction of cdHa can be extended, as we shall mention later, to complete (infinite) lattices by requiring the property of infinite distributivity.

As mentioned in the introduction, Simplification Logic for implications has been successfully extended in several ways. In particular, in [10], an extension to represent the unknown information was provided by considering an *ad hoc* instance of a weakening of complete dual Heyting algebras. This weakening of the cdHas was presented as follows:

Definition 6. A *weak complete dual Heyting algebra* (*weak-cdHa* for short) is an algebra $(L, \leqslant, \smallsetminus)$ such that $(L, \leqslant)$ is a complete lattice and the following properties hold:

`[w1]` $y \vee x \neq \top$ implies $(y \vee x) \smallsetminus x \leqslant y$ for all $x, y \in L$.
`[w2]` $x \smallsetminus y \leqslant x$ for all $x, y \in L$.
`[w3]` $x \smallsetminus y = \bot$ if and only if $x \leqslant y$ for all $x, y \in L$.
`[w4]` $x \vee y = x \vee (y \smallsetminus x)$ for all $x, y \in L$.

In addition, in [10] we also proved that any cdHa is a weak-cdHa, providing a counterexample for the opposite implication.

2.2. Formal Concept Analysis

We outline in this subsection only a few key ideas of Formal Concept Analysis (FCA). For more details, we recommend [19–21].

Initially, in classic FCA, the starting point is a formal context $\mathbb{K} = (G, M, I)$, where G and M are both finite and non-empty sets (the elements of which are called objects and attributes, respectively), and $I \subseteq G \times M$ establishes the relationship between each object and its corresponding attributes. The information stored in the context is treated in two ways: extracting knowledge in terms of *formal concepts* or in terms of *attribute implications*.

Definition 7. Let $\mathbb{K} = (G, M, I)$ be a formal context, and $\mathcal{P}(G)$ and $\mathcal{P}(M)$ be the powerset of objects and attributes respectively. The *concept forming operators* are defined as follows:

$^{\uparrow}: \mathcal{P}(G) \to \mathcal{P}(M)$ where $A^{\uparrow} = \{m \in M : g\ I\ m \text{ for all } g \in A\}$.
$^{\downarrow}: \mathcal{P}(M) \to \mathcal{P}(G)$ where $B^{\downarrow} = \{g \in G : g\ I\ m \text{ for all } m \in B\}$.

A pair $(A, B) \in \mathcal{P}(G) \times \mathcal{P}(M)$ is said to be a *formal concept* if $A^{\uparrow} = B$ and $B^{\downarrow} = A$.

So a formal concept is a pair (A, B) such that A is the set of objects that have all the attributes in B, and B is the set of attributes that are common to all the objects in A.

Theorem 8. The pair $({}^{\uparrow}, {}^{\downarrow})$ is a Galois connection between the boolean algebras $(\mathcal{P}(G), \subseteq)$ and $(\mathcal{P}(M), \subseteq)$ and, therefore, both compositions ${}^{\uparrow\circ\downarrow}$ and ${}^{\downarrow\circ\uparrow}$ are closure operators on $(\mathcal{P}(G), \subseteq)$ and $(\mathcal{P}(M), \subseteq)$, respectively.

Corollary 9. *The formal concepts are the fixed pairs of the Galois connection and, with the order $\leqslant$, defined as*

$$(A_1, B_1) \leqslant (A_2, B_2) \text{ iff } A_1 \subseteq A_2 \text{ (or equivalently, iff } B_2 \subseteq B_1\text{)},$$

forms a complete lattice, which is denoted by $\underline{\mathfrak{B}}(\mathbb{K})$ *and named* concept lattice.

FCA provides an alternative and equivalent way of representing knowledge using attribute implications in addition to the concept lattice. An attribute implication $A \to B$ is a pair of subsets of M, and it holds in the context $\mathbb{K}$ if each object having all the attributes in A also has all the attributes in B.

Definition 10. Let $\mathbb{K} = (G, M, I)$ be a formal context, and $A, B \in \mathcal{P}(M)$. We say that an *attribute implication $A \to B$ holds in* $\mathbb{K}$ if $A^{\downarrow} \subseteq B^{\downarrow}$ or, equivalently, $B \subseteq A^{\downarrow\uparrow}$.

A set of implications Σ is said to be an implicational system for a formal context $\mathbb{K}$ if every implication $A \to B \in \Sigma$ holds in $\mathbb{K}$. We say that an implication $A \to B$ (semantically) follows from a set of implications Σ when, for all formal context $\mathbb{K}$, if Σ is an implicational system for $\mathbb{K}$, then $A \to B$ holds in $\mathbb{K}$. An implicational system for $\mathbb{K}$ is said to be complete if any valid implication for $\mathbb{K}$ follows from Σ.

3. Simplification Logic in a more general framework

Classical attribute implications are pairs of subsets in the attribute power set, which is a boolean algebra. Fuzzy and graded attribute implications are pairs of fuzzy sets, i.e. elements in the L-power set, being L the corresponding structure of membership values (usually some kind of lattice). Here, we present a generalised view of these approaches by considering implications as pairs of elements in a given set L that can be instantiated as a classical power set, a graded power set or the particular lattice that we need to consider. Thus, given a set L, the *language* is defined as follows:

$$\mathcal{L}_L = \{a \to b : \ a, b \in L\}.$$

The expressions $a \to b$ are called *implications* and the elements a and b are called the *premise* and the *conclusion* of the implication, respectively.

The following definition introduces the *semantics*.

Definition 11. An order $\leqslant$ such that $(L, \leqslant)$ is a complete lattice induces the *model theory* as follows: for all $d \in L$, $D \subseteq L$ and $a \to b \in \mathcal{L}_L$, we say that

- d is a model of $a \to b$, denoted by $d \vDash a \to b$, if $a \leqslant d$ implies $b \leqslant d$.
- D is a model of $a \to b$, denoted by $D \vDash a \to b$, if $d \vDash a \to b$ for all $d \in D$.

The set of models of an implication $a \to b \in \mathcal{L}_L$ will be denoted by $\mathscr{M}(a \to b)$. Thus, $D \vDash a \to b$ iff $D \subseteq \mathscr{M}(a \to b)$.

The study of model theory is closely connected to the concepts of closure system and closure operator, which are interchangeable (see Section 2.1).

Proposition 12. *Let $(L, \leqslant)$ be a complete lattice. For each $D \subseteq L$, consider $\mathsf{c}_D : L \to L$ defined as $\mathsf{c}_D(x) = \bigwedge\{d \in D \mid x \leqslant d\}$ for all $x \in L$. Then,*

1. *The operator c_D is a closure operator.*
2. *The closure system induced by c_D is $\{\bigwedge X \mid X \subseteq D\}$.*

In addition, for all $a, b \in L$, $D \vDash a \to b$ if and only if $b \leqslant \mathsf{c}_D(a)$.

