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Abstract	The problem of constructing an exact solution of singular integro-differential equations related to problems of adhesive interaction between elastic thin semi-infinite homogeneous patch and elastic plate is investigated. For the patch loaded with horizontal forces the usual model of the uniaxial stress state is valid. Using the methods of the theory of analytic functions and integral transformation, the singular integro-differential equation is reduced to the Riemann boundary value problem of the theory of analytic functions. The exact solution of this problem and asymptotic estimates of tangential contact stresses are obtained.

Keywords (separated by '-')	Adhesive contact problem - Elastic patch - Integro-differential equation - Integral transformation - Riemann problem
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Exact solutions of some singular integro-differential equations related to adhesive contact problems of elasticity theory

Nugzar Shavlakadze, Nana Odishelidze and Francisco Criado-Aldeanueva

Abstract. The problem of constructing an exact solution of singular integro-differential equations related to problems of adhesive interaction between elastic thin semi-infinite homogeneous patch and elastic plate is investigated. For the patch [1] loaded with horizontal forces the usual model of the uniaxial stress state is valid. Using the methods of the theory of analytic [2] functions and integral transformation, the singular integro-differential equation is reduced to the Riemann boundary value problem of the theory of analytic functions. The exact solution of this problem and asymptotic estimates of tangential [3] contact stresses are obtained.

Mathematics Subject Classification. 74B05, 74K20, 74K15.

Keywords. Adhesive contact problem, Elastic patch, Integro-differential equation, Integral transformation, Riemann problem.

1. Introduction

Exact or approximate solutions of static contact problems for different domains, reinforced by elastic thin inclusions and patches of variable stiffness were obtained, and the behavior of the contact stresses at the ends of the contact line has been investigated as a function of the geometrical and physical parameters of these elements [1–7,14–24]. One model assumes continuous interaction, while the other assumes the adhesive contact of thin-shared elements (stringers or inclusions) with massive deformable bodies. In [11], a finite-length stringer is attached to a thin elastic sheet subjected to plane stress. The two different materials are joined along the entire stringer length by a thin uniform elastic adhesive layer assumed to be in pure shear state. The bending is neglected and the interaction between the sheet and the stringer is idealized as a line loading of the sheet. In [9], an elastic semi-infinite plate is strengthened by an elastic finite stringer. The contact between the plate and the stringer is achieved by a thin glue layer. Asymptotic estimates, exact and approximate solutions of the associated integro-differential equation are obtained.

In the present paper, the exact solution of singular integro-differential equations related to the problems of adhesive interaction between an elastic thin semi-infinite homogeneous patch and an elastic plate is obtained. From the physical point of view, consideration of the case of the adhesive interaction between the plate and the stringer is interesting since the boundedness of the unknown tangential contact stress near the ends of the stringer is proved by rigorous mathematical methods. The limit transition (case A) corresponds to the case of rigid contact, in which the obtained solution, i.e. tangential contact stress, has a singularity at the end of the stringer.

2. Formulation of the problem and reduction to the integro-differential equation

Let a semi-infinite patch with modulus of elasticity $E_1(x)$, thickness $h_1(x)$ and Poisson's coefficient ν_1 be attached to the plate (E_2, ν_2), which is in the condition of a plane deformation. It is assumed that the

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horizontal stresses with intensity $\tau_0(x)$ act on the patch along the ox-axis. In the horizontal direction, the patch is compressed or stretched like a rod being in uniaxial stress state. The contact between the plate and patch is achieved by a thin glue layer with width h_0 and Lame's constants λ_0, μ_0.

The adhesive contact condition has the form [11]

$$u_1(x) - u_2(x,0) = k_0\tau(x), \qquad x > 0 \tag{1}$$

where $u_2(x,y)$ and $u_1(x)$ are displacements of the plate points and displacements of the patch points, respectively. $\tau(x)$ is unknown tangential contact stresses and $k_0 = h_0/\mu_0$.

We have to define the law of distribution of tangential contact stresses $\tau(x)$ on the line of contact, the asymptotic behavior of these stresses at the end of the patches.

