Z. Angew. Math. Phys. _####################_

https://doi.org/10.1007/s00033-021-01471-4

Zeitschrift für angewandte Mathematik und Physik ZAMP

The investigation of singular integro-differential equations relating to adhesive contact problems of the theory of viscoelasticity

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Abstract. The exact and approximate solutions of singular integro-differential equations relating to the problems of interaction of an elastic thin finite or infinite non-homogeneous patch with a plate are considered, provided that the materials of plate and patch possess the creep property. Using the method of orthogonal polynomials the problem is reduced to the [1] infinite system of Volterra integral equations, and using the method of integral transformations this problem is reduced to [2] the different boundary value problems of the theory of analytic functions. An asymptotic analysis is also performed. [3]

The considerable development of the hereditary theory of Bolzano–Volterra mechanics has been defined by various technical applications in the theory of metals, plastics and concrete and in mining engineering. The fundamentals of the theory of viscoelasticity, the methods for solving linear and nonlinear problems of the theory of creep, the problems of mechanics of inhomogeneously ageing viscoelastic materials, some boundary value problems of the theory of growing solids, the contact and mixed problems of the theory of viscoelasticity for composite inhomogeneously ageing and nonlinearly-ageing bodies are considered in [1–4].

The full investigation of various possible forms of viscoelastic relations and of some aspects of the general theory of viscoelasticity are studied in [5–8]. Research on the field of creep materials can be found in [9–12].

Contact and mixed boundary value problems on the transfer of the load from elastic thin-walled elements (stringers, inclusions, patches) to massive deformable (including aging viscoelastic) bodies, as well as on the indentation of a rigid stamp into the surface of a viscoelastic body, represent an urgent problem both in theoretical and applied aspects. Problems of this type are often encountered in engineering applications and lead themselves to rigorous mathematical research due to their applied significance.

Exact and approximate solutions to static contact problems for different domains, reinforced with non-homogeneous elastic thin inclusions and patches were obtained, and the behavior of the contact stresses at the ends of the contact line were investigated in [13–16]. One type of analysis assumes continuous interaction and the other the adhesive contact of thin-shared elements (stringers or inclusions) with massive deformable bodies. As is known, stringers and inclusions, such as rigid punches and cuts, are areas of stress concentration. Therefore, the study of the problems of stress concentration and the development of various methods for its reduction is of great importance in engineering practice.

In work [17] we consider integro-differential equations with a variable coefficient relating to the interaction of an elastic thin finite inclusion and plate, when the inclusion and plate materials possess the creep property. Here continuous contact between inclusion and plate is considered. The solutions to integro-differential equations of the first order are obtained on the basis of investigations of different boundary value problems of the theory of analytic functions. The asymptotic behavior of unknown contact stresses is established.

In this paper, in contrast to work [17], contact with a thin layer of glue is studied when the patch, plate and adhesive materials have the property of creep. A second-order singular integro-differential equation

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was obtained. Here the asymptotic analysis was also carried out and approximate and exact solutions were obtained for various cases.

1. Formulation of the problems and reduction to integral equations

Let a finite or infinite non-homogeneous patch with modulus of elasticity E_1, thickness $h_1(x)$ and Poisson's coefficient ν_1 be attached to the plate (E_2, ν_2), which occupies the entire complex plane and is in the condition of a plane deformation. It is assumed that the patch, as thin element, is glued to the plate along the real axis, has no bending rigidity, is in the uniaxial stressed state and is subject only to tension. The tangential stress $q_0(x)H(t-t_0)$ acts on the line of contact between the inclusion and the plate from t_0 ($H(t)$ is the unit Heaviside function). The one-dimensional contact between the plate and patch is affected by a thin layer of glue with thickness h_0 and modulus of shear G_0.

It is assumed that the plate, patch and glue layer materials have the creep property which is characterized by the non-homogeneity of the ageing process and has different creep measures $C_i(t,\tau) = \varphi_i(\tau)[1-e^{-\gamma(t-\tau)}]$, where $\varphi_i(\tau)$ are the functions that define the ageing process of the plate, patch and glue layer materials; the age of the different materials is $\tau_i(x) = \tau_i = \text{const}$; $\gamma = \text{const} > 0$, $i = 1, 2, 3$.

Besides, the plate Poisson's coefficients for elastic-instant deformation $\nu_2(t)$ and creep deformation $\nu_2(t,\tau)$ are the same and constant: $\nu_2(t) = \nu_2(t,\tau) = \nu_2 = \text{const}$.

Assuming that every element of the glue layer is under the condition of pure shear, the contact condition has the form [18]

$$u_1(t,x) - u_2(t,x,0) = k_0(I-L_3)q(t,x), \qquad |x| \le 1, \tag{1}$$

where $u_2(t,x,y)$ is the displacement of the plate points along the ox-axis and $k_0{:=}h_0/G_0$, $u_1(t,x)$ is the displacement of the inclusion points along the ox-axis, I is the unit operator.

We have to define the law of distribution of tangential contact stresses $q(t,x)$ on the line of contact and the asymptotic behavior of these stresses at the end of the patch.

To define the unknown contact stresses we obtain the following integral equation (see [1–4])

$$\begin{aligned} &\frac{2(1-\nu_2^2)}{\pi E_2} \ (I-L_2)\int_{-1}^{1}\frac{q(t,y)\,\mathrm{d}y}{y-x} \\ &\quad = \frac{1}{E(x)}(I-L_1)\int_{-1}^{x}[q(t,y)-q_0(y)H(t-t_0)]\,dy - k_0(I-L_3)q'(t,x), \qquad |x|<1, \\ &\int_{-1}^{1}[q(t,y)-q_0(y)H(t-t_0)]\,\mathrm{d}y = 0 \end{aligned} \tag{2}$$

where time operators $L_i = 1,2,3$ act on an arbitrary function in the following manner:

$$\begin{aligned} &(I-L_i)\psi(t) = \psi(t) - \int_{\tau_i^0}^{t} K_i(t+\rho_i, \tau+\rho_i)\psi(\tau)\,\mathrm{d}\tau, \qquad \rho_i = \tau_i - \tau_i^0, \qquad i = 1,2,3, \\ &K_i(t,\tau) = E_i\frac{\partial C_i(t,\tau)}{\partial\tau}, \qquad i = 1,2, \qquad K_3(t,\tau) = G_0\frac{\partial C_3(t,\tau)}{\partial\tau}, \\ &\omega(t,\tau) = \varphi_3(\tau)[1-e^{-\gamma(1-\tau)}], \qquad E(x) = \frac{E_1 h_1(x)}{1-\nu_1^2}, \end{aligned}$$

where $\tau_i^0 = t_0$ is the instant of load application.