Proof. Items 1 and 2 follow from [22, Theorems 19 and 20], but, in order to make the paper self-contained, we provide a direct proof in Appendix A. The last part of the proposition is proved as follows:

Assume that $D \vDash a \to b$. Then, $\{d \in D \mid a \leqslant d\} \subseteq \{d \in D \mid b \leqslant d\}$ because $a \leqslant d$ implies $b \leqslant d$ for all $d \in D$. Therefore, by item 1, we have that $b \leqslant \mathsf{c}_D(b) \leqslant \mathsf{c}_D(a)$.

Conversely, assume that $b \leqslant \mathsf{c}_D(a) = \bigwedge\{d \in D \mid a \leqslant d\}$. Then, by the definition of infimum and transitivity, we have that $a \leqslant d$ implies $b \leqslant d$ for all $d \in D$. That is, $D \vDash a \to b$. $\square$

Previous Definition 11 (Model Theory) and Proposition 12 (characterisation of the closure operator) are the base for the following. Definition 13 introduces the notion of semantic entailment, establishing a connection between an implication and a set of implications by means of its models. Then in Proposition 14, the semantic entailment relationship is characterised by means of the models of the theory. Definition 15 introduces the so-called complete theories and, finally, Proposition 16 relates all these previous notions and results.

Definition 13. Let $(L, \leqslant)$ be a complete lattice, $a \to b \in \mathcal{L}_L$ and $\Sigma \subseteq \mathcal{L}_L$. We say that Σ *semantically entails* $a \to b$, denoted by $\Sigma \vDash a \to b$, if

$$\mathscr{M}(\Sigma) \overset{def}{=} \bigcap_{x \to y \in \Sigma} \mathscr{M}(x \to y) \subseteq \mathscr{M}(a \to b).$$

Proposition 14. *Consider a complete lattice $(L, \leqslant)$ and $\Sigma \subseteq \mathcal{L}_L$. The set $\mathscr{M}(\Sigma) \subseteq L$ is a closure system and, for all $a, b \in L$,*

$$\Sigma \vDash a \to b \quad \text{if and only if} \quad b \leqslant \mathsf{c}_{\mathscr{M}(\Sigma)}(a).$$

Hereinafter, for readability matter, we write c_Σ instead of $\mathsf{c}_{\mathscr{M}(\Sigma)}$.

Proof. It is straightforward that $X \subseteq \mathscr{M}(\Sigma)$ implies $\bigwedge X \in \mathscr{M}(\Sigma)$. Thus, $\mathscr{M}(\Sigma)$ is a closure system.

Assume that $\Sigma \vDash a \to b$ and consider $\mathsf{c}_\Sigma(x) = \bigwedge\{d \in \mathscr{M}(\Sigma) \mid x \leqslant d\}$. Since $\mathscr{M}(\Sigma)$ is a closure system, we have that $a \leqslant \mathsf{c}_\Sigma(a) \in \mathscr{M}(\Sigma)$ and therefore $b \leqslant \mathsf{c}_\Sigma(a)$. Conversely, assume that $b \leqslant \mathsf{c}_\Sigma(a) = \bigwedge\{d \in \mathscr{M}(\Sigma) \mid a \leqslant d\}$. Then, $d \in \mathscr{M}(\Sigma)$ and $a \leqslant d$ implies $b \leqslant c_\Sigma(a) \leqslant d$. That is, $\mathscr{M}(\Sigma) \subseteq \mathscr{M}(a \to b)$. □

Definition 15. Consider a complete lattice $(L, \leqslant)$ and $D \subseteq L$. A set $\Sigma \subseteq \mathcal{L}_L$ is said to be a *complete theory* for D if, for all $a \to b \in \mathcal{L}_L$, we have that

$$\Sigma \vDash a \to b \qquad \text{if and only if} \qquad D \vDash a \to b.$$

Straightforwardly, closures operators introduced in Propositions 12 and 14 agree for complete theories, and vice versa, as the following proposition states.

Proposition 16. *Consider a complete lattice $(L, \leqslant)$, $D \subseteq L$ and $\Sigma \subseteq \mathcal{L}_L$. Then, Σ is a complete theory for D if and only if $\mathsf{c}_\Sigma(x) = \mathsf{c}_D(x)$ for all $x \in L$, or, equivalently, $\mathscr{M}(\Sigma) = \{\bigwedge X \mid X \subseteq D\}$.*

The semantics we have presented here is strongly inspired by the essence of the majority of the generalisations of FCA, specifically in the treatment of attribute implications (see Section 2.2).

Example 1. Classical formal concept analysis is an illustrative example of this framework. It can be described in terms of our framework as follows: Consider a formal context $\mathbb{K} = (G, M, I)$ and the family of sets $\mathcal{D} = \{\{g\}^\uparrow \mid g \in G\} \subseteq \mathcal{P}(M)$. For all $A, B \in \mathcal{P}(M)$, we have

$$\begin{aligned} \mathcal{D} \vDash A \to B & \text{ iff, for all } g \in G, \{g\}^\uparrow \vDash A \to B \\ & \text{ iff, for all } g \in G, A \subseteq \{g\}^\uparrow \text{ implies } B \subseteq \{g\}^\uparrow \\ & \text{ iff, for all } g \in G, g \in A^\downarrow \text{ implies } g \in B^\downarrow \\ & \text{ iff } A^\downarrow \subseteq B^\downarrow \\ & \text{ iff } A \to B \text{ holds in } \mathbb{K}. \end{aligned}$$

In addition, Proposition 12 defines a closure operator in $(\mathcal{P}(M), \subseteq)$, i.e. $\mathsf{c}_\mathcal{D}(X) = \bigcap\{D \in \mathcal{D} \colon X \subseteq D\}$ for each $X \in \mathcal{P}(M)$, which coincides with the closure operator ${}^{\downarrow\circ\uparrow}$; and then their corresponding closure systems also coincide. Therefore, the mapping $(A, B) \mapsto B$ is an isomorphism between the concept lattice $\underline{\mathfrak{B}}(\mathbb{K})$ and the lattice $\big(\{\bigcap X \mid X \subseteq \mathcal{D}\}, \subseteq\big)$.

Finally, a set of implications Σ is a complete implicational system for the formal context $\mathbb{K}$ if and only if it is a complete theory for $\mathcal{D}$, and, therefore, $\mathscr{M}(\Sigma) = \{\bigcap X \mid X \subseteq \mathcal{D}\}$.

4. Axiomatic system

After the definition of the language and the semantics, we introduce the inference engine; i.e. the axiomatic system.

