Smooth actions of connected compact Lie groups with a free point are determined by two vector fields

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ABSTRACT

Consider a smooth action $G \times M \to M$ of a compact connected Lie group G on a connected manifold M. Assume the existence of a point of M whose isotropy group has a single element (a free point). Then we prove that there exist two complete vector field X, X_1 such that their group of automorphisms equals G regarded as a group of diffeomorphisms of M (the existence of a free point implies that the action of G is effective). Moreover, some examples of effective actions with no free point where this result fails are exhibited.

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1. Introduction

In our two preceding works [9,10], focused on the inverse Galois problem's framework, we proved that any effective action on a manifold by a finite group or a torus can be described by a single vector field. However, it is crucial to note that the generality of this result does not extend to compact connected Lie groups, as exemplified in Example 6.3 of [10].

As a result, a compelling and challenging question naturally arises: how do we determine the actions of these compact connected Lie groups? To address this intriguing inquiry, we embark on exploring the possibility of employing a family of vector fields as a potential solution.

In this paper, we demonstrate that for any connected compact Lie group smoothly acting on a connected manifold, the group can always be equated to the group of automorphisms of a couple of vector fields defined on this manifold, given the existence of a free point (Theorem 1.1). Furthermore, we provide illustrative examples of Lie group effective actions lacking a free point, and in which the action cannot be described by any family of vector fields (Theorem 6.3).

To ensure clarity and consistency throughout the manuscript, we begin by establishing the notation and conventions that will be employed throughout the paper. Our fundamental reference sources for differential topology, differential geometry, and Lie group actions are as follows: [4] for differential topology, [5] for differential geometry, and [6] and [11] for Lie groups actions. It is assumed that the reader is already acquainted with our two preceding papers [9,10].

Throughout this work, we consider manifolds (without boundary) and their associated objects to be real and of class C^∞, with actions on the left, unless otherwise stated.

Consider a diffeomorphism $F : M \to M'$ and a vector field X on M. The notation F_*X represents the vector field on M' defined as follows:

$$(F_*X)(y) = F_*(X(F^{-1}(y)))$$

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for every $y \in M'$. Here, $F_* : TM \to TM'$ refers to the differential of the mapping F that acts on tangent vectors between the tangent bundles TM and TM'.

Let $\mathcal{F}$ be a family of vector fields on an m-manifold M. The group of automorphisms of $\mathcal{F}$, denoted as $\mathrm{Aut}(\mathcal{F})$, comprises the subgroup of diffeomorphisms of M that preserve each element of $\mathcal{F}$.

Now, let us consider an action of a Lie group $\mathbf{G}$ on the manifold M. A point $p \in M$ is termed *free* if its isotropy group (or stabilizer) reduces to the neutral element $\mathbf{e}$ of $\mathbf{G}$. Observe that if p is free, then all points within its orbit are also free. It is worth noting that the existence of a free point guarantees the action's effectiveness, and conversely, this holds true when $\mathbf{G}$ is a torus (as stated in Proposition 7.1 of [10]). Thus, for torus actions, there is no difference between being effective and the existence of free points.

Furthermore, it is important to recall that if the action is effective, then $\mathbf{G}$ can be treated as a subgroup of the group $\mathrm{Diff}(M)$, which consists of diffeomorphisms of M.

Then, our main result is:

Theorem 1.1. *Consider an action of a connected compact Lie group $\mathbf{G}$ on a connected manifold M. If the action of $\mathbf{G}$ has a free point then there exist two complete vector fields X, X_1 such that $\mathbf{G} = \mathrm{Aut}(X, X_1)$.*

On the other hand, Theorem 6.3 demonstrates that the aforementioned hypothesis of the existence of a free point cannot be substituted by merely assuming the action to be effective (further details are provided below).

Consider a family of vector fields, denoted as $\mathcal{F}$, defined on the manifold M. In the context of this paper, we shall use the terms $\mathcal{F}$ *determines* or $\mathcal{F}$ *describes* to refer to the action of a transformation group $\mathbf{G}$, if and only if $\mathbf{G}$ is the automorphism group of $\mathcal{F}$, expressed as $\mathbf{G} = \mathrm{Aut}(\mathcal{F})$. It is important to note that under such circumstances, every element belonging to the family $\mathcal{F}$ is inherently $\mathbf{G}$-invariant.

Organization of the paper: This paper is structured into eight sections, with the first section serving as the introduction. Sections 2 and 3 present the essential results concerning Lie groups and vector fields, which are needed later on. The main result is demonstrated in Sections 4 and 5.

In Section 6, it is shown that the assumption of the existence of a free point cannot be omitted. Specifically, it is showed that the natural action of $\mathrm{U}(n)$ on $\mathbb{C}^n \equiv \mathbb{R}^{2n}$, $n \geq 2$, is described by two vector fields. However, no family of $\mathrm{U}(n)$-invariant vector fields can completely determine the natural action of $\mathrm{U}(n)$ on $\mathbb{C}^n \smallsetminus \{0\}$ (see Example 6.4).

Section 7 presents additional examples of compact connected linear groups. For instance, we consider the natural action of $\mathrm{SO}(n)$ for $n \geq 3$ and $\mathrm{SU}(m)$ for $m \geq 3$. In these cases, it is shown that the action cannot be described by invariant vector fields. However, a different outcome is observed in the case of the symplectic group, where its natural action can always be determined using two invariant vector fields. We end this section with a result on the stability of some determinable actions (Proposition 7.3 and Remark 7.4).

The concluding section of this paper presents a curated list of open problems and questions that naturally arise from our work.

2. Preliminaries on Lie groups

In this section, we will present some fundamental concepts related to Lie groups, which will be crucial for our subsequent discussions.

Throughout the remainder of this paper $\mathbf{G}$ will be a connected compact Lie group of dimension n, $\mathbf{e}$ its neutral element, and $\mathcal{G}$ its Lie algebra of left-invariant vector fields. When required, we will treat $\mathcal{G}$ as $T_\mathbf{e}\mathbf{G}$ using the vector isomorphism $X \in \mathcal{G} \mapsto X(\mathbf{e}) \in T_\mathbf{e}\mathbf{G}$. We say that a couple X, Y of elements of $\mathcal{G}$ is *dense* if the connected Lie subgroup $\mathbf{H}$ corresponding to the Lie subalgebra $\mathcal{H}$ spanned by X and Y is dense within $\mathbf{G}$.

Note that if the couple X, Y is dense and $f : \mathbf{G} \to \mathbb{R}$ is a function such that $Xf = Yf = 0$, then f is constant.

We show that dense couples do always exist.