According to the equilibrium equation of patch elements and Hooke's law, we have:

$$\frac{\mathrm{d}u_1(x)}{\mathrm{d}x} = \frac{1}{E(x)}\int_0^x [\tau(t) - \tau_0(t)]\mathrm{d}t, \qquad x > 0, \tag{2}$$

and the equilibrium equation of the patch has the form

$$\int_0^\infty [\tau(t) - \tau_0(t)]\mathrm{d}t = 0,$$

$$E(x) = \frac{E_1(x)h_1(x)}{1-\nu_1^2} \tag{3}$$

According to known results for plate [12], the horizontal deformation of the points of the OX axis has the form

$$\frac{\mathrm{d}u_2(x,0)}{\mathrm{d}x} = \frac{b}{\pi}\int_0^\infty \frac{\tau(t)\mathrm{d}t}{t-x} \tag{4}$$

where

$$b = \frac{2(1-\nu_2^2)}{E_2}$$

Introducing the notation

$$g(x) = \int_0^x [\tau(t) - \tau_0(t)]\mathrm{d}t,$$

from (1), (2) and (4) we obtain the following integro-differential equation

$$\frac{g(x)}{E(x)} - \frac{b}{\pi}\int_0^\infty \frac{g'(t)\,\mathrm{d}t}{t-x} - k_0 g''(x) = f(x), \qquad x > 0 \tag{5}$$

where

$$f(x) = \frac{b}{\pi}\int_0^\infty \frac{\tau_0(t)\,\mathrm{d}t}{t-x} + k_0\tau_0'(x)$$

with the condition

$$g(\infty) = 0 \tag{6}$$

Thus, the above posed boundary contact problem is reduced to the singular integro-differential equation (5) with the condition (6).

The solutions of Eq. (5) under the condition (6) can be sought in the class of functions: $g, g' \in H[0,\infty)$, $g'' \in H(0,\infty)$ [13].

We assume that the function $\tau_0(x)$ is continuous in the Holder's sense and $\tau_0(x)$ has a first-order continuous derivative.

3. Exact solution of singular integro-differential equations

Suppose that a plate on a semi-infinite interval is reinforced with a homogeneous patch and is free of external loads. The contact between the plate and the patch is carried out through a thin layer of glue.

The problem means determination of contact stresses when a horizontal force T applies at one end of the patch (at a point $x = 0$). $E(x) = E_0 = \text{const}$.

Equation (5) and the boundary conditions (6) take the form

$$\varphi(x) - \frac{\lambda}{\pi}\int_0^\infty \frac{\varphi'(t)\mathrm{d}t}{t-x} - k\varphi''(x) = 0, \qquad x > 0 \tag{7}$$

$$\varphi(0) = T, \quad \varphi(\infty) = 0, \tag{8}$$

where

$$\varphi(x) = T - \int_0^x \tau(t)\mathrm{d}t, \quad \lambda = bE_0, \quad k = k_0E_0.$$

The solution of Eq. (7) is sought in the class of functions $\varphi, \varphi' \in H[0,\infty)$, $\varphi'' \in H(0,\infty)$.

By a generalized Fourier transform with the convolution theorem [8], from (7), (8) we arrive at a Riemann problem

$$\Phi^+(s)\left[1 + \lambda|s| + ks^2\right] = F^-(s) - i\lambda T\mathrm{sgn}s - k\varphi'(0) + iksT, \tag{9}$$

where $\Phi^+(s)$ and $F^-(s)$ are Fourier transforms of functions

$$\varphi_0(x) = \begin{cases}\varphi(x), & x \geq 0\\ 0, & x < 0\end{cases} \qquad \text{and} \qquad f(x) = \begin{cases}0, & x \geq 0\\ \frac{\lambda}{\pi}\int\limits_{-\infty}^{\infty}\frac{\varphi_0'(t)\mathrm{d}t}{t-x} - k\varphi_0''(x), & x < 0\end{cases}$$