Introducing the notation

$$\varphi(t,x) = \int_{-1}^{x}[q(t,y)-q_0(y)H(t-t_0)]\,\mathrm{d}y, \qquad \lambda = \frac{2(1-\nu_2^2)}{E_2}$$

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from (2) we obtain the following two-dimensional integro-differential equation

$$\frac{\lambda}{\pi}(I-L_2)\int_{-1}^{1}\frac{\varphi'(t,y)\,\mathrm{d}y}{y-x}=\frac{1}{E(x)}(I-L_1)\varphi(t,x)-k_0(I-L_3)\varphi''(t,x)+g(t,x),\qquad |x|<1,$$

$$g(t,x)=-\frac{\lambda}{\pi}(1-E_2\varphi_2(t)(1-e^{-\gamma(t-t_0)}))\int_{-1}^{1}\frac{q_0(y)\,\mathrm{d}y}{y-x}-k_0q_0'(x)(1-G_0\varphi_3(t)(1-e^{-\gamma(t-t_0)}))\quad(3)$$

with conditions

$$\varphi(t,1)=0,\qquad t\geq t_0 \tag{4}$$

Thus, the above posed boundary contact problem is reduced to the solution to singular integro-differential equation (SIDE) with condition (4). From the symmetry of the problem, we assume, that $E(x)$ and $q_0(x)$ are even and odd functions, respectively. The solution of Eq. (3) under condition (4) with respect to variable x can be sought in the class of even functions. Moreover, we assume that function $q_0(x)$ is continuous in Holder's sense (hereinafter, H) and is continuous up to the first order derivative on an interval $[-1,1]$, i.e. $q_0\in C^1([-1,1])$.

2. The asymptotic investigation

Under the assumption that

$$E(x)=(1-x^2)^{\omega}b_0(x), \tag{5}$$

where $\omega=\text{const}\geq 0$, $b_0(x)=b_0(-x)$, $b_0\in C([-1,1])$, $b_0(x)\geq c_0=\text{const}>0$, the solution to problem (3), (4) will be sought in the class of even function whose derivative with respect to variable x can be represented as follows:

$$\varphi'(t,x)=(1-x^2)^{\alpha}g_0(t,x),\qquad \alpha>-1, \tag{6}$$

where $g_0(t,x)=-g_0(t,-x)$, $g_0\in C^1([-1,1])$, $g_0(t,x)\neq 0$, $x\in[-1,1]$. $\varphi'(t,x)$ represents the unknown tangential contact stress.

Introducing the notation

$$\Phi_0(x,t)=\int_{-1}^{1}\frac{(1-s^2)^{\alpha}g_0(t,s)}{s-x}\,\mathrm{d}s$$

by virtue of the well-known asymptotic formula [28] we have for $-1<\alpha<0$

$$\Phi_0(x,t)=\mp\pi\,\mathrm{ctg}\pi\alpha\,g_0(t,\mp 1)2^{\alpha}(1\pm x)^{\alpha}+\Phi_{\pm}(x,t),\qquad x\to\mp 1;$$
$$\Phi_{\mp}(x,t)=\Phi_{\mp}^{*}(x,t)(1\pm x)^{\alpha_{\pm}},\qquad \alpha_{\pm}=\text{const}>\alpha$$

and for $\alpha=0$

$$\Phi_0(x,t)=\mp g_0(t,\mp 1)\ln(1\pm x)+\widetilde{\Phi}_{\pm}(x,t),\qquad x\to\mp 1$$

Functions $\Phi_{\mp}^{*}(x,t)$ and $\widetilde{\Phi}_{\mp}(x,t)$ satisfy (H)'s condition in a neighborhood of the points $x=\mp 1$, respectively.

In case $\alpha>0$ function $\Phi_0(x,t)$ belongs to the (H) class in a neighborhood of the points $x=\pm 1$.

In addition, we have [22]

$$\int_{-1}^{x}(1-s^2)^{\alpha}g_0(t,s)\,\mathrm{d}s=\frac{2^{\alpha}(1\pm x)^{\alpha+1}}{\alpha+1}g_0(t,\mp 1)\,F(\alpha+1,-\alpha,2+\alpha,(1\pm x)/2)+G_{\mp}(x,t),\qquad x\to\mp 1,$$
$$\lim_{x\to\mp 1}G_{\mp}(x,t)(1\pm x)^{-(\alpha+1)}=0$$

where $F(a,b,c,x)$ is a hypergeometric Gaussian function.

The case $-1<\alpha<0$ is not of interest, since negative values of the indicator α contradict the physical meaning of condition (1).

Let $0 \le \alpha \le 1$, then in a neighborhood of the points $x = -1$ equation (3) can be written in the following form

$$(I - L_2)\Psi(x,t) + \frac{2^\alpha (1+x)^{2+\varepsilon}(I-L_1)g_0(-1,t)}{2^\omega(\alpha+1)(1+x)^\omega b_0(-1)} + (I-L_1)G_-(x,t)(1+x)^{1+\varepsilon-\alpha}$$
$$-k_0 2^\alpha (1+x)^\varepsilon (I-L_3)\widetilde{g}_0(-1,t) = g(-1,t)(1+x)^{1+\varepsilon-\alpha}$$
$$\Psi(x,t) = \begin{cases} \lambda g_0(-1,t)(1+x)^{1+\varepsilon}\ln(1+x) - \frac{\lambda}{\pi}(1+x)^{1+\varepsilon}\widetilde{\Phi}_-(x,t), & \text{for } \alpha = 0 \\ -\frac{\lambda}{\pi}(1+x)^{1+\varepsilon-\alpha}\Phi_0(x,t), & \text{for } \alpha \neq 0 \end{cases} \tag{7}$$

where ε is an arbitrarily small positive number. When passing to limit $x \to -1$, the analysis of the obtained equations leads to the necessity of satisfying inequality $2 + \varepsilon > \omega$, i.e. $\omega \le 2$.

In case $\alpha > 1$ from (7) it follows that $\alpha = \omega - 1$.

An analogous result is obtained in the neighborhood of the point $x = 1$.

The obtained results can be formulated as follows:

Theorem 1. *Assuming that (5) holds, if problem (3),(4) has the solution in the form (6), then:*

- *If $\omega > 2$ then $\alpha = \omega - 1$, $(\alpha > 1)$*
- *If $\omega \le 2$ then $0 \le \alpha \le 1$.*

Conclusion. If the patch rigidity varies by the law

$$E(x) = (1-x^2)^{n+1/2} b_0(x),$$

where $b_0(x) > 0$ for $|x| \le 1$, $b_0(x) = b_0(-x)$, $n \ge 0$ is integer, then from the above asymptotic analysis, we obtain:

$$\alpha = n - \frac{1}{2}, \qquad \text{for } n = 2, 3, \ldots$$

and $0 < \alpha < 1$ for $n = 0$ or $n = 1$ (the same result is obtained for $E(x) = b_0(x) > 0$ or $E(x) = \text{const}$, $|x| \le 1$).

