An axiomatic approach to measure the 'Leave No One Behind' principle

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Abstract

'Leave no one behind' has become an important principle of the entire 2030 Sustainable Development Agenda and the Sustainable Development Goals that brings inequality to the centre stage. Although a precise definition of what it is to lag behind is essential to assess sustainable economic growth and social progress a discussion on the desirable properties of such a measure is absent in the literature. This paper fills that gap by proposing and discussing a number of normative and analytical properties a measure of the `Leave no one behind´ principle should satisfy. The axioms proposed are necessary and sufficient to characterize the fuzzy measure by Garcia-Pardo et al. (2021) and together with other additional axioms make the structure of this measure more transparent.

Keywords: axiomatic characterization, fuzzy sets, 'to leave no one left behind' principle, economic growth and social progress

JEL classification: D31, D63

Highlights:

- Leaving no one behind is the central focus of the Sustainable Development Goals.
- The Leaving no one behind principle suggests going beyond the averages.
- The ambiguity of the left behind principle makes it convenient to use a fuzzy approach.
- We discuss desirable properties of a measure of the extent an individual lags behind.
- A subset of axioms characterizes the fuzzy set proposed by García-Pardo et al. (2021).

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1. **Introduction**

Leaving no one behind (LNOB) constitutes a central, crosscutting focus of the entire 2030 Sustainable Development Agenda and the Sustainable Development Goals (SDGs; see UN2015), bringing inequalities to centre stage. In committing to the realization of the 2030 Agenda for Sustainable Development, Member States endeavoured to reach those who are furthest behind first.

It is clear that a precise understanding and identification of those who are left behind and the quantification of the extent to which each individual is left behind is crucial in order to move from aspirational language to implementing specific and effective actions based on equality and non-discrimination to assure economic growth and social progress. The LNOB principle suggests going beyond the averages, but as Klasen and Fleurbaey (2018) have acknowledged, there is no precise formulation of this concept in practice.

There are proposals in the literature regarding pro-poor growth related to the measurement of the LNOB principle, where an absolute (Ravallion, 2004) and a relative approach (Kakwani and Pernia, 2000) can be identified. The World Bank (2013) proposes the measure of 'shared prosperity' as the increase in income of the poorest 40% of the population. More recently, Ravallion (2016) has highlighted the relevance of judging progress by success in raising the distribution floor. D'Ambrosio and Frick (2012) and Bossert and D'Ambrosio (2020) use a dynamic approach to identify individuals who lag behind as those who are overtaken in the final period by another individual who was further behind in the distribution in the initial period.

These proposals have some limitations to our understanding. Some are global measures that do not identify those who lag behind, nor do they quantify the degree to which they lag, others ignore what happens beyond a threshold, while yet others only identify who is exceeded in the distribution. When a threshold is imposed, any action that improves the situation of those above the threshold does not influence the measure and can even ignore situations in which inequality increases in favour of the very rich (Fleurbaey, 2018). Therefore, these measures are not sensitive to redistribution from the middle to the better-off positions.

Using a fuzzy approach, García-Pardo, et al. (2021) propose a measure of the LNOB principle that avoids the use of thresholds. They start by defining a fuzzy set, and then assign the degree to which an individual belongs to the set using a definition of fuzzy set based on the evaluation of an individual's achievement against the achievements of other individuals in society, which are taken as a sort of benchmark. Specifically, they use a transformation of the Yitzhaki (1979) and Hey and Lambert (1980) indexes of individual deprivation.

Given that the LNOB concept is ambiguous, it is desirable to preserve this ambiguity when measuring it; hence, the use of a fuzzy approach is appropriate. In doing so, Sen's (1997, p. 121) idea that 'a precise representation of that ambiguous concept must preserve that ambiguity' is satisfied. There has been no rigorous discussion on desirable axioms for a measure of the extent to which an individual is left behind in a fuzzy environment in the literature. The purpose of this paper is to present and discuss a number of properties that such a measure should satisfy. We show that a subset of these properties is sufficient to characterize the fuzzy set proposed by García-Pardo et al. (2021).