4. Case A

(a) For $k \geq 0$, the coefficient of the problem (9) can be represented in the form

$$1 + \lambda|s| + ks^2 = \frac{1 + \lambda|s| + ks^2}{\sqrt{1+\lambda^2s^2}\sqrt{1+\widetilde{k}^2s^2}}\sqrt{\lambda s + i}\sqrt{\lambda s - i}\sqrt{\widetilde{k}s + i}\sqrt{\widetilde{k}s - i}, \quad \widetilde{k} = \frac{k}{\lambda}$$

and we consider the canonical solution of the problem of linear conjugation

$$X^+(s) = \frac{1 + \lambda|s| + ks^2}{\sqrt{1+\lambda^2s^2}\sqrt{1+\widetilde{k}^2s^2}}X^-(s) \tag{10}$$

Everywhere, we mean by functions of type $\sqrt{\lambda z + i}$ and $\sqrt{\lambda z - i}$ the branches that are analytic in planes with cuts along the rays, drawn from the points $z = -i/\lambda$ and $z = i/\lambda$, respectively, in the OX direction, and which take positive and negative values, respectively, on the upper side of the cut. With this choice of branches, the function $\sqrt{1 + \lambda^2z^2}$ is analytic in the strip $-1/\lambda < \Im z < 1/\lambda$ and takes a positive value on the real axis.

The function

$$X(z)=\exp\left\{\frac{1}{2\pi i}\int\limits_{-\infty}^{\infty}\ln\frac{1+\lambda|s|+ks^2}{\sqrt{1+\lambda^2s^2}\sqrt{1+\widetilde{k}^2s^2}}\frac{\mathrm{d}s}{s-z}\right\}$$

satisfies the boundary condition (10), does not vanish anywhere and $X^{\pm}(\infty)=1$.

Representing the boundary condition (9) in the form

$$\begin{aligned}&\Phi^+(s)\sqrt{(\lambda s+i)(\widetilde{k}s+i)}X^+(s)\\&=\frac{F^-(s)X^-(s)}{\sqrt{(\lambda s-i)(\widetilde{k}s-i)}}+\frac{(-i\lambda T\mathrm{sgn}s-k\varphi'(0)+ikTs)X^-(s)}{\sqrt{(\lambda s-i)(\widetilde{k}s-i)}}\end{aligned}$$

we get

$$\begin{aligned}&\Phi(z)\sqrt{(\lambda z+i)(\widetilde{k}z+i)}X(z)\\&=-\frac{\lambda T}{2\pi}\int\limits_{-\infty}^{\infty}\frac{X^-(s)\mathrm{sgn}s}{\sqrt{(\lambda s-i)(\widetilde{k}s-i)}}\frac{\mathrm{d}s}{s-z}-\frac{k\varphi'(0)}{2\pi i}\int\limits_{-\infty}^{\infty}\frac{X^-(s)}{\sqrt{(\lambda s-i)(\widetilde{k}s-i)}}\frac{\mathrm{d}s}{s-z}\\&\quad+\frac{kT}{2\pi}\int\limits_{-\infty}^{\infty}\frac{sX^-(s)}{\sqrt{(\lambda s-i)(\widetilde{k}s-i)}}\frac{\mathrm{d}s}{s-z}\end{aligned}$$

and based on the well-known Cauchy formula

$$\begin{aligned}\Phi(z)&=\frac{-\lambda T}{\pi X(z)\sqrt{(\lambda z+i)(\widetilde{k}z+i)}}\int\limits_{0}^{\infty}\frac{X^-(s)}{\sqrt{(\lambda s-i)(\widetilde{k}s-i)}}\frac{\mathrm{d}s}{s-z}\\&\quad+\frac{i\sqrt{k}T}{2X(z)\sqrt{(\lambda z+i)(\widetilde{k}z+i)}},\qquad \mathrm{Im}z>0\end{aligned}$$

The boundary value of function $K(z)=-T-iz\Phi(z)$ is Fourier transform of function $\varphi'(x)$.