3. An approximate solution to SIDE (3)

From the relation

$$\frac{1}{\pi}\int_{-1}^{1} \frac{(1-s)^\alpha (1+s)^\beta P_m^{(\alpha,\beta)}(s)\,ds}{s-x} = \text{ctg}\pi\alpha (1-x)^\alpha (1+x)^\beta P_m^{(\alpha,\beta)}(x) -$$
$$\frac{2^{\alpha+\beta}\Gamma(\alpha)\Gamma(\beta+m+1)}{\pi\Gamma(\alpha+\beta+m+1)} F(m+1, -\alpha-\beta-m, 1-\alpha, (1-x)/2)$$

obtained by Tricomi [19] for orthogonal Jacobi polynomials $P_m^{(\alpha,\beta)}(x)$ and from the well-known equality (see [20]).

$$m! P_m^{(\alpha,\beta)}(1-2x) = \frac{\Gamma(\alpha+m+1)}{\Gamma(1+\alpha)} F(\alpha+\beta+m+1, -m, 1+\alpha, x)$$

we get the following spectral relation for the Hilbert singular operator

$$\int_{-1}^{1} \frac{(1-s^2)^{n-1/2} P_m^{(n-1/2,n-1/2)}(s)\,\mathrm{d}s}{s-x} = (-1)^n 2^{2n-1}\pi P_{m+2n-1}^{(1/2-n,1/2-n)}(x), \tag{8}$$

where $\Gamma(z)$ is the known Gamma function.

1. On the basis of the above asymptotic analysis performed in the cases

$$n = 0; n = 1; \qquad E(x) = b_0(x) > 0; \qquad E(x) = \text{const}, \qquad |x| \le 1;$$

the solution to equation (3) will be sought in the form

$$\varphi'(t,x) = \sqrt{1-x^2}\sum_{k=1}^{\infty} X_k(t) P_k^{(1/2,1/2)}(x), \tag{9}$$

where function $X_k(t)$ has to be defined for $k = 1, 2, \ldots$.

Using relation (8) and the Rodrigues formula (see [21]) for (9) we obtain

$$\int_{-1}^{1} \frac{\sqrt{1-t^2}P_k^{(1/2,1/2)}(t)\,\mathrm{d}t}{t-x} = -2\pi P_{k+1}^{(-1/2,-1/2)}(x),$$

$$\varphi(t,x) = -(1-x^2)^{3/2}\sum_{k=1}^{\infty}\frac{X_k(t)}{2k}P_{k-1}^{(3/2,3/2)}(x),$$

$$\varphi''(t,x) = -2(1-x^2)^{-1/2}\sum_{k=1}^{\infty} kX_k(t)P_{k+1}^{(-1/2,-1/2)}(x). \tag{10}$$

Substituting relation (9), (10) into equation (3), we have

$$-\frac{(1-x^2)^{3/2}}{E_1(x)}(I-L_1)\sum_{r=1}^{\infty}\frac{X_k(t)}{2k}P_{k-1}^{(3/2,3/2)}(x) - 2\lambda_0(I-L_2)\sum_{k=1}^{\infty}X_k(t)P_{k+1}^{(-1/2,-1/2)}(x) +$$
$$2k_0(1-x^2)^{-1/2}(I-L_3)\sum_{k=1}^{\infty}kX_k(t)P_{k+1}^{(-1/2,-1/2)}(x) = g(t,x), \qquad |x|\le 1. \tag{11}$$

Multiplying both parts of equality (11) by $P_{m+1}^{(-1/2,-1/2)}(x)$ and integrating in the interval $(-1, 1)$, we obtain an infinite system of Volterra's linear integral equations

$$k_0 m\left(\frac{\Gamma(m+3/2)}{\Gamma(m+2)}\right)^2 (I-L_3)X_m(t) - \sum_{k=1}^{\infty}R_{mk}^{(2)}(I-L_2)X_k(t) - \sum_{k=1}^{\infty}\frac{R_{mk}^{(1)}}{k}(I-L_1)X_k(t) = g_m(t),$$
$$m = 1, 2, \ldots \tag{12}$$

where

$$R_{mk}^{(1)} = \frac{1}{2}\int_{-1}^{1}\frac{(1-x^2)^{3/2}}{E(x)}P_{k-1}^{(3/2,3/2)}(x)P_{m+1}^{(-1/2,-1/2)}(x)\,\mathrm{d}x,$$

$$R_{mk}^{(2)} = -2\lambda\int_{-1}^{1}P_{k+1}^{(-1/2,-1/2)}(x)P_{m+1}^{(-1/2,-1/2)}(x)\,\mathrm{d}x$$

$$g_m(t) = \int_{-1}^{1} g(t,x)P_{m+1}^{(-1/2,-1/2)}(x)\,\mathrm{d}x.$$

Introducing the notation

$$T_m(t) = \omega_m\left[k_0X_m(t) - \sum_{k=1}^{\infty}\frac{R_{mk}^{(1)}}{k\omega_k}X_k(t) - \sum_{k=1}^{\infty}\frac{R_{mk}^{(2)}}{\omega_k}X_k(t)\right],$$

where

$$\omega_m = m\left(\frac{\Gamma(m+3/2)}{\Gamma(m+2)}\right)^2 \to 1, \qquad m\to\infty$$

system (12) will take the form

$$T_m(t) - k_0 \int_{t_0}^{t} K_3(t-\tau)X_k(\tau)\,d\tau + \sum_{k=1}^{\infty} \frac{R_{mk}^{(1)}}{k\omega_k} \int_{t_0}^{t} K_1(t-\tau)X_k(\tau)\,\mathrm{d}\tau$$

$$+ \sum_{k=1}^{\infty} \frac{R_{mk}^{(2)}}{\omega_k} \int_{t_0}^{t} K_2(t-\tau)X_k(\tau)\,\mathrm{d}\tau = g_m(t), \qquad m = 1, 2, \dots \tag{13}$$

