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The adhesive contact problem for a piecewise-homogeneous orthotropic plate with an elastic patch

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Abstract
A piecewise-homogeneous elastic orthotropic plate, reinforced with a finite patch of the wedge-shaped, which meets the interface at a right angle and is loaded with tangential and normal forces is considered. Using methods of the theory of analytic functions, the problem is reduced to the system of singular integro-differential equations (SIDE) with fixed singularity. Under tension-compression of patch using an integral transformation a Riemann problem is obtained, the solution of which is presented in explicit form. The tangential contact stresses along the contact line are determined and their asymptotic behavior in the neighborhood of singular points is established.

Keywords
Contact problem, orthotropic plate, elastic inclusion, integro-differential equation, integral transformation, Riemann problem, asymptotic estimates

2010 Mathematics Subject Classification: 74B05, 74K20, 74K15

1. Introduction

The solutions of static contact problems for different domains, reinforced with elastic thin inclusions and patches of variable stiffness and the behavior of the contact stresses at the ends of the contact line, have been investigated as a function of the law of variation of the geometrical and physical parameters

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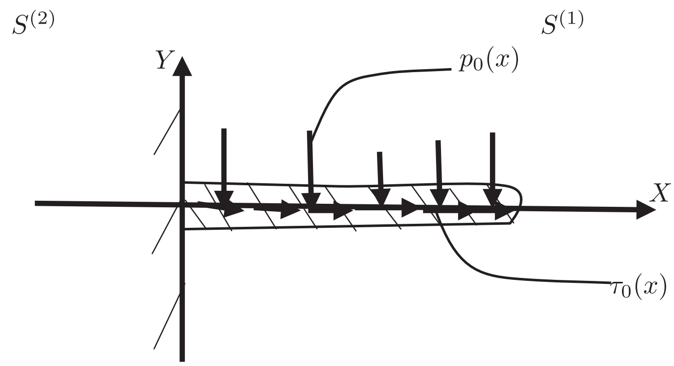

Figure 1. Problem statement. Graphical sketch.

of these thin-walled elements [1–13]. The first fundamental problem for a piecewise-homogeneous plane, when a crack of finite length arrives at the interface of two bodies at the right angle, was solved in Khrapkov [14]; a similar problem for a piecewise-homogeneous plane when acted upon by symmetrical normal stresses at the crack sides was solved in Bantsuri [15] and Ungiadze [16], as well as the contact problems for a piecewise-homogeneous plane with a semi-infinite and finite inclusion were solved in Bantsuri and Shavlakadze [17], Shavlakadze et al. [18], and Shavlakadze et al. [19].[AQ: 2]

2. Problem statement and its reduction to the system of SIDE

It is considered a piecewise-homogeneous orthotropic plate in the condition of plane deformation, which consists of two half-planes of dissimilar materials and reinforced with a finite or half infinite patch (inclusion) with modulus of elasticity $E_1(x)$, thickness $h_1(x)$, and Poisson's coefficient ν_1. It is assumed that the horizontal and vertical stresses with intensity $\tau_0(x)$ and $p_0(x)$ act on the patch along the OX-axis (the functions $\tau_0(x)$ and $p_0(x)$ are bounded functions on the finite interval) (Figure 1).

The patch in the vertical direction bends like a beam (it has a finite bending stiffness) and also in the horizontal direction the patch is compressed or stretched like a rod being in uniaxial stress state.

The contact between the plate and patch is performed by a thin glue layer with width h_0 and Lame's constants λ_0, μ_0. The contact conditions for the sandwich components have the form [20]

$$u_1(x) - u^{(1)}(x,0) = k_0\tau(x), \qquad v_1(x) - v^{(1)}(x,0) = m_0 p(x), \qquad 0<x<1 \tag{1}$$

where $u^{(1)}(x,y)$, $v^{(1)}(x,y)$ are displacement components of the plate points and $u_1(x)$, $v_1(x)$ displacements of the patch points along the OX-axis:

$$k_0 := h_0/\mu_0, \qquad m_0 := h_0/(\lambda_0 + 2\mu_0)$$

We have to define the law of distribution of tangential and normal contact stresses $\tau(x)$ and $p(x)$ on the contact line and the asymptotic behavior of these stresses at the ends of the patches.

According to the equilibrium equation of patch element and Hooke's law, one obtains:

$$\begin{gathered} \frac{du_1(x)}{dx} = \frac{1}{E(x)}\int_0^x [\tau(t) - \tau_0(t)]\,dt, \\ \frac{d^2}{dx^2} D(x)\frac{d^2 v_1(x)}{dx^2} = p_0(x) - p(x), \qquad 0<x<1 \end{gathered} \tag{2}$$

and the equilibrium equation of the patch has the form

$$\int_0^1 [\tau(t) - \tau_0(t)]\,dt = 0, \qquad \int_0^1 [p(t) - p_0(t)]\,dt = 0, \qquad \int_0^1 t[p(t) - p_0(t)]\,dt = 0,$$

where

$$E(x) = \frac{E_1(x)h_1(x)}{1 - \nu_1^2}, \qquad D(x) = \frac{E_1(x)h_1^3(x)}{1 - \nu_1^2}.$$