We investigate the behavior at infinity of the following function:

$$\begin{aligned}K(z)&=-T-\frac{\lambda Tz}{\pi iX(z)\sqrt{(\lambda z+i)(\widetilde{k}z+i)}}\int\limits_{0}^{\infty}\frac{X^-(s)\mathrm{d}s}{\sqrt{(\lambda s-i)(\widetilde{k}s-i)}(s-z)}\\&\quad+\frac{z\sqrt{k}T}{2X(z)\sqrt{(\lambda z+i)(\widetilde{k}z+i)}}\end{aligned}\tag{11}$$

Introducing the notations

$$\begin{aligned}K_1(z)&=\frac{\lambda Tz}{\pi iX(z)\sqrt{(\lambda z+i)(\widetilde{k}z+i)}}\int\limits_{0}^{\infty}\frac{X^-(s)\,\mathrm{d}s}{\sqrt{(\lambda s-i)(\widetilde{k}s-i)}(s-z)},\\K_2(z)&=\frac{z\sqrt{k}T}{2X(z)\sqrt{(\lambda z+i)(\widetilde{k}z+i)}}\end{aligned}$$

and the change of variable $z=-1/\xi$, $s=-1/t_0$ for function $K_1(z)$ gives

$$K_1^*(\xi)=\frac{-\lambda T\xi}{\pi iX^*(\xi)\sqrt{(\lambda-i\xi)(\widetilde{k}-i\xi)}}\int\limits_{-\infty}^{0}\frac{X^{-*}(t_0)\,\mathrm{d}t_0}{\sqrt{(\lambda+it_0)(\widetilde{k}+it_0)(t_0-\xi)}}, \tag{12}$$

where $K_1^*(\xi)=K_1(z)$, $X^*(\xi)=X(z)$. By virtue of known results [13], $K_1^*(\xi)=O(\xi ln\xi)$, $\xi\to 0$ and, respectively, $K_1(z)=O(|z|^{-(1-\epsilon)})$, $|z|\to\infty$ (ϵ arbitrarily small positive number).

Since $K_2(\infty)=T/2$, , the function $\widetilde{K}_1^+(z)=K(z)+T/2$ is holomorphic in half-plate Im$z>0$ and vanishes at infinity as $|z|^{-(1-\epsilon)}$, $|z|\to\infty$.

Consequently, unknown tangential contact stresses are determined by the formula

$$\tau(x)=\varphi'(x)=\frac{1}{2\pi}\int\limits_{-\infty}^{\infty}\widetilde{K}_1(t)e^{-itx}\,\mathrm{d}t, \tag{13}$$

and it is bounded, when $x\to 0^+$.

By limiting transition $k\to 0$ from (12), we have

$$K_1^*(\xi)=\frac{\lambda T\sqrt{\xi}}{\pi iX^*(\xi)\sqrt{\lambda-i\xi}}\int\limits_{-\infty}^{0}\frac{X^{-*}(t_0)\mathrm{d}t_0}{\sqrt{t_0(\lambda+it_0)}(t_0-\xi)},$$

$$K_1^*(\xi)=-T+O(\xi^{1/2-\delta}),\qquad 0<\delta<1/2,\qquad \xi\to 0,$$

$$K_1(z)+T=O(|z|^{-(1/2+\delta)}),\qquad |z|\to\infty,\qquad K_2(z)=0.$$

The tangential contact stresses are determined by the formula

$$\tau(x)=\varphi'(x)=\frac{1}{2\pi}\int\limits_{-\infty}^{\infty}\widetilde{K}_2(t)e^{-itx}\mathrm{d}t$$

where the function $\widetilde{K}_2^+(z)=K(z)+2T$ is holomorphic in half-plate Im$z>0$ and vanishes at infinity as $|z|^{-(1/2+\delta)}$, $|z|\to\infty$.

Therefore, tangential contact stresses $\tau(x)$, when $x\to 0^+$, has a singularity less than 1/2. This result matches to results from [10,16,18].

5. Case B

(b) Let $k>k_1>0$, then the solution of problem (9) can be represented in another form. The coefficient of the problem can be written in the form

$$1+\lambda|s|+ks^2=\frac{1+\lambda|s|+ks^2}{1+ks^2}(1-i\sqrt{k}s)(1+i\sqrt{k}s)$$

and the canonical solution of the problem of linear conjugation

$$X^+(s)=\frac{1+\lambda|s|+ks^2}{1+ks^2}X^-(s)$$

has the form

$$X(z)=\exp\left\{\frac{1}{2\pi i}\int\limits_{-\infty}^{\infty}\ln\frac{1+\lambda|s|+ks^2}{1+ks^2}\frac{\mathrm{d}s}{s-z}\right\}.$$

The function $X(z)$ does not vanish anywhere and $X^\pm(\infty)=1$.

