manuscript No.
(will be inserted by the editor)

Multipliers and integration operators between conformally invariant spaces

Daniel Girela · Noel Merchán

Received: date / Accepted: date

Abstract In this paper we are concerned with two classes of conformally invariant spaces of analytic functions in the unit disc $\mathbb{D}$, the Besov spaces B^p $(1 \leq p < \infty)$ and the Q_s spaces $(0 < s < \infty)$. Our main objective is to characterize for a given pair (X, Y) of spaces in these classes, the space of pointwise multipliers $M(X, Y)$, as well as to study the related questions of obtaining characterizations of those g analytic in $\mathbb{D}$ such that the Volterra operator T_g or the companion operator I_g with symbol g is a bounded operator from X into Y.

Keywords Möbius invariant spaces · Besov spaces · Q_s spaces · multipliers · integration operators · Carleson measures

Mathematics Subject Classification (2010) 30H25 · 47B38

1 Introduction

Let $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ denote the open unit disc of the complex plane $\mathbb{C}$ and let $\mathcal{H}ol(\mathbb{D})$ be the space of all analytic functions in $\mathbb{D}$ endowed with the topology of uniform convergence on compact subsets.

This research is supported in part by a grant from "El Ministerio de Ciencia, Innovación y Universidades", Spain (PGC2018-096166-B-I00) and by grants from la Junta de Andalucía (FQM-210 and UMA18-FEDERJA-002).

D. Girela
Departamento de Análisis Matemático, Estadística e Investigación Operativa, y Matemática Aplicada,
Universidad de Málaga, 29071 Málaga, Spain
E-mail: girela@uma.es

N. Merchán
Departamento de Matemática Aplicada,
Universidad de Málaga, 29071 Málaga, Spain
E-mail: noel@uma.es

If $0 < r < 1$ and $f \in \mathcal{H}ol(\mathbb{D})$, we set

$$M_p(r,f) = \left(\frac{1}{2\pi}\int_0^{2\pi} |f(re^{it})|^p\,dt\right)^{1/p}, \quad 0 < p < \infty,$$
$$M_\infty(r,f) = \sup_{|z|=r} |f(z)|.$$

If $0 < p \leq \infty$ the Hardy space H^p consists of those $f \in \mathcal{H}ol(\mathbb{D})$ such that

$$\|f\|_{H^p} \stackrel{\text{def}}{=} \sup_{0<r<1} M_p(r,f) < \infty.$$

We mention [18] for the theory of H^p-spaces.

If $0 < p < \infty$ and $\alpha > -1$, the weighted Bergman space A^p_α consists of those $f \in \mathcal{H}ol(\mathbb{D})$ such that

$$\|f\|_{A^p_\alpha} \stackrel{\text{def}}{=} \left((\alpha+1)\int_{\mathbb{D}} (1-|z|)^\alpha |f(z)|^p\,dA(z)\right)^{1/p} < \infty.$$

The unweighted Bergman space A^p_0 is simply denoted by A^p. Here, $dA(z) = \frac{1}{\pi}dx\,dy$ denotes the normalized Lebesgue area measure in $\mathbb{D}$. We refer to [19], [36] and [58] for the theory of these spaces.

We let $\mathrm{Aut}(\mathbb{D})$ denote the set of all disc automorphisms, that is, of all one-to-one analytic maps φ from $\mathbb{D}$ onto itself. It is well known that $\mathrm{Aut}(\mathbb{D})$ coincides with the set of all Möbius transformations from $\mathbb{D}$ onto itself:

$$\mathrm{Aut}(\mathbb{D}) = \{\lambda\varphi_a : |\lambda| = 1,\, a \in \mathbb{D}\},$$

where $\varphi_a(z) = (a-z)/(1-\overline{a}z)$ $(z \in \mathbb{D})$.

A linear space X of analytic functions in $\mathbb{D}$ is said to be *conformally invariant* or *Möbius invariant* if whenever $f \in X$, then also $f \circ \varphi \in X$ for any $\varphi \in \mathrm{Aut}(\mathbb{D})$ and, moreover, X is equipped with a semi-norm ρ for which there exists a positive constant C such that

$$\rho(f \circ \varphi) \leq C\rho(f), \quad \text{whenever } f \in X \text{ and } \varphi \in \mathrm{Aut}(\mathbb{D}).$$

The articles [8] and [44] are fundamental references for the theory of Möbius invariant spaces which have attracted much attention in recent years (see, e.g., [3, 16, 17, 30, 47, 57, 58]).

The *Bloch space* $\mathcal{B}$ consists of all analytic functions f in $\mathbb{D}$ such that

$$\rho_{\mathcal{B}}(f) \stackrel{\text{def}}{=} \sup_{z\in\mathbb{D}} (1-|z|^2)\,|f'(z)| < \infty.$$

The Schwarz-Pick lemma easily implies that $\rho_{\mathcal{B}}$ is a conformally invariant semi-norm, thus $\mathcal{B}$ is a conformally invariant space. It is also a Banach space with the norm $\|\cdot\|_{\mathcal{B}}$ defined by $\|f\|_{\mathcal{B}} = |f(0)| + \rho_{\mathcal{B}}(f)$. The little Bloch space $\mathcal{B}_0$ is the set of those $f \in \mathcal{B}$ such that $\lim_{|z|\to 1}(1-|z|^2)|f'(z)| = 0$. Alternatively, $\mathcal{B}_0$ is the closure of the polynomials in the Bloch norm. A classical reference for the theory of Bloch functions is [7]. Rubel and Timoney [44] proved that $\mathcal{B}$ is the largest "reasonable" Möbius invariant space. More precisely, they proved the following result.

Theorem A *Let X be a Möbius invariant linear space of analytic functions in $\mathbb{D}$ and let ρ be a Möbius invariant seminorm on X. If there exists a non-zero decent linear functional L on X which is continuous with respect to ρ, then $X \subset \mathcal{B}$ and there exists a constant $A > 0$ such that $\rho_{\mathcal{B}}(f) \le A\rho(f)$, for all $f \in X$.*

Here, a linear functional L on X is said to be decent if it extends continuously to $\mathcal{H}ol(\mathbb{D})$.

The space $BMOA$ consists of those functions f in H^1 whose boundary values have bounded mean oscillation on the unit circle $\partial\mathbb{D}$ as defined by F. John and L. Nirenberg. There are many characterizations of $BMOA$ functions. Let us mention the following:

If $f \in \mathcal{H}ol(\mathbb{D})$, then $f \in BMOA$ if and only if $\|f\|_{BMOA} \overset{def}{=} |f(0)| + \rho_*(f) < \infty$, where

$$\rho_*(f) = \sup_{a\in\mathbb{D}} \|f \circ \varphi_a - f(a)\|_{H^2}.$$

It is well known that $H^\infty \subset BMOA \subset \mathcal{B}$ and that $BMOA$ equipped with the seminorm ρ_* is a Möbius invariant space. The space $VMOA$ consists of those $f \in BMOA$ such that $\lim_{|a|\to 1} \|f \circ \varphi_a - f(a)\|_{H^2} = 0$, it is the closure of the polynomials in the $BMOA$-norm. We mention [28] as a general reference for the space $BMOA$.

Other important Möbius invariant spaces are the Besov spaces and the Q_s spaces.

For $1 < p < \infty$, the *analytic Besov space* B^p is defined as the set of all functions f analytic in $\mathbb{D}$ such that $f' \in A^p_{p-2}$. All B^p spaces $(1 < p < \infty)$ are conformally invariant with respect to the semi-norm ρ_{B^p} defined by

$$\rho_{B^p}(f) \overset{\text{def}}{=} \|f'\|_{A^p_{p-2}}$$

(see [8, p. 112] or [16, p. 46]) and Banach spaces with the norm $\|\cdot\|_{B^p}$ defined by $\|f\|_{B^p} = |f(0)| + \rho_{B^p}(f)$. An important and well-studied case is the classical *Dirichlet space* B^2 (often denoted by $\mathcal{D}$) of analytic functions whose image has a finite area, counting multiplicities.