Suppose an elastic body S occupies the plate of complex variable $z = x + iy$, which contains an elastic patch along the segment $l_1 = (0, 1)$ and consists of two half-planes of dissimilar materials

$$S^{(1)} = \{z | \operatorname{Re} z > 0, z \notin 0, 1]\}, \qquad S^{(2)} = \{z | \operatorname{Re} z < 0\}$$

joined along the OY axis. Quantities and functions, referred to the half-plane $S^{(k)}$, will be denoted by the index k $(k = 1, 2)$, while the boundary values of the other functions on the upper and lower sides of the patch will be denoted by a plus or minus sign, respectively. We will assume that the left and right half-planes are homogeneous and the principal directions of elasticity coincide with the coordinate axes.

At the interface of the two materials, we have the continuity conditions

$$\sigma_x^{(1)} = \sigma_x^{(2)}, \qquad \tau_{xy}^{(1)} = \tau_{xy}^{(2)}, \qquad u^{(1)} = u^{(2)}, \qquad v^{(1)} = v^{(2)}$$

where $\sigma_x^{(k)}$, $\tau_{xy}^{(k)}$ are the stress components and $u^{(k)}$, $v^{(k)}$ are the displacement components $(k = 1, 2)$.

The boundary conditions of the components of the stress and displacement fields in the half-plane $S^{(1)}$ have the form

$$\begin{aligned} \sigma_y^{(1)+} - \sigma_y^{(1)-} &= p(x), \qquad \tau_{xy}^{(1)+} - \tau_{xy}^{(1)-} = \tau(x), \\ u^{(1)+} &= u^{(1)-}, \qquad v^{(1)+} = v^{(1)-}, \end{aligned} \qquad 0 < x < 1. \tag{3}$$

Using Lekhnitskii's formulae [21], the components of stress and displacement are represented in the form

$$\begin{aligned} &\sigma_x^{(k)} = -2\operatorname{Re}[\beta_k^2\Phi_k(z_k) + \gamma_k^2\Psi_k(\zeta_k)] \; \sigma_y^{(k)} = 2\operatorname{Re}[\Phi_k(z_k) + \Psi_k(\zeta_k)] \\ &\tau_{xy}^{(k)} = 2\operatorname{Im} = 2\operatorname{Im}[\beta_k\Phi_k(z_k) + \gamma_k\Psi_k(\zeta_k)] \\ &u^{(k)} = 2\operatorname{Re}[\rho_k\varphi_k(z_k) + r_k\psi_k(\zeta_k)] \; v^{(k)} = -2\operatorname{Im}[\beta_k r_k\varphi_k(z_k) + \gamma_k\rho_k\psi_k(\zeta_k)] \\ &z_k = x + i\beta_k y, \; \zeta_k = x + i\gamma_k y, \; \Phi_k(z_k) = \varphi_k'(z_k), \; \Psi_k(\zeta_k) = \psi_k'(\zeta_k), \; k = 1, 2 \end{aligned}$$

here $\pm i\beta_k$, $\pm i\gamma_k$ are the roots of the characteristic equation

$$\mu^4 + \left(\frac{E_k}{G_k} - 2\nu_k\right)\mu^2 + \frac{E_k}{E_k^*} = 0, \qquad (\beta_k > \gamma_k).$$

(E_k, E_k^*) are Young's modulus with respect to the principal (OX, OY) directions, respectively, G_k are the shear modulus, and ν_k are Poisson's ratios.

The problem with conditions 1–3 is reduced to the problem of finding of functions $\Phi_k(z_k)$ $\Psi_k(\zeta_k)$, $(k = 1, 2)$ which are holomorphic in the regions $S^{(k)}$, respectively, and satisfy the following boundary conditions:

$$\begin{aligned} 2\operatorname{Re}[\Phi_1^+(x) - \Phi_1^-(x) + \Psi_1^+(x) - \Psi_1^-(x)] &= p(x) \\ 2\operatorname{Im}[\beta_1(\Phi_1^+(x) - \Phi_1^-(x)) + \gamma_1(\Psi_1^+(x) - \Psi_1^-(x)] &= \tau(x) \\ \operatorname{Re}[\rho_1(\Phi_1^+(x) - \Phi_1^-(x)) + r_1(\Psi_1^+(x) - \Psi_1^-(x))] &= 0 \\ \operatorname{Im}[\beta_1 r_1(\Phi_1^+(x) - \Phi_1^-(x)) + \gamma_1\rho_1(\Psi_1^+(x) - \Psi_1^-(x))] &= 0 \end{aligned} \qquad 0 < x < 1 \tag{4}$$