The paper is organized as follows. In section 2 preliminary definitions are presented. Section 3 proposes and discusses several axioms to properly address the extent to which an individual is left behind. Section 4 provides the characterization of the LNOB fuzzy measure by García-Pardo et al. (2021) and section 5 discusses further additional properties. Finally, Section 6 concludes.

2. **Preliminary definitions**

We consider a complete residuated lattice $\mathbb{L} = (L, \leq, \otimes, \rightarrow, 0, 1)$, that is, an algebra where $(L, \leq, 0, 1)$ is a complete lattice, the least element is 0 and the greatest element is 1, $(L, \otimes, 1)$ is a commutative monoid and $(\otimes, \rightarrow)$ is an adjoint couple, $x \leq y \rightarrow z$ if and only if $x \otimes y \leq z$ for all $x, y, z \in L$ ($\leq$ denoted the lattice ordering).[3] We denote the supremum and infimum operation in the lattice with the symbols $\vee$ and $\wedge$, respectively.

An $\mathbb{L}$-fuzzy set on U is a mapping $A: U \rightarrow L$ where, for all $u \in U$, $A(u)$ is called the degree of membership of u in A. The set of all $\mathbb{L}$-fuzzy[4] sets on U is denoted by L^U.

Let $A, B \in L^U$, where A is equal to B and denoted $A = B$ if $A(u) = B(u)$ for all $u \in U$, and where A is included in B and denoted $A \subseteq B$ if $A(u) \leq B(u)$ for all $u \in U$.

Theorem 1. [Zadeh (1965)] Let U be a population set and consider $A: U \rightarrow [0,1]$ a fuzzy set. Then, the following assertions are equivalent:

i. A is convex.
ii. $A(\lambda i + (1-\lambda)j) \geq A(i) \wedge A(j)$ for all $i, j \in U$ and for all $\lambda \in [0,1]$.

García-Pardo et al. (2021) measure the degree an individual is 'left behind', as follows.

Definition 1. Let $\mathbb{L} = (L, \leq, \otimes, \rightarrow, 0, 1)$ be a complete residuate and h be a continuous[5] dimension. Given a population set U composed of $n \geq 2$ individuals, for each individual $i \in U$, the fuzzy set LB_h is defined as the mapping $LB_h: U \rightarrow L$,

$$LB_h(i) = \frac{\frac{1}{n}\sum_{j=i+1}^{n}(x_{h,j} - x_{h,i})}{\mu_h} = \left(1 - F(x_{h,i})\right)\frac{\left(\mu^+_{x_{h,i}} - x_{h,i}\right)}{\mu_h} \tag{1}$$

where $x_{h,i}$ is the value of the dimension h of individual with ranking i with $x_{h,i} < x_{h,i+1}$, $F(x_{h,i})$ is the distribution function, μ_h is the mean of the distribution of dimension h, and $\mu^+_{x_{h,i}}$ is the mean of dimension h for individuals with values greater than $x_{h,i}$.

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[3] For more details, see, for example, Bêlohlávek (2002), Birkhoff (1967) and Davey and Priestley (2002).

[4] From now on, when no confusion arises, we will omit the prefix $\mathbb{L}$.

[5] This definition applies to continuous dimensions and discrete (non-continuous) dimensions, see Garcia-Pardo et al. (2021) for more details.

Thus, $LB_h(i)$ represents *the degree the individual* $i \in U$ *is 'left behind' in dimension* h.

The following propositions show that, for $0 \leq x_{h,i}$, LB_h is defined on $L = [0,1]$ and is a decreasing mapping in the values $x_{h,i}$.

Proposition 1 [García-Pardo et al. (2021)]. Let U be a population set, $\mathbb{L}$ be a residuated lattice, and h be a continuous dimension. Then, the map $LB_h: U \to L$, where $LB_h(i) = \left(1 - F(x_{h,i})\right)\frac{\left(\mu^{+}_{x_{h,i}} - x_{h,i}\right)}{\mu_h}$, verifies that $0 \leq LB_h(i) \leq 1$ for all $i \in U$.

Proposition 2 [García-Pardo et al. (2021)]. Let U be a population set, $\mathbb{L}$ be a residuated lattice, and h be a continuous dimension. Consider the map $LB_h: U \to L$ such that

$LB_h(i) = \left(1 - F(x_{h,i})\right)\frac{\left(\mu^{+}_{x_{h,i}} - x_{h,i}\right)}{\mu_h}$; if $x_{h,i} \leq x_{h,j}$, then $LB_h(j) \leq LB_h(i)$ for all $i, j \in U$.