Proposition 2.1. *In the algebra $\mathcal{G}$ of a connected compact Lie group $\mathbf{G}$ there exist dense couples.*

Proof. According to a classical result by Schreier and Ulam [7], if $\mathbf{G}$ is both connected and compact, then there always exist elements $\mathbf{g}$ and $\mathbf{h}$ in $\mathbf{G}$ such that the group generated by $\mathbf{g}$ and $\mathbf{h}$ is dense in $\mathbf{G}$. In this case, since the exponential map of $\mathbf{G}$ is surjective onto the group, there exist X and Y in the Lie algebra $\mathcal{G}$ such that $\exp(X) = \mathbf{g}$ and $\exp(Y) = \mathbf{h}$. It is evident that the couple X, Y is dense. $\square$

Given an abstract group G acting on a set S on the left (resp. right), and $g \in G$, $L_g : S \to S$ (resp. $R_g : S \to S$) denotes the induced by left (resp. right) multiplication by g. Observe that if $S = G$ endowed with the natural left and right G-actions, then the maps L_g and R_h commute for any $g, h \in G$. Therefore:

Lemma 2.2. *Let* **G** *be a Lie group and* $\mathbf{g} \in \mathbf{G}$. *If* X *is a left-invariant vector field on* **G**, *then* $(R_{\mathbf{g}})_* X$ *is the left-invariant vector field with initial condition*

$$((R_{\mathbf{g}})_* X)(\mathbf{e}) = (\mathbb{A}_{\mathbf{g}^{-1}})_*(X(\mathbf{e}))$$

where $\mathbb{A}_{\mathbf{h}} = L_{\mathbf{h}} \circ R_{\mathbf{h}^{-1}}$.
  Similarly, if X' *is a right invariant vector field on* **G**, *then* $(L_{\mathbf{g}})_* X'$ *is right invariant and*

$$((L_{\mathbf{g}})_* X')(\mathbf{e}) = (\mathbb{A}_{\mathbf{g}})_*(X'(\mathbf{e})).$$

Let Z be a vector field defined on a product manifold $P_1 \times P_2$. We say that Z is *horizontal* if $(\pi_2)_* Z = 0$, and *strongly horizontal* or *s-horizontal* if Z is horizontal and preserves the foliation given by the second factor (roughly speaking, Z is tangent to the first factor and independent of the "second variable"). In a similar way one can define the notions of *vertical* and *s-vertical* vector field on $P_1 \times P_2$.

Observe that given a vector field Z_1 on P_1 (resp. Z_2 on P_2) there exists one and only one vector field on $P_1 \times P_2$, which we will still denote as Z_1 (resp. Z_2), such that it is s-horizontal (resp. s-vertical) and $(\pi_1)_* Z_1 = Z_1$ (resp. $(\pi_2)_* Z_2 = Z_2$).

In a more general setting, given a map $\pi : P \to Q$ between manifolds, a vector field Y on P is called *vertical* if $\pi_* Y = 0$.

Let A be an open subset of $\mathbb{R}^k$ and **G** be a Lie group. Consider a vector field H on A and a map $\varphi : A \to T_{\mathbf{e}} \mathbf{G}$. Let $F : A \times \mathbf{G} \to A \times \mathbf{G}$ be the diffeomorphism given by $F(x, \mathbf{g}) = (x, \mathbf{g} \cdot \exp(\varphi(x)))$. Then:

Lemma 2.3. *If* φ *takes values in an abelian subalgebra* $\mathcal{G}'$ *of* $\mathcal{G} \equiv T_{\mathbf{e}} \mathbf{G}$, *and* H *is thought as an s-horizontal vector field on* $A \times \mathbf{G}$, *then* $F_* H = H + V$ *where* V *is vertical, and every* $V(x, \_)$, $x \in A$, *is the left-invariant vector field with initial condition* $V(x, \mathbf{e}) = (\varphi_* H)(x)$.

Proof. Consider the obvious left and right **G**-actions on $A \times \mathbf{G}$ given by $\mathbf{h} \cdot (x, \mathbf{g}) = (x, \mathbf{hg})$ and $(x, \mathbf{g}) \cdot \mathbf{h} = (x, \mathbf{gh})$, and observe these actions commute too. Since $F(x, \mathbf{g}) = R_{\exp(\varphi(x))}(x, \mathbf{g})$, then F and $L_{\tilde{\mathbf{g}}}$, $\tilde{\mathbf{g}} \in \mathbf{G}$, commute. Therefore:

$$(L_{\tilde{\mathbf{g}}})_*(F_* H) = F_*((L_{\tilde{\mathbf{g}}})_* H) = F_* H$$

hence $(L_{\tilde{\mathbf{g}}})_* V = V$, which implies that every $V(x, \_)$ is left-invariant.

Now we compute the initial conditions that determine $V(x, \_)$. Let $\mathbf{G}'$ be the connected Lie subgroup of **G** with algebra $\mathcal{G}'$; set $\mathbf{G}_0 = \overline{\mathbf{G}'}$. Then $\mathbf{G}_0$ is a connected abelian closed Lie subgroup of **G** and a regular submanifold. As $\exp(\varphi(A)) \subset \mathbf{G}_0$, it is enough to compute the initial condition for $F : A \times \mathbf{G}_0 \to A \times \mathbf{G}_0$. In other words, we may assume that **G** is abelian and connected by replacing it with $\mathbf{G}_0$ if needed, thus we assume that $\mathbf{G} = \mathbb{R}^r \times \mathbb{T}^s$.

On $\mathbb{R}^r \times \mathbb{T}^s$ consider coordinates $(y, \theta) = (y_1, \dots, y_r, \theta_1, \dots, \theta_s)$ and the vector fields $\partial/\partial y_1, \dots, \partial/\partial y_r, \partial/\partial \theta_1, \dots, \partial/\partial \theta_s$. Then a vector field U on $\mathbb{R}^r \times \mathbb{T}^s$ is invariant (left or right) whenever there exist $a_1, \dots, a_r, b_1, \dots, b_s \in \mathbb{R}$ such that $U = \sum_{j=1}^r a_j \partial/\partial y_j + \sum_{\ell=1}^s b_\ell \partial/\partial \theta_\ell$.

If $T_{\mathbf{e}}(\mathbb{R}^r \times \mathbb{T}^s)$ is thought as $\mathbb{R}^r \times \mathbb{R}^s$ through $\partial/\partial y_1, \dots, \partial/\partial y_r, \partial/\partial \theta_1, \dots, \partial/\partial \theta_s$, then $\exp(v, w) = (v, p(w))$ where $p : \mathbb{R}^s \to \mathbb{T}^s \equiv (\mathbb{R}^s / 2\pi \mathbb{Z}^s)$ is the canonical projection. Thus $F(x, (y, \theta)) = (x, y + \varphi_1(x), \theta + p(\varphi_2(x)))$ where $\varphi_1 = \pi_1 \circ \varphi$ and $\varphi_2 = \pi_2 \circ \varphi$.