As we mentioned in the introduction, the set difference in the boolean algebra (classical implications) was extended by using complete dual Heyting algebras in the fuzzy case. In this work, we generalise it further in order to extend the treatment of unknown information to graded environments. For this aim, complete dual Heyting algebras, used in [7], are replaced by weak complete dual

Heyting algebras. Using this algebraic framework, in the following, we introduce the inference rules (Definition 17) and the notion of syntactic derivation (Definition 18), and we prove the soundness of the axiomatic system (Theorem 19).

Definition 17. Let $(L, \leq, \smallsetminus)$ be a weak-cdHa. *Simplification Axiomatic System* consists of the rules $\{[\texttt{Inc}], [\texttt{Un}], [\texttt{Simp}]\}$ where [Inc], [Un] and [Simp] are, respectively, defined as follows: for all $a, b, c, d \in L$:

Inclusion: Infer $a \vee b \to a$.
Union: From $a \to b$ and $a \to c$ infer $a \to b \vee c$.
Simplification: From $a \to b$ and $c \to d$ infer $a \vee (c \smallsetminus b) \to d$.

The axiomatic system induces a syntactic derivation mechanism in the usual way:

Definition 18. An implication $a \to b \in \mathcal{L}_L$ is considered to be syntactically derived, or inferred, from $\Sigma \subseteq \mathcal{L}_L$ if there exists a sequence $(x_i \to y_i \mid 1 \leqslant i \leqslant n)$ such that $x_n = a$, $y_n = b$ and, for all $1 \leqslant i \leqslant n$, the implication $x_i \to y_i$ belongs to Σ or is obtained by applying one of the Simplification Axiomatic System rules to implications in $\{x_j \to y_j \mid 1 \leqslant j < i\}$. This situation is denoted by $\Sigma \vdash a \to b$, and the sequence $x_1 \to y_1, \cdots, x_n \to y_n$ is said to be a proof for $\Sigma \vdash a \to b$.

The keystones of an axiomatic system are soundness and completeness, i.e. whether (syntactic) derivation and (semantic) entailment coincide. We will now concentrate on soundness and, leaving completeness for Section 5.

Theorem 19 *(Soundness).* Let $(L, \leq, \smallsetminus)$ be a weak-cdHa. For all implication $a \to b \in \mathcal{L}_L$ and all set $\Sigma \subseteq \mathcal{L}_L$, we have that $\Sigma \vdash a \to b$ implies $\Sigma \vDash a \to b$.

Proof. It is enough to prove the soundness of the three primitive rules introduced in Definition 17.

First, the soundness of the inclusion rule is a straightforward consequence of the fact that $\mathcal{M}(a \vee b \to a) = L$.

Second, for the union rule, if $m \in \mathcal{M}(a \to b) \cap \mathcal{M}(a \to c)$ (i.e. $a \leqslant m$ implies $b \leqslant m$ and $c \leqslant m$) then, obviously, m is model for $a \to b \vee c$.

Finally, we prove the soundness of the simplification rule. Consider $m \in \mathcal{M}(a \to b) \cap \mathcal{M}(c \to d)$ and assume that $a \vee (c \smallsetminus b) \leqslant m$. First, we have that $a \leqslant m$ and, since $m \in \mathcal{M}(a \to b)$, we have that $b \leqslant m$. Therefore, $a \vee b \vee (c \smallsetminus b) \leqslant m$ and, by [w4], $a \vee b \vee c \leqslant m$. Thus, $c \leqslant m$ and, since $m \in \mathcal{M}(c \to d)$, we conclude that $d \leqslant m$. □

4.1. The Simplification paradigm

Notice that, for the proof of the soundness, the only property we need, other than the lattice properties, is [w4]. However, the logics belonging to the Simplification family share a fundamental trait that we name the *Simplification paradigm*: their inference rules can be interpreted as equivalence rules, which enables the simplification of a set of implications while retaining the entirety of the knowledge. To achieve this goal we need all the properties that define weak-cdHas, as we will see in this section.

On the other hand, as we have already commented, the results presented here are a generalisation of those presented in [10]. There we introduced the Simplification logic built on just one particular weak-cdHa and some of the results presented used some particular features of that particular case. Here, we extend these results to any weak-cdHa. Obviously, once it is proved that the axiomatic system is correct in this general framework, any result obtained by syntactic derivation and using properties of (all) weak-cdHas will still hold. Thus, some of the proofs given in [10] can be considered in this framework.

Before providing the equivalence rules we present some inference rules that are derived from the axiomatic system in this framework.

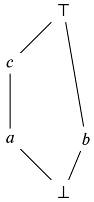
Proposition 20. *From Simplification Axiomatic System, the following inference rules can be derived: for all $a, b, c, d \in L$,*

[Aug] *If $a \leqslant c$ and $d \leqslant c \vee b$, then* $\{a \to b\} \vdash c \to d$.
[Comp] $\{a \to b, c \to d\} \vdash a \vee c \to b \vee d$.
[Tran] $\{a \to b, b \to c\} \vdash a \to c$.
[Frag] $\{a \to b \vee c\} \vdash a \to b$.

These inference rules are named augmentation, composition, fragmentation *and* transitivity, *respectively.*

Proof. The proof for [Aug] is those given in [10, Proposition 14] that is based on the axiomatic system and [w1]. It can also be read in Appendix A, in order to make this paper self-contained. The following sequence proves [Comp]:

(1) $a \to b$ By hypothesis.
(2) $c \to d$ By hypothesis.



(a) $(L, \leqslant)$

$\smallsetminus$	$\bot$	a	b	c	$\top$
$\bot$	$\bot$	$\bot$	$\bot$	$\bot$	$\bot$
a	a	$\bot$	a	$\bot$	$\bot$
b	b	b	$\bot$	b	$\bot$
c	c	c	a	$\bot$	$\bot$
$\top$	$\top$	$\top$	$\top$	$\top$	$\bot$

(b) Difference operator

Fig. 1. A weak-cdHa $(L, \leq, \smallsetminus)$.

(3) $a \vee c \to b$ By applying [Aug] to (1).

(4) $a \vee c \to d$ By applying [Aug] to (2).

(5) $a \vee c \to b \vee d$ By applying [Un] to (3) and (4).

From [w3], we have that [Tran] is a particular case of [Simp]. Finally, the following sequence proves [Frag]:

(1) $a \to b \vee c$ By hypothesis.

(2) $b \vee c \to b$ By [Inc].