The space B^1 requires a special definition: it is the space of all analytic functions f in $\mathbb{D}$ for which $f'' \in A^1$. Although the semi-norm ρ defined by $\rho(f) = \|f''\|_{A^1}$ is not conformally invariant, the space itself is. An alternative definition of B^1 with a conformally invariant semi-norm is given in [8], where it is also proved that B^1 is contained in any Möbius invariant space. A lot of information on Besov spaces can be found in [8,16,17,37,56,58]. Let us recall that

$$VMOA \subsetneq \mathcal{B}_0, \quad BMOA \subsetneq \mathcal{B},$$
$$B^1 \subsetneq B^p \subsetneq B^q \subsetneq VMOA \subsetneq BMOA, \quad 1 < p < q < \infty.$$

If $0 \le s < \infty$, we say that $f \in Q_s$ if f is analytic in $\mathbb{D}$ and

$$\sup_{a\in\mathbb{D}} \int_{\mathbb{D}} |f'(z)|^2 g(z,a)^s \, dA(z) < \infty,$$

where $g(z,a) = \log(|1 - \overline{a}z|/|a - z|)$ is the Green function of $\mathbb{D}$. These spaces were introduced by Aulaskari and Lappan [12] while looking for characterizations of

Bloch functions (see [50] for the case $s = 2$). For $s > 1$, Q_s is the Bloch space, $Q_1 = BMOA$, and

$$\mathcal{D} \subsetneq Q_{s_1} \subsetneq Q_{s_2} \subsetneq BMOA, \qquad 0 < s_1 < s_2 < 1.$$

It is well known [14,46] that for every s with $0 \le s < \infty$, a function $f \in \mathcal{H}ol(\mathbb{D})$ belongs to Q_s if and only if

$$\rho_{Q_s}(f) \stackrel{\text{def}}{=} \left(\sup_{a\in\mathbb{D}} \int_{\mathbb{D}} |f'(z)|^2 (1 - |\varphi_a(z)|^2)^s \, dA(z) \right)^{1/2} < \infty.$$

All Q_s spaces ($0 \le s < \infty$) are conformally invariant with respect to the seminorm ρ_{Q_s}. They are also Banach spaces with the norm $\| \cdot \|_{Q_s}$ defined by $\|f\|_{Q_s} = |f(0)| + \rho_{Q_s}(f)$. We mention [52,53] as excellent references for the theory of Q_s-spaces.

Let us recall the following two facts which were first observed in [10].

$$\text{If } 0 < p \le 2, \text{ then } B^p \subset Q_s \text{ for all } s > 0. \tag{1}$$

$$\text{If } 2 < p < \infty, \text{ then } B^p \subset Q_s \text{ if and only if } 1 - \frac{2}{p} < s. \tag{2}$$

For g analytic in $\mathbb{D}$, the Volterra operator T_g is defined as follows:

$$T_g(f)(z) \stackrel{\text{def}}{=} \int_0^z g'(\xi) f(\xi) \, d\xi, \quad f \in \mathcal{H}ol(\mathbb{D}), \quad z \in \mathbb{D}.$$

We define also the companion operator I_g by

$$I_g(f)(z) \stackrel{\text{def}}{=} \int_0^z g(\xi) f'(\xi) d\xi, \quad f \in \mathcal{H}ol(\mathbb{D}), \quad z \in \mathbb{D}.$$

The integration operators T_g and I_g have been studied in a good number of papers. Let us just mention here that Pommerenke [43] proved that T_g is bounded on H^2 if and only if $g \in BMOA$ and that Aleman and Siskakis [4] characterized those $g \in \mathcal{H}ol(\mathbb{D})$ for which T_g is bounded on H^p ($p \ge 1$), while Aleman and Cima characterized in [1] those $g \in \mathcal{H}ol(\mathbb{D})$ for which T_g maps H^p into H^q. Aleman and Siskakis [5] studied the operators I_g and T_g acting on Bergman spaces.

For $g \in \mathcal{H}ol(\mathbb{D})$, the multiplication operator M_g is defined by

$$M_g(f)(z) \stackrel{\text{def}}{=} g(z) f(z), \quad f \in \mathcal{H}ol(\mathbb{D}), \ z \in \mathbb{D}.$$

If X and Y are two Banach spaces of analytic function in $\mathbb{D}$ continuously embedded in $\mathcal{H}ol(\mathbb{D})$ and $g \in \mathcal{H}ol(\mathbb{D})$ then g is said to be a multiplier from X to Y if $M_g(X) \subset Y$. The space of all multipliers from X to Y will be denoted by $M(X, Y)$ and $M(X)$ will stand for $M(X, X)$. Using the closed graph theorem we see that for the three operators T_g, I_g, M_g, we have that if one of them maps X into Y then it is continuous from X to Y. We remark also that

$$T_g(f) + I_g(f) = M_g(f) - f(0)g(0). \tag{3}$$

Thus if two of the operators T_g, I_g, M_g are bounded from X to Y so is the third one.

It is well known that if X is nontrivial then $M(X) \subset H^\infty$ (see, e. g., [2, Lemma 1. 1] or [48, Lemma 1. 10]), but $M(X,Y)$ need not be included in H^∞ if $Y \not\subset X$. However, when dealing with Möbius invariant spaces we have the following result.

Proposition 1 *Let X and Y be two Möbius invariant spaces of analytic functions in $\mathbb{D}$ equipped with the seminorms ρ_X and ρ_Y, respectively. Suppose that there exists a non-trivial decent linear functional L on Y which is continuous with respect to ρ_Y. Let $g \in \mathcal{H}ol(\mathbb{D})$. Then the following statements hold.*

(i) If M_g is continuous from (X,ρ_X) into (Y,ρ_Y), then $g \in H^\infty$.
(ii) If I_g is continuous from (X,ρ_X) into (Y,ρ_Y), then $g \in H^\infty$.

Before embarking into the proof of Proposition 1, let us mention that, as usual, throughout the paper we shall be using the convention that $C = C(p,\alpha,q,\beta,\dots)$ will denote a positive constant which depends only upon the displayed parameters $p,\alpha,q,\beta\dots$ (which sometimes will be omitted) but not necessarily the same at different occurrences. Moreover, for two real-valued functions E_1, E_2 we write $E_1 \lesssim E_2$, or $E_1 \gtrsim E_2$, if there exists a positive constant C independent of the arguments such that $E_1 \le CE_2$, respectively $E_1 \ge CE_2$. If we have $E_1 \lesssim E_2$ and $E_1 \gtrsim E_2$ simultaneously then we say that E_1 and E_2 are equivalent and we write $E_1 \asymp E_2$. Also, if $1 < p < \infty$, p' will stand for its conjugate exponent, that is, $\frac{1}{p} + \frac{1}{p'} = 1$.

Proof of Proposition 1. Since X is conformally invariant, $\mathrm{Aut}(\mathbb{D}) \subset X$ [8, p. 114] and

$$\rho_X(\varphi_a) \asymp 1, \quad a \in \mathbb{D}. \tag{4}$$

Suppose that M_g is continuous from (X,ρ_X) into (Y,ρ_Y). Using this, Theorem A, and (4) we obtain

$$\rho_{\mathcal{B}}(g\,\varphi_a) \lesssim \rho_Y(g\,\varphi_a) \lesssim \rho_X(\varphi_a) \lesssim 1, \quad a \in \mathbb{D}.$$

This implies that

$$(1-|a|^2)\left|g'(a)\varphi_a(a) + g(a)\varphi_a'(a)\right| \lesssim 1, \quad a \in \mathbb{D}.$$

Since $\varphi(a) = 0$ and $\varphi_a'(a) = -(1-|a|^2)^{-1}$, it follows that

$$|g(a)| \lesssim 1, \quad a \in \mathbb{D},$$

that is, $g \in H^\infty$.

Similarly, if we assume that I_g is continuous from (X,ρ_X) into (Y,ρ_Y), we obtain

$$\rho_{\mathcal{B}}\left(I_g(\varphi_a)\right) \lesssim 1, \quad a \in \mathbb{D}.$$

This implies that

$$(1-|a|^2)\left|\left(I_g(\varphi_a)\right)'(a)\right| = (1-|a|^2)|\varphi_a'(a)||g(a)| = |g(a)| \lesssim 1, \quad a \in \mathbb{D}.$$

□

For notational convenience, set

$$\mathcal{BQ} = \{Q_s : 0 \le s < \infty\} \cup \{B^p : 1 \le p < \infty\}.$$

The main purpose of this paper is characterizing, for a given pair of spaces $X, Y \in \mathcal{BQ}$, the functions $g \in \mathcal{H}ol(\mathbb{D})$ such that the operators M_g, T_g and/or I_g map X into Y. When X and Y are Besov spaces this question has been extensively studied (see, e. g. [9,26,32,45,49,59]). Thus we shall concentrate ourselves to study these operators when acting between a certain Besov space B^p and a certain Q_s space and when acting between Q_{s_1} and Q_{s_2} for a certain pair of positive numbers s_1, s_2.

2 Multipliers and integration operators from Besov spaces into Q_s-spaces

For $\alpha > 0$, *the α-logarithmic Bloch space* $\mathcal{B}_{\log,\alpha}$ is the Banach space of those functions $f \in \mathcal{H}ol(\mathbb{D})$ which satisfy

$$\|f\|_{\log,\alpha} \stackrel{\text{def}}{=} |f(0)| + \sup_{z\in\mathbb{D}} (1-|z|^2) \left(\log \frac{2}{1-|z|^2}\right)^{\alpha} |f'(z)| < \infty. \tag{5}$$

For simplicity, the space $\mathcal{B}_{\log,1}$ will be denoted by $\mathcal{B}_{\log}$.