With the aim of simplifying and clarifying the description of the properties and the results in this work, from now on, and without loss of generality, we consider only one dimension and refer to it as income.

3. Properties of the measures of the Leaving no one behind (LNOB) principle

The set of all possible fuzzy sets to measure the extent an individual is left behind can be rather large and could contain many mappings. A number of desirable properties could be accepted as "basic" properties of a fuzzy set to measure the LNOB principle and as such serve to reduce the number of allowable mappings. The choice of properties is always based on (subjective) value judgements. This section proposes such basic properties.

We consider a population set U and $A: U \to [0,1]$ a general fuzzy set composed of $n \geq 2$ individuals identical in everything but income. The income distribution is given by the vector $X = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n_+$, where x_i is the income of individual with ranking i, with $0 \leq x_1 < x_2 < \cdots < x_n$. Let F(.) be its distribution function and $0 < \mu_X < \infty$ its mean income. When we need to distinguish between the fuzzy set applied to different distributions, we use the sub-index, such that A_X denotes the fuzzy set applied from the universe set U with income distribution X. The proposed desirable properties are:

A1: A is anonymous.

Let $X = (x_i)_{i \in U}$ and $X' = (x'_i)_{i \in U}$ be two income distributions such that $X' = (x'_i)_{i \in U}$ is obtained by a permutation[6] from X, then $A_X = A_{X'}$.

That is, the degree of the membership of the fuzzy set should not be affected by any characteristic other than income.

A2: A is scale invariant.

[6] This axiom implies that $A_X(i) = A_{X'}(i)$ since the income vectors are ordered.

Let $Y = \alpha\, X = (\alpha x_i)_{i \in U}$ be an income distribution where $\alpha \in \mathbb{R}^+$, then $A_X = A_Y$.

A2 allows the fuzzy set to remain unaltered under equi-proportionate variations in all values, that is, the mapping which defines A is homogeneous of degree zero in the income distribution (we are measuring a relative concept).

A3: A is replication invariant.

Consider a new population set U^ℓ that is generated by the ℓ-fold replication of the population set U, that is, $U^\ell = \{\underbrace{U, U, \dots, U}_{\ell-\text{times}}\} = \{\underbrace{1,1, \dots, 1}_{\ell-\text{times}}, \dots, \underbrace{n, n, \dots, n}_{\ell-\text{times}}\}$ with income distribution

$$Y = X^\ell = (\underbrace{(x_\mathrm{i})_{i \in U}, (x_\mathrm{i})_{i \in U}, \dots, (x_\mathrm{i})_{i \in U}}_{\ell-\text{times}}) = \left(\underbrace{x_1, x_1, \dots, x_1}_{\ell-\text{times}}, \dots, \underbrace{x_n, x_n, \dots, x_n}_{\ell-\text{times}}\right) \in \mathbb{R}^{\ell \mathrm{n}},$$

then $A_X(i) = A_Y(ki)$ for all $i \in U$ and $k = 1, \dots, \ell$.

This property allows comparisons among populations in which the number of individuals is different and implies that the union of identical populations does not change the degree an individual is left behind. Specifically, this property describes the same idea of Dalton's principle of population for inequality indexes (Dalton, 1920).

A4: A decreases with income**.**

For all $i, j \in U$, if $x_i \leq x_j$, then $A(j) \leq A(i)$.

Given any distribution, by *A4* an individual is further behind the smaller her/his income.

A5: A is (strong) convex in income

Given a non-empty interval $I \subset [0, x_n]$, for all $i, j \in U$ with $x_i, x_j \in I^0$ and $x_i < x_j$, A is a convex function in income that satisfies: $A(j) \geq A(i) + \frac{i-n}{n}(x_j - x_i)$.