(3) $a \to b$ By applying [Tran] to (1) and (2). □

The subsequent proposition offers a set of equivalences that can be utilized for implementing the Simplification paradigm (simplifying the set of implications and preserving equivalency). As usual, two sets Σ_1 and Σ_2 are said to be equivalent, denoted by $\Sigma_1 \equiv \Sigma_2$, if it holds that $\Sigma_1 \vdash x \to y$ implies $\Sigma_2 \vdash x \to y$ and vice versa. Furthermore, this is true if $\Sigma_1 \vdash x \to y$ for all $x \to y \in \Sigma_2$ and $\Sigma_2 \vdash x \to y$ for all $x \to y \in \Sigma_1$.

Proposition 21. *The following equivalence rules hold for any* $a, b, c, d \in L$*:*

[FragEq]: $\{a \to b\} \equiv \{a \to b \smallsetminus a\}$.
[UnEq]: $\{a \to b,\ a \to c\} \equiv \{a \to b \vee c\}$.
[⊥-Eq]: $\{a \to \bot\} \equiv \varnothing$.
[SimpEq]: $\{a \to b,\ c \to d\} \equiv \{a \to b,\ c \smallsetminus b \to d \smallsetminus b\}$ *when* $a \leqslant c \smallsetminus b$.

The proof for this proposition follows the same scheme as the proof of [10, Proposition 15], which is based on the axiomatic system and the properties [w2], [w3] and [w4]. To make the paper self-contained, we include the proof of Proposition 21 in Appendix A.

The Simplification paradigm is behind the previous proposition because if you read each equivalence from left to right, it enables to reduce the number of implications or to substitute the premise/conclusion by a lower element in the complete lattice. Notice that if the general case is instantiated and associated with a (generalised) power set, the number of attributes in the premise/conclusion is reduced.

Example 2. Let $(L, \leq, \smallsetminus)$ be the weak-cdHa where $(L, \leq)$ is the lattice given by Fig. 1a and $\smallsetminus$ is the difference operator shown in the table in Fig. 1b. Then, the set $\{c \to a, a \to b, a \to c\}$ can be simplified, by applying the equivalence rules, as follows:

1. By using [FragEq] and [⊥-Eq] we have that $\{c \to a\} \equiv \{c \to \bot\} \equiv \varnothing$.
2. By using [UnEq] we have that $\{a \to b, a \to c\} \equiv \{a \to \top\}$.

Thus, we have that $\{c \to a, a \to b, a \to c\} \equiv \{a \to \top\}$.

5. Completeness of the axiomatic system

To ensure completeness, we must first address a preliminary question. The semantic component relies on the complete lattice structure, which guarantees the existence of arbitrary suprema and infima. In contrast, the syntactic component involves only binary operations and finite sequences of implications (i.e., proofs). In order to make the results as general as possible, we now introduce some conditions that will allow us to overcome these difficulties for infinite lattices. Our approach is based on the well-known definition of compactness.

Definition 22. Consider a weak-cdHa $(L, \leqslant, \smallsetminus)$. An element $k \in L$ is called *compact* if, whenever $J \subseteq L$ satisfies $k \leqslant \bigvee J$, there exists a finite subset $F \subseteq J$ such that $k \leqslant \bigvee F$. We denote by K the set of all compact elements in L.

We say that $(L, \leqslant, \smallsetminus)$ is *algebraic* if it satisfies the following two conditions:

1. For every $a \in L$, there exists a subset $X \subseteq K$ such that $a = \bigvee X$.
2. For all $a, b \in K$, the element $a \smallsetminus b$ belongs to K as well.

Note that, apart from what was previously stated, as it is well-known, K is closed for finite suprema, that is, if $x, y \in K$ then $x \vee y \in K$.

In the following we will assume that $(L, \leqslant, \smallsetminus)$ is an algebraic weak-cdHa. We will also restrict the language to $\mathcal{L}_K = \{a \to b \mid a, b \in K\}$. Note that this restriction does not affect the semantics, and moreover, since K is closed for both operations $\vee$ and $\smallsetminus$, any implication inferred from implications in $\mathcal{L}_K$ also belongs to $\mathcal{L}_K$. Recall that $K = L$ when L is finite.

Now, we introduce some preliminary notions and results that allow us to make the proof of completeness.

For each set of implications $\Sigma \subseteq \mathcal{L}_K$ and each $a \in L$, we define the set

$$\mathcal{Y}_\Sigma(a) = \{y \in K \mid \Sigma \vdash x \to y \text{ for some } x \in K \text{ with } x \leqslant a\} \tag{5}$$

and, by using it, we define the mapping

$$[\ ]_\Sigma : L \to L \quad \text{where} \quad [a]_\Sigma = \bigvee \mathcal{Y}_\Sigma(a) \quad \text{for all } a \in L. \tag{6}$$

As a consequence of Theorem 26, which we will prove below, in the general case, the mapping is recognised as a closure operator with respect to Σ, and we have given it the name of *syntactic closure*. Previously, Proposition 24 below states its extensiveness and Theorem 25 proves the completeness of Simplification Logic in the compact framework.

Lemma 23. *Let $\Sigma \subseteq \mathcal{L}_K$, $b \to c \in \mathcal{L}_K$ and $a \in L$. If $\Sigma \vdash b \to c$ and $b \leqslant [a]_\Sigma$, then $c \in \mathcal{Y}_\Sigma(a)$.*

Proof. Since $b \in K$ and $b \leqslant \bigvee \mathcal{Y}_\Sigma(a)$, there exists finite subset $\{y_1, \dots, y_n\} \subseteq \mathcal{Y}_\Sigma(a) \subseteq K$ such that $b \leqslant \bigvee_{i=1}^n y_i$ and, for each $1 \leqslant i \leqslant n$, there exists $x_i \in K$ with $x_i \leqslant a$ and $\Sigma \vdash x_i \to y_i$. Thus, by applying [Comp] n times, we obtain that $\Sigma \vdash \bigvee_{i=1}^n x_i \to \bigvee_{i=1}^n y_i$. In addition, by [Aug], we have that $\Sigma \vdash \bigvee_{i=1}^n x_i \to b$ and, by [Tran] with $\Sigma \vdash b \to c$, we have that $\Sigma \vdash \bigvee_{i=1}^n x_i \to c$. Finally, since $\bigvee_{i=1}^n x_i \leqslant a$, we conclude that $c \in \mathcal{Y}_\Sigma(a)$. □

The previous lemma yields a useful immediate consequence, which is that

$$\Sigma \vdash b \to c \text{ implies } [a]_\Sigma \in \mathcal{M}(b \to c) \text{ for all } a \in L. \tag{7}$$

Proposition 24. *Let $\Sigma \subseteq \mathcal{L}_K$. The mapping $[\ ]_\Sigma : L \to L$ is extensive. Furthermore,*

$$\Sigma \vdash b \to c \quad \textit{if and only if} \quad c \leqslant [b]_\Sigma.$$

Proof. Since $(L, \leqslant, \smallsetminus)$ is algebraic, for any $a \in L$ there is a subset $X \subseteq K$ such that $a = \bigvee X$. On the other hand, for all $x \in X$, we have that $x \leqslant a$ and, by [Inc], $\Sigma \vdash x \to x$. Therefore, $X \subseteq \mathcal{Y}_\Sigma(a)$ and, thus, $a = \bigvee X \leqslant \bigvee \mathcal{Y}_\Sigma(a) = [a]_\Sigma$, i.e. the operator $[\ ]_\Sigma$ is extensive.