In condition $G_0\varphi_3(t) = E_1\varphi_1(t) = E_2\varphi_2(t)$ system (13) reduces to the following ordinary differential equation of second order

$$\ddot{T}_m(t) + \gamma(1 + G_0\varphi_3(t))\dot{T}_m(t) = \ddot{g}_m(t) + \gamma\dot{g}_m(t), \tag{14}$$

with initial conditions:

$$T_m(t_0) = 0, \qquad \dot{T}_m(t_0) = \dot{g}_m(t_0)$$

The solution to this differential equation gives an infinite system of linear algebraic equations with respect to $X_m(t)$, $m = 1, 2, \dots$

$$k_0 X_m(t) - \sum_{k=1}^{\infty} \frac{R_{mk}^{(1)}}{k\omega_k} X_k(t) - \sum_{k=1}^{\infty} \frac{R_{mk}^{(2)}}{\omega_k} X_k(t) = \frac{T_m(t)}{\omega_m} \tag{15}$$

where

$$T_m(t) = \dot{g}_m(t_0) \int_{t_0}^{t} \frac{d\tau}{\alpha(\tau)} + \int_{t_0}^{t} \frac{d\tau}{\alpha(\tau)} \int_{t_0}^{\tau} [\ddot{g}_m(s) + \gamma\dot{g}_m(s)]\alpha(s)\,\mathrm{d}s,$$

$$\alpha(t) = \exp \int_{t_0}^{t} \gamma(1 + G_0\varphi_3(s))\,\mathrm{d}s$$

Let us investigate system (15) for regularity in the class of bounded sequences using the known relations for the Chebyshev first-order polynomials and for the Gamma function [5]

$$P_m^{(-1/2,-1/2)}(x) = \frac{\Gamma(m+1/2)}{\sqrt{\pi}\,\Gamma(m+1)} T_m(x), \qquad T_m(\cos(\theta)) = \cos m\theta, \qquad \lim_{m\to\infty} m^{b-a} \frac{\Gamma(m+a)}{\Gamma(m+b)} = 1$$

we have

$$R_{mk}^{(2)} = -\frac{2\lambda\alpha(k)\beta(m)}{\pi\sqrt{(k+1)(m+1)}} \int_0^{\pi} \cos(k+1)\theta \cos(m+1)\theta \sin\theta\, d\theta$$

$$= -\frac{2\lambda\alpha(k)\beta(k)}{\pi\sqrt{(k+1)(m+1)}} \times \begin{cases} 1 - \frac{1}{(2m+3)(2m+1)}, & k = m \\ -\frac{(-1)^{k+m}+1}{2} \left[\frac{1}{(k+m+3)(k+m+1)} + \frac{1}{(k-m+1)(k-m-1)} \right], & k \neq m, \end{cases}$$

$$= \begin{cases} O(m^{-1}), & k = m, \quad m \to \infty \\ O(m^{-5/2}), O(k^{-5/2}), & k \neq m, \quad k, m \to \infty, \end{cases},$$

where $\alpha(k), \beta(m) \to 1$, when $k, m \to \infty$.

By virtue of the Darboux asymptotic formula (see [8]), we obtain analogous estimates for

$$R_{mk}^{(1)} = \begin{cases} O(m^{-1}), & k = m, \quad m \to \infty, \\ O(m^{-5/2}), O(k^{-1/2}), & k \neq m, \quad k, m \to \infty \end{cases}$$

and the right-hand side $T_m(t)/\omega_m$ of equation (15) satisfies at least the estimate

$$\frac{T_m(t)}{\omega_m} = O(m^{-1/2}), m \to \infty$$

2. If $n = 2$ the solution to equation (3) will be sought in the form

$$\varphi'(t,x) = (1-x^2)^{3/2} \sum_{k=1}^{\infty} Y_k(t) P_k^{(3/2,3/2)}(x), \tag{16}$$

where numbers Y_k have to be defined for $k = 1, 2, \ldots$.

Using the relation arising from (8) and from the Rodrigues formula (see [21]) for the orthogonal Jacobi polynomials, we get

$$\frac{1}{\pi}\int_{-1}^{1} \frac{(1-x^2)^{3/2} P_k^{(3/2,3/2)}(t)\,dt}{t-x} = -2\pi P_{k+1}^{(-3/2,-3/2)}(x),$$

$$\varphi(t,x) = -(1-x^2)^{5/2} \sum_{k=1}^{\infty} \frac{Y_k(t)}{2k} P_{k-1}^{(5/2,5/2)}(x),$$

$$\varphi''(t,x) = -2(1-x^2)^{1/2} \sum_{k=1}^{\infty} k Y_k(t) P_{k+1}^{(1/2,1/2)}(x). \tag{17}$$

Similarly as for system (15), we obtain

$$\delta_m Y_m(t) - \sum_{k=1}^{\infty} \left(R_{mk}^{(3)} + \frac{R_{mk}^{(4)}}{k} \right) Y_k(t) = \widetilde{T}_m(t), \qquad m = 1, 2, \ldots \tag{18}$$

where

$$R_{mk}^{(3)} = -2\lambda \int_{-1}^{1} P_{k+1}^{(-3/2,-3/2)}(x) P_{m+1}^{(1/2,1/2)}(x)\,dx,$$

$$R_{mk}^{(4)} = \frac{1}{2} \int_{-1}^{1} \frac{1}{b_0(x)} P_{k-1}^{(5/2,5/2)}(x) P_{m+1}^{(1/2,1/2)}(x)\,dx,$$

$$\widetilde{g}_m(t) = \int_{-1}^{1} g(t,x) P_{m+1}^{(1/2,1/2)}(x)\,\mathrm{d}x$$

$$\delta_m = 4k_0 m \left(\frac{\Gamma(m+5/2)}{\Gamma(m+3)} \right)^2 \to 1, \qquad m \to \infty,$$

$$\widetilde{T}_m(t) = \dot{\widetilde{g}}_m(t_0) \int_{t_0}^{t} \frac{d\tau}{\alpha(\tau)} + \int_{t_0}^{t} \frac{d\tau}{\alpha(\tau)} \int_{t_0}^{\tau} [\ddot{\widetilde{g}}_m(s) + \gamma \dot{\widetilde{g}}_m(s)] \alpha(s)\,\mathrm{d}s.$$

Using again the Darboux formula, and the known relation for the Chebyshev second-order polynomials (see [21,22])

$$P_m^{(1/2,1/2)}(x) = \frac{\Gamma(m+3/2)}{\sqrt{\pi}\,\Gamma(m+2)} U_m(x), \qquad U_m(\cos\theta) = \frac{\sin(n+1)\theta}{\sin\theta},$$

we obtain the following estimates:

$$R_{mk}^{(3)} = \begin{cases} O(m^{-1}), & k = m, \quad m \to \infty, \\ O(m^{-5/2}), O(k^{-5/2}), & k \neq m, \quad k, m \to \infty, \end{cases}$$

$$R_{mk}^{(4)} = \begin{cases} O(m^{-1}), & k = m, \quad m \to \infty, \\ O(m^{-1/2}), O(k^{-1/2}), & k \neq m, \quad k, m \to \infty, \end{cases},$$

$$\widetilde{g}_m = O(m^{-1/2}), \qquad m \to \infty.$$

Thus, systems (15) and (18) are quasi-completely regular for any positive values of parameters k_0 and λ in the class of bounded sequences.