It is clear that $B_{\log,\alpha} \subset \mathcal{B}_0$, for all $\alpha > 0$. Using the characterization of $VMOA$ in terms of Carleson measures [28, p. 102], it follows easily that

$$B_{\log,\alpha} \subset VMOA, \quad \text{for all } \alpha > 1/2.$$

In particular, $\mathcal{B}_{\log} \subset VMOA$.

Brown and Shields [15] showed that $M(\mathcal{B}) = \mathcal{B}_{\log} \cap H^\infty$. The spaces $M(B^p, \mathcal{B})$ $(1 \le p < \infty)$ were characterized in [25]. Namely, Theorem 1 of [25] asserts that $M(B^1, \mathcal{B}) = H^\infty$ and

$$M(B^p, \mathcal{B}) = H^\infty \cap \mathcal{B}_{\log,1/p'}, \quad 1 < p < \infty, \tag{6}$$

where p' is the exponent conjugate to p, that is, $\frac{1}{p} + \frac{1}{p'} = 1$.

In this section we extend these results. In particular, we shall obtain for any pair (p, s) with $2 < p < \infty$ and $0 < s < \infty$ a complete characterization of the space of multipliers $M(B^p, Q_s)$.

Let us start with the case $s \ge 1$ which is the simplest one.

Theorem 1 *Let $g \in \mathcal{H}ol(\mathbb{D})$. Then:*

(i) I_g maps B^1 into $\mathcal{B}$ if and only if $g \in H^\infty$.
(ii) M_g maps B^1 into $\mathcal{B}$ if and only if $g \in H^\infty$.
(iii) T_g maps B^1 into $\mathcal{B}$ if and only if $g \in \mathcal{B}$.

Proof. If $I_g(B^1) \subset \mathcal{B}$ then, using Proposition 1, it follows that $g \in H^\infty$.

To prove the converse it suffices to recall that $B^1 \subset \mathcal{B}$. Indeed, suppose that $g \in H^\infty$ and take $f \in B^1$. Then

$$(1-|z|^2)\left|(I_g(f))'(z)\right| = (1-|z|^2)|f'(z)||g(z)| \le \|f\|_{\mathcal{B}}\|g\|_{H^\infty}.$$

Thus $I_g(f) \in \mathcal{B}$.

Hence (i) is proved. Now, (ii) is contained in [25, Theorem 1].

It remains to prove (iii). If $T_g(B^1) \subset \mathcal{B}$ then $T_g(1) = g - g(0) \in \mathcal{B}$ and, hence $g \in \mathcal{B}$. Conversely, if $g \in \mathcal{B}$ and $f \in \mathcal{B}^1$ then, using the fact that $B^1 \subset H^\infty$, we obtain

$$(1-|z|^2)\left|(T_g(f))'(z)\right| = (1-|z|^2)|g'(z)||f(z)| \le \|g\|_{\mathcal{B}}\|f\|_{H^\infty}.$$

Thus $T_g(f) \in \mathcal{B}$. Hence (iii) is also proved. □

Theorem 2 *Suppose that* $1 < p < \infty$, $\frac{1}{p} + \frac{1}{p'} = 1$, *and let* $g \in \mathcal{H}ol(\mathbb{D})$. *Then:*

(i) I_g *maps* B^p *into* $\mathcal{B}$ *if and only if* $g \in H^\infty$.
(ii) M_g *maps* B^p *into* $\mathcal{B}$ *if and only if* $g \in H^\infty \cap \mathcal{B}_{\log,1/p'}$.
(iii) T_g *maps* B^p *into* $\mathcal{B}$ *if and only if* $g \in \mathcal{B}_{\log,1/p'}$.

Proof. If I_g maps B^p into $\mathcal{B}$ then Proposition 1 implies that $g \in H^\infty$. Conversely, using that $B^p \subset \mathcal{B}$, we see that if $g \in H^\infty$ and $f \in B^p$ then

$$(1-|z|^2)\left|(I_g(f))'(z)\right| = (1-|z|^2)|f'(z)||g(z)| \le \|f\|_{\mathcal{B}}\|g\|_{H^\infty}.$$

Hence, $I_g(f) \in \mathcal{B}$. Thus (i) is proved and (ii) reduces to (6).

Finally, (iii) follows from the following more precise result.

Theorem 3 *Suppose that* $1 < p < \infty$, $\frac{1}{p} + \frac{1}{p'} = 1$, *and let* $g \in \mathcal{H}ol(\mathbb{D})$. *Then the following conditions are equivalent.*

(a) T_g *maps* B^p *into* $\mathcal{B}$.
(b) $g \in \mathcal{B}_{\log,1/p'}$.
(c) T_g *maps* B^p *into* $\mathcal{B}_0$.

Proof of Theorem 3. (a) ⇒ (b) Suppose (a). By the closed graph theorem T_g is a bounded operator from B^p into $\mathcal{B}$, hence

$$(1-|z|^2)|g'(z)f(z)| \lesssim \|f\|_{B^p}, \quad z \in \mathbb{D},\ f \in B^p. \tag{7}$$

For $a \in \mathbb{D}$ with $a \neq 0$, set

$$f_a(z) = \left(\log \frac{1}{1-|a|^2}\right)^{-1/p} \log \frac{1}{1-\overline{a}\,z}, \quad z \in \mathbb{D}. \tag{8}$$

It is readily seen that $f_a \in B^p$ for all a and that $\|f_a\|_{B^p} \asymp 1$. Using this and taking $f = f_a$ and $z = a$ in (7), we obtain

$$(1-|a|^2)|g'(a)|\left(\frac{1}{1-|a|^2)}\right)^{1/p'} \lesssim 1,$$

that is $g \in \mathcal{B}_{\log,1/p'}$.

(b) ⇒ (c) Suppose (b) and take $f \in B^p$. It is well known that

$$|f(z)| = \mathrm{o}\left(\left(\log \frac{1}{1-|z|^2}\right)^{1/p'}\right), \quad \text{as } |z| \to 1,$$

(see, e. g., [37, 56]). This and (b) immediately yield that $T_g(f) \in \mathcal{B}_0$.

The implication (c) ⇒ (a) is trivial. Hence the proof of Theorem 3 is finished and, consequently, Theorem 2 is also proved. □

Let us turn now to the case $0 < s \leq 1$. We shall consider first the Volterra operators T_g. For $0 < s < \infty$ and $\alpha > 0$ we set

$$Q_{s,\log,\alpha} = \left\{ f \in \mathcal{H}ol(\mathbb{D}) : \sup_{a\in\mathbb{D}} \left(\log \frac{2}{1-|a|} \right)^{2\alpha} \int_{\mathbb{D}} |f'(z)|^2 (1 - |\varphi_a(z)|^2)^s \, dA(z) < \infty \right\}.$$

We have the following results.

Theorem 4 *Suppose that $0 < s \leq 1$ and let $g \in \mathcal{H}ol(\mathbb{D})$. Then:*

(i) T_g maps B^1 into Q_s if and only if $g \in Q_s$.
(ii) If $1 < p < \infty$, $0 < s \leq 1$, and T_g maps B^p into Q_s, then $g \in Q_{s,\log,1/p'}$.
(iii) If $1 < p < \infty$, then T_g maps B^p into $Q_1 = BMOA$ if and only if $g \in Q_{1,\log,1/p'}$.
(iv) If $2 < p < \infty$, $0 < s < 1$, and $1 - \frac{2}{p} < s$ then T_g maps B^p into Q_s if and only if $g \in Q_{s,\log,1/p'}$.

Before we get into the proofs of these results we shall introduce some notation and recall some results which will be needed in our work.

If $I \subset \partial\mathbb{D}$ is an interval, $|I|$ will denote the length of I. The *Carleson square* $S(I)$ is defined as $S(I) = \{re^{it} : e^{it} \in I, \quad 1 - \frac{|I|}{2\pi} \leq r < 1\}$. Also, for $a \in \mathbb{D}$, the Carleson box $S(a)$ is defined by

$$S(a) = \left\{ z \in \mathbb{D} : 1 - |z| \leq 1 - |a|, \; \left| \frac{\arg(a\bar{z})}{2\pi} \right| \leq \frac{1-|a|}{2} \right\}.$$

If $s > 0$ and μ is a positive Borel measure on $\mathbb{D}$, we shall say that μ is an s-Carleson measure if there exists a positive constant C such that

$$\mu\left(S(I)\right) \leq C|I|^s, \quad \text{for any interval } I \subset \partial\mathbb{D},$$

or, equivalently, if there exists $C > 0$ such that

$$\mu\left(S(a)\right) \leq C(1 - |a|)^s, \quad \text{for all } a \in \mathbb{D}.$$

A 1-Carleson measure will be simply called a Carleson measure.