$$\begin{aligned} \mathrm{Re}\,[\beta_1^2\Phi_1(t_1)+\gamma_1^2\Psi_1(\sigma_1)] &= \mathrm{Re}\,[\beta_2^2\Phi_2(t_2)+\gamma_2^2\Psi_2(\sigma_2)] \\ \mathrm{Im}\,[\beta_1\Phi_1(t_1)+\gamma_1\Psi_1(\sigma_1)] &= \mathrm{Im}\,[\beta_2\Phi_2(t_2)+\gamma_2\Psi_2(\sigma_2)] \\ \mathrm{Im}\,[\rho_1\beta_1\Phi_1(t_1)+r_1\gamma_1\Psi_1(\sigma_1)] &= \mathrm{Im}\,[\rho_2\beta_2\Phi_2(t_2)+r_2\gamma_2\Psi_2(\sigma_2)] \\ \mathrm{Re}\,[\beta_1^2 r_1\Phi_1(t_1)+\gamma_1^2\rho_1\Psi_1(\sigma_1)] &= \mathrm{Re}\,[\beta_2^2 r_2\Phi_2(t_2)+\gamma_2^2\rho_2\Psi_2(\sigma_2)] \end{aligned} \tag{5}$$

where $t_k=i\beta_k y$, $\sigma_k=i\gamma_k y$, $\rho_k=-(\beta_k^2+\nu_k)/E_k$, $r_k=-(\gamma_k^2+\nu_k)/E_k$, $k=1,2$.

System (4) has the unique solution:

$$\begin{aligned} \Phi_1^+(x)-\Phi_1^-(x) &= \frac{-r_1\beta_1 p(x)+i\rho_1\tau(x)}{2\beta_1(\rho_1-r_1)} \\ \Psi_1^+(x)-\Psi_1^-(x) &= \frac{\rho_1\gamma_1 p(x)-ir_1\tau(x)}{2\gamma_1(\rho_1-r_1)} \end{aligned} \qquad 0<x<1 \tag{6}$$

In view of the fact that $\tau(x)=0$, $p(x)=0$ when $x>1$, the general solution of problem (6) can be represented in the form [22]

$$\begin{aligned} \Phi_1(z_1) &= \frac{ir_1}{4\pi(\rho_1-r_1)}\int_0^1\frac{N_1(t)\,dt}{t-z_1}+w_1(z_1)\equiv ir_1w_0(z_1)+w_1(z_1), \\ \Psi_1(\zeta_1) &= -\frac{i\rho_1}{4\pi(\rho_1-r_1)}\int_0^1\frac{N_2(t)\,dt}{t-\zeta_1}+w_2(\zeta_1)\equiv -i\rho_1w_0(\zeta_1)+w_2(\zeta_1), \\ N_1(t) &= p(t)-i\frac{\rho_1}{r_1\beta_1}\tau(t), \qquad N_2(t)=p(t)-i\frac{r_1}{\rho_1\gamma_1}\tau(t), \end{aligned} \tag{7}$$

where $w_1(z_1)$ and $w_2(\zeta_1)$ are unknown analytic functions in the half-planes $\mathrm{Re}\,z_1>0$, $\mathrm{Re}\,\zeta_1>0$, respectively, which will be defined using the conditions (5).

Let us substitute the boundary values of functions $\Phi_1(z_1)$ and $\Psi_1(\zeta_1)$, expressed by formulae (7), into equalities (5) and then the obtained expressions are multiplied by $\frac{1}{2\pi i}\frac{dt}{t-z}$, $t=iy$, $z=x+iy$, $x>0$ and integrated along the imaginary axis. It is known that if $\Phi(z)$ is a holomorphic function in the half-plane $\mathrm{Im}\,z>0$ ($\mathrm{Im}\,z<0$), then $\overline{\Phi(iy)}$ is the boundary value of the function $\overline{\Phi(-\bar z)}$, which is holomorphic in the half-plane $\mathrm{Im}\,z<0$ ($\mathrm{Im}\,z>0$). As a result, using Cauchy's theorem and formula, we obtain the system:

$$\begin{aligned} \beta_1^2w_1(\beta_1z)+\gamma_1^2w_2(\gamma_1z)-\beta_2^2\overline{\Phi_2(-\beta_2\bar z)}-\gamma_2^2\overline{\Psi_2(-\gamma_2\bar z)} &= -ir_1\beta_1^2\overline{w_0(-\beta_1\bar z)}+i\rho_1\gamma_1^2\overline{w_0(-\gamma_1\bar z)} \\ \beta_1w_1(\beta_1z)+\gamma_1w_2(\gamma_1z)+\beta_2\overline{\Phi_2(-\beta_2\bar z)}+\gamma_2\overline{\Psi_2(-\gamma_2\bar z)} &= ir_1\beta_1\overline{w_0(-\beta_1\bar z)}-i\rho_1\gamma_1\overline{w_0(-\gamma_1\bar z)} \\ \rho_1\beta_1w_1(\beta_1z)+r_1\gamma_1w_2(\gamma_1z)+\rho_2\beta_2\overline{\Phi_2(-\beta_2\bar z)}+\gamma_2r_2\overline{\Psi_2(-\gamma_2\bar z)} &= ir_1\rho_1\beta_1\overline{w_0(-\beta_1\bar z)}-i\rho_1r_1\gamma_1\overline{w_0(-\gamma_1\bar z)} \\ \beta_1^2r_1w_1(\beta_1z)+\gamma_1^2\rho_1w_2(\gamma_1z)-\beta_2^2r_2\overline{\Phi_2(-\beta_2\bar z)}-\gamma_2^2\rho_2\overline{\Psi_2(-\gamma_2\bar z)} &= -ir_1^2\beta_1^2\overline{w_0(-\beta_1\bar z)}+i\rho_1^2\gamma_1^2\overline{w_0(-\gamma_1\bar z)} \end{aligned}$$