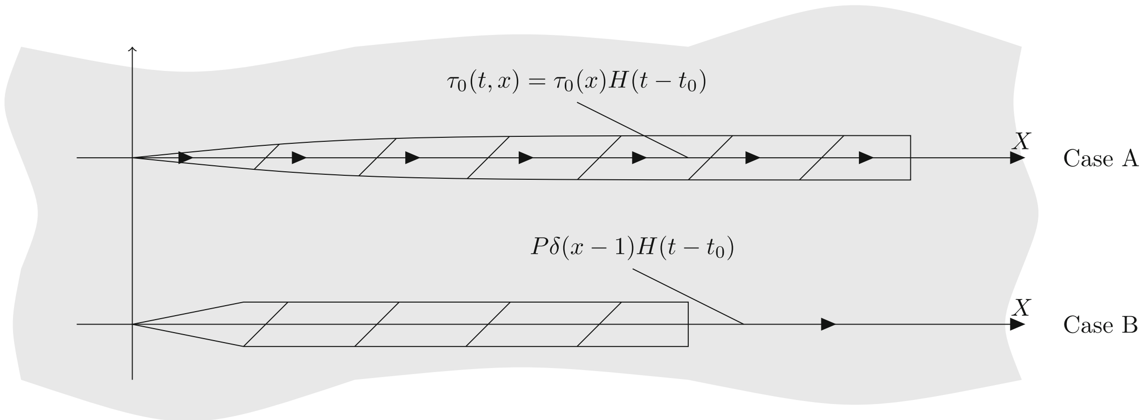

Fig. 1. Graph of cases A (upper) and B (lower)

On the basis of the Hilbert alternatives [23,24], if the determinants of the corresponding finite systems of linear algebraic equations are other than zero, then systems (15) and (18) will have unique solutions in the class of bounded sequences. Therefore, by the equivalence of system (15) (or (18)) and SIDE (3) the latter has a unique solution.

4. Exact solution to SIDE (3)

Case A. Suppose that a plate on a semi-infinite segment is reinforced by an inhomogeneous patch whose rigidity changes by the law $E(x) = hx^2$, $h = \text{const} > 0$. The patch is loaded by a tangential force of intensity $\tau_0(t,x) = \tau_0(x)H(t-t_0)$ and the plate is free from external loads (see Fig. 1). We have to define the law of distribution of tangential contact stresses $\tau(t,x)$ on the line of contact and the asymptotic behaviour of these stresses at the end of the patch.

$$\tau_0, \tau_0' \in H([0,\infty)), \qquad \tau_0(0) = 0, \qquad \tau_0'(x) = O(x^{-2}), \qquad x \to \infty, \qquad \int_0^\infty \tau_0(x)\,\mathrm{d}x = 0.$$

To determine the unknown contact stresses we obtain the following integral equation

$$\begin{aligned}
&(I-L_1)\frac{\eta_1(t,x)}{hx^2} - \frac{\lambda}{\pi}(I-L_2)\int_0^\infty \frac{\eta_1'(t,y)\,\mathrm{d}y}{y-x} - k_0(I-L_3)\eta_1''(t,x) = g_1(t,x), \qquad x > 0,\\
&\eta_1(t,0) = 0, \qquad \eta_1(t,\infty) = 0,\\
&\eta_1(t,x) = \int_0^x [\tau(t,y) - \tau_0(t,y)]\,dy, \qquad g_1(t,x) = k_0\tau_0'(t,x) + \frac{\lambda}{\pi}\int_0^\infty \frac{\tau_0(t,y)\,dy}{y-x}\\
&g_1 \in H((0,\infty)), \qquad g_1(t,x) = O(1), \quad x \to 0_+, \qquad g_1(t,x) = O(x^{-2}), \quad x \to \infty\\
&\eta_1, \eta_1' \in H([0,\infty)), \qquad \eta_1'' \in H((0,\infty))
\end{aligned} \tag{19}$$

The change of the variables $x = e^\xi$, $y = e^\zeta$ in equation (19) gives

$$\begin{aligned}
&(I-L_1)\ \frac{\varphi_0(t,\xi)}{he^\xi} - \frac{\lambda}{\pi}(I-L_2)\int_{-\infty}^\infty \frac{\varphi_0'(t,\zeta)\,d\zeta}{e^{\zeta-\xi}-1} - k_0e^{-\xi}(I-L_3)[\varphi_0''(t,\xi) - \varphi_0'(t,\xi)]\\
&\qquad = e^\xi g_0(t,\xi), \qquad |\xi| < \infty
\end{aligned} \tag{20}$$

where $\varphi_0(t,\xi) = \eta_1(t,e^\xi)$, $g_0(t,\xi) = g_1(t,e^\xi)$, $|g_0(t,\xi)| \le ce^{-|\xi|}$, $|\xi| \to \infty$.

Subjecting both parts of equation (20) to Fourier's transformation with respect to ξ [25] and using the convolution theorem under condition $E_1\varphi_1(\tau) = G_0\varphi_3(\tau)$, we obtain following boundary condition of the Carleman-type problem for a strip

$$(I - L_1)\Phi(t, s+i) + \frac{\lambda hs\mathrm{cth}\pi s}{1 + k_0hs(s+i)}(I - L_2)\Phi(t,s) = \frac{F(t,s)}{1 + k_0hs(s+i)}, \qquad |s| < \infty \tag{21}$$

where

$$\Phi(t,s) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\varphi_0(t,\xi)e^{i\xi s}\,\mathrm{d}\xi, \qquad F(t,s) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}e^{\xi}g_0(t,\xi)e^{i\xi s}\,\mathrm{d}\xi$$

Function $F(t,z)$ is holomorphic on strip $-1 < \mathrm{Im}\, z < 1$.

The Carleman-type problem for a strip is formulated as follows:

Find a function which is analytic on strip $-1 < \mathrm{Im}\, z < 1$, (with the exception of a finite number of points lying on strip $-1 < \mathrm{Im}\, z < 0$, at which it has poles), continuously extendable on strip boundary, vanishes at infinity and satisfies condition (21) [26,27].

If we find function $\Phi(t,z)$ which is holomorphic on strip $0 < \mathrm{Im}\, z < 1$, extends continuously on the strip boundary and satisfies condition (21), then the solution of the problem is the function

$$\Phi_0(t,z) = \begin{cases} \Phi(t,z), & 0 \le \mathrm{Im}\, z < 1 \\ \frac{-\Phi(t,z+i)+F_0(t,z)}{G(x)}, & -1 < \mathrm{Im}\, z < 0 \end{cases}$$

where

$$G(z) = \frac{\lambda hz\mathrm{cth}\pi z}{1 + k_0hz(z+i)}, \qquad F_0(t,z) = \frac{F(t,z)}{1 + k_0hz(z+i)}$$