Finally, a straightforward computation shows that $V(x, \mathbf{e}) = (\varphi_* H)(x)$. $\quad\square$

The following is a restatement of Exercise 9 found in [11, p. 134]:

Lemma 2.4. *Let* **G** *be a connected Lie group and* $\psi : \mathbf{G} \to \mathbf{G}$ *be a diffeomorphism such that:*

(a) ψ *maps left-invariant vector fields to left-invariant vectors fields (or alternatively right ones to right ones).*
(b) $\psi(\mathbf{e}) = \mathbf{e}$.

  Then ψ *is a Lie group isomorphism.*

3. Some useful results on vector fields

In this section, we introduce key results pertaining to vector fields that will be utilized subsequently. We follow the convention established in Section 2, and employ the notation defined therein. Moreover, from now on $\xi = \sum_{j=1}^k x_j \partial/\partial x_j$ denotes the radial vector field of $\mathbb{R}^k$, where $k \geq 1$, endowed with the canonical coordinates $(x_1, \dots, x_k)$ of $\mathbb{R}^k$. As needed, we may also view ξ as an s-horizontal vector field on $\mathbb{R}^k \times \mathbf{G}$.

Let P be a regular submanifold of a manifold Q and Z be a vector field defined on an open subset of Q that includes P. We say that Z *is tangent to* P *at order* 1 if:

(1) Z vanishes at P.
(2) For every vector field Y defined on an open subset B of Q, the vector field $[Z, Y]$ is tangent to P at $B \cap P$.

On **G** consider a left-invariant vector field V and think of it as an s-vertical vector field on $\mathbb{R}^k \times$ **G**. Set $X = \xi + V$ on $\mathbb{R}^k \times$ **G**. Then $\{0\} \times$ **G** is the set of those points whose X-trajectory has compact adherence. Therefore, $\{0\} \times$ **G** is an invariant of X, and if Y is a vector field commuting with X, then Y has to be tangent to $\{0\} \times$ **G**.

On the other hand, consider a vertical vector field V_1 such that each $V_1(x, \_)$, $x \in \mathbb{R}^k$, is left-invariant, and consider a function $h \colon \mathbb{R}^k \to \mathbb{R}$ with compact support such that $j_0^4 h = 0$ but $j_0^5 h \neq 0$. Set $X_1 = V_1 + (h \circ \pi_1) X$.

Let $\mathcal{L}(X, X_1)$ be the set of those vector fields Y on $\mathbb{R}^k \times$ **G** such that:

(a) $[X, Y] = 0$
(b) $[X_1, Y]$ is tangent to $\{0\} \times$ **G** at order 1.

Finally, let $\{\theta_1, \ldots, \theta_n\}$ be a basis of the algebra of right invariant vector fields of **G**, and think of each θ_r, $r = 1, \ldots, n$, as an s-vertical vector field on $\mathbb{R}^k \times$ **G**. Then:

Lemma 3.1. *If $V, V_1(0, \_)$ is a dense couple of $\mathcal{G}$, then $\mathcal{L}(X, X_1)$ is a Lie subalgebra of the Lie algebra of vector fields on $\mathbb{R}^k \times$ **G** of dimension $k^2 + n$ with basis*

$$\left\{ x_j \frac{\partial}{\partial x_\ell}, \theta_r \right\}, \quad j, \ell = 1, \ldots, k; \ r = 1, \ldots, n.$$

Proof. First note that, for each $Y \in \mathcal{L}(X, X_1)$, $[X_1, Y]$ is tangent to $\{0\} \times$ **G** at order 1 if and only if $[V_1, Y]$ is so. Therefore, we shall prove the result assuming this last hypothesis.

Let $Y = \sum_{j=1}^k f_j(x, \mathbf{g}) \partial / \partial x_j + \sum_{r=1}^n \varphi_r(x, \mathbf{g}) \theta_r \in \mathcal{L}(X, X_1)$. Then X and Y commute, and Y has to be tangent to $\{0\} \times$ **G**, thus $f_j(0, \mathbf{g}) = 0$, for every $\mathbf{g} \in$ **G** and every $j = 1, \ldots, k$.

Since V, V_1 are left-invariant, then X, V, and V_1 commute with $\theta_1, \ldots, \theta_n$. Now, from $[X, Y]_{|\{0\} \times \mathbf{G}} = [V_1, Y]_{|\{0\} \times \mathbf{G}} = 0$ it follows that $V \cdot \varphi_r$ and $V_1 \cdot \varphi_r$, $r = 1, \ldots, n$, vanish on $\{0\} \times$ **G**. But V, V_1 is a dense couple so each φ_r is constant on $\{0\} \times$ **G**.

A computation shows that

$$[X, Y] = \widetilde{Y} + \sum_{r=1}^n (X \cdot \varphi_r) \theta_r$$

where $\widetilde{Y}$ is a functional combination of $\partial / \partial x_1, \ldots, \partial / \partial x_k$. Hence $X \cdot \varphi_r = 0$, i.e. φ_r is constant along the X-trajectories. But the α-limits of X-trajectories are included in $\{0\} \times$ **G**, set where φ_r is constant, so each φ_r, $r = 1, \ldots, n$, is constant.

Replacing Y by $Y - \sum_{r=1}^n \varphi_r \theta_r$ allows us to assume $Y = \sum_{j=1}^k f_j(x, \mathbf{g}) \partial / \partial x_j$ where each f_j vanishes on $\{0\} \times$ **G**.

In turn from $[X, Y] = 0$ it follows $X \cdot f_j = f_j$, $j = 1, \ldots, k$.

On the other hand

$$[V_1, Y] = \sum_{j=1}^k (V_1 \cdot f_j) \partial / \partial x_j + W$$

where W is vertical. Thus $V_1 \cdot f_j = 0$, $j = 1, \ldots, k$, on $\{0\} \times$ **G** since $[V_1, Y]$ is tangent to this submanifold. Moreover as $[V_1, Y]$ is tangent to $\{0\} \times$ **G** at order 1, $U \cdot (V_1 \cdot f_j) = 0$, $j = 1, \ldots, k$, on $\{0\} \times$ **G** for every vector field $U = \sum_{\ell=1}^k a_\ell \partial / \partial x_\ell$ with $a_1, \ldots, a_k \in \mathbb{R}$.

In order to finish the proof, we must show that every f_j is independent of $\mathbf{g}$ and linear on x. In other words, given $f \colon \mathbb{R}^k \times$ **G** $\to \mathbb{R}$ such that $f(\{0\} \times$ **G**$) = (V_1 \cdot f)(\{0\} \times$ **G**$) = 0$, $X \cdot f = f$ and $(U \cdot (V_1 \cdot f_j))(\{0\} \times$ **G**$) = 0$ for every constant vector field U, we need to prove that f is independent of $\mathbf{g}$ and linear on x.