Assume now that $\Sigma \vdash b \to c$. By extensiveness, $b \leqslant [b]_\Sigma$ and, by Lemma 23, we conclude that $c \leq [b]_\Sigma$. Conversely, if $c \leq [b]_\Sigma = \bigvee \mathcal{Y}_\Sigma(b)$, since $c \in K$, there exists a finite subset $\{y_1, \dots, y_n\} \subseteq \mathcal{Y}_\Sigma(b) \subseteq K$ such that $c \leqslant \bigvee_{i=1}^n y_i$ and, for each $1 \leqslant i \leqslant n$, there exists $x_i \in K$ with $x_i \leqslant b$ and $\Sigma \vdash x_i \to y_i$. Thus, by applying [Comp] n times, we obtain that $\Sigma \vdash \bigvee_{i=1}^n x_i \to \bigvee_{i=1}^n y_i$. Finally, by [Aug], we have that $\Sigma \vdash b \to c$. □

Theorem 25 *(Completeness).* Let $\Sigma \subseteq \mathcal{L}_K$ and $b \to c \in \mathcal{L}_K$. If $\Sigma \vDash b \to c$ then $\Sigma \vdash b \to c$.

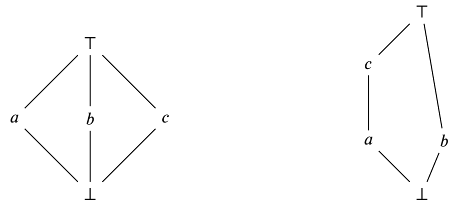
Proof. First, $[b]_\Sigma \in \mathcal{M}(\Sigma)$ because, for all $x \to y \in \Sigma$, we have that $\Sigma \vdash x \to y$ and, by (7), we conclude that $[b]_\Sigma \in \mathcal{M}(x \to y)$. Second, from $\Sigma \vDash b \to c$, we have that $[b]_\Sigma \in \mathcal{M}(b \to c)$. Finally, since $b \leqslant [b]_\Sigma$, we conclude that $c \leqslant [b]_\Sigma$ and, by Proposition 24, $\Sigma \vdash b \to c$. □

Combining the previous theorem with Theorem 19, we conclude that the fragment $\mathcal{L}_K$ is both sound and complete within its axiomatic system. That is, for all $\Sigma \subseteq \mathcal{L}_K$ and $a \to b \in \mathcal{L}_K$,

$$\Sigma \vDash a \to b \quad \text{if and only if} \quad \Sigma \vdash a \to b.$$

Note that in some cases $L = K$; for example, when L is finite, which is very common in practice. In any case, the following theorem extends the good connection between the semantic and syntactic facets to all the elements in L.

Theorem 26. Let $\Sigma \subseteq \mathcal{L}_K$. For all $a \in L$, one has $\mathrm{c}_\Sigma(a) = [a]_\Sigma$.



(a) $(L,\leqslant_1)$ (b) $(L,\leqslant_2)$

Fig. 2. Diamond and Pentagon lattices.

Proof. First, we prove that $c_\Sigma([a]_\Sigma) = c_\Sigma(a)$. On the one hand, by extensiveness of $[\ ]_\Sigma$ and isotonicity of c_Σ, we have that $c_\Sigma(a) \leqslant c_\Sigma([a]_\Sigma)$. On the other hand, if $d \in \mathscr{M}(\Sigma)$ and $a \leqslant d$, then, by Theorem 19, for all $x, y \in K$ such that $x \leqslant a$ and $\Sigma \vdash x \to y$, we have that $y \leqslant d$. Therefore, d is an upper bound of $\mathcal{Y}_\Sigma(a)$ and $[a]_\Sigma \leqslant d$. Thus, $\{d \in \mathscr{M}(\Sigma) \mid a \leqslant d\} \subseteq \{d \in \mathscr{M}(\Sigma) \mid [a]_\Sigma \leqslant d\}$ and, then, $c_\Sigma([a]_\Sigma) \leqslant c_\Sigma(a)$.

Finally, we prove $c_\Sigma([a]_\Sigma) = [a]_\Sigma$. On the one hand, since c_Σ is extensive, we have that $[a]_\Sigma \leqslant c_\Sigma([a]_\Sigma)$. On the other hand, by (7) and extensiveness of $[\ ]_\Sigma$, we have that $[a]_\Sigma \in \mathscr{M}(\Sigma)$ and $a \leqslant [a]_\Sigma$. Therefore, $c_\Sigma([a]_\Sigma) = \bigwedge\{d \in \mathscr{M}(\Sigma) \mid a \leqslant d\} \leqslant [a]_\Sigma$. □

The function $[\ ]_\Sigma$ being a closure operator is a direct outcome of the preceding theorem, as previously noted.

6. Characterisation of the weak complete dual Heyting algebras

In the previous sections, we have presented the semantics based on an arbitrary complete lattice. Then we have introduced the axiomatic system after endowing the complete lattice with a new operation, called difference, which provides a weak-cdHa structure. Finally, we have also shown that, in order to guarantee the completeness of the axiomatic system, we need the weak-cdHa to be algebraic. The aim of this section is to find, given a complete lattice $(L,\leqslant)$, necessary and sufficient conditions that must be satisfied to establish an operation $\smallsetminus$ that allows $(L,\leqslant,\smallsetminus)$ to be a weak-cdHa. We will also study the uniqueness or not of such an operation in search of the algebraicity condition.

As mentioned in Section 2, every cdHa is a weak-cdHa. So, (2) is a sufficient condition, but not a necessary one. The following example shows two lattices in which Condition (2) is not satisfied and in which, not only can we define an operation that converts them to weak-cdHa, but there is more than one.

Example 3. Consider the complete lattices depicted in Fig. 2a and Fig. 2b, denoted by $(L,\leqslant_1)$ and $(L,\leqslant_2)$, and known as Diamond and Pentagon lattices, respectively.