These concepts were generalized in [55] as follows: If μ is a positive Borel measure in $\mathbb{D}$, $0 \leq \alpha < \infty$, and $0 < s < \infty$, we say that μ is an α-logarithmic s-Carleson measure if there exists a positive constant C such that

$$\frac{\mu\left(S(I)\right)\left(\log \frac{2\pi}{|I|}\right)^\alpha}{|I|^s} \leq C, \quad \text{for any interval } I \subset \partial\mathbb{D}$$

or, equivalently, if

$$\sup_{a\in\mathbb{D}} \frac{\mu\left(S(a)\right)\left(\log \frac{2}{1-|a|^2}\right)^\alpha}{(1-|a|^2)^s} < \infty.$$

Carleson measures and logarithmic Carleson measures are known to play a basic role in the study of the boundedness of a great number of operators between analytic function spaces. In particular we recall the Carleson embedding theorem for Hardy spaces which asserts that if $0 < p < \infty$ and μ is a positive Borel measure on $\mathbb{D}$ then μ is a Carleson measure if and only if the Hardy space H^p is continuously embedded in $L^p(d\mu)$ (see [18, Chapter 9]).

In the next theorem we collect a number of known results which will be needed in our work.

Theorem B *(i) If $0 < s \le 1$ and $f \in \mathcal{H}ol(\mathbb{D})$, then $f \in Q_s$ if and only if the Borel measure μ on D defined by*

$$d\mu(z) \,=\, (1-|z|^2)^s |f'(z)|^2 \, dA(z)$$

is an s-Carleson measure.

(ii) If $0 \le \alpha < \infty$, $0 < s < \infty$, and μ is a positive Borel measure on $\mathbb{D}$ then μ is an α-logarithmic s-Carleson measure if and only if

$$\sup_{a\in\mathbb{D}} \left(\log \frac{2}{1-|a|^2}\right)^{\alpha} \int_{\mathbb{D}} \left(\frac{1-|a|^2}{|1-\overline{a}\,z|^2}\right)^{s} d\mu(z) \,<\, \infty.$$

(iii) If $1 < p \le 2$ then $B^p \subset Q_s$ for all $s > 0$.

(iv) If $2 < p < \infty$ and $1 - \frac{2}{p} < s$, then $B^p \subset Q_s$.

(v) For $s > -1$, we let $\mathcal{D}_s$ be the space of those functions $f \in \mathcal{H}ol(\mathbb{D})$ for which

$$\|f\|_{\mathcal{D}_s} \overset{def}{=} |f(0)| + \left(\int_{\mathbb{D}} (1-|z|^2)^s |f'(z)|^2 \, dA(z)\right)^{1/2} \,<\, \infty.$$

Suppose that $0 < s < 1$ and $\alpha > 1$, and let μ be a positive Borel measure on $\mathbb{D}$. If μ is an α-logarithmic s Carleson measure, then μ is a Carleson measures for $\mathcal{D}_s$, that is, $\mathcal{D}_s$ is continuously embedded in $L^2(d\mu)$.

Let us mention that (i) is due to Aulaskari, Stegenga and Xiao [13], (ii) is due to Zhao [55], (iii) and (iv) were proved by Aulaskari and Csordas in [10], and (v) is due to Pau and Peláez [41, Lemma 1].

Using Theorem B (ii) and the fact that

$$1-|\varphi_(z)|^2 \,=\, \frac{(1-|a|^2)(1-|z|^2)}{|1-\overline{a}\,z|^2},$$

we see that for a function $f \in \mathcal{H}ol(\mathbb{D})$ we have that $f \in Q_{s,\log,\alpha}$ if and only if the measure μ defined by $d\mu(z) = (1-|z|^2)^s|f'(z)|^2\,dA(z)$ is a 2α-logarithmic s-Carleson measure.

Proof of Theorem 4 (i). Suppose that T_g maps B^1 into Q_s. Since the constant functions belong to B^1, we have that $T_g(1) = g - g(0) \in Q_s$ and, hence, $g \in Q_s$.

To prove the converse, suppose that $g \in Q_s$. Then the measure μ defined by

$$d\mu(z) \,=\, (1-|z|^2)^s |g'(z)|^2 \, dA(z)$$

is an s-Carleson measure. Take now $f \in B^1$, then $f \in H^\infty$ and, hence,

$$(1-|z|^2)^s \left|(T_g(f))'(z)\right|^2 \,=\, (1-|z|^2)^s |g'(z)|^2 |f(z)|^2 \,\le\, \|f\|^2_{H^\infty} (1-|z|^2)^s |g'(z)|^2.$$

Since μ is an s-Carleson measure, it follows readily that the measure ν given by $d\nu(z) = (1-|z|^2)^s \left|(T_g(f))'(z)\right|^2 dA(z)$ is also an s-Carleson measure and, hence, $T_g(f) \in Q_s$. □

Proof of Theorem 4 (ii).

Suppose that $0 < s \le 1$, $1 < p < \infty$, and that T_g maps B^p into Q_s. For $a \in \mathbb{D} \setminus \{0\}$, set

$$f_a(z) \,=\, \left(\log \frac{1}{1-|a|^2}\right)^{-1/p} \log \frac{1}{1-\overline{a}\,z}, \quad z \in \mathbb{D},$$

as in (8). We have that $\|f_a\|_{B^p} \asymp 1$ and it is also clear that

$$|f_a(z)| \asymp \left(\log \frac{1}{1-|a|^2}\right)^{1/p'}, \quad z \in S(a).$$

Using these facts, we obtain

$$\begin{aligned}
&\frac{\left(\log \frac{1}{1-|a|^2}\right)^{2/p'}}{(1-|a|^2)^s} \int_{S(a)} (1-|z|^2)^s |g'(z)|^2 \, dA(z) \\
&\asymp \frac{1}{(1-|a|^2)^s} \int_{S(a)} (1-|z|^2)^s |g'(z) f_a(z)|^2 \, dA(z) \\
&= \frac{1}{(1-|a|^2)^s} \int_{S(a)} (1-|z|^2)^s \left| \left(T_g(f_a)\right)'(z)\right|^2 \, dA(z).
\end{aligned}$$

The fact that T_g is a bounded operator from B^p into Q_s, implies that the measures $(1-|z|^2)^s \left| \left(T_g(f_a)\right)'(z)\right|^2 dA(z)$ are s-Carleson measures with constants controlled by $\|T_g\|^2$. Then it follows that the measure $(1-|z|^2)^s |g'(z)|^2 \, dA(z)$ is a $2/p'$-logarithmic s-Carleson measure and, hence, $g \in Q_{s,\log,1/p'}$. $\square$

Proof of Theorem 4 (iii) and (iv). In view of (ii) we only have to prove that if $g \in Q_{s,\log,1/p'}$ then T_g maps B^p into Q_s.

Hence, take $g \in Q_{s,\log,1/p'}$ and set

$$K(g) = \sup_{a \in \mathbb{D}} \left(\log \frac{2}{1-|a|}\right)^{2/p'} \int_{\mathbb{D}} |g'(z)|^2 (1-|\varphi_a(z)|^2)^s \, dA(z),$$

and take $f \in B^p$. Set $F = T_g(f)$, we have to prove that $F \in Q_s$ or, equivalently, that the measure μ_F defined by

$$d\mu_F(z) = (1-|z|^2)^s |F'(z)|^2 \, dA(z)$$

is an s-Carleson measure. Let $a \in \mathbb{D}$. Using the well known fact that

$$1-|a|^2 \asymp |1-\overline{a}\, z|, \quad z \in S(a),$$

we obtain

$$\begin{aligned}
&\frac{1}{(1-|a|^2)^s} \int_{S(a)} |F'(z)|^2 (1-|z|^2)^s \, dA(z) \asymp \int_{S(a)} |F'(z)|^2 \frac{(1-|z|^2)^s (1-|a|^2)^s}{|1-\overline{a}\, z|^{2s}} \, dA(z) \\
&= \int_{S(a)} |f(z)|^2 |g'(z)|^2 (1-|\varphi_a(z)|^2)^s \, dA(z) \\
&\leq 2 \int_{\mathbb{D}} |f(a)|^2 |g'(z)|^2 (1-|\varphi_a(z)|^2)^s \, dA(z) \\
&\quad + 2 \int_{\mathbb{D}} |f(z)-f(a)|^2 |g'(z)|^2 (1-|\varphi_a(z)|^2)^s \, dA(z) \\
&= 2T_1(a) + 2T_2(a). \qquad (9)
\end{aligned}$$