This axiom implies that A is a convex function in income[7] and, operating this expression, we get that that $\frac{A(i)-A(j)}{(x_j - x_i)} \leq \frac{n-i}{n}$. Thus, the average rate of decrease in $[x_i, x_j]$, $\frac{A(i)-A(j)}{(x_j - x_i)}$, is smaller than the proportion of individuals with incomes greater than individual i. In other words, the average rate of decrease decreases as the ranking of individual i increases.

Given that we are measuring the \`leave no one behind principle´ for the individual i it seems appropriate to link this change in the mapping to the survival function, $1 - F(x_i)$, such that the degree to which the mapping is curved reduces with the proportion of individuals richer than the one for which we are assessing the degree she/he falls behind.

A6: Minimality. $A(n) = 0$.

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[7] Let $f\colon D \subseteq \mathbb{R} \to \mathbb{R}$ be a convex map if and only if, for any $x, y \in I \subset D$, f verifies that $f(x) \geq f(y) + f'(y)(x - y)$.

That is, as we are assuming a relative assessment of the position of the individuals, in each population there is always at least one individual that is not left behind, since he/she is the richest, and consequently the degree the richest individual is left behind is 0, or in other words, the richest individual is not left behind.

A7: Minimality (bis). For a distribution where μ_X is the mean and $x_i = \mu_X$ for all $i \in U$, then $A(i) = 0$.

When there is no inequality in the income distribution the minimum value of the fuzzy set is achieved. That is, no individual is left behind when incomes are equally distributed.

A8: Maximality. For a distribution where $x_1 = 0$ and $x_i \neq x_1$ for all $i \neq 1$, then $A(1) = 1$.

That is, when all individuals have an income greater than 0, except the individual that has 0 income, that individual is totally left behind and the mapping takes the value of 1.

Remark 1:

The mapping used to define fuzzy set A is differentiable and continuous on its domain from axiom *A5*. The continuity of the map with respect to changes in income is an important property. Although apparently it is an analytical requirement or an operational property, it conveys to an intuitive idea: the continuity of this mapping establishes that small changes in the income distribution induce small changes in the degree of membership of the fuzzy set. On the other hand, the differentiability, obviously more demanding than continuity, requires that small changes in the variable induce small changes not only to the values of the fuzzy set but also to its rate of change.

4. **Characterization**

In this section, we start proving that the fuzzy set $LB: U \rightarrow [0,1]$ from Definition 1 verifies the desirable properties.

Proposition 3. Let U be a population set composed of $n \geq 2$ individuals. The income distribution is given by the vector $X = (x_1, x_2, \dots, x_n) \in \mathbb{R}_+^n$, where x_i is the income of individual with ranking i, with $0 \leq x_1 < x_2 < \cdots < x_n$ and where $0 < \mu_X < \infty$ is the mean of the distribution X. The fuzzy set $LB_X(i) = \left(1 - F(x_i)\right)\frac{\left(\mu_{x_i}^{+} - x_i\right)}{\mu_X}$ verifies axioms *A1* to *A8*.

Proof. *A1, A4, A5, A6 and A8* are straightforward by Definition 1 and by Proposition 2.

A2 Scale invariance

Consider $\alpha \in \mathbb{R}^+$ and $i \in U$ such that $y_i = \alpha x_i$. Let us prove that $LB_X(i) = LB_Y(i)$. By definition, we have that

$$F(x_i) = F(y_i), \tag{2}$$

$$\mu_Y = \frac{\sum_{j=1}^{n} \alpha x_j}{n} = \alpha\mu_X \text{ and} \tag{3}$$

$$\mu_{y_i}^{+} = \sum_{j=i+1}^{n} \frac{\alpha x_j}{n-i} = \alpha\mu_{x_i}^{+}. \tag{4}$$

Therefore, using equations (2), (3) and (4), we obtain

$$LB_Y(i) = \big(1 - F(y_i)\big)\frac{\left(\mu_{y_i}^{+} - y_i\right)}{\mu_Y} = \big(1 - F(x_i)\big)\frac{\left(\alpha\mu_{x_i}^{+} - \alpha x_i\right)}{\alpha\mu_X}$$

$$= \big(1 - F(x_i)\big)\frac{\left(\mu_{x_i}^{+} - x_i\right)}{\mu_X} = LB_X(i).$$

A3: LB *is replication invariant.*

Consider the distribution vector $Y = (\underbrace{x_1, x_1, \dots, x_1}_{\ell-\text{times}}, \dots, \underbrace{x_n, x_n, \dots, x_n}_{\ell-\text{times}}) \in \mathbb{R}_+^{\ell n}$ generated by the ℓ-fold replication of an original income distribution X with $\ell \in \mathbb{N}$.