Solving this system for functions $w_1(\beta_1z)$ and $w_2(\gamma_1z)$, and replacing z by z_1/β_1 and ζ_1/γ_1, respectively, one obtains

$$w_1(z_1)=\frac{iI_1}{\Delta}\overline{w_0(-\overline{z_1})}+\frac{iI_2}{\Delta}\overline{w_0\left(-\frac{\gamma_1}{\beta_1}\overline{z_1}\right)}, \qquad w_2(\zeta_1)=\frac{iI_1^*}{\Delta}\overline{w_0\left(-\frac{\beta_1}{\gamma_1}\overline{\zeta_1}\right)}+\frac{iI_2^*}{\Delta}\overline{w_0(-\overline{\zeta_1})} \tag{8}$$

for functions $\Phi_2(-\beta_2z)$ and $\Psi_2(-\gamma_2z)$ with this notation $-\beta_2z=z_2$, $-\gamma_2z=\zeta_2$, we have

$$\begin{aligned} \Phi_2(z_2) &= -\frac{iI_3}{\Delta}w_0\left(\frac{\beta_1}{\beta_2}z_2\right)-\frac{iI_4}{\Delta}w_0\left(\frac{\gamma_1}{\beta_2}z_2\right), \\ \Psi_2(\zeta_2) &= -\frac{iI_3^*}{\Delta}w_0\left(\frac{\beta_1}{\gamma_2}\zeta_2\right)-\frac{iI_4^*}{\Delta}w_0\left(\frac{\gamma_1}{\gamma_2}\zeta_2\right), \end{aligned}$$

where

$$I_1 = -\Delta_{11} r_1 \beta_1^2 + \Delta_{21} r_1 \beta_1 + \Delta_{31} r_1 \rho_1 \beta_1 - \Delta_{41} \beta_1^2 r_1^2, \quad I_3 = -\Delta_{13} r_1 \beta_1^2 + \Delta_{23} r_1 \beta_1 + \Delta_{33} r_1 \rho_1 \beta_1 - \Delta_{43} \beta_1^2 r_1^2$$
$$I_2 = \Delta_{11} \rho_1 \gamma_1^2 - \Delta_{21} \rho_1 \gamma_1 - \Delta_{31} \rho_1 r_1 \gamma_1 + \Delta_{41} \rho_1^2 \gamma_1^2, \quad I_4 = \Delta_{13} \rho_1 \gamma_1^2 - \Delta_{23} \rho_1 \gamma_1 - \Delta_{33} \rho_1 r_1 \gamma_1 + \Delta_{43} \rho_1^2 \gamma_1^2$$
$$I_1^* = -\Delta_{12} r_1 \beta_1^2 + \Delta_{22} r_1 \beta_1 + \Delta_{32} r_1 \rho_1 \beta_1 - \Delta_{42} \beta_1^2 r_1^2, \quad I_3^* = -\Delta_{14} r_1 \beta_1^2 + \Delta_{24} r_1 \beta_1 + \Delta_{34} r_1 \rho_1 \beta_1 - \Delta_{44} \beta_1^2 r_1^2$$
$$I_2^* = \Delta_{12} \rho_1 \gamma_1^2 - \Delta_{22} \rho_1 \gamma_1 - \Delta_{32} \rho_1 r_1 \gamma_1 + \Delta_{42} \rho_1^2 \gamma_1^2, \quad I_4^* = \Delta_{14} \rho_1 \gamma_1^2 - \Delta_{24} \rho_1 \gamma_1 - \Delta_{34} \rho_1 r_1 \gamma_1 + \Delta_{44} \rho_1^2 \gamma_1^2$$
$$\Delta = \begin{vmatrix} \beta_1^2 & \gamma_1^2 & -\beta_2^2 & -\gamma_2^2 \\ \beta_1 & \gamma_1 & \beta_2 & \gamma_2 \\ \rho_1 \beta_1 & r_1 \gamma_1 & \rho_2 \beta_2 & r_2 \gamma_2 \\ \beta_1^2 r_1 & \gamma_1^2 \rho_1 & -\beta_2^2 r_2 & -\gamma_2^2 \rho_2 \end{vmatrix}$$

Δ_{ij} $(i, j = 1, 2, 3, 4)$ are the cofactors of the corresponding matrix elements.
Boundary condition (2) when $0 < x < 1$ is equivalent to the relations:

$$\frac{1}{E(x)} \int_0^x [\tau_1(t) - \tau_1^0(t)]\, dt - [\rho_1 \Phi_1(x) + \rho_1 \overline{\Phi_1(x)} + r_1 \Psi_1(x) + r_1 \overline{\Psi_1(x)}] = k_0 \tau'(x)$$
$$\frac{1}{D(x)} \int_0^x dt \int_0^t [p_1^0(\tau) - p_1(\tau)]\, d\tau - i \frac{d}{dx} [\beta_1 r_1 \Phi_1(x) - \beta_1 r_1 \overline{\Phi_1(x)} + \gamma_1 \rho_1 \Psi_1(x) - \gamma_1 \rho_1 \overline{\Psi_1(x)}] = m_0 p_1''(x) \tag{9}$$