Using the fact that

$$|f(a)-f(0)| \lesssim \|f\|_{B^p}\left(\log\frac{2}{1-|a|^2}\right)^{1/p'}, \tag{10}$$

we obtain

$$T_1(a) \lesssim \|f\|_{B^p}^2\left(\log\frac{2}{1-|a|^2}\right)^{2/p'}\int_{\mathbb{D}}|g'(z)|^2(1-|\varphi_a(z)|^2)^s\,dA(z) \lesssim\ K(g)\|f\|_{B^p}^2. \tag{11}$$

To estimate $T_2(a)$ we shall treat separately the cases $s=1$ and $0<s<1$.
Let us start with the case $s=1$. Then

$$T_2(a)\ =\ \int_{\mathbb{D}}|f(z)-f(a)|^2|g'(z)|^2(1-|\varphi_a(z)|^2)\,dA(z).$$

Making the change of variable $w=\varphi(z)$ in the last integral, we obtain

$$T_2(a)\ =\ \int_{\mathbb{D}}|(f\circ\varphi_a)(w)-f(a)|^2|(g\circ\varphi_a)'(w)|^2(1-|w|^2)\,dA(w).$$

Since $Q_{1,\log,1/p'}\subset Q_1=BMOA$, $g\in BMOA$ and then it follows that, for all $a\in\mathbb{D}$, $g\circ\varphi_a\in BMOA$ and $\rho_*(g\circ\varphi_a)=\rho_*(g)$. This gives that all the measures $(1-|w|^2)|(g\circ\varphi_a)'(w)|^2\,dA(w)$ $(a\in\mathbb{D})$ are Carleson measures with constants controlled by $\|g\|_{BMOA}^2$. Then, using the Carleson embedding theorem for H^2 and the fact that B^p is continuously embedded in $BMOA$, it follows that

$$T_2(a)\ \lesssim \|g\|_{BMOA}^2\|f\circ\varphi_a\ -f(a)\|_{H^2}^2\ \lesssim \|g\|_{BMOA}^2\|f\|_{BMOA}^2 \lesssim \|g\|_{BMOA}^2\|f\|_{B^p}^2.$$

Putting together this, (9), and (11), we see that the measure μ_F is a Carleson measure. This finishes the proof of part (iii).

To finish the proof of part (iv) we proceed to estimate $T_2(a)$ assuming that $2<p<\infty$, $0<s<1$, and $1-\frac{2}{p}<s$. Notice that

$$T_2(a)\ =\ (1-|a|^2)^s\int_{\mathbb{D}}\left|\frac{f(z)-f(a)}{(1-\overline{a}\,z)^s}\right|^2|g'(z)|^2(1-|z|^2)^s\,dA(z).$$

Since $0<s<1$, $2/p'>1$, and the measure $(1-|z|^2)^s|g'(z)|^2\,dA(z)$ is a $2/p'$-logarithmic s-Carleson measure, using Theorem B (v), it follows that

$$T_2(a)\ \lesssim\ (1-|a|^2)^s\left(|f(a)-f(0)|^2\ +\ \int_{\mathbb{D}}\left|\left(\frac{f(z)-f(a)}{(1-\overline{a}\,z)^s}\right)'\right|^2(1-|z|^2)^s\,dA(z)\right).$$

The growth estimate (10) and simple computations yield

$$\begin{aligned}
T_2(a) &\lesssim \|f\|_{B^p}^2(1-|a|^2)^s\left(\log\frac{2}{1-|a|^2}\right)^{2/p'}\ +\ \int_{\mathbb{D}}|f'(z)|^2(1-|\varphi_a(z)|^2)^s\,dA(z)\\
&\quad+\ \int_{\mathbb{D}}\frac{|f(z)-f(a)|^2}{|1-\overline{a}\,z|^2}(1-|\varphi_a(z)|^2)^s\,dA(z)\\
&\lesssim \|f\|_{B^p}^2\ +\ \int_{\mathbb{D}}|f'(z)|^2(1-|\varphi_a(z)|^2)^s\,dA(z)\ +\ \int_{\mathbb{D}}\frac{|f(z)-f(a)|^2}{|1-\overline{a}\,z|^2}(1-|\varphi_a(z)|^2)^s\,dA(z).
\end{aligned}$$

By Theorem B (iv), our assumptions on s and p imply that B^p is continuously embedded in Q_s. Hence, $f \in Q_s$. This implies that

$$\int_{\mathbb{D}} |f'(z)|^2 (1-|\varphi_a(z)|^2)^s \, dA(z) \le \|f\|_{Q_s}^2 \lesssim \|f\|_{B^p}^2$$

and that

$$\int_{\mathbb{D}} \frac{|f(z)-f(a)|^2}{|1-\overline{a}\,z|^2} (1-|\varphi_a(z)|^2)^s \, dA(z) \lesssim \|f\|_{Q_s}^2 \lesssim \|f\|_{B^p}^2,$$

by a result proved by Pau and Peláez in [41, pp. 551–552]. Consequently, we have proved that $T_2(a) \lesssim \|f\|_{B^p}^2$. This, together with (9) and (11), shows that μ_F is an s-Carleson measure as desired. Thus the proof is also finished in this case. □

The case when $1 < p \le 2$ and $0 < s < 1$ remains open. This is so because if we set $\alpha = 2/p'$, then $\alpha \le 1$ and, hence, α is not in the conditions of Theorem B (v). We can prove the following result.

Theorem 5 *Suppose that $1 < p \le 2$ and $0 < s < 1$, and let $g \in \mathcal{H}ol(\mathbb{D})$. The following statements hold.*

(i) If T_g maps B^p into Q_s then $g \in Q_{s,\log,1/p'}$.
(ii) If $\alpha > 1/2$ and $g \in Q_{s,\log,\alpha}$ then T_g maps B^p into Q_s.

Proof. (i) follows from part (ii) of Theorem 4.

Let us turn to prove (ii). Suppose that $0 < s < 1$, $\alpha > 1/2$, and $g \in Q_{s,\log,\alpha}$. Set

$$K(g) = \sup_{a\in\mathbb{D}} \left(\log \frac{2}{1-|a|}\right)^{2\alpha} \int_{\mathbb{D}} |g'(z)|^2 (1-|\varphi_a(z)|^2)^s \, dA(z),$$

and take $f \in B^p$. Set $F = T_g(f)$, we have to prove the $F \in Q_s$ or, equivalently, that the measure μ_F defined by

$$d\mu_F(z) = (1-|z|^2)^s |F'(z)|^2 \, dA(z)$$

is an s-Carleson measure. Now we argue as in the proof of Theorem 4 (iv). For $a \in \mathbb{D}$, we obtain

$$\frac{1}{(1-|a|^2)^s} \int_{S(a)} |F'(z)|^2 (1-|z|^2)^s \, dA(z) \lesssim 2T_1(a) + 2T_2(a), \tag{12}$$

where $T_1(a)$ and $T_2(a)$ are defined as in the proof of Theorem 4. Using (10) and the fact that $\frac{1}{p'} \le \frac{1}{2} < \alpha$, we obtain

$$|f(a)-f(0)| \lesssim \|f\|_{B^p} \left(\log \frac{2}{1-|a|^2}\right)^{\alpha}.$$

This yields

$$T_1(a) \lesssim \|f\|_{B^p}^2 \left(\log \frac{2}{1-|a|^2}\right)^{2\alpha} \int_{\mathbb{D}} |g'(z)|^2 (1-|\varphi_a(z)|^2)^s \, dA(z) \lesssim K(g)\|f\|_{B^p}^2. \tag{13}$$

To estimate $T_2(a)$, observe that the measure $(1-|z|^2)^s |g'(z)|^2 \, dA(z)$ is a 2α-logarithmic s-Carleson measure. Since $2\alpha > 1$, using Lemma 1 of [41], this implies that the measure $(1-|z|^2)^s|g'(z)|^2\,dA(z)$ is a Carleson measure for $\mathcal{D}_s$. Then, arguing as in the proof of Theorem 4 (iv), we obtain $T_2(a)\lesssim \|f\|^2_{B^p}$. This, together with (13) and (12), implies that the measure μ_F is an s-Carleson measure. □

Regarding the operators I_g and M_g we have the following results.

Theorem 6 *Let $g\in\mathcal{H}ol(\mathbb{D})$, then:*

(1) If $1<p\le 2$ and $0<s\le 1$ then:

(1a) I_g maps B^p into Q_s if and only if $g\in H^\infty$.

(1b) If M_g maps B^p into Q_s then $g\in Q_{s,\log,1/p'}\cap H^\infty$.

(1c) If $g\in Q_{s,\log,\alpha}\cap H^\infty$ for some $\alpha>1/2$ then M_g maps B^p into Q_s.