Representing the function $G(s)$ in the form

$$G(s) = \frac{\lambda s}{ik_0(s^2+1)}\frac{k_0h(s^2+1)\mathrm{cth}\pi s\mathrm{th}\frac{\pi}{2}s}{1 + k_0hs(s+i)}\frac{\mathrm{sh}\frac{\pi}{2}(s+i)}{\mathrm{sh}\frac{\pi}{2}s} = \frac{\lambda s}{ik_0(s^2+1)}G_0(s)\frac{\mathrm{sh}\frac{\pi}{2}(s+i)}{\mathrm{sh}\frac{\pi}{2}s},$$

where

$$G_0(s) = \frac{k_0h(s^2+1)\mathrm{cth}\pi s\mathrm{th}\frac{\pi}{2}s}{1 + k_0hs(s+i)}.$$

and remarking that the index of function $G_0(s)$ on $(-\infty,\infty)$ is equal to zero and $G_0(s) \to 1$, $s \to \pm\infty$, function $\ln G_0(s)$ is integrable on the axis and we can write it in the form

$$G_0(s) = \frac{X_0(s+i)}{X_0(s)}, \qquad |s| < \infty, \tag{22}$$

where

$$X_0(z) = \exp\left\{\frac{1}{2i}\int_{-\infty}^{\infty}\ln G_0(s)\mathrm{cth}\pi(s-z)\,\mathrm{d}s\right\}.$$

Function $X_0(z)$ is holomorphic on strip $0 < \mathrm{Im}\, z < 1$ and bounded on the closed strip.

Substituting (22) in condition (21) and introducing the notations

$$\Psi(t,z) = \frac{z\Phi(t,z)}{X_1(z)}, \qquad \lambda_0 = \frac{k_0}{\lambda}, \qquad X_1(z) = X_0(z)X(z)\mathrm{sh}\frac{\pi z}{2},$$

$$X(z) = \lambda_0^{iz}\Gamma(2+iz), \qquad F(t,z) = \frac{(z+i)F_0(t,z)}{X_1(z+i)},$$

we have

$$(I - L_1)\Psi(t, s+i) + (I - L_2)\Psi(t,s) = F(t,s), \qquad |s| < \infty, \tag{23}$$

Using Stirling's formula [22] for the Gamma-function, the following estimate is valid

$$|X(z)| = O(|s|^{3/2-\omega})e^{-\pi|s|/2}, \qquad |X_1(z)| = O(|\delta|^{3/2-\omega}), \quad z = s + i\omega, \quad 0 \le \omega \le 1.$$

Applying the Fourier transformation to (23), we obtain the Volterra's integral equation of second kind

$$[e^{w}(I-L_1)+(I-L_2)]\widehat{\Phi}_1(t,w)=\widehat{F}(t,w)$$

where $\hat{\Phi}_1(t,\omega)$, $\hat{F}(t,\omega)$ are the Fourier transformations of functions $\Psi(t,s)$, $F_1(t,s)$, respectively.

Since function $F(t,z)$ is analytic on strip $-1<\operatorname{Im} z<1$ and $F(t,z)\to 0$ uniformly, for $|z|\to\infty$, function $\widehat{F}(t,w)$ exponentially vanishes at infinity, i.e. $|\widehat{F}(t,w)|<c\exp(-|w|)$, $|w|\to\infty$.

It is easy to show that integral equation (19) can equivalently be reduced to the following differential equation of second order

$$\ddot{\widehat{\Phi}}_1(t,w)+\gamma a(t,w)\dot{\widehat{\Phi}}_1(t,w)=g(t,w) \tag{24}$$

with the initial conditions

$$\begin{aligned}
&\widehat{\Phi}_1(t_0,w)=\widehat{F}_1(t_0,w)(1+e^{w})^{-1},\\
&\dot{\widehat{\Phi}}_1(t_0,w)=\left[\dot{\widehat{F}}_1(t_0,w)-\gamma\widehat{F}_1(t_0,w)(e^{w}\varphi_1(t_0)+\varphi_2(t_0))(1+e^{w})^{-1}\right](1+e^{w})^{-1}
\end{aligned}$$

where

$$\begin{aligned}
a(t,w)&=1+(E_1e^{w}\varphi_1(t)+E_2\varphi_2(t))(1+e^{w})^{-1},\\
g(t,w)&=g_0(t,w)(1+e^{w})^{-1},\\
g_0(t,w)&=\ddot{\widehat{F}}_1(t,w)+\gamma\dot{\widehat{F}}_1(t,w).
\end{aligned}$$

Integrating differential equation (24) and fulfilling the initial conditions, for function $\widehat{\Phi}_1(t,w)$ we obtain the expression

$$\widehat{\Phi}_1(t,w)=\{\widehat{F}_1(t,w)+F_1(t,t_0,w)\}(1+e^{w})^{-1} \tag{25}$$

where

$$\begin{aligned}
F_1(t,t_0,w)&=\gamma\widehat{F}_1(t_0,w)(e^{w}\varphi_1(t_0)+\varphi_2(t_0))(1+e^{w})^{-1}\int_{\tau_0}^{t}\exp(-\gamma b(w,\tau,t_0)\,\mathrm{d}\tau\\
&\quad-\gamma\int_{\tau_0}^{t}\exp(-\gamma b(w,\tau,t_0)\,d\tau\int_{\tau_0}^{\tau}(\alpha(q,w)-1)\exp(\gamma b(w,q,t_0)\dot{\widehat{F}}_1(q,w)\,\mathrm{d}q,\\
b(w,\tau,t_0)&=\int_{\tau_0}^{\tau}a(p,w)\,\mathrm{d}p=(\tau-t_0)+(E_1e^{w}\psi_1(\tau,t_0)+E_2\psi_2(\tau,t_0))(1+e^{w})^{-1},\\
\psi_1(\tau,t_0)&=\int_{t_0}^{\tau}\varphi_1(p)\,\mathrm{d}p,\qquad \psi_2(\tau,t_0)=\int_{t_0}^{\tau}\varphi_2(p)\,\mathrm{d}p
\end{aligned}$$

Function $\widehat{\Phi}_1(t,w)$ given by (25) has the same property as function $\widehat{F}_1(t,w)$ when $|w|\to\infty$.

By the inverse transformation of equality (25) and using the generalized Parseval's formula we obtain

$$\begin{aligned}
\Phi(t,z)&=\frac{X_1(z)}{iz}\sqrt{\frac{2}{\pi}}\int_{-\infty}^{\infty}\frac{F(t,s)(is+1)\,\mathrm{d}s}{X_1(s-i)\mathrm{sh}\pi(s-z)}\\
&\quad+\frac{X_1(z)}{iz}\gamma(e^{w}\varphi_2(\tau_0)+\varphi_1(\tau_0))\int_{t_0}^{t}Q_1(\tau,z)\,\mathrm{d}\tau-\frac{X_1(z)}{iz}\gamma\int_{t_0}^{t}d\tau\int_{t_0}^{\tau}Q_2(\tau,q,z)\,\mathrm{d}q
\end{aligned} \tag{26}$$

where

$$\begin{aligned}
Q_1(\tau,z)&=\int_{-\infty}^{\infty}\frac{\exp(-\gamma b(w,\tau,\tau_0)\widehat{F}_1(\tau_0,w)e^{-iwz}\,\mathrm{d}w}{(1+e^{w})^2},\\
Q_2(\tau,q,z)&=\int_{-\infty}^{\infty}\frac{\exp(-\gamma b(w,\tau,\tau_0)(\alpha(q,w)-1)\exp(\gamma b(w,q,\tau_0)\dot{\widehat{F}}_1(q,w)e^{-iwz}\,\mathrm{d}w}{1+e^{w}}
\end{aligned}$$

Thus functions $F(t,z)$, $Q_1(\tau,z)$, $Q_2(\tau,q,z)$ are analytic on strip $-1 < \operatorname{Im} z < 1$ and vanish uniformly $|\operatorname{Re} z| \to \infty$. The function defined by (26) is holomorphic on strip $-1 < \operatorname{Im} z < 0$, and continuously extendable on the strip boundary.