First, we consider the case $k = 1$. Since f vanishes on $\{0\} \times$ **G**, then $f = x\varphi$ for some function φ. Now $X \cdot f = f$ becomes $X \cdot \varphi = 0$, which implies that φ is constant along the X-trajectories.

Set $U = \partial / \partial x$. Then

$$\frac{\partial (V_1 \cdot f)}{\partial x} = V_1 \cdot \varphi + x \frac{\partial (V_1 \cdot \varphi)}{\partial x}$$

vanishes on $\{0\} \times$ **G**, hence $V_1 \cdot \varphi$ does so.

As $X = V$ on $\{0\} \times$ **G**, and the couple $V, V_1(0, \_)$ is dense, then φ must be constant on $\{0\} \times$ **G**. But the α-limits of all the X-trajectories are included in $\{0\} \times$ **G**, so φ is constant and $f = ax$ for some $a \in \mathbb{R}$. That is, f is independent of $\mathbf{g}$ and linear on x, what concludes the case $k = 1$.

We consider now the case $k \geq 2$. Let E be any vector line in $\mathbb{R}^k$, thus $E \cong \mathbb{R}$. Since:

- X and V_1 are tangent to $E \times \mathbf{G}$,
- the restriction of ξ to E is still the radial vector field
- and one may choose U to be a constant vector field tangent to E,

we apply the 1-dimensional case above to get that $f \colon E \times \mathbf{G} \to \mathbb{R}$ is independent of $\mathbf{g}$ and linear on E

Finally, as the union of all the vector lines E equals $\mathbb{R}^k$, it follows that f is independent of $\mathbf{g}$ and linear on $\mathbb{R}^k$. Indeed, clearly f is homogeneous of degree 1 and therefore linear (see Remark 3.2 below). □

Remark 3.2. Let $A \subset \mathbb{R}^k$ be an open ball centered at the origin and radius $r \in (0, \infty]$. If $\varphi \colon A \to \mathbb{R}$ is homogeneous of degree d then $\partial\varphi/\partial x_1, \ldots, \partial\varphi/\partial x_k$ are homogeneous of degree $d - 1$. Therefore, when $d = 1$ the partial derivative $\partial\varphi/\partial x_1, \ldots, \partial\varphi/\partial x_k$ are constant and φ has to be linear.

Proposition 3.3. Let $F \colon \mathbb{R}^k \times \mathbf{G} \to \mathbb{R}^k \times \mathbf{G}$ be an automorphism of $X = \xi + V$ that preserves $X_1 = V_1 + (h \circ \pi_1)X$ on an open neighborhood of $\{0\} \times \mathbf{G}$, that is, there exists A, an open set neighborhood of $\{0\} \times \mathbf{G}$, such that $F_* X_1(x, \mathbf{g}) = X_1\big(F(x, \mathbf{g})\big)$ for any $(x, \mathbf{g}) \in A$. If $V, V_1(0, \_)$ is a dense couple, then there exist an isomorphism $\varphi \colon \mathbb{R}^k \to \mathbb{R}^k$ and an element $\lambda \in \mathbf{G}$ such that

$$F(x, \mathbf{g}) = (\varphi(x), \lambda\mathbf{g})$$

for all $(x, \mathbf{g}) \in \mathbb{R}^k \times \mathbf{G}$.

Proof. The diffeomorphism F induces a Lie algebra automorphism of $\mathcal{L}(X, X_1)$. Let $\mathcal{L}_1$ denote the ideal with basis $\{x_j \partial/\partial x_\ell \mid j, \ell = 1, \ldots, k\}$, while $\mathcal{L}_2$ denotes the ideal with basis $\{\theta_1, \ldots, \theta_n\}$. Then $[\mathcal{L}_1, \mathcal{L}_2] = 0$ and $\mathcal{L}(X, X_1) = \mathcal{L}_1 \oplus \mathcal{L}_2$.

Note that $Y \in \mathcal{L}(X, X_1)$ has a zero if and only if $Y \in \mathcal{L}_1$. Thus $\mathcal{L}_1$ is an invariant of F. Moreover, $\mathcal{L}_1$ gives rise to the foliation $\mathbf{g} = $ constant (first define it on $(\mathbb{R}^k - \{0\}) \times \mathbf{G}$ and then extend it by continuity to $\mathbb{R}^k \times \mathbf{G}$). Of course this foliation is an invariant of F.

On the other hand, F has to map $\mathcal{L}_2$ to an ideal $\mathcal{L}_2'$ such that $\mathcal{L}(X, X_1) = \mathcal{L}_1 \oplus \mathcal{L}_2'$ and $[\mathcal{L}_1, \mathcal{L}_2'] = 0$. Therefore, there exist $Z_1, \ldots, Z_n \in \mathcal{L}_1$ such that $\{\theta_1 + Z_1, \ldots, \theta_n + Z_n\}$ is a basis of $\mathcal{L}_2'$ and $[Z_r, \mathcal{L}_1] = 0$, $r = 1, \ldots, n$. This implies that $Z_r = b_r \xi$, $b_r \in \mathbb{R}$, since the center of $\mathcal{L}_1$ is spanned by ξ.

But the closure of the trajectories of any element in $\mathcal{L}_2$ is compact, hence the closure of the trajectories of any element in $\mathcal{L}_2'$ has to be so. In the case of $\theta_r + Z_r$, this last assertion is true only if $Z_r = 0$. In short $\mathcal{L}_2' = \mathcal{L}_2$ and $\mathcal{L}_2$ is an invariant of F. Consequently, the foliation $x = $ constant associated to $\mathcal{L}_2$ is also an invariant of F.

From the invariance of the foliations associated to $\mathcal{L}_1$ and $\mathcal{L}_2$ respectively, that is the foliations given by the factors of $\mathbb{R}^k \times \mathbf{G}$, it follows the existence of two map $\varphi \colon \mathbb{R}^k \to \mathbb{R}^k$ and $\psi \colon \mathbf{G} \to \mathbf{G}$ such that $F(x, \mathbf{g}) = (\varphi(x), \psi(\mathbf{g}))$.

As F preserves X, φ must preserve ξ, which implies that φ is linear. Obviously φ is a bijection hence an isomorphism.

In turn ψ induces a Lie algebra automorphism of the ideal $\mathcal{L}_2$, that is of the Lie algebra of right invariant vector fields. Moreover, composing ψ on the left with $L_{\lambda^{-1}}$, where $\lambda = \psi(\mathbf{e})$, one may suppose $\psi(\mathbf{e}) = \mathbf{e}$. In this case by Lemma 2.4 ψ is a Lie group isomorphism. Therefore, ψ induces an isomorphism of $\mathcal{G}$.