(2) If $2<p<\infty$ and $1-\frac{2}{p}<s\le 1$ then:

(2a) I_g maps B^p into Q_s if and only if $g\in H^\infty$.

(2b) M_g maps B^p into Q_s if and only if $g\in Q_{s,\log,1/p'}\cap H^\infty$.

(3) If $2<p<\infty$ and $0<s\le 1-\frac{2}{p}$ then:

(3a) I_g maps B^p into Q_s if and only if $g\equiv 0$.

(3b) M_g maps B^p into Q_s if and only if $g\equiv 0$.

Proof of Parts (1) and (2) of Theorem 6. Using Proposition 1 it follows that if either I_g or M_g maps B^p into Q_s for any pair (s,p) with $0<s<\infty$ and $1<p<\infty$ then $g\in H^\infty$.

Suppose now that s and p are in the conditions of (1) or (2) and that $g\in H^\infty$. Take $f\in B^p$. We have to prove $I_g(f)\in Q_s$ or, equivalently, that the measure

$$(1-|z|^2)^s|f'(z)|^2|g(z)|^2\,dA(z)\text{is an } s\text{-Carleson measure.} \tag{14}$$

Using (1) and (2), we see that $B^p\subset Q_s$. Hence $f\in Q_s$ which is the same as saying that $(1-|z|^2)^s|f'(z)|^2\,dA(z)$ is an s-Carleson measure. This and the fact that $g\in H^\infty$ trivially yield (14). Thus (1a) and (2a) are proved. Then (1b), (1c), and (2b) follow using Proposition 1, the fact that if two of the operators T_g, I_g, M_g map B^p into Q_s so does the third one, Theorem 4, and Theorem 5. □

In order to prove Theorem 6 (3), for $2<p<\infty$ we shall consider the function F_p defined by

$$F_p(z)=\sum_{k=1}^{\infty}\frac{1}{k^{1/2}2^{k/p}}\,z^{2^k},\quad z\in\mathbb{D}. \tag{15}$$

Using [10, Corollary 7] or [14, Theorem 6], we see that $F_p\in B^p$ and $F_p\notin Q_{1-\frac{2}{p}}$. Hence

$$F_p\in B^p\setminus Q_s,\quad 0<s\le 1-\frac{2}{p},\quad 2<p<\infty. \tag{16}$$

Let us estimate the integral means $M_2(r,F_p')$. We have

$$zF_p'(z)=\sum_{k=1}^{\infty}2^{k/p'}k^{-1/2}z^{2^k},\quad z\in\mathbb{D}$$

and, hence,

$$M_2(r,F_p')^2\gtrsim\sum_{k=1}^{\infty}2^{2k/p'}k^{-1}\,r^{2^{k+1}},\quad 0<r<1.$$

Set $r_n = 1 - 2^{-n}$ $(n = 1, 2, \dots)$. Then

$$M_2(r_n, F_p')^2 \gtrsim \sum_{k=1}^{\infty} 2^{2k/p'} k^{-1} r_n^{2^{k+1}}$$
$$\gtrsim 2^{2n/p'} n^{-1} r_n^{2^{n+1}} \gtrsim 2^{2n/p'} n^{-1} \asymp \frac{1}{(1-r_n)^{2/p'} \log \frac{2}{1-r_n}}, \quad n = 1, 2, \dots.$$

This readily yields

$$M_2(r, F_p')^2 \gtrsim \frac{1}{(1-r)^{2/p'} \log \frac{2}{1-r}}, \quad 0 < r < 1. \tag{17}$$

Proof of part (3) of Theorem 6. Suppose that $2 < p < \infty$ and $0 < s \le 1 - \frac{2}{p}$ and $g \in \mathcal{H}ol(\mathbb{D})$ is not identically zero.

Suppose first that either I_g or M_g maps B^p into Q_s. We know that then $g \in H^\infty$ and then, by Fatou's theorem and the Riesz uniqueness theorem, we know that g has a finite non-tangential limit $g(e^{i\theta})$ for almost every $\theta \in [0, 2\pi]$ and that $g(e^{i\theta}) \neq 0$ for almost every θ. Then it follows that there exist $C > 0$, $r_0 \in (0, 1)$, and a measurable set $E \subset [0, 2\pi]$ whose Lebesgue measure $|E|$ is positive such that

$$|g(re^{i\theta})| \ge C, \quad \theta \in E, \quad r_0 < r < 1. \tag{18}$$

Since F_p is given by a power series with Hadamard gaps, Lemma 6. 5 in [60, Vol. 1, p. 203] implies that

$$\int_E |F_p'(re^{i\theta})|^2 \, d\theta \asymp M_2(r, F_p')^2, \quad 0 < r < 1. \tag{19}$$

Using the fact that $s \le 1 - \frac{2}{p}$, (18), (19), and (17), we obtain

$$\int_0^1 (1-r)^s M_2(r, F_p' g)^2 \, dr \ \ge \int_{r_0}^1 (1-r)^{1-\frac{2}{p}} M_2(r, F_p' g)^2 \, dr$$
$$\gtrsim \int_{r_0}^1 (1-r)^{1-\frac{2}{p}} \int_E |F_p'(re^{i\theta})|^2 \, |g(re^{i\theta})|^2 \, d\theta \, dr \gtrsim \int_{r_0}^1 (1-r)^{1-\frac{2}{p}} \int_E |F_p'(re^{i\theta})|^2 \, d\theta \, dr$$
$$\gtrsim \int_{r_0}^1 (1-r)^{1-\frac{2}{p}} M_2(r, F_p')^2 \, dr \gtrsim \int_{r_0}^1 \frac{dr}{(1-r) \log \frac{2}{1-r}} = \infty. \tag{20}$$

If we assume that I_g maps B^p into Q_s then $I_g(F_p) \in Q_s$ and then, using [11, Proposition 3. 1], it follows that

$$\int_0^1 (1-r)^s M_2(r, F_p' g)^2 \, dr \ < \infty.$$

This is in contradiction with (20).

Assume now that M_g maps B^p into Q_s. Since 1 and F_p belong to B^p, we have that g and $F_p \, g$ belong to Q_s and then, by [11, Proposition 3. 1],

$$\int_0^1 (1-r)^s \, M_2(r, g')^2 \, dr \ < \ \infty \tag{21}$$

and

$$\int_0^1 (1-r)^s \, M_2(r, \, (F_p g)')^2 \, dr \; < \; \infty. \tag{22}$$

Notice that $F_p \in H^\infty$ and then

$$M_2(r, F_p \, g') \; \lesssim \; M_2(r, g'), \quad 0 < r < 1.$$

This and (21) imply that

$$\int_0^1 (1-r)^s \, M_2(r, F_p' \, g)^2 \, dr \; < \; \infty. \tag{23}$$

We have arrived to a contradiction because it is clear that (20) and (23) cannot be simultaneously true. □

In the other direction we have the following result.

Theorem 7 *Suppose that* $0 < s < \infty$ *and* $1 \le p < \infty$ *and let* $g \in \mathcal{H}ol(\mathbb{D})$. *Then the following conditions are equivalent*

(i) M_g *maps* Q_s *into* B^p.
(ii) $g \equiv 0$.

Proof. Suppose that $g \not\equiv 0$. Choose an increasing sequence $\{r_n\}_{n=1}^\infty \subset (0,1)$ with $\lim\{r_n\} = 1$ and a sequence $\{\theta_n\}_{n=1}^\infty \subset [0, 2\pi]$ such that

$$|g(r_n e^{i\theta_n})| = M_\infty(r_n, g), \quad n = 1, 2, \dots .$$

For each n set

$$f_n(z) \; = \; \log \frac{1}{1 - e^{-i\theta_n} z}, \quad z \in \mathbb{D}.$$

Notice that $M(r_1, g) > 0$ and that the sequence $\{M(r_n, g)\}$ is increasing. Set

$$f_n(z) = \log \frac{1}{1 - e^{-i\theta_n} z}, \quad z \in \mathbb{D}, \quad n = 1, 2, \dots .$$

We have that $f_n \in Q_s$ for all n and

$$\|f_n\|_{Q_s} \asymp 1.$$

Assume that M_g maps Q_s into B^p. Then, by the closed graph theorem, M_g is bounded operator from Q_s into B^p. Hence the sequence $\{g \, f_n\}_{n=1}^\infty$ is a bounded sequence on B^p, that is,

$$\|g \, f_n\|_{B^p} \lesssim 1.$$

Then it follows that

$$\begin{aligned} M(r_1, g) \log \frac{1}{1 - r_n} \le & M(r_n, g) \log \frac{1}{1 - r_n} = |g(r_n e^{i\theta_n}) f_n(r_n e^{i\theta_n})| \\ \lesssim & \|g \, f_n\|_{B^p} \left(\log \frac{1}{1 - r_n} \right)^{1/p'} \lesssim \left(\log \frac{1}{1 - r_n} \right)^{1/p'} . \end{aligned}$$

This is a contradiction. □

3 Multipliers and integration operators between Q_s spaces

As we mentioned above the space of multipliers $M(\mathcal{B}) = M(Q_s)$ $(s > 1)$ was characterized by Brown and Shields in [15]. Ortega and Fàbrega [40] characterized the space $M(BMOA) = M(Q_1)$. Pau and Peláez [41] and Xiao [54] characterized the spaces $M(Q_s)$ $(0 < s < 1)$ closing a conjecture formulated in [51]. Indeed, Theorem 1 of [41] and Theorem 1. 2 of [54] assert the following.