Representing the boundary condition (9) in the form

$$\Phi^+(s)(1-i\sqrt{k}s)X^+(s)=\frac{F^-(s)X^-(s)}{1+i\sqrt{k}s}-\frac{i\lambda T\mathrm{sgn}s+k\varphi'(0)-ikTs}{\sqrt{1+i\sqrt{k}s}}$$

we get

$$\Phi(z)(1-i\sqrt{k}z)X(z)=-\frac{\lambda T}{2\pi}\int_{-\infty}^{\infty}\frac{X^-(s)\mathrm{sgn}s}{1+i\sqrt{k}s}\frac{\mathrm{d}s}{s-z}$$

$$-\frac{k\varphi'(0)}{2\pi i}\int_{-\infty}^{\infty}\frac{X^-(s)}{1+i\sqrt{k}s}\frac{\mathrm{d}s}{s-z}+\frac{kT}{2\pi}\int_{-\infty}^{\infty}\frac{sX^-(s)}{1+i\sqrt{k}s}\frac{\mathrm{d}s}{s-z}$$

and based on the well-known Cauchy formula

$$\Phi(z)=-\frac{\lambda T}{\pi X(z)(1-i\sqrt{k}z)}\int_{0}^{\infty}\frac{X^-(s)}{1+i\sqrt{k}s}\frac{\mathrm{d}s}{s-z}+\frac{\sqrt{k}T}{2X(z)(1-i\sqrt{k}z)},\qquad \mathrm{Im}z>0$$

The boundary value of function $K^\circ(z)=-T-iz\Phi(z)$ is Fourier transform of function $\varphi'(x)$.

We investigate the behavior of the function

$$K^0(z)=-T+\frac{i\lambda Tz}{\pi X(z)(1-i\sqrt{k}z)}\int_{0}^{\infty}\frac{X^-(s)\mathrm{d}s}{(1+i\sqrt{k}s)(s-z)}-\frac{iz\sqrt{k}T}{2X(z)(1-i\sqrt{k}z)}$$

at infinity. Introducing the notations $K_1^0(z)=\frac{i\lambda Tz}{\pi X(z)(1-i\sqrt{k}z)}\int_0^\infty\frac{X^-(s)\mathrm{d}s}{(1+i\sqrt{k}s)(s-z)}$, $K_2^0(z)=\frac{-iz\sqrt{k}T}{2X(z)(1-i\sqrt{k}z)}$, and the change of variable $z=-1/\xi$, $s=-1/t_0$ in function $K_1^0(z)$ gives

$$K_1^{0*}(\xi)=\frac{\lambda T\xi}{\pi X^*(\xi)(\xi+i\sqrt{k})}\int_{-\infty}^{0}\frac{X^{-*}(t_0)\mathrm{d}t_0}{(t_0-i\sqrt{k})(\xi-t_0)},$$

where $K_1^*(\xi)=K_1(z)$, $X^*(\xi)=X(z)$.

It is obvious that $K_1^{0*}(\xi)=O(\xi ln\xi)$, $\xi\to 0$ and $K_1^0(z)=O(|z|^{-(1-\epsilon)})$ $|z|\to\infty$, $K_2^0(\infty)=\frac{T}{2}$ (ϵ arbitrarily small positive number).

Consequently, unknown contact stresses are determined by the formula

$$\tau(x)=\varphi'(x)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\widetilde{K}_3(t)e^{-itx}\mathrm{d}t, \tag{14}$$

where the function $\widetilde{K}_3^+(z)=K^0(z)+T/2$ is holomorphic in half-plate $\mathrm{Im}z>0$ and vanishes at infinity as $|z|^{-(1-\epsilon)}$, $|z|\to\infty$. Therefore, tangential contact stresses, defined by formula (14), are bounded when $x\to 0^+$.