As $F_* X = X$, $F_* X_1 = X_1$ around $\{0\} \times \mathbf{G}$, and $X = V$ and $X_1 = V_1(0, \_)$ on $\{0\} \times \mathbf{G}$, one concludes that $\psi_* V = V$ and $\psi_* V_1(0, \_) = V_1(0, \_)$.

Let $\mathcal{H}$ be the Lie subalgebra of $\mathcal{G}$ spanned by $V, V_1(0, \_)$, and $\mathbf{H}$ the connected Lie subgroup of $\mathbf{G}$ corresponding to $\mathcal{H}$. Clearly $\psi_* Y = Y$ for all $Y \in \mathcal{H}$, so ψ equals the identity on $\mathbf{H}$. Finally, since $V; V_1(0, \_)$ is a dense couple $\overline{\mathbf{H}} = \mathbf{G}$, hence $\psi = \mathrm{Id}$, and the result follows. □

Remark 3.4. Notice that if F is like in Proposition 3.3 and φ is shown to be a multiple of the identity, then $\varphi = \mathrm{Id}$.

Indeed, as F preserves the couple X, X_1 around $\{0\} \times \mathbf{G}$, then close to the origin φ has to preserve the function $h \colon \mathbb{R}^k \to \mathbb{R}$. Set $\varphi = a\,\mathrm{Id}$. Consider a vector line E in $\mathbb{R}^k$ such that the jet of order 5 at the origin of $h_{|E}$ does not vanish. Endow E with the coordinate y. Then $h_{|E}(y) = y^5 \tilde{h}(y)$ where $\tilde{h}(0) \neq 0$.

As h is φ-invariant near the origin one has $h_{|E}(ay) = h_{|E}(y)$, hence $(ay)^5 \tilde{h}(ay) = y^5 \tilde{h}(y)$; finally computing the fifth derivative at the origin yields $5! \, a^5 \tilde{h}(0) = 5! \, \tilde{h}(0)$, so $a = 1$.

Note that the support of h around the origin can be taken as small as desired. Even more, h can be replaced by hh_1 in the expression $X_1 = V_1 + hX$ provided that $h_1(0) \neq 0$.

Let $A \subset \mathbb{R}^k$ be open, and W be a vertical vector field defined on $A \times \mathbf{G}$. It what follows we say that W is *left (respectively right) invariant* if each $W(x, \_)$, $x \in A$, is left (right) invariant.

Lemma 3.5. Let $\widetilde{X}$ be the vector field on $\mathbb{R}^k \times \mathbf{G}$ given by $\widetilde{X} = \widetilde{\xi} + \widetilde{V}$ where $\widetilde{\xi} = \sum_{j=1}^{k} f_j(x) \partial/\partial x_j$, and $\widetilde{V}$ is a vertical left-invariant vector field. Assume that:

(1) On $\mathbb{R}^k$ it holds:
  (a) $\widetilde{\xi}$ is complete.

(b) $\widetilde{\xi}(0) = 0$ and its linear part at the origin is a positive multiple of identity.

(c) The outset of the origin equals $\mathbb{R}^k$, that is, the α-limit of every trajectory of $\widetilde{\xi}$ is the origin.

(2) There is an abelian subalgebra $\mathcal{G}' \subset \mathcal{G}$ such that $\widetilde{V}(x, \_) \in \mathcal{G}'$ for every $x \in \mathbb{R}^k$.

Then there exist a diffeomorphism $F : \mathbb{R}^k \times \mathbf{G} \to \mathbb{R}^k \times \mathbf{G}$, $b \in \mathbb{R}^+$, and $V \in \mathcal{G}$ such that:

(i) F commutes with the (natural) left action of $\mathbf{G}$ on $\mathbb{R}^k \times \mathbf{G}$ and $V = \widetilde{V}(0, \_)$.

(ii) $F_* \widetilde{X} = b\xi + V$ when V is thought as an s-vertical vector field.

Proof. The first part of the proof of Lemma 2.4 in [10] (see also [8] and [9, Proposition 2.1]) allows us to assume $\widetilde{\xi} = \xi$. (Note that this fact is essentially a consequence of the Sternberg linearization theorem stated as Theorem 3.7 just after the end of this proof.)

Define $\psi : \mathbb{R}^k \to \mathcal{G}' \subset \mathcal{G} \equiv T_e \mathbf{G}$ by setting $\psi(x) = \widetilde{V}(x, \mathbf{e})$. By Lemma 2.1 in [10] there is $\varphi : \mathbb{R}^k \to \mathcal{G}'$ such that $(\varphi_* \xi)(x) = \psi(0) - \psi(x)$, $x \in \mathbb{R}^k$.

Now consider the diffeomorphism $F : \mathbb{R}^k \times \mathbf{G} \to \mathbb{R}^k \times \mathbf{G}$ given by $F(x, \mathbf{g}) = (x, \mathbf{g} \cdot \exp(\varphi(x)))$. By Lemma 2.3, $F_* \xi = \xi + \theta$ where θ is a vertical left-invariant vector field with initial condition

$$\theta(x, \mathbf{e}) = (\varphi_* \xi)(x) = \psi(0) - \psi(x), \quad x \in \mathbb{R}^k.$$

Moreover, since $\mathcal{G}'$ is abelian then $F_* \widetilde{V} = \widetilde{V}$ by Lemma 2.2. Therefore:

$$F_*(\xi + \widetilde{V})(x, \mathbf{e}) = \xi(x) + \psi(0) - \psi(x) + \widetilde{V}(x, \mathbf{e}) = \xi(x) + \psi(0). \quad \square$$

Remark 3.6. Observe that if $\widetilde{X}$ matches the hypotheses of Lemma 3.5, then $\widetilde{X}' = (h \circ \pi_1)\widetilde{\xi} + (h_1 \circ \pi_1)\widetilde{V}$, where $h : \mathbb{R}^k \to \mathbb{R}$ is a positive and bounded function and any $h_1 : \mathbb{R}^k \to \mathbb{R}$, fulfills the hypotheses of Lemma 3.5 too.

We will now rephrase the classical Sternberg linearization theorem [8, Theorem 1] within the context of vector fields.

Theorem 3.7 (Sternberg linearization theorem). Let $0 \in D \subset \mathbb{R}^n$ be an open set, and let $X = X_0 + X_1$ be a vector field on D where

$$X_0 = \sum_{i,j=1}^n a_{ij} x_i \frac{\partial}{\partial x_j}, \ a_{ij} \in \mathbb{R}, \quad and \quad X_1 = \sum_{j=1}^n f_j \frac{\partial}{\partial x_j}$$

with $f_j(0) = 0$, $j = 1, \ldots, n$, and $(\partial f_j / \partial x_i)(0) = 0$, $i, j = 1, \ldots, n$. Let $\lambda_1, \ldots, \lambda_n$ be the eigenvalues of the square matrix (a_{ij}) defined by the coefficients of X_0. Then if

$$\lambda_j \neq \sum_{r=1}^n k_r \lambda_r$$

for every $j = 1, \ldots, n$ and every family $k_1, \ldots, k_n$ of non-negative integers such that $k_1 + \cdots + k_n > 1$, there exist two open sets $0 \in D_1 \subset D$, $0 \in D_2 \subset D$ and a diffeomorphism $F : D_1 \to D_2$ such that $F(0) = 0$ and $F_* X = X_0$.