If function $F(t,z)$ (or $F_0(t,z)$) exponentially vanishes at infinity, then it is easy to prove that function $\Phi(t,z)$ has the same property. The inverse Fourier's transformation gives

$$\tau(t,x) = \tau_0(t,x) + \eta_1'(t,x) = \tau_0(t,x) + \frac{x^{-1}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} is\Phi(t,s)e^{-is\ln x}\,\mathrm{d}s. \tag{27}$$

Taking into account Cauchy's formula, we get

$$\begin{aligned}\tau(t,x) &= \tau_0(t,x) + \frac{ix^{-1}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} (s+i)\Phi(t,s+i)e^{-i(s+i)\ln x}\,\mathrm{d}s \\ &= \tau_0(t,x) + \frac{i}{\sqrt{2\pi}} \int_{-\infty}^{\infty} (s+i)\Phi(t,s+i)e^{-is\ln x}\,\mathrm{d}s.\end{aligned}$$

Consequently, for the tangential contact stresses have

$$\tau(t,x) = \tau_0(t,x) + \begin{cases} O(1), & x \to 0_+ \\ O(x^{-1-\delta}), & x \to \infty, \quad \delta > 0 \end{cases} \tag{28}$$

The obtained results can be formulated as

Theorem 2. *If $E(x) = h_0x^2$, $x > 0$, $h_0 = const > 0$, integro-differential equation (19) has the solution, which is represented effectively by (27) and admits estimate (28).*

Conclusion 1. Thus, when the rigidity of half infinite patch changes with parabolic law the tangential contact stresses at the thin end of inclusion has no singularities, it is bounded.

Case B. Suppose that on the finite segment of OX axis, the plate is reinforced by an inhomogeneous patch whose rigidity changes by the law $E(x) = hx$, $h = \text{const} > 0$ (for example, a wedge shaped inclusion). The contact between the plate and the patch is achieved by a thin glue layer with rigidity $k_0(x) = k_0x$, $0 < x < 1$, $k_0 = \text{const} > 0$.

The patch is loaded by a horizontal force $P\delta(x-1)H(t-t_0)$ and the plate is free from external loads (see Fig. 1).

To define the unknown contact stresses we obtain the following integral equation

$$\begin{aligned}&(I-L_1)\ \frac{\eta_2(t,x)}{E(x)} - \frac{\lambda}{\pi}(I-L_2)\int_0^1 \frac{\eta_2'(t,y)\,\mathrm{d}y}{y-x} - (I-L_3)(k_0(x)\eta_2'(t,x))' = 0, \qquad 0<x<1,\\ &\eta_2(t,0)=0, \qquad \eta_2(t,1)=P, \qquad \eta_2(t,x)=\int_0^x \tau(t,y)\,dy,\\ &\eta_2 \in H([0,1)), \qquad \eta_2' \in C((0,1)), \qquad \sup_{x\in(0,1)} |\eta_2'(x)| < \infty.\end{aligned} \tag{29}$$

The change of variables $x = e^{\xi}$, $y = e^{\zeta}$ in equation (29) gives

$$\begin{aligned}&(I-L_1)\ \frac{\psi(t,\xi)}{h} + \frac{\lambda}{\pi}(I-L_2)\int_{-\infty}^0 \frac{\psi'(t,\zeta)\,d\zeta}{1-e^{-(\xi-\zeta)}} - k_0(I-L_3)\psi''(t,\xi) = 0, \qquad \xi<0,\\ &\psi(t,-\infty)=0, \qquad \psi(t,0)=P, \qquad \psi(t,\xi)=\eta_2(t,e^{\xi}).\end{aligned} \tag{30}$$

Applying Fourier's transformation to both parts of equation (30) and using the convolution theorem we obtain the following boundary condition of the Riemann problem [25]

$$\begin{aligned}\Psi^+(t,s) = (I-L_1)\Phi^-(t,s) + \lambda \mathrm{hscth}\pi s(I-L_2)\Phi^-(t,s) + k_0hs^2(I-L_3)\Phi^-(t,s) + g_{01}(t,s),\\ -\infty < s < \infty,\end{aligned} \tag{31}$$

where

$$\Phi^-(t,s)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{0}\psi(t,\zeta)e^{is\zeta}\,d\zeta,\qquad g_{01}(t,s)=\frac{1}{\sqrt{2\pi}}(Pi\lambda h(\mathrm{cth}\pi s)_- + Pik_0hs - k_0h\psi'(t,0)),$$

$$\Psi^+(t,s)=\frac{h}{\pi}\int_0^{\infty}\psi^+(t,\zeta)e^{is\zeta}\,d\zeta,\qquad \psi^+(t,\xi)=\begin{cases}0, & \xi<0\\ \frac{\lambda}{\pi}\int_{-\infty}^{0}\frac{\psi'(t,\zeta)\,d\zeta}{1-e^{-(\xi-\zeta)}}-k_0\psi''(t,\xi), & \xi>0\end{cases}$$

Equation (31) under condition

$$G_0\varphi_3(t)=E_1\varphi_1(t)=E_2\varphi_2(t)$$

takes the form

$$\Psi^+(t,s)=(1+\pi\lambda s\mathrm{cth}\pi s+k_0hs^2)[\ddot{\Phi}(t,s)+\gamma(1+E_1\varphi_1(t+\rho_1))\dot{\Phi}(t,s)]^- + g_{01}(t,s) \tag{32}$$