Substituting expressions (7) and (8) into (9), one obtains

$$\frac{\psi(x)}{E(x)} - \frac{1}{2\pi} \int_0^1 Q(t, x) \psi'(t)\, dt - k_0 \psi''(x) = f_1(x),$$
$$\frac{\varphi(x)}{D(x)} + \frac{1}{2\pi} \frac{d}{dx} \int_0^1 R(t, x) \varphi''(t)\, dt + m_0 \varphi^{IV}(x) = f_2(x), \tag{10}$$
$$\psi(1) = 0, \qquad \varphi(1) = 0, \qquad \varphi'(1) = 0$$

where

$$Q(t, x) = \frac{\lambda_1}{t - x} + \frac{\lambda_2}{t + x} + \frac{\lambda_3}{\beta_1 t + \gamma_1 x} + \frac{\lambda_4}{\gamma_1 t + \beta_1 x}$$
$$R(t, x) = \frac{k_1}{t - x} + \frac{k_2}{t + x} + \frac{k_3}{\beta_1 t + \gamma_1 x} + \frac{k_4}{\gamma_1 t + \beta_1 x}$$
$$\psi(x) = \int_0^t [\tau(t) - \tau_0(t)]\, dt, \qquad \varphi(x) = \int_0^x dt \int_0^t [p_0(t) - p(\tau)]\, d\tau,$$
$$f_1(x) = \frac{1}{2\pi} \int_0^1 Q(t, x) \tau_0(t)\, dt + k_0 \frac{d}{dx} \tau_0(x), \qquad f_2(x) = m_0 \frac{d^2}{dx^2} p_0(x) + \frac{1}{2\pi} \frac{d}{dx} \int_0^1 R(t, x) p_0(t)\, dt$$
$$\lambda_1 = \frac{\rho_1^2 \gamma_1 - r_1^2 \beta_1}{(\rho_1 - r_1) \beta_1 \gamma_1}, \lambda_2 = \frac{\rho_1^2 \gamma_1 I_1 + r_1^2 \beta_1 I_2^*}{\Delta \beta_1 \gamma_1 (\rho_1 - r_1)}, \lambda_3 = \frac{-I_2 \rho_1^2}{\Delta r_1 (\rho_1 - r_1)}, \lambda_4 = \frac{-I_1^* r_1^2}{\Delta \rho_1 (\rho_1 - r_1)}$$
$$k_1 = \frac{\beta_1 r_1^2 + \gamma_1 \rho_1^2}{\rho_1 - r_1}, k_2 = \frac{\beta_1 r_1 I_1 + \gamma_1 \rho_1 I_2^*}{\Delta(\rho_1 - r_1)}, k_3 = \frac{\beta_1^2 r_1 I_2}{\Delta(\rho_1 - r_1)}, k_4 = \frac{\gamma_1^2 \rho_1 I_1^*}{\Delta(\rho_1 - r_1)}$$

3. Exact solution of equation (10)

Let the patch be loaded by a tangential force $P\delta(x-1)$ and the plate be free from external loads. ($\delta(x)$ is Dirac function.) Stiffness of the patch and glue varies linearly, i.e., $E(x) = hx$, $k_0(x) = k_0 x$, $0 < x < 1$ (Figure 2). Equation (10) and the corresponding boundary conditions take the form

$$\begin{aligned}&\frac{\psi(x)}{E(x)}-\frac{1}{2\pi}\int_0^1 Q(t,x)\psi'(t)\,dt-(k_0(x)\psi'(x))'=0; \qquad 0<x<1\\&\psi(1)=P, \qquad \psi(x)=\int_0^x \tau(t)dt\end{aligned} \tag{11}$$

The solution of equation (11) is sought in the class of functions

$$\psi, \psi' \in H([0,1]), \qquad \psi'' \in H((0,1))$$

The change of variables $x=e^{\xi}$, $t=e^{\zeta}$ in equation (11) gives

$$\frac{\psi_0(\xi)}{h}-\frac{1}{2\pi}\int_{-\infty}^0 Q(e^{\zeta-\xi},1)\psi_0'(\zeta)\,d\zeta-k_0\psi_0''(\xi)=0, \xi<0,$$
$$\psi_0(-\infty)=0, \psi_0(0)=P, \psi_0(\xi)=\psi(e^{\xi})$$