Theorem C *Suppose that* $0 < s \le 1$ *and let* g *be an analytic function in the unit disc* $\mathbb{D}$*. Then:*

(i) T_g *maps* Q_s *into itself if and only if* $g \in Q_{s,\log,1}$.
(ii) I_g *maps* Q_s *into itself if and only if* $g \in H^\infty$.
(ii) M_g *maps* Q_s *into itself if and only if* $g \in Q_{s,\log,1} \cap H^\infty$.

We shall prove the following results.

Theorem 8 *Suppose that* $0 < s_1 \le s_2 \le 1$ *and let* $g \in \mathcal{H}ol(\mathbb{D})$*. Then:*

(i) T_g *maps* Q_{s_1} *into* Q_{s_2} *if and only if* $g \in Q_{s_2,\log,1}$.
(ii) I_g *maps* Q_{s_1} *into* Q_{s_2} *if and only if* $g \in H^\infty$.
(iii) M_g *maps* Q_{s_1} *into* Q_{s_2} *if and only if* $g \in Q_{s_2,\log,1} \cap H^\infty$.

Theorem 9 *Suppose that* $0 < s_1 < s_2 \le 1$ *and let* $g \in \mathcal{H}ol(\mathbb{D})$*. Then the following conditions are equivalent:*

(i) I_g *maps* Q_{s_2} *into* Q_{s_1}.
(ii) M_g *maps* Q_{s_2} *into* Q_{s_1}.
(iii) $g \equiv 0$.

Proof of Theorem 8. For $a \in \mathbb{D}$ we set

$$h_a(z) = \log \frac{2}{1 - \overline{a}\, z}, \quad z \in \mathbb{D}.$$

Then $h_a \in Q_{s_1}$ for all $a \in \mathbb{D}$ and

$$\|h_a\|_{Q_{s_1}} \asymp 1. \tag{24}$$

• If T_g maps Q_{s_1} into Q_{s_2} then T_g is a bounded operator from Q_{s_1} into Q_{s_2}. Using this and (24), it follows that for all $a \in \mathbb{D}$ the measure $(1-|z|^2)^{s_2}|g'(z)|^2|h_a(z)|^2\, dA(z)$ is an s_2-Carleson measure and that

$$\int_{S(a)} (1 - |z|^2)^{s_2}|g'(z)|^2|h_a(z)|^2\, dA(z) \lesssim (1 - |a|^2)^{s_2}, \quad a \in \mathbb{D}. \tag{25}$$

Since

$$|h_a(z)| \asymp \log \frac{2}{1 - |a|^2}, \quad z \in S(a),$$

(25) implies that

$$\left(\log \frac{2}{1 - |a|^2}\right)^2 \int_{S(a)} (1 - |z|^2)^{s_2}|g'(z)|^2\, dA(z) \lesssim (1 - |a|^2)^{s_2}.$$

This is the same as saying that the measure $(1-|z|^2)^{s_2}|g'(z)|^2\,dA(z)$ is a 2-logarithmic s_2-Carleson measure or, equivalently, that $g\in Q_{s_2,\log,1}$.

If $g\in Q_{s_2,\log,1}$ then, by Theorem C, T_g maps Q_{s_2} into itself. Since $Q_{s_1}\subset Q_{s_2}$, it follows trivially that T_g maps Q_{s_1} into Q_{s_2}. Hence (i) is proved

• Proposition 1 shows that if I_g maps Q_{s_1} into Q_{s_2} then $g\in H^\infty$.

Conversely, suppose that $g\in H^\infty$. In order to prove that I_g maps Q_{s_1} into Q_{s_2}, we have to prove that for any $f\in Q_{s_1}$ the measure $(1-|z|^2)^{s_2}|g(z)|^2|f'(z)|^2\,dA(z)$ is an s_2-Carleson measure. So, take $f\in Q_{s_1}$. Then $(1-|z|^2)^{s_1}|f'(z)|^2\,dA(z)$ is an s_1-Carleson measure. Then it follows that

$$\begin{aligned}&\int_{S(a)}(1-|z|^2)^{s_2}|g(z)|^2|f'(z)|^2\,dA(z)\\ &\le \|g\|_{H^\infty}^2(1-|a|^2)^{s_2-s_1}\int_{S(a)}(1-|z|^2)^{s_1}|f'(z)|^2\,dA(z)\\ &\lesssim (1-|a|^2)^{s_2}.\end{aligned}$$

This shows that $(1-|z|^2)^{s_2}|g(z)|^2|f'(z)|^2\,dA(z)$ is an s_2-Carleson measure as desired, finishing the proof of (ii).

• If M_g maps Q_{s_1} into Q_{s_2} then, Proposition 1, $g\in H^\infty$. Then (i) implies that I_g maps Q_{s_1} into Q_{s_2}. Since $M_g(f)=I_g(f)+T_g(f)+f(0)g(0)$, it follows that T_g maps Q_{s_1} into Q_{s_2}. Then (i) yields $g\in Q_{s_2,\log,1}$. Then we have that $g\in Q_{s_2,\log,1}\cap H^\infty$.

Conversely, if $g\in Q_{s_2,\log,1}\cap H^\infty$ then (i) and (ii) immediately give that both T_g and I_g map Q_{s_1} into Q_{s_2} and then so does M_g. □

Some results from [11] will be used to prove Theorem 9. As we have already noticed if $0<s\le 1$ and $f\in Q_s$ then $\int_0^1(1-r)^sM_2(r,f')^2\,dr<\infty$. Using ideas from [27], Aulaskari, Girela and Wulan [11, Theorem 3. 1] proved that this result is sharp in a very strong sense.

Theorem D *Suppose that $0<s\le 1$ and let φ be a positive increasing function defined in $(0,1)$ such that*

$$\int_0^1(1-r)^s\,\varphi(r)^2\,dr<\infty.$$

Then there exists a function $f\in Q_s$ given by a power series with Hadamard gaps such that $M_2(r,f')\ge\varphi(r)$ for all $r\in(0,1)$.

Proof of Theorem 9. Suppose that $g\not\equiv 0$ and that either I_g or M_g maps Q_{s_2} into Q_{s_1}. By Proposition 1, $g\in H^\infty$ and then it follows that there exist $C>0$, $r_0\in(0,1)$, and a measurable set $E\subset[0,2\pi]$ whose Lebesgue measure $|E|$ is positive such that

$$|g(re^{i\theta})|\ge C,\quad \theta\in E,\quad r_0<r<1.$$

• Suppose that I_g maps Q_{s_2} into Q_{s_1}. Then we use Theorem D to pick a function $F\in Q_{s_2}$ given by a power series with Hadamard gaps so that

$$M_2(r,F')\ \ge\ \frac{1}{(1-r)^{(1+s_1)/2}},\quad 0<r<1. \tag{26}$$

Since $I_g(F)\in Q_{s_1}$,

$$\int_0^1(1-r)^{s_1}M_2(r,F'g)^2\,dr\ <\ \infty. \tag{27}$$

However, using Lemma 6. 5 in [60, Vol. 1, p. 203] and (26), it follows that

$$\begin{aligned}\int_0^1 (1-r)^{s_1} M_2(r, F'g)^2\, dr &\gtrsim \int_{r_0}^1 (1-r)^{s_1} \int_E |F'(re^{i\theta})|^2 |g(re^{i\theta})|^2\, d\theta\, dr \\ &\gtrsim \int_{r_0}^1 (1-r)^{s_1} \int_E |F'(re^{i\theta})|^2\, d\theta\, dr \\ &\asymp \int_{r_0}^1 (1-r)^{s_1} M_2(r, F')^2\, dr \\ &\gtrsim \int_{r_0}^1 (1-r)^{-1}\, dr \\ &= \infty.\end{aligned}$$

This is in contradiction with (27).