Thus, it is proved the following theorem

Theorem 1. *Integro-differential equation (7)–(8) has the solution, which is represented effectively by formulas (13)–(14) and* $\varphi'(x)=O(1)$, $x\to 0^+$.

In Table 1 and Fig. 1, the dependence of the tangential contact stress $\tau(x)$ at the point $x=0$ with the value of k is presented for the following physical and geometric parameters of the problem: module of elasticity $E_2=95\cdot 10^9$ Pa and Poisson's coefficient $\nu_2=0.3$ of semi-plate material; module of elasticity $E_1=120\cdot 10^9$ Pa and Poisson's coefficient $\nu_1=0.5$ of stringer material and thickness of stringer $h_1=5\cdot 10^{-2}$ m; in the table value of number $k=h_0E_0/\mu_0$ is defined for various values of h_0, μ_0, for

TABLE 1. *Dependence of tangential contact stress $\tau(x)$ at $x = 0$ with the value of k*

k (m)	$\tau(0)$ (N/Dm^2)
$3.42 \cdot 10^{-2}$	1.5138
$1.52 \cdot 10^{-2}$	1.6279
$0.55 \cdot 10^{-2}$	2.5806
$0.25 \cdot 10^{-2}$	3.5236
$1.42 \cdot 10^{-3}$	6.9176
$0.55 \cdot 10^{-3}$	20.864
$0.25 \cdot 10^{-3}$	35.768
$1.0 \cdot 10^{-4}$	42.616
$0.7 \cdot 10^{-4}$	46.568
$1.0 \cdot 10^{-5}$	56.545
$0.5 \cdot 10^{-5}$	58.325

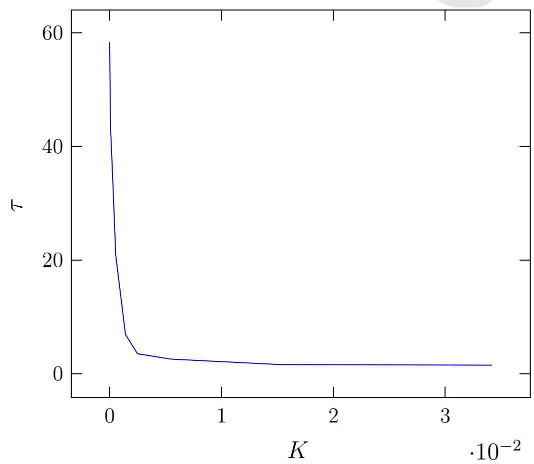

FIG. 1. Dependence of tangential contact stress $\tau(x)$ at $x = 0$ with the value of k

example, we obtain $k = 3.42 \cdot 10^{-2}$ m in case of shear module $\mu_0 = 0.117 \cdot 10^9$ Pa and thickness of glue layer $h_0 = 5 \cdot 10^{-4}$ m.

Calculations show that a decrease of thickness or an increase of the shear modulus of the adhesive material, i.e. a decrease of the number k, corresponds to tend to the rigid contact of the stringer with the plate, at which the tangential contact stress tends to infinity at the endpoints of the stringer.

6. Conclusion

In this paper, the well-known method of Wiener–Hopf is used for solving a Riemann problem. We made the factorizations for specific coefficients related to the investigated integro-differential equation, whose effective solutions and the asymptotic estimates are obtained.

The case $k = 0$ corresponds to the absolute rigid contact between the elastic plate and patch. $k \to 0$ means that adhesive contact tends to rigid contact between the elastic plate and patch. In case B, the integro differential equation is considered for $k > k_1 > 0$. The corresponding factorization of the coefficient of the Riemann problem was carried out, and the exact solution and asymptotic estimates were obtained.

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In case A, the same integro-differential equation is considered for $k \geq 0$. Here, the other factorization of the coefficient is carried out to make a limit transition ($k \to 0$) and to compare the result with the known results from [10,16,18].

The mechanical result of this study is the following: in condition of the rigid contact ($k = 0$) between elastic plate and patch, the tangential contact stresses at the end (in point $x = 0$) of elastic patch have a singularity less than 1/2; the tangential contact stresses are bounded in case of the adhesive contact among them ($k \neq 0$).

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