Subjecting both parts of this equation to generalized Fourier transform [23], one obtains the following condition of Riemann boundary value problem

$$\Phi^+(s)=G(s)\Psi^-(s)+g(s), \qquad -\infty<s<\infty, \tag{12}$$

where

$$G(s)=1+\frac{h\lambda_1 s}{2}\text{cth}\pi s-\frac{h\lambda_2 s}{2\text{sh}\pi s}-\frac{h\lambda_3 s e^{i\mu s}}{2\text{sh}\pi s}-\frac{h\lambda_4 s e^{-i\mu s}}{2\text{sh}\pi s}+k_0hs^2, \qquad \mu=\ln\frac{\beta_1}{\gamma_1}$$
$$\Psi^-(s)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^0 \psi_0^-(\zeta)e^{is\zeta}\,d\zeta,$$
$$\sqrt{2\pi}g(s)=\frac{Pi}{2}\left(\lambda_1 h\text{cth}\pi s-\frac{\lambda_2 h}{\text{sh}\pi s}-\frac{\lambda_3 h e^{i\mu s}}{\text{sh}\pi s}-\frac{\lambda_4 h e^{-i\mu s}}{\text{sh}\pi s}\right)_- +Pik_0hs-k_0h\psi'_0(0)$$
$$\varphi^+(\xi)=\begin{cases}0, & \xi<0\\ -\frac{h}{2\pi}\int_{-\infty}^0 Q(e^{\zeta-\xi},1)\psi_{0'}(\zeta)\,d\zeta-hk_0\psi_{0''}(\xi), & \xi>0\end{cases},$$
$$\Phi^+(s)=\frac{1}{\sqrt{2\pi}}\int_0^\infty \varphi^+(\zeta)e^{is\zeta}\,d\zeta$$

By virtue of functions $\Psi^-(s)$, $\Phi^+(s)$ definition, they will be boundary values of the functions which are holomorphic in the lower and upper half-planes, respectively.

The problem can be formulated as follows: it should be determined by the functions $\Phi^+(z)$, holomorphic in the half-plane $\text{Im}\, z>0$ and the function $\Psi^-(z)$, holomorphic in the half-plane $\text{Im}\, z<1$ (with the exception of a finite number of zeros of function $G(z)$), which are vanishing at infinity and are continuous on the real axis by condition (12).

Condition (12) can be represented as

$$\frac{\Phi^+(s)}{s+i}=\frac{G(s)}{1+s^2}\Psi^-(s)(s-i)+\frac{g(s)}{s+i} \tag{13}$$

Introducing the notation $G_0(s)=(k_0h)^{-1}G(s)(1+s^2)^{-1}$, it can be shown that $\text{Re}\, G_0(s)>0$, $G_0(\infty)=G_0(-\infty)=1$, therefore $\text{Ind}G_0(s)=0$.

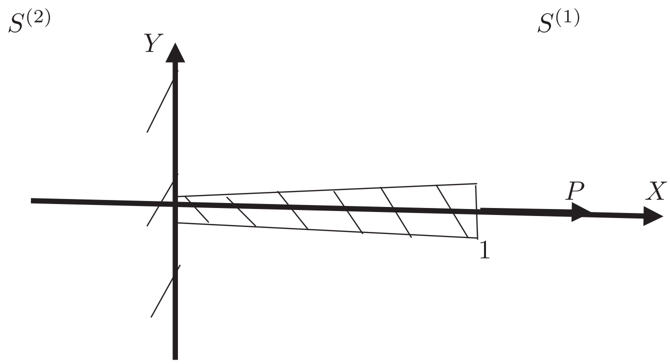

Figure 2. Exact solution. Graphical sketch.

The unique solution of problem (13) has the form [22]

$$\Psi^-(z)=\frac{\tilde{X}(z)}{k_0h(z-i)},\quad \operatorname{Im} z\leqslant 0;\qquad \Phi^+(z)=\tilde{X}(z)(z+i),\quad \operatorname{Im} z>0,$$
$$\Psi^-(z)=(\Phi^+(z)-g(z))G^{-1}(z),\quad 0<\operatorname{Im} z<1, \tag{14}$$

where

$$\tilde{X}(z)=\frac{X(z)}{2\pi i}\int_{-\infty}^{\infty}\frac{g(t)}{X^+(t)(t+i)(t-z)}\,dt,\qquad X(z)=\exp\left\{\frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{\ln G_0(t)}{t-z}\,dt\right\}.$$

It can be shown that $\Psi^-(x+i0)=\Psi^-(x-i0)$, and the function $\Psi^-(z)$ is holomorphic in the half-plane $\operatorname{Im} z<1$, except points that are zeros of the function $G(z)$ in the strip $0<\operatorname{Im} z<1$.

The boundary value of the function $K(z)=\frac{P}{\sqrt{2\pi}}-iz\Psi^-(z)$ is the Fourier transform of the function $\psi'(e^\xi)$. The function $K(z)$ can be represented as

$$\begin{aligned}K(z)&=\frac{P}{\sqrt{2\pi}}-\frac{\lambda PizX(z)}{2\pi\sqrt{2\pi}k_0(z-i)}\int_{-\infty}^{\infty}\frac{\operatorname{cth}\pi t}{X^+(t)(t+i)(t-z)}\,dt\\&\quad-\frac{PizX(z)}{2\pi\sqrt{2\pi}(z-i)}\int_{-\infty}^{\infty}\frac{t}{X^+(t)(t+i)(t-z)}\,dt+\frac{zX(z)}{2\pi\sqrt{2\pi}(z-i)}\psi'(0)\int_{-\infty}^{\infty}\frac{1}{X^+(t)(t+i)(t-z)}\,dt\\&=\frac{P}{2\pi}+K_1(z)+K_2(z)+K_3(z),\qquad \operatorname{Im} z<0\end{aligned} \tag{15}$$