Both lattices do not satisfy Condition (2) because, in both,

$$\min\{x \mid c \leqslant a \vee x\} = \min\{b, c, \top\}$$

do not exist. However, over $(L,\leqslant_1)$, not only can we define one weak-cdHa but we can define, at least, two different weak-cdHas: $(L,\leqslant_1,\smallsetminus_1)$ and $(L,\leqslant_1,\smallsetminus_2)$ where

$\smallsetminus_1$	$\bot$	a	b	c	$\top$
$\bot$	$\bot$	$\bot$	$\bot$	$\bot$	$\bot$
a	a	$\bot$	a	a	$\bot$
b	b	b	$\bot$	b	$\bot$
c	c	c	c	$\bot$	$\bot$
$\top$	$\top$	$\top$	$\top$	$\top$	$\bot$

$\smallsetminus_2$	$\bot$	a	b	c	$\top$
$\bot$	$\bot$	$\bot$	$\bot$	$\bot$	$\bot$
a	a	$\bot$	a	a	$\bot$
b	b	b	$\bot$	b	$\bot$
c	c	c	c	$\bot$	$\bot$
$\top$	$\top$	b	c	b	$\bot$

Analogously, $(L,\leqslant_2,\smallsetminus_3)$ and $(L,\leqslant_2,\smallsetminus_4)$ are also two different weak-cdHas over the same lattice $(L,\leqslant_2)$, where

$\smallsetminus_3$	$\bot$	a	b	c	$\top$
$\bot$	$\bot$	$\bot$	$\bot$	$\bot$	$\bot$
a	a	$\bot$	a	$\bot$	$\bot$
b	b	b	$\bot$	b	$\bot$
c	c	c	c	$\bot$	$\bot$
$\top$	$\top$	$\top$	$\top$	$\top$	$\bot$

$\smallsetminus_4$	$\bot$	a	b	c	$\top$
$\bot$	$\bot$	$\bot$	$\bot$	$\bot$	$\bot$
a	a	$\bot$	a	$\bot$	$\bot$
b	b	b	$\bot$	b	$\bot$
c	c	c	a	$\bot$	$\bot$
$\top$	$\top$	$\top$	$\top$	$\top$	$\bot$

Furthermore, Remark 1 establishes that a necessary condition for $(L,\leqslant,\smallsetminus)$ being a cdHa is that $(L,\leqslant)$ need to be a distributive lattice. In addition, infinite distributivity is a sufficient condition to build a cdHa, which is also a weak-cdHa. However, it is not a necessary condition for being a weak-cdHa, as previous Example 3 shows.

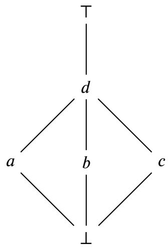


Fig. 3. The Diamond lattice with one extra vertex.

We have shown examples where, over the same lattice, different operations define different weak-cdHas, contrary to the situation with cdHas. Our current inquiry is focused on the possibility of defining an operation $\smallsetminus : L \times L \to L$ for all lattices $(L, \leqslant)$ to create a weak-cdHa structure $(L, \leqslant, \smallsetminus)$. However, our following counterexample states that this is not true.

Example 4. Consider the complete lattice $(L, \leqslant)$ depicted in Fig. 3. Assume that there exists $\smallsetminus : L \times L \to L$ such that $(L, \leqslant, \smallsetminus)$ is a weak-cdHa.

First, by [w4], we have that $d = d \vee a = a \vee (d \smallsetminus a)$ and, therefore,

$$d \smallsetminus a \in \{c, b, d\}. \tag{8}$$

Second, by [w1], we have that $d \smallsetminus a = (a \vee b) \smallsetminus a \leqslant b$, $d \smallsetminus a = (a \vee c) \smallsetminus a \leqslant c$, and $d \smallsetminus a = (a \vee d) \smallsetminus a \leqslant d$. As a consequence $d \smallsetminus a \leqslant b \wedge c \wedge d = \bot$, which contradicts (8).

Notice that, as expected, the lattice of the previous example is not distributive.

In summary, we have seen that there are lattices in which we can define more than one weak-cdHas and there are other lattices in which we can not define any. Now, the following theorem of characterisation of weak-cdHas will later provide us, sufficient and necessary conditions for the existence and uniqueness issues (see Corollary 28, Corollary 29 and Theorem 30, respectively.)

Theorem 27 *(Characterisation).* Consider a complete lattice $(L, \leqslant)$ and a difference operation $\smallsetminus : L \times L \to L$. Then, $(L, \leqslant, \smallsetminus)$ is a weak-cdHa if and only if the following conditions are satisfied:

$$x \smallsetminus y = \min\{z \in L \mid z \vee y = x \vee y\} \text{ for all } x, y \in L \text{ with } x \nparallel y \text{ and } x \neq \top. \tag{9}$$

$$\top \smallsetminus \top = \bot. \tag{10}$$

$$\top \smallsetminus y \in \{z \in L \mid z \vee y = \top\} \text{ for all } y \in L \text{ with } y \neq \top. \tag{11}$$

$$x \smallsetminus y \in \{z \in L \mid z \leqslant x \text{ and } z \vee y = x \vee y\} \text{ for all } x, y \in L \text{ with } x \parallel y. \tag{12}$$

Proof. First, let's suppose that $(L, \leqslant, \smallsetminus)$ is a weak-cdHa and we prove that it holds the four assertions:

To prove (9), consider $x, y \in L$ such that x and y are comparable, being $x \neq \top$, and distinguish two cases:

- If $x \vee y = y$, by [w3], one has that $x \smallsetminus y = \bot = \min\{z \in L \mid z \vee y = x \vee y\}$.
- In a different situation, i.e. if $x \vee y = x \neq \top$, then $x \smallsetminus y \in \{z \in L \mid z \vee y = x \vee y\}$ by [w4]. In addition, for all $z \in \{z \in L \mid z \vee y = x \vee y\}$, one has that $z \vee y = x \neq \top$ and, by [w1], $x \smallsetminus y \leqslant z$. Consequently, $x \smallsetminus y = \min\{z \in L \mid z \vee y = x \vee y\}$.

(10) is straightforward from [w3].

(11) is equal to $(\top \smallsetminus y) \vee y = \top$ for all $y \in L$ with $y \neq \top$, which is a particular case of [w4].

Finally, to prove (12) assume that $x, y \in L$ being x not comparable with y. By [w2] and [w4], one has that $x \smallsetminus y \in \{z \in L \mid z \leqslant x \text{ and } z \vee y = x \vee y\}$.

Conversely, consider a complete lattice $(L, \leqslant)$ and a difference operation $\smallsetminus : L \times L \to L$ fulfilling (9)–(12), and let us prove that $(L, \leqslant, \smallsetminus)$ is a weak-cdHa, i.e. [w1], [w2], [w3] and [w4] hold.