Note that given any manifold S, the projection map $\pi_1 : S \times \mathbf{G} \to S$ is a $\mathbf{G}$-principal fibre bundle for the natural left $\mathbf{G}$-action on $S \times \mathbf{G}$.

Consider a connection $\widetilde{\mathcal{D}}$ on a $\mathbf{G}$-principal fibre bundle $\pi : P \to Q$ and a point $q \in Q$. One will say that $\widetilde{\mathcal{D}}$ is a *product around* $\pi^{-1}(q)$ if there are an open set $q \in A \subset Q$ and a fibre bundle isomorphism (over Id: $A \to A$) between $\pi : \pi^{-1}(A) \to A$ and $\pi_1 : A \times \mathbf{G} \to A$ such that $\widetilde{\mathcal{D}}$, regarded on $\pi_1 : A \times \mathbf{G} \to A$, is given by $\widetilde{\mathcal{D}}(u, \mathbf{g}) = T_u A \times \{0\} \subset T_{(u,\mathbf{g})}$. Equivalently, the connection $\widetilde{\mathcal{D}}$ is a product around $\pi^{-1}(q)$ if $\widetilde{\mathcal{D}}$ is *flat* in a neighborhood of q [5, Section II.9].

Lemma 3.8. Let $\pi : E \to \mathbb{R}^k$ be a $\mathbf{G}$-principal fibre bundle endowed with a connection $\mathcal{D}$. Let ξ' be the lift of ξ to E by means of $\mathcal{D}$. If around $\pi^{-1}(0)$ the connection $\mathcal{D}$ is a product then there exists a fibre bundle isomorphism (over the identity) $F : E \to \mathbb{R}^k \times \mathbf{G}$ such that $F_* \xi' = \sum_{j=1}^k x_j \partial / \partial x_j$.

Proof. Since around $\pi^{-1}(0)$ the connection $\mathcal{D}$ is a product then there is an open ball $B_{2a}(0)$ such that, as principal fibre bundles, $\pi : \pi^{-1}(B_{2a}(0)) \to B_{2a}(0)$ is identified (over the identity) to $\pi_1 : B_{2a}(0) \times \mathbf{G} \to B_{2a}(0)$ while $\mathcal{D}$ is given by the first factor. Note that on $B_{2a}(0) \times \mathbf{G}$ one has $\xi' = \sum_{j=1}^k x_j \partial / \partial x_j$.

Define $\tau : B_{2a}(0) \to \pi^{-1}(B_{2a}(0)) \equiv B_{2a}(0) \times \mathbf{G}$ by setting $\tau(x) = (x, \mathbf{e})$. Let $\Phi' : \mathbb{R} \times E \to E$ be the flow of ξ' (clearly this vector field is complete since $\mathbf{G}$ is compact). Recall that the flow of ξ is $\Phi(t, x) = e^t x$.

Now define $\tau' \colon \mathbb{R}^k \smallsetminus \overline{B}_a(0) \to E$ by setting

$$\tau'(x) = \Phi'\left(\log(\|x\| - a), \tau\left(\frac{ax}{\|x\|}\right)\right).$$

Since $\Phi(\log(\|x\| - a), ax/\|x\|) = x$ and ξ is the projection of ξ', it follows that $\tau'(x) \in \pi^{-1}(x)$; that is τ' is a section.

As τ and τ' agree on $B_{2a}(0) \smallsetminus \overline{B}_a(0)$ they define a global section $s \colon \mathbb{R}^k \to E$.

By construction $s(\mathbb{R}^k)$, which is a closed regular submanifold of E, is union of integral curves of ξ', so ξ' is tangent to $s(\mathbb{R}^k)$.

Consider $\psi \colon \mathbb{R}^k \times \mathbf{G} \to E$ given by $\psi(x, \mathbf{g}) = \mathbf{g} \cdot s(x)$. Then $\pi_1 = \pi \circ \psi$. Thus $(\psi^{-1})_* \xi'$ projects by π_1 in ξ. Moreover, $(\psi^{-1})_* \xi'$ is $\mathbf{G}$-invariant and tangent to $\mathbb{R}^k \times \{\mathbf{e}\}$ since ξ' is $\mathbf{G}$-invariant and tangent to $s(\mathbb{R}^k)$. Therefore, necessarily $(\psi^{-1})_* \xi' = \sum_{j=1}^{k} x_j \partial/\partial x_j$, and it suffices to set $F = \psi^{-1}$. $\square$

Remark 3.9. As the referee kindly pointed us, it is possible to give a geometric description of the proof above: we use flatness of the connection $\mathcal{D}$ around 0 to find a covariantly constant trivialization of π in a neighborhood of 0, and then we extend this trivialization by parallel transport along radial directions.

Lemma 3.10. *Let Y be a vector field on $\mathbb{R}^k \times \mathbf{G}$ such that $Y = \xi + W$ where W is vertical and left invariant. Given $0 < a < b$ there exists a principal bundle automorphism (over the identity) $F \colon \mathbb{R}^k \times \mathbf{G} \to \mathbb{R}^k \times \mathbf{G}$ such that $F_* Y = \xi + \widetilde{W}$ where:*

(a) $\widetilde{W}$ *is vertical and left-invariant.*
(b) $\widetilde{W}(x, \_) = 0$ *if $x \notin B_b(0)$.*
(c) F *equals the identity on $\overline{B}_a(0) \times \mathbf{G}$, and therefore $\widetilde{W}(x, \_) = W(x, \_)$ if $x \in \overline{B}_a(0)$.*

Proof. Take $c \in (a, b)$. Consider a vertical left-invariant vector field W', with support included in $B_b(0) \times \mathbf{G}$, such that $W' = W$ on $\overline{B}_c(0) \times \mathbf{G}$. Let Φ' be the flow of $\xi + W - W'$ and let $s \colon \mathbb{R}^k \smallsetminus \overline{B}_a(0) \to \mathbb{R}^k \times \mathbf{G}$ be the section given by

$$s(x) = \Phi'\left(\log(\|x\| - a), \left(\frac{ax}{\|x\|}, \mathbf{e}\right)\right).$$

As $s(x) = (x, \mathbf{e})$ when $x \in B_c(0) \smallsetminus \overline{B}_a(0)$, we can extend s to $\mathbb{R}^k$ by setting $s(x) = (x, \mathbf{e})$ on $\overline{B}_a(0)$.