• Suppose now that M_g maps Q_{s_2} into Q_{s_1}. Take $\varepsilon > 0$ with $s_2 - s_1 - \varepsilon > 0$ and use Theorem D to pick a function $H \in Q_{s_2}$ given by a power series with Hadamard gaps so that

$$M_2(r, H') \geq \frac{1}{(1-r)^{(1+s_1+\varepsilon)/2}}, \quad 0 < r < 1. \tag{28}$$

Since $gH \in Q_{s_1}$ we have that

$$\int_0^1 (1-r)^{s_1} M_2(r, g'H + gH')^2\, dr < \infty. \tag{29}$$

Using Lemma 6. 5 in [60, Vol. 1, p. 203] and (28), we obtain as above that

$$\begin{aligned}\int_0^1 (1-r)^{s_1+\varepsilon} M_2(r, H'g)^2\, dr &\gtrsim \int_{r_0}^1 (1-r)^{s_1+\varepsilon} \int_E |H'(re^{i\theta})|^2\, d\theta\, dr \\ &\gtrsim \int_{r_0}^1 (1-r)^{s_1+\varepsilon} M_2(r, H')^2\, dr \\ &\gtrsim \int_{r_0}^1 \frac{dr}{1-r} \\ &= \infty.\end{aligned} \tag{30}$$

Notice that $g \in Q_{s_1}$. Using this and the fact that

$$|H(z)| \lesssim \log \frac{2}{1-|z|}, \quad z \in \mathbb{D},$$

it follows that

$$\begin{aligned}&\int_0^1 (1-r)^{s_1+\varepsilon} M_2(r, Hg')^2\, dr \lesssim \int_0^1 (1-r)^{s_1+\varepsilon} \left(\log \frac{2}{1-r}\right)^2 M_2(r, g')^2\, dr \\ &\lesssim \int_0^1 (1-r)^{s_1+\frac{\varepsilon}{2}} M_2(r, g')\, dr < \infty.\end{aligned} \tag{31}$$

We have arrived to a contradiction because (29), (30), and (31) cannot hold simultaneously. □

Remark 1 The implication (ii) $\Rightarrow$ (iii) in Theorem 9 was obtained by Pau and Peláez [42, Corollary 4] using the fact that there exists a function $f \in Q_{s_2}$, $f \not\equiv 0$, whose sequence of zeros is not a Q_{s_1}-zero set.

This idea gives also the following:

$$M(\mathcal{B}, Q_s) = \{0\}, \quad 0 < s \le 1.$$

Indeed, it is well known that there exists a function $f \in \mathcal{B}$, $f \not\equiv 0$, whose sequence of zeros does not satisfy the Blaschke condition [7,31]. If $g \not\equiv 0$ were a multiplier from $\mathcal{B}$ into Q_s for some $s \le 1$ then the sequence of zeros of fg would satisfy the Blaschke condition. But this is not true because all the zeros of f are zeros of gf.

4 Some further results

The inner-outer factorization of functions in the Hardy spaces plays an outstanding role in lots of questions. In many cases the outer factor O_f of f inherits properties of f. Working in this setting the following concepts arise as natural and quite interesting.

A subspace X of H^1 is said to have the f-property (also called the property of division by inner functions) if $h/I \in X$ whenever $h \in X$ and I is an inner function with $h/I \in H^1$.

Given $v \in L^\infty(\partial\mathbb{D})$, the Toeplitz operator T_v associated with the symbol v is defined by

$$T_v f(z) = P(vf)(z) = \frac{1}{2\pi i}\int_{\partial\mathbb{D}} \frac{v(\xi)f(\xi)}{\xi - z}\, d\xi, \quad f \in H^1, \quad z \in \mathbb{D}.$$

Here, P is the Szegö projection.

A subspace X of H^1 is said to have the K-property if $T_{\overline{\psi}}(X) \subset X$ for any $\psi \in H^\infty$.

These notions were introduced by Havin in [34]. It was also pointed out in [34] that the K-property implies the f-property: indeed, if $h \in H^1$, I is inner and $h/I \in H^1$ then $h/I = T_{\overline{I}}h$.

In addition to the Hardy spaces H^p $(1 < p < \infty)$ many other spaces such as the Dirichlet space [34,38], several spaces of Dirichlet type including all the Besov spaces B^p $(1 < p < \infty)$ [20–22,39], the spaces $BMOA$ and $VMOA$ [35], and the Q_s spaces $(0 < s < 1)$ [23] have the K-property. The Hardy space H^1, H^∞ and $VMOA \cap H^\infty$ are examples of spaces which have the f-property bur fail to have the K-property [35].

The first example of a subspace of H^1 not possessing the f-property is due to Gurarii [33] who proved that the space of analytic functions in $\mathbb{D}$ whose sequence of Taylor coefficients is in ℓ^1 does not have the f-property. Anderson [6] proved that the space $\mathcal{B}_0 \cap H^\infty$ does not have the f-property. Later on it was proved in [29] that if $1 \le p < \infty$ then $H^p \cap \mathcal{B}$ fails to have the f-property also.

Since as we have already mentioned the Besov spaces B^p $(1 < p < \infty)$ and the Q_s spaces $(0 < s \le 1)$ have the K-property (and, also, the f-property), it seems natural to investigate whether the spaces of multipliers and the spaces $Q_{s,\log,\alpha}$ that we have considered in our work have also these properties. We shall prove the following results.

Theorem 10 *The spaces of multipliers* $M(B^p, Q_s)$ $(0 < s \le 1, 1 \le p < \infty)$, $M(Q_{s_1}, Q_{s_2})$ $(0 < s_1, s_2 \le 1)$, *and* $M(B^p, B^q)$ $(1 \le p, q < \infty)$ *have the* f*-property.*

Theorem 11 *For* $\alpha > 0$ *and* $0 < s < 1$ *the space* $Q_{s,\log,\alpha}$ *has the* K*-property.*

Theorem 10 follows readily from the following result.

Lemma 1 *Let* X *and* Y *be to Banach spaces of analytic functions which are continuously contained in* H^1*. Suppose that* X *contains the constants functions and that* Y *has the* f*-property. Then the space of multipliers* $M(X, Y)$ *also has the* f*-property.*

Proof. Since X contains the constants functions $M(X, Y) \subset Y \subset H^1$.

Suppose that $F \in M(X, Y)$, I is an inner function, and $F/I \in H^1$. Take $f \in X$. Then $fF \in Y \subset H^1$ and then $fF/I \in H^1$. Since Y has the f-property, it follows that $fF/I \in Y$. Thus, we have proved that $F/I \in M(X, Y)$. □

Theorem 11 will follows from a characterization of the spaces $Q_{s,\log,\alpha}$ in terms of pseudoanalytic continuation. We refer to Dyn'kin's paper [24] for similar descriptions of classical smoothness spaces, as well as for other important applications of the pseudoanalytic extension method.

Let, $\mathbb{D}_-$ denotes the region $\{z \in \mathbb{C} : |z| > 1\}$, and write

$$z^* \stackrel{\text{def}}{=} 1/\overline{z}, \quad z \in \mathbb{C} \setminus \{0\}.$$

We shall need the Cauchy-Riemann operator

$$\overline{\partial} = \frac{\partial}{\partial \overline{z}} \stackrel{\text{def}}{=} \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right), \quad z = x + iy.$$

The following result is an extension of [23, Theorem 1].

Theorem 12 *Suppose that* $0 < s < 1$, $\alpha > 0$, *and* $f \in \cap_{0<q<\infty} H^q$*. Then the following conditions are equivalent.*

(i) $f \in Q_{s,\log,\alpha}$.

(ii) $$\sup_{|a|<1} \left(\log \frac{2}{1-|a|}\right)^{2\alpha} \int_{\mathbb{D}} |f'(z)|^2 \left(\frac{1}{|\varphi_a(z)|^2} - 1\right)^s dA(z) < \infty.$$

(iii) There exists a function $F \in C^1(\mathbb{D}_-)$ *satisfying*

$$F(z) = O(1), \quad \text{as } z \to \infty,$$

$$\lim_{r \to 1^+} F(re^{i\theta}) = f(e^{i\theta}), \quad \text{a.e. and in } L^q([-\pi, \pi]) \text{ for all } q \in [1, \infty)\ ,$$

$$\sup_{|a|<1} \left(\log \frac{2}{1-|a|}\right)^{2\alpha} \int_{\mathbb{D}_-} \left|\overline{\partial} F(z)\right|^2 \left(|\varphi_a(z)|^2 - 1\right)^s dA(z) < \infty.$$

Theorem 12 can be proved following the arguments used in the proof of [23, Theorem 1], we omit the details. Once Theorem 12 is established, noticing that if $\alpha > 0$ and $0 < s < 1$ then $Q_{s,\log\alpha} \subset Q_s \subset BMOA$, Theorem 11 can be proved following the steps in the proof of [23, Theorem 2]. Again, we omit the details.

Acknowledgements. We wish to thank the referees for reading carefully the paper and making a number of nice suggestions to improve it.

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