Let $x, y \in L$ with $x \vee y \neq \top$. By (9), $(x \vee y) \smallsetminus y = \min\{z \in L \mid z \vee y = x \vee y\}$. Therefore, $(x \vee y) \smallsetminus y \leqslant z$ for all $z \in \{z \in L \mid z \vee y = x \vee y\}$ and, in particular, $(x \vee y) \smallsetminus y \leqslant x$, i.e. [w1] holds.

In all the cases, (9)–(12), it is straightforward that [w2] holds. Notice that (9)–(12) exhaustively describe all the situations for $x \smallsetminus y$, depicting a classification in four disjoints cases.

Let's prove [w3]. On the one hand, assume that $x \leqslant y$. If $x = \top$, by (10), $\top \smallsetminus y = \top \smallsetminus \top = \bot$. In other case, $x \smallsetminus y = \min\{z \in L \mid z \vee y = x \vee y\}$, by (9), and this minimum element is $\bot$ because $\bot \vee y = y = x \vee y$. Thus, $x \leqslant y$ implies $x \smallsetminus y = \bot$.

On the other hand, assume $x \nleqslant y$ and prove $x \smallsetminus y \neq \bot$. If $y < x \neq \top$, by (9), $x \smallsetminus y = \min\{z \in L \mid z \vee y = x \vee y\}$, which is not $\bot$ because $\bot \vee y = y \neq x \vee y = x$. Analogously, by (11), it is proved that $y < x = \top$ implies $x \smallsetminus y \neq \bot$. Finally, if x is not comparable with y, by (12), $x \smallsetminus y \in \{z \in L \mid z \leqslant x \text{ and } z \vee y = x \vee y\}$ and, therefore, $x \smallsetminus y \neq \bot$ because $\bot \vee y \neq x \vee y$.

Finally, it is straightforward that [w4] holds in all the cases (9)–(12). □

Given a complete lattice, since the sets $\{z \in L \mid z \leqslant x \text{ and } z \vee y = x \vee y\}$ and $\{z \in L \mid z \vee x = \top\}$ are always non empty, we can always define a $\smallsetminus$ operation holding (10)–(12). Thus, we focus on (9); i.e., on the existence of $\min\{z \in L \mid z \vee y = x \vee y\}$ for all $x, y \in L$ with x comparable with y and being $x \neq \top$. In addition, if $x \leqslant y$, that minimum always exists and it is $\bot$. Otherwise, if x is $\vee$-irreducible, $\{z \in L \mid z \vee y = x \vee y\} = \{x\}$ and the minimum also exists. In summary, there exists just one situation where (9) are not guaranteed: $x > y$ and x is a $\vee$-reducible element. The following corollary presents this situation that we have justified above:

Corollary 28. *Consider a complete lattice $(L, \leqslant)$. There exists a difference operation $\smallsetminus : L \times L \to L$ such that $(L, \leqslant, \smallsetminus)$ is a weak-cdHa if and only if the following holds:*

$$\min\{z \in L \mid z \vee y = x\} \text{ exists for all } \vee\text{-reducible } x \neq \top \text{ and all } y < x \tag{13}$$

Once we have characterised the existence, we focus on the uniqueness issue. Recall that in Example 3 we show two lattices where uniqueness does not fulfil. In the first one, $\{z \in L \mid z \vee a = \top\}$ is not a singleton (see (11)) whereas in the second one, $\{z \in L \mid z \leqslant c \text{ and } z \vee b = c \vee b\}$ is either not a singleton (see (12)).

Since (9)–(12) exhaustively describe all the situations for $x \smallsetminus y$, depicting a classification in four disjoints cases, we have the following corollary from Theorem 27.

Corollary 29. *Consider a complete lattice $(L, \leqslant)$ such that there exists a difference operation $\smallsetminus : L \times L \to L$ satisfying that $(L, \leqslant, \smallsetminus)$ is a weak-cdHa. Then, this operation $\smallsetminus$ is the unique one with $(L, \leqslant, \smallsetminus)$ being a weak-cdHa if and only if the following properties hold:*

$$\{z \in L \mid z \vee y = \top\} = \{\top\} \text{ for all } y \in L \text{ with } y \neq \top. \tag{14}$$

$$\{z \in L \mid z \leqslant x \text{ and } z \vee y = x \vee y\} = \{x\} \text{ for all } x, y \in L \text{ with } x \parallel y. \tag{15}$$

The subsequent theorem provides a criterion that establishes both the necessary and sufficient conditions for the uniqueness of the weak-cdHas.

Theorem 30 *(Uniqueness)*. Consider a complete lattice $(L, \leqslant)$ satisfying (13). There is just one difference operation $\smallsetminus : L \times L \to L$ being $(L, \leqslant, \smallsetminus)$ a weak-cdHa if and only if the following conditions hold:

$$\top \text{ is } \vee\text{-irreducible.} \tag{16}$$

$$\text{For all } x, y, z \in L,\ x \parallel y \parallel z \text{ and } z \vee y = x \vee y \text{ implies } z = x. \tag{17}$$

Proof. From the $\vee$-irreducibleness definition, (16) and (14) are equivalent. It is also straightforward that (17) implies (15). We conclude the proof showing that (13), (14) (or, equivalently, (16)) and (15) imply (17).

Let $x, y, z \in L$ with $x \parallel y \parallel z$ and $z \vee y = x \vee y$. Consider $w = x \vee y$, which is $\vee$-reducible (because, the opposite contradicts that $x \parallel y$) and, therefore, by (16), we have that $w \neq \top$ and $y < w$. In addition, by (13) we have that there exists $v = \min\{t \mid t \vee y = w\}$. Thus, $v \leqslant x$ and $v \leqslant z$ because $x, z \in \{t \mid t \vee y = w\}$. Furthermore, $v \vee y = w = x \vee y$. From (15), we have that $v = x$ and, as consequence, $x \leqslant z$.

Repeating each step of the previous paragraph, but swapping the roles of x and z, we have that $z \leqslant x$, concluding that $z = x$. □

Notice that (17) can be replaced by the following equivalent condition: for all $\vee$-reducible element $x \in L$ and for any $y \in L$ we have that $\{z \in L \mid z \vee y = x\}$ contains, at most, two elements.

7. Conclusions and further works

The focus of this paper is to develop a very general framework for dealing with unknown information. First, we provide a more general language that can be instantiated, for example, as graded information, positive, negative and unknown information, etc. Then we consider the structure of complete lattice as the layout for building the semantics. Later, a key point in the simplification paradigm is defined: the difference operation on the complete lattice. This operation must satisfy certain properties in order to achieve the soundness of Simplification logic and to provide a foundation for the corresponding inference engine. The paper places special emphasis on the weak complete dual Heyting algebra (weak-cdHa), which is a weakening of the properties of the complete dual Heyting algebra in an adequate way to preserve the necessary properties. Requiring the underlying lattice to be algebraic, we can prove that the corresponding Simplification logic is complete.