The problem can be formulated as follows: it is required to obtain function $\Psi^+(z)$, holomorphic in the $\operatorname{Im} z>0$ half-plane, which vanishes at infinity, and function $\Phi^-(z)$ holomorphic in the $\operatorname{Im} z<1$ half-plane (with the exception of a finite number roots of function $G_1(z)$), which vanishes at infinity. Both are continuous on the real axis and satisfy condition (32) [25]. Boundary condition (32) is represented in the form

$$\frac{\Psi^+(t,s)}{s+i}=\frac{G_1(s)}{1+s^2}[\ddot{\Phi}(t,s)+\gamma(1+E_1\varphi_1(t+\rho_1))\dot{\Phi}(t,s)]^-\cdot(s-i)+\frac{g_{01}(t,s)}{s+i}.$$

$$G_1(s)=1+\lambda hs\mathrm{cth}\pi s+k_0hs^2,$$

$$G_{01}(s)=(k_0h)^{-1}G_1(s)(1+s^2)^{-1},\quad \operatorname{Re}G_{01}(s)>0,$$

$$G_{01}(\infty)=G_{01}(-\infty)=1,\quad \mathrm{Ind}G_{01}(s)=0. \tag{33}$$

Introducing the notation

$$[\ddot{\Phi}(t,s)+\gamma(1+E_1\varphi_1(t+\rho_1))\dot{\Phi}(t,s)]^-=K^-(t,s)$$

the solution of this problem has the form [28]

$$K^-(t,z)=\frac{\widetilde{X}(t,z)}{k_0h(z-i)},\qquad \operatorname{Im}z\le 0,\qquad \Psi^+(t,z)=\widetilde{X}(t,z)(z+i),\qquad \operatorname{Im}z>0,$$

$$K^-(t,z)=(\Psi^+(t,z)-g_{01}(t,z))G_1^{-1}(z),\qquad 0<\operatorname{Im}z<1, \tag{34}$$

where

$$\widetilde{X}(t,z)=X(z)\left\{\frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{g_{01}(t,y)\,\mathrm{d}y}{X^+(y)(y+i)(y-z)}\right\},\qquad X(z)=\exp\left\{\frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{\ln G_{01}(y)\,\mathrm{d}y}{y-z}\right\}.$$

we have the following differential equation

$$\ddot{\Phi}^-(t,s)+\gamma(1+E_1\varphi_1(t+\rho_1))\dot{\Phi}^-(t,s)=K^-(t,s) \tag{35}$$

with the initial conditions

$$\Phi^-(t_0,s)=K^-(t_0,s),\qquad \dot{\Phi}^-(t_0,s)=K^-(t_0,s)\gamma E_1\varphi_1(t_0+\rho_1)$$

Integrating differential equation (35) and fulfilling the initial condition, for function $\Psi^-(t,s)$ we obtain the expression

$$\Phi^-(t,s)=K^-(t,s)(1+T(t)) \tag{36}$$

where

$$T(t)=\gamma E_1\varphi_1(t_0+\rho_1)\int_{t_0}^{t}\exp(-\gamma b(\tau,t_0)\,\mathrm{d}\tau+\int_{t_0}^{t}[\exp(-\gamma b(\tau,t_0)\int_{t_0}^{\tau}\exp(\gamma b(p,t_0)\,dp]\,\mathrm{d}\tau,$$

$$b(\tau,t_0)=\int_{t_0}^{\tau}\alpha(q)\,\mathrm{d}q,\qquad \alpha(q)=1+E_1\varphi_1(q+\rho_1)$$

The boundary value of function $Q^{-}(t,z)=\frac{P}{2\sqrt{2\pi}}-iz\Phi^{-}(t,z)$ is the Fourier transform of function $\Psi'(t,e^{\zeta})$. Therefore, we get

$$\tau(t,x)=\eta_2'(t,x)=\frac{1}{\sqrt{2\pi}x}\int_{-\infty}^{\infty}Q^{-}(t,s)e^{-is\ln x}\,\mathrm{d}s, \tag{37}$$

$$\tau(t,x)=O(1), \qquad x\to 1_{-} \tag{38}$$

$$\tau(t,x)=O(x^{y_0-1}), \qquad x\to 0_{+}, \qquad y_0>1/\sqrt{k_0h} \tag{39}$$

Remark 2. If $k_0h\le 1$, then $\tau(t,x)=O(1)$, $x\to 0_{+}$.

Remark 3. If $k_0h=4$, then $G_1(i/2)=0$ and $\tau(t,x)=O(x^{-1/2})$, $x\to 0_{+}$.

Thus, the following theorem is proven:

Theorem 3. *Integro-differential equation* (29) *has the solution, which is represented effectively by formula* (37) *and admits estimates* (38), (39).

5. Discussion and numerical results

Asymptotic estimates for the solution to integro-differential equation (2) are obtained. A method of reduction for infinite regular systems of linear algebraic equations is justified. For any law of variation of the stiffness of the patch, tangential contact stresses have finite values at the ends of patches.

To obtain numerical results, specific values of the aging functions of the plane, patch and glue are considered in the form

$$\varphi_1(t)=0.0098\varphi_3(t)$$
$$\varphi_2(t)=0.00123\varphi_3(t)$$
$$\varphi_3(t)=0.09\cdot 10^{-10}+\frac{4.82\cdot 10^{-10}}{t}$$

The numerical values of the remaining parameters of the problem are taken as follows:

$$E_1=120\cdot 10^{9}\mathrm{MPa},\quad \nu_1=0.5,\quad E_2=95\cdot 10^{9}\mathrm{MPa},\quad \nu_2=0.3,$$
$$G_0^{(1)}=0.117\cdot 10^{9}\mathrm{MPa},\quad (G_0^{(2)}=11.7\cdot 10^{9}\mathrm{M}Pa),\quad h_0=5\cdot 10^{-4}\mathrm{m},\quad h_1(x)=h_1=5\cdot 10^{-2}\mathrm{m},$$
$$\gamma=0.026\,\mathrm{day}^{-1},\quad q_0^{(1)}(x)=10^{5}\sqrt{1-x^2}\,\mathrm{N},\quad (q_0^{(2)}(x)=10^{7}\sqrt{1-x^2}\,\mathrm{N}),\quad \rho_i=0\quad (i=1,2,3),$$
$$t_0=45\,\mathrm{days}, t^{(1)}=2.5\cdot 10^{3}\,\mathrm{days}, (t^{(2)}=9\cdot 10^{3}\mathrm{days})$$

The shortened finite systems of linear equations corresponding to systems (15) and (18), consisting of 10 and 12 equations have been solved. The results of the calculation show that an increase in the number of equations in the systems led to a change only in the seventh decimal place in the solutions.

Increasing the shear modulus of the glue causes the increase of the sought contact stresses, and the increase of the time value is corresponded by a decrease of the values of these stresses.

For comparison, the following should be noted: in contrast to a number of works in which a rigid contact between two interacting materials is considered and where unknown contact stresses have singularities at the ends of the contact line (i.e. stress concentrations arise), in this work, the contact between two bodies with viscoelastic (creep) properties is carried out using a thin layer of glue and, therefore, the found contact stresses at the ends of the contact line turned out to be limited (finite).

Obviously, the absence of stress concentration in the deformable body is extremely important from a engineering point of view.

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(Received: October 6, 2020; revised: January 4, 2021; accepted: January 6, 2021)