Let us study the behavior at infinity of each of these integrals, the first of which gives

$$K_1(z)=-\frac{\lambda PizX(z)}{2\pi\sqrt{2\pi}k_0(z-i)}\left\{\int_{-\infty}^{\infty}\frac{[\operatorname{cth}\pi t-\operatorname{sgn}t]\,dt}{X^+(t)(t+i)(t-z)}+\int_{-\infty}^{\infty}\frac{\operatorname{sgn}t\,dt}{X^+(t)(t+i)(t-z)}\right\}$$

Here, the first term tends to zero at infinity, and the second term

$$\tilde{K}_1(z)=-\frac{\lambda PizX(z)}{2\pi\sqrt{2\pi}k_0(z-i)}\int_{-\infty}^{\infty}\frac{\operatorname{sgn}t\,dt}{X(t)(t+i)(t-z)}$$

as a result of the change of variables $z=-1/\xi$, $t=-1/t_0$ can be represented in the form

$$\tilde{K}_1^*(\xi)=-\frac{\lambda PX^*(\xi)\xi}{\pi\sqrt{2\pi}k_0(1+i\xi)}\int_0^{\infty}\frac{1}{X^{+*}(t_0)(1-it_0)(t_0-\xi)}\,dt_0$$

where $\tilde{K}_1^*(\xi)=\tilde{K}_1(\xi)$, $X^*(\xi)=X(\xi)$. Applying the formulas of Muskhelishvili [22] in the neighborhood of the point $\xi=0$, we will have $\tilde{K}_1^*(\xi)=O(\xi\ln\xi)$.

Therefore, the function $\tilde{K}_1(z)$ (i.e., $K_1(z)$) at infinity vanishes by no more than one order: $|K_1(z) = O(|z|^{-(1-\epsilon)}), |z| \to \infty$ (ε is an arbitrary positive number).

Based on the well-known Cauchy theorem, from the second and third integrals of formula (15) one obtains

$$K_2(z) = \frac{PzX(z)}{2\sqrt{2\pi}(z-i)}, \quad K_3(z) = 0, \quad \operatorname{Im} z < 0, \quad \text{and} \quad K_2^-(\infty) = \frac{P}{2\sqrt{2\pi}}$$

Thus, from here one concludes that the function

$$M(z) = K(z) - \frac{P}{2\sqrt{2\pi}}, \qquad \operatorname{Im} z < 0$$

is holomorphic in a half-plane $\operatorname{Im} z < 0$, vanishes at infinity as $O(|z|^{-(1-\epsilon)})$. Its boundary value is the Fourier transform of a function $\varphi'(e^{\xi})$, which is continuous on the half-line $\xi \leqslant 0$ (except maybe the point $\xi = 0$ where it may have a discontinuity of the first kind). Thus, by the inverse Fourier transform, we obtain the expression for the sought function

$$\tau(x) = \psi'(x) = \frac{1}{\sqrt{2\pi}x} \int_{-\infty}^{\infty} M^-(t) e^{-it \ln x} \, dt. \tag{16}$$

Based on the formulas (14), the behavior of the function (16) in a neighborhood of a point $x = 1$ has the form

$$\psi'(x) = O(1), \qquad x \to 1-. \tag{17}$$

Let us study the behavior of the function in a neighborhood of the point $x = 0$.

We conclude that the boundary value of function

$$Q(z) = \frac{P}{\sqrt{2\pi}} - iz(\Psi^+(z) - g(z))G^{-1}(z), \qquad 0 < \operatorname{Im} z < 1,$$

is the Fourier transform of a function $\varphi'(e^{\xi})$ and the function $Q_0(z) = Q(z) - \frac{P}{2\sqrt{2\pi}}$ is holomorphic in the half-plane $\operatorname{Im} z > 0$ (except the points, where the function $G(z)$ has roots) and vanishes at infinity with order no less than $|z|^{-1}$.

It is proved that the function $G(z)$ has no zeros in the strip $0 < \operatorname{Im} z \leqslant 1$. Let $z_0 = \omega_0 + i\tau_0$ be a zero of function $G(z)$ with minimal imaginary part in the half-plane $\operatorname{Im} z > 0$. Therefore, applying the Cauchy's residue theorem to the function $e^{-i\xi z} Q_0(z)$ for a rectangle $D(N)$ with a boundary $L(N)$, that consists of segments

$$[-N, N], \quad [N + i0, N + i\beta_0], \quad [N + i\beta_0, -N + i\beta_0], \quad [-N + i\beta_0, -N + i0], \quad \beta_0 > \tau_0$$

we will obtain

$$\int_{L(N)} Q^-(t) e^{-it\xi} \, dt = \int_{-N}^{N} Q_0^-(t) e^{-it\xi} \, dt - e^{-\beta_0 \xi} \int_{-N}^{N} Q_0^-(t + i\beta_0) e^{-it\xi} \, dt + \rho(N, \xi) = K_0 e^{\tau_0 \xi}$$

where $\rho(N, \xi) \to 0$, $N \to \infty$. Passing to the limit in the last equality and returning to the old variables, we have

$$\tau(x) = \psi'(x) = O(x^{\tau_0 - 1}), \qquad x \to 0+, \tau_0 > 1. \tag{18}$$

Thus, the integro-differential equation (11) has a unique solution, which is represented explicitly by formula (16) and satisfies estimates (17) and (18).