By definition of F, μ_X and $\mu_{x_i}^{+}$, we have that $F(y_i) = F(x_i)$, $\mu_Y = \frac{\sum_{j=1}^{n} \ell x_j}{\ell n} = \mu_X$ and $\mu_{y_i}^{+} = \sum_{j=i+1}^{n} \frac{\ell x_j}{\ell(n-i)} = \mu_{x_i}^{+}$ for all $i \in U$. Hence, $LB_Y(i) = LB_X(i)$ for all $i \in U$.

A7: *Minimality (bis)*

Consider the income distribution with $x_i = \mu_X$ for all $i \in U$, then, we have that

$$LB(i) = \frac{1}{\mu_X}\left(\frac{\sum_{j=i+1}^{n} \mu_X}{n-i} - \mu_X\right)\frac{n-i}{n} = \frac{1}{\mu_X}(\mu_x - \mu_x)\frac{n-i}{n} = 0. \qquad \square$$

The axioms proposed above restrict the set of fuzzy sets for measuring the LNOB principle. Some are sufficient to characterize one index. We are able to establish the following result.

Theorem 2. Let U be a population set composed of $n \geq 2$ individuals. The income distribution is given by the vector $X = (x_1, x_2, \dots, x_n) \in \mathbb{R}_+^n$ with $0 \leq x_1 < x_2 < \cdots < x_n$ where x_i is the income of individual with ranking i and $0 < \mu_X < \infty$ is the average of X. A fuzzy set $A: U \longrightarrow [0,1]$ verifies the axioms of strong convex in income, minimality and maximality if and only if $A(i) = LB(i) = \big(1 - F(x_i)\big)\frac{\left(\mu_{x_i}^{+} - x_i\right)}{\mu_X}$.

Proof. We start to prove the necessary condition. Since A verifies the *strong convexity axiom,* for any non-empty interval $I \subset [0, x_n]$ and for all $i, j \in U$ with $x_i, x_j \in I^0$ such that $x_i < x_j$, we have that $\frac{A(i)-A(j)}{(x_j - x_i)} \leq \frac{n-i}{n}$. Therefore, for any $i \in U$ with $x_i < x_{i+1}$ and $0 \leq \beta \leq 1$, we have that

$$A(i) - A(i+1) = \beta(x_{i+1} - x_i)\,\frac{n-i}{n}. \tag{5}$$

Thus, using expression (5) recurrently we obtain

$$A(i) = A(i+2) + \beta(x_{i+2} - x_{i+1})\frac{n-(i+1)}{n} + \beta(x_{i+1} - x_i)\frac{n-i}{n}$$

$$\ldots$$

$$A(i) = A(n) + \beta \sum_{j=i+1}^{n}\left(x_j - x_{j-1}\right)\frac{n-(j-1)}{n}.$$

The *minimality* axiom implies that

$$A(i) = \beta \sum_{j=i+1}^{n}\left(x_j - x_{j-1}\right)\frac{n-(j-1)}{n}. \tag{6}$$

Now expanding expression (6), we obtain

$$A(i) = \beta\ (\ (x_{i+1} - x_i)\frac{n-i}{n} + (x_{i+2} - x_{i+1})\frac{n-(i+1)}{n} + \cdots + (x_n - x_{n-1})\frac{1}{n}\ )$$

$$= \beta\left(\frac{x_{i+1}}{n} + \frac{x_{i+2}}{n} + \cdots + \frac{x_n}{n} - x_i\frac{n-i}{n}\right) = \beta\ \left(\frac{\sum_{j=i+1}^{n} x_j}{n-i}\ \frac{n-i}{n} - x_i\frac{n-i}{n}\right).$$

By *maximality*, for a distribution where $x_1 = 0$ and $x_i \neq x_1$ for all $i \neq 1$, we have that

$$A(1) = \beta\ \left(\frac{\sum_{j=2}^{n} x_j}{n-1} - 0\right)\frac{n-1}{n} = \beta\frac{\sum_{j=1}^{n} x_j}{n} = \beta\ \mu_X = 1.$$

Thus $\beta = \frac{1}{\mu_x}$.