Now consider the diffeomorphism $\psi \colon \mathbb{R}^k \times \mathbf{G} \to \mathbb{R}^k \times \mathbf{G}$ given by $\psi(x, \mathbf{g}) = \mathbf{g} \cdot s(x)$. The same reasoning as in the proof of Lemma 3.8 shows that

$$(\psi^{-1})_* (\xi + W - W') = \sum_{j=1}^{k} x_j \partial/\partial x_j.$$

Therefore,

$$(\psi^{-1})_* Y = (\psi^{-1})_* (\xi + W - W') + (\psi^{-1})_* W' = \xi + (\psi^{-1})_* W'.$$

For finishing, set $F = \psi^{-1}$ and $\widetilde{W} = (\psi^{-1})_* W'$. $\square$

Remark 3.11. If $Y = \xi + W$, and W vanishes around $\{0\} \times \mathbf{G}$, then there is an F as in Lemma 3.10 such that $F_* Y = \xi$.

Indeed, choose $b > 0$ in such a way that W vanishes on $B_b(0) \times \mathbf{G}$ and set $W' = 0$ in the proof above.

4. The free case

In this section we prove Theorem 1.1 under the following two additional hypotheses:

- The action of the n-dimensional connected compact Lie group $\mathbf{G}$ on the m-manifold M is free.
- $k = m - n \geq 1$.

Observe that the free $\mathbf{G}$-action on M gives rise to a $\mathbf{G}$-principal fibre bundle $\pi \colon M \to B$ where B is a connected k-manifold.

Let $\mu \colon B \to \mathbb{R}$ be a proper and non-negative Morse function. Let C denote the set of critical points of μ, which is closed and discrete, hence countable. From the paracompactness of B, it follows that there exists a locally finite family $\{A_p\}_{p \in C}$ of disjoint open sets such that $p \in A_p$, $p \in C$.

According to [10, Section 3], there exists a Riemannian metric $\tilde{g}$ on B such that the gradient vector field Y of μ is complete and, moreover, around each $p \in C$ there are coordinates $(x_1, \ldots, x_n)$ with $p \equiv 0$ and $Y = \sum_{j=1}^{k} \lambda_j x_j \partial/\partial x_j$ where:

(1) $\lambda_1 = \cdots = \lambda_k > 0$ if p is a source, i.e. p is a minimum of μ.

(2) $\lambda_1 = \cdots = \lambda_k < 0$ if p is a sink, i.e. p is a maximum of μ.

(3) Some λ_j are positive and the others negative if p is a saddle point.

Indeed, the scalars $\lambda_1, \ldots, \lambda_k$ do depend on the point p. However, to maintain a more concise notation, we have omitted this detail.

Furthermore, it is important to emphasize that these properties remain valid even when the vector field Y is scaled by a positive bounded function. This is because their dependence is solely on the Sternberg linearization theorem (see Theorem 3.7).

Let I be the set of sources of Y, i.e. I is the set of local minima of μ. For each $i \in I$, let S_i denote the Y-outset of i. Then [9, Lemma 3.3] becomes:

Lemma 4.1. *The family* $\{S_i\}_{i \in I}$ *is locally finite and* $\bigcup_{i \in I} S_i$ *a dense set in* B.

In what follows we assume that the locally finite family $\{A_p\}_{p \in C}$ defined above verifies $A_i \subset S_i$ by replacing A_i by $A_i \cap S_i$ if necessary.

Shrinking each A_p, $p \in C \smallsetminus I$, allows us to identify A_p with the ball $B_{\varepsilon_p}(0)$, with $p \equiv 0$ and $Y = \sum_{j=1}^k \lambda_j x_j \partial/\partial x_j$, thus the $\mathbf{G}$-principal fibre bundle $\pi : \pi^{-1}(A_p) \to A_p$ becomes $\pi_1 : B_{\varepsilon_p}(0) \times \mathbf{G} \to B_{\varepsilon_p}(0)$.

As the family $\{A_p\}_{p \in C}$ is locally finite and its elements are disjoint, then it is possible to find a connection $\mathcal{D}$ on $\pi : M \to B$ with the following properties:

(1) For every $p \in C \smallsetminus I$ there exists $0 < \delta_p \le \varepsilon_p$ such that the connection $\mathcal{D}$ on $B_{\delta_p}(0) \times \mathbf{G}$ equals the one induced by the first factor.

(2) $\mathcal{D}$ is a product on each fibre $\pi^{-1}(i)$, $i \in I$.

Let Y' denote the lift of Y to M by means of $\mathcal{D}$. By construction Y' is $\mathbf{G}$-invariant and complete (recall $\mathbf{G}$ is compact). On the other hand, given that Property (1) above holds for $\mathcal{D}$, whenever $p \in C \smallsetminus I$, we can assert that $Y' = \sum_{j=1}^k \lambda_j x_j \partial/\partial x_j$ on $B_{\delta_p}(0) \times \mathbf{G}$.

As it is well known, the ω-limit of a regular trajectory of the gradient vector field Y is empty, a saddle or a sink. Therefore (see [10, p. 884] for the definition of *order*):

Lemma 4.2. *The* ω-*limit of a regular trajectory of* Y' *is empty or a zero of* Y' *of order* 1.

Now, according to [9, Proposition 2.1], each outset S_i may be identified to $\mathbb{R}^k$ in such a way that $Y = a_i \sum_{j=1}^k x_j \partial/\partial x_j$, $a_i > 0$. In turn, by Lemma 3.8, $\pi : \pi^{-1}(S_i) \to S_i \equiv \mathbb{R}^k$ can be identified, as $\mathbf{G}$-principal fibre bundle, to $\pi_1 : \mathbb{R}^k \times \mathbf{G} \to \mathbb{R}^k$ in such a way that $Y' = a_i \sum_{j=1}^k x_j \partial/\partial x_j$ (where $a_i \sum_{j=1}^k x_j \partial/\partial x_j$ is regarded as an s-horizontal vector field).

Moreover, we can assume that A_i is identified with an open set in $\mathbb{R}^k$ that includes the closed ball $\overline{B}_3(0)$. This can be achieved by applying an appropriate dilation to both instances of $\mathbb{R}^k$, the total space and the base space, as needed.

Let K_i denote the compact set $\overline{B}_3(0) \times \mathbf{G}$ when regarded within $\pi^{-1}(S_i)$.

4.1. Construction of Y''

The next step is to modify Y' by adding a vertical component to it. More precisely, considering that the family $\pi^{-1}(S_i) i \in I$ is locally finite and its elements are disjoint (due to the properties of the family $S_i i \in I$), we will construct the new vector field by augmenting Y' on each $\pi^{-1}(S_i)$ with a left-invariant vertical vector field whose support is contained within K_i.