4. Discussion and numerical results

Asymptotic estimates for the solution of integro-differential equation (11) are obtained by formulas (17) and (18). Numerical calculations made in MATLAB show that for any value of the elastic and geometrical parameters, the function $G(z)$ has no zeros in the strip $0 < \operatorname{Im} z \leqslant 1$, the latter providing finite values of tangential contact stresses at the ends of the patch.

Thus, the tangential contact stresses are bounded at the end of the patch and the intensity factor of contact stresses is equal to zero.

Under conditions of rigid contact between the plate and the patch, the contact stress in the neighborhood of the ends of the patch can be significantly increased, i.e., the contact stress can have a singularity.

In this case, the normal interatomic distance increases, the grip strength between atoms begin to decrease in the neighborhood of the ends of the inclusion and a precondition for the appearance of a crack is created. When a crack appears, energy is released and the stresses begin to subside. Under the conditions of adhesive contact of the plate with the patch, the latter phenomenon is excluded.

Obviously, the absence of stress concentration in the deformable body is extremely important from an engineering point of view.

Numerical calculations (Cases 1–3) for different values of the parameters (close to natural) of the plate (E_1, E_1^*, E_2, E_2^*, G_1, G_2, ν_1, ν_2) and patch (h) show that $\tau_0 > 1$ and the contact stress increases insignificantly (with an accuracy of 10^{-9}) depending on the increase of the parameter k_0 (this means an increase in the thickness h_0 or a decrease in the shear modulus μ_0 of the adhesive, $k_0 := h_0/\mu_0$) in the neighborhood of the end of the patch.

4.1. Case 1

$$h_0 = 5 \cdot 10^{-n}, \quad n = 4, 3, 2 \qquad \mu_0 = 0.117 \cdot 10^9 \qquad E_1 = 55.917 \cdot 10^9$$
$$E_1^* = 36.735 \cdot 10^9 \qquad G_1 = 5.592 \cdot 10^9 \qquad G_2 = 4.902 \cdot 10^9$$
$$\nu_1 = 0.32 \qquad \nu_2 = 0.3 h = 0.1$$
$$E_2 = 19.236 \cdot 10^9 \qquad E_2^* = 30.145 \cdot 10^9.$$

k_0	ω_0	τ_0
$42.7 \cdot 10^{-13}$, $(n = 4)$	0.000000001107485	7.718681569000190
$42.7 \cdot 10^{-12}$, $(n = 3)$	0.000000001107485	7.718681568951642
$42.7 \cdot 10^{-11}$, $(n = 2)$	0.000000001107487	7.718681568465962

4.2. Case 2

$$h_0 = 5 \cdot 10^{-n}, n = 4, 3, 2 \qquad \mu_0 = 0.117 \cdot 10^9 \qquad E_1 = 23.517 \cdot 10^9$$
$$E_1^* = 40.125 \cdot 10^9 \qquad G_1 = 4.905 \cdot 10^9 \qquad G_2 = 8.315 \cdot 10^9$$
$$\nu_1 = 0.25 \qquad \nu_2 = 0.38 h = 0.1$$
$$E_2 = 58.124 \cdot 10^9 \qquad E_2^* = 32.245 \cdot 10^9.$$

k_0	ω_0	τ_0
$42.7 \cdot 10^{-13}$, $(n = 4)$	–0.000000000273508	6.715298333139011
$42.7 \cdot 10^{-12}$, $(n = 3)$	–0.000000000273507	6.715298333099307
$42.7 \cdot 10^{-11}$, $(n = 2)$	–0.000000000273506	6.715298332702201

4.3. Case 3

$$h_0 = 5 \cdot 10^{-n}, \quad n = 4, 3, 2 \qquad \mu_0 = 0.117 \cdot 10^9 \qquad E_1 = 28.155 \cdot 10^9$$
$$E_1^* = 30.475 \cdot 10^9 \qquad G_1 = 6.149 \cdot 10^9 \qquad G_2 = 5.850 \cdot 10^9$$
$$\nu_1 = 0.25 \qquad \nu_2 = 0.08 h = 0.1$$
$$E_2 = 35.180 \cdot 10^9 \qquad E_2^* = 51.556 \cdot 10^9.$$

k_0	ω_0	τ_0
$42.7 \cdot 10^{-13}$, $(n=4)$	0.427105973827816	9.275927911785338
$42.7 \cdot 10^{-12}$, $(n=3)$	0.427105973921047	9.275927911742233
$42.7 \cdot 10^{-11}$, $(n=2)$	0.427105974853223	9.275927911311051

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