Therefore, by Definition 1 we obtain that

$$A(i) = \frac{1}{\mu_x}\left(\frac{\sum_{j=i+1}^{n} x_j}{n-i}\ \frac{n-i}{n} - x_i\frac{n-i}{n}\right) = \frac{1}{\mu_x}\left(\mu_{x_i}^{+} - x_i\right)(1 - F(x_i) = LB(i).$$

Finally, the sufficiency condition follows from Proposition 3. □

The axioms used are independent. The Appendix demonstrates that, for each axiom, there exists a fuzzy set that satisfies the remaining ones, which differs from the solutions characterized.

5. Further properties of LB

In this section we provide three additional properties of the fuzzy set LB. First, all individuals in the population contribute to the degree individual i is 'left behind'; second, the degree individual i is 'left behind' does not change with a rank-preserving transfer among two individuals, both richer or both poorer than individual i; third, the degree individual i is 'left behind' decreases with a rank-preserving transfer from an individual richer than individual i to anybody poorer than individual i.

Property 1. All individuals in the population contribute to the degree individual i is left behind.

1.a For any $i \in U$, if there is a rank preserving increase in x_i such that the vector of the new distribution $Y = (x_1, x_2,, \dots, x_i + \delta, \dots, x_k, \dots, x_n) \in \mathbb{R}_+^n$, with $0 < \delta$, then $LB_Y(i) < LB_X(i)$, as it is a decreasing mapping.

Proof: For all $i \in U$, $F(x_i) = F(y_i)$, $\mu_Y = \mu_X + \frac{\delta}{n}$ and $\mu_{y_i}^+ = \mu_{x_i}^+$. Using the definition of LB and the previous expressions, we have that

$$LB_Y(i) = \frac{1}{\mu_X + \frac{\delta}{n}}\left(\mu_{x_i}^+ - (x_i + \delta)\right)\left(1 - F(x_i)\right) < \frac{1}{\mu_X}\left(\mu_{x_i}^+ - (x_i + \delta)\right)\left(1 - F(x_i)\right) < \frac{1}{\mu_X}\left(\mu_{x_i}^+ - x_i\right)\left(1 - F(x_i)\right) = LB_X(i). \quad \square$$

1.b For any $i \in U$ with $x_i < x_k$. If there is a rank preserving increase in x_k of δ units, with $0 < \delta$, such that the vector of the new distribution $Y = (x_1, x_2, \dots, x_i, \dots, x_k + \delta, \dots, x_n) \in \mathbb{R}_+^n$, then $LB_X(i) < LB_Y(i)$.

Proof: For all $i \in U$, $F(x_i) = F(y_i)$, $\mu_Y = \mu_X + \frac{\delta}{n}$ and $\mu_{y_i}^+ = \mu_{x_i}^+ + \frac{\delta}{n-i}$. Using the definition of LB and the previous expressions, we have that

$$LB_Y(i) = \frac{1}{\mu_X + \frac{\delta}{n}}\left(\mu_{x_i}^+ + \frac{\delta}{n-i} - x_i\right)\left(1 - F(x_i)\right)$$

$$= LB_X(i)\frac{\mu_X}{\mu_X + \frac{\delta}{n}} + \frac{\delta}{n\left(\mu_X + \frac{\delta}{n}\right)}.$$

$$LB_Y(i) - LB_X(i) = LB_X(i)\left(\frac{\mu_X}{\mu_X + \frac{\delta}{n}} - 1\right) + \frac{\delta}{n\left(\mu_X + \frac{\delta}{n}\right)}$$

$$= \frac{\delta}{n\left(\mu_X + \frac{\delta}{n}\right)}\left(1 - LB_X(i)\right).$$

Since $LB_X(i) \leq 1$, we obtain that $LB_Y(i) < LB_X(i)$, for all $i \in U$. $\square$

1.c For any $i \in U$ with $x_k < x_i$. If there is a rank preserving increase in x_k of δ units, with $0 < \delta$, such that the vector of the new distribution $Y = (x_1, x_2, \dots, x_k + \delta, \dots, x_i, \dots, x_n) \in \mathbb{R}_+^n$, then $LB_Y(i) < LB_X(i)$.