Let $Z = \xi + U$ be a vector field on $\mathbb{R}^k \times \mathbf{G}$ where:

(1) U is s-vertical and left-invariant.

(2) $U(0, \mathbf{e})$ belongs to some dense couple of $\mathcal{G} \equiv T_{\mathbf{e}}\mathbf{G}$.

Let $F : \mathbb{R}^k \times \mathbf{G} \to \mathbb{R}^k \times \mathbf{G}$ be the principal bundle automorphisms (thus a diffeomorphism) as provided by Lemma 3.10 for $a = 1$, $b = 2$ and Z. Set $F_* Z = \xi + \widetilde{U}$. Observe that the support of $\widetilde{U}$ is included in $\overline{B}_3(0) \times \mathbf{G}$. Let $\widetilde{U}_i$ be the vector field on $\pi^{-1}(S_i)$ which is identified to $\widetilde{U}$ on $\mathbb{R}^k \times \mathbf{G}$.

Define the vector field Y'' by setting:

- $Y'' = Y' + a_i \widetilde{U}_i$ on $\pi^{-1}(S_i)$, $i \in I$.
- $Y'' = Y'$ on $M - \bigcup_{i \in I} \pi^{-1}(S_i)$.

Then Y'' is left-invariant and complete ($\pi_* Y'' = Y$ and $\mathbf{G}$ is compact). Moreover, Lemma 4.2 still holds for Y'' because $Y'' = Y'$ on $M \smallsetminus \bigcup_{i \in I} K_i$.

The identification of $\pi^{-1}(S_i)$ with $\mathbb{R}^k \times \mathbf{G}$ may be composed with F^{-1} in order to obtain a second identification of $\pi^{-1}(S_i)$ with $\mathbb{R}^k \times \mathbf{G}$ such that $Y'' = a_i(\sum_{j=1}^k x_j \partial/\partial x_j + U)$.

It is easily seen that $q \in \pi^{-1}(I)$ if and only if the closure of the Y''-trajectory of q is a torus of dimension ≥ 1. Thus, $\pi^{-1}(I)$ is an invariant of Y'', but it's important to note that not necessarily each $\pi^{-1}(i)$, $i \in I$, is an invariant of this vector field. To avoid this issue, we will need to modify Y''.

4.2. Construction of X

We first assume $k \geq 2$. For every source $i \in I$ choose $P_i \subset A_i$ a set of $k+1$ points, all them close enough to i but different from it, in such a way that the linear α-limits of the Y-trajectories of these points are in general position (see [9, pp. 319–320] and [10, p. 844] for definitions). Let $P = \bigcup_{i \in I} P_i$.

Fix an injective set theoretical map $\eta \colon P \to \mathbb{N} \smallsetminus \{0\}$. As $\{A_i\}_{i \in I}$ is still locally finite, there is a bounded function $\tau \colon B \to \mathbb{R}$ that is positive on $B \smallsetminus P$ and has a zero of order $2\eta(p)$ at every $p \in P$.

Finally define $X = (\tau \circ \pi)Y''$, which is clearly a $\mathbf{G}$-invariant and complete vector field.

Let R_i, $i \in I$, be the outset of i with respect to τY. Given a point $a \in \mathbb{R}^k$, let $[a, \infty) \subset \mathbb{R}^k$ denote the ray $\{ta \colon t \in [1, \infty)\}$. Identifying S_i with $\mathbb{R}^k$ in the usual way implies that $S_i \smallsetminus R_i = \bigcup_{p \in P_i}[p, \infty)$. Therefore:

Lemma 4.3. *The family $\{R_i\}_{i \in I}$ is locally finite, and the set $\bigcup_{i \in I} R_i$ is dense in B. Consequently, the family $\{\pi^{-1}(R_i)\}_{i \in I}$ is also locally finite, and the set $\bigcup_{i \in I} \pi^{-1}(R_i)$ is dense in M.*

Now, we identify $\pi^{-1}(S_i)$ with $\mathbb{R}^k \times \mathbf{G}$ so that $Y'' = a_i(\sum_{j=1}^k x_j \partial/\partial x_j + U)$. Then $\pi^{-1}(S_i) \smallsetminus \pi^{-1}(R_i) = (\bigcup_{p \in P_i}[p, \infty)) \times \mathbf{G}$. According to [9, Proposition 2.1], which can also be considered as a consequence of Theorem 3.7, for each $i \in I$ there exists a diffeomorphism $f_i \colon R_i \to \mathbb{R}^k$ such that

$$(f_i)_*(a_i \tau \sum_{j=1}^k x_j \partial/\partial x_j) = b_i \sum_{j=1}^k x_j \partial/\partial x_j$$

where $b_i = a_i \tau(0) > 0$. Therefore, the diffeomorphism $F_i \colon R_i \times \mathbf{G} \to \mathbb{R}^k \times \mathbf{G}$, given by $F_i(x, \mathbf{g}) = (f_i(x), \mathbf{g})$, allows us to identify $\pi \colon \pi^{-1}(R_i) \to R_i$ with $\pi_1 \colon \mathbb{R}^k \times \mathbf{G} \to \mathbb{R}^k$, as $\mathbf{G}$-principal fibre bundles and over f_i, in such a way that

$$X = b_i \sum_{j=1}^k x_j \partial/\partial x_j + \widetilde{W}_i$$

where $\widetilde{W}_i$ is vertical and left-invariant, $\widetilde{W}_i(0, \mathbf{e})$ belongs to some dense couple and every $\widetilde{W}_i(x, \mathbf{e})$, $x \in \mathbb{R}^k$, is proportional to $\widetilde{W}_i(0, \mathbf{e})$.

By Lemma 3.5 applied to $b_i \sum_{j=1}^k x_j \partial/\partial x_j + \widetilde{W}_i$ there exist a diffeomorphism $\widetilde{F} \colon \mathbb{R}^k \times \mathbf{G} \to \mathbb{R}^k \times \mathbf{G}$, which commutes with the left action of $\mathbf{G}$, and $V_i \in \mathcal{G}$ such that $V_i = \widetilde{W}_i(0, \_)$ and

$$\widetilde{F}_* \left(b_i \sum_{j=1}^k x_j \partial/\partial x_j + \widetilde{W}_i \right) = b_i \sum_{j=1}^k x_j \partial/\partial x_j + V_i.$$

Composing, on the left, the foregoing identification with $\widetilde{F}$ gives rise to a second identification of $\pi \colon \pi^{-1}(R_i) \to R_i$ with $\pi_1 \colon \mathbb{R}^k \times \mathbf{G} \to \mathbb{R}^k$, over some $\tilde{f}_i \colon R_i \to \mathbb{R}^k$ and as $\mathbf{G}$-principal fibre bundles, such that $X = b_i \sum_{j=1}^k x_j \partial/\partial x_j + V