Finally, we have characterised the weak-cdHas in order to distinguish the lattices in which we can define this structure from those in which we cannot. In addition, we also characterise the properties of the difference operation to build such a structure in a consistent way, and provide the conditions that ensure the unicity of the difference operation.

As further work, first, in the formal framework, it is relevant to study which weak-cdHas are algebraic and which are not. In addition, from the point of view of the applicability of this new framework, we propose to apply it to different purposes. In particular, we propose to extend the work done in [10] to the fuzzy FCA framework; in this line, one of the points to consider is [23]. Our approach is to incorporate the unknown information in the formal context by considering pairs of degrees in the following way: the first one will be the degree to which we know that an object has the attribute, while the second one will be the degree to which we know that an object does not have the attribute. Thus, this paper is the needed bridge to move from the particular case presented in [10] to a more general framework. In particular, we are interested to consider the multi-adjoint framework which is now a very popular issue [24,25]. We also plan to study the relationship with other related approaches [26]. Another interesting topic is the study of the impact of lattice distributivity [27,28] on the Simplification Logic.

CRediT authorship contribution statement

Pablo Cordero: Writing – review & editing, Writing – original draft, Validation, Supervision, Formal analysis, Conceptualization. **Manuel Enciso:** Writing – review & editing, Writing – original draft, Supervision, Formal analysis, Conceptualization. **Ángel Mora:** Writing – review & editing, Writing – original draft, Supervision, Formal analysis, Conceptualization. **Francisco Pérez-Gámez:** Writing – review & editing, Writing – original draft, Supervision, Formal analysis, Conceptualization.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Data availability

No data was used for the research described in the article.

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Appendix A. Some proofs

Proof for item 1 in Proposition 12. First, $\mathtt{c}_D$ is inflationary because $x \leqslant \mathtt{c}_D(x) = \bigwedge\{d \in D \mid x \leqslant d\}$ for all $x \in L$. The operator $\mathtt{c}_D$ is also isotone because, for all $x, y \in L$, $x \leqslant y$ implies $\{d \in D \mid y \leqslant d\} \subseteq \{d \in D \mid x \leqslant d\}$, and therefore we have that $\mathtt{c}_D(x) = \bigwedge\{d \in D \mid x \leqslant d\} \leqslant \bigwedge\{d \in D \mid y \leqslant d\} = \mathtt{c}_D(y)$. Finally, $\mathtt{c}_D$ is idempotent because, since it is inflationary and isotone, we have that $\mathtt{c}_D(x) \leqslant \mathtt{c}_D(\mathtt{c}_D(x))$, and, since $\{d \in D \mid x \leqslant d\} \subseteq \{d \in D \mid \mathtt{c}_D(x) \leqslant d\}$, we also have that $\mathtt{c}_D(\mathtt{c}_D(x)) = \bigwedge\{d \in D \mid \mathtt{c}_D(x) \leqslant d\} \leqslant \bigwedge\{d \in D \mid x \leqslant d\} = \mathtt{c}_D(x)$. So, $\mathtt{c}_D$ is a closure operator. □

Proof for item 2 in Proposition 12. The closure operator $\mathtt{c}_D$ induces the closure system $\mathtt{c}_D(L) = \{x \in L \mid \mathtt{c}_D(x) = x\}$. It is straightforward that $\mathtt{c}_D(z) \in \{\bigwedge X \mid X \subseteq D\}$ for all $z \in L$. Conversely, let $z = \bigwedge X$ for some $X \subseteq D$, then $X \subseteq \{x \in D \mid z \leqslant x\}$ and $\mathtt{c}_D(z) \leqslant \bigwedge X = z$. In addition, since $\mathtt{c}_D$ is inflationary, we conclude that $\mathtt{c}_D(z) = z$ and $z \in c_D(L)$. □

Proof of the derivability of `[Aug]` in Proposition 20. The following sequence is a proof for `[Aug]`:

$\varphi_1 = a \to b$ By hypothesis.

$\varphi_2 = c \vee b \to d$ By `[Inc]`.

$\varphi_3 = a \vee ((c \vee b) \smallsetminus b) \to d$ By using `[Simp]` to φ_1 and φ_2.

$\varphi_4 = c \to a \vee ((c \vee b) \smallsetminus b)$ By `[Inc]` and `[w1]`.

$\varphi_5 = c \to d$ By using `[Simp]` to φ_4 and φ_3. □

Proof of Proposition 21. First, a proof for $a \to b \vdash a \to b \smallsetminus a$ is the following sequence:

$\varphi_1 = a \to b$ By hypothesis.

$\varphi_2 = b \to b \smallsetminus a$ By `[w2]` and `[Inc]`.

$\varphi_3 = a \to b \smallsetminus a$ Applying `[Simp]` to φ_1 and φ_2 using `[w3]`.

The opposite direction can be proved to apply [Aug] to $a \to b \smallsetminus a$ (which is the hypothesis) and using [w4].

Second, to prove that from $\{a \to b,\ a \to c\}$ we can derive $\{a \to b \vee c\}$ we use [Un] to both hypothesis. The opposite direction is straightforward from [Aug].

[⊥-Eq] is due to [FragEq] and [Inc].

Finally, the following sequence proves that from $\{a \to b,\ c \to d\}$ we can derive $\{a \to b,\ c \smallsetminus b \to d \smallsetminus b\}$ when $a \leq c \smallsetminus b$. We start proving that from $a \to b$ and $c \to d$ we derive $a \to b$ and $c \smallsetminus b \to d \smallsetminus b$ if $a \leq c \smallsetminus b$:

$\varphi_1 = a \to b$ By hypothesis.

$\varphi_2 = c \smallsetminus b \to c$ By using [Aug], $a \leq c \smallsetminus b$ and $c \leq c \vee (c \smallsetminus b)$.

$\varphi_3 = c \to d$ By hypothesis.

$\varphi_4 = c \smallsetminus b \to d$ By using [Simp] to φ_2 and φ_3.

$\varphi_5 = c \smallsetminus b \to d \smallsetminus b$ By using [Aug] to φ_4 and [w2].

To prove the opposite direction, we use the following sequence.

$\varphi_1 = a \to b$ By hypothesis.

$\varphi_2 = c \smallsetminus b \to d \smallsetminus b$ By hypothesis.

$\varphi_3 = c \smallsetminus b \to d \vee b$ Applying [Un] to φ_1 and φ_2.

$\varphi_4 = c \to d$ By using [Aug] to φ_3 and [w2].

Notice that in φ_3 we use $a \leq c \smallsetminus b$ and $(d \smallsetminus b) \vee b = d \vee b$ by [w4]. □

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