Proof: For all $i \in U$, $F(x_i) = F(y_i)$, $\mu_Y = \mu_X + \frac{\delta}{n}$ and $\mu_{y_i}^+ = \mu_{x_i}^+$. Using the previous expressions, we have that

$$LB_Y(i) = \frac{1}{\mu_X + \frac{\delta}{n}}\left(\mu_{x_i}^+ - x_i\right)\left(1 - F(x_i)\right) = LB_X(i)\frac{\mu_X}{\mu_X + \frac{\delta}{n}}$$

Therefore, $LB_Y(i) < LB_X(i)$, for all $i \in U$. $\square$

Property 2. A rank-preserving transfer among two individuals, both richer or both poorer than individual i, leaves $LB(i)$ unchanged.

Note that this transfer does not affect the mean income nor the mean income of individuals with an income greater than x_i, nor the income nor the rank of i .

Property 3. A rank-preserving transfer of δ units from an individual richer than individual i to anybody poorer than individual i such that, for any $i \in U$ with $x_l < x_i < x_k$, there is an increase in x_l of δ units and a decrease in x_k of δ units, with $0 < \delta$, the vector of the new distribution $Y = (x_1, x_2, \dots, x_l + \delta, \dots, x_i, \dots, x_k - \delta, \dots, x_n) \in \mathbb{R}^n_+$ and then $LB_Y(i) < LB_X(i)$.

Note that this transfer does not affect the mean income but reduces the mean income of individuals with an income greater than x_i and does not change the income or the rank of i.

Proof: Consider $l, i, k \in U$ with $x_l < x_i < x_k$, $0 < \delta$. By definition of F, μ_X and $\mu^+_{x_i}$, we have that $F(x_i) = F(y_i)$, $\mu_Y = \mu_X$ and $\mu^+_{y_i} = \mu^+_{x_i} - \frac{\delta}{n-i}$. Therefore,

$$LB_Y(i) = \frac{1}{\mu_X}\left(\mu^+_{x_i} - \frac{\delta}{n-i} - x_i\right)\left(1 - F(x_i)\right) < LB_X(i). \quad \square$$

5. Conclusions

In this study we propose desirable properties for the definitions of fuzzy sets that allow us to measure the extent to which an individual is 'left behind' in any dimension and provide an axiomatization to measure the 'Leave no one behind' principle. Only one fuzzy set satisfies all the desirable properties, the one proposed by Garcia-Pardo et al. (2021).

This measure has proved to be useful as it can incorporate both quantitative and qualitative variables and combine them in a multidimensional setting, thus making use of the rich fuzzy structure of operations on residuated lattices. Moreover, an overall fuzzy measure of the LNOB principle that measures the extent to which individuals in a society are left behind is possible by averaging. The natural continuation of this research is to provide a unified axiomatic framework for the multidimensional fuzzy measurement of the LNOB principle.

Appendix A

The following definitions of fuzzy sets $LB^k; U \to [0,1]$ satisfy axioms *A5*, *A6* and *A8* but one.

Let U be a population set composed of $n \geq 2$ individuals and the income distribution be given by the vector $X = (x_1, x_2, \dots, x_n) \in \mathbb{R}^n_+$, where x_i is the income of individual with ranking i, with $0 \leq x_1 < x_2 < \dots < x_n$ and where $0 < \mu_X < \infty$ is the mean of the distribution X, then

$LB^1(i) = \frac{1}{n}\sum_{j=i+1}^{n} \frac{x_j - x_i}{\mu_X}\delta$ with $0 < \delta < 1$ (not *A8: Maximality).*

$LB^2(i) = \frac{1}{n}\sum_{j=i+1}^{n} \frac{x_j - x_i}{\mu_X} + \delta x_1$ with $\delta < \frac{1}{n\mu_X}$ (not *A6: Minimality).*

$LB^3(i) = \frac{1}{n}\sum_{j=i+1}^{n} \frac{x_j}{\mu_X}$ (not *A5: Strong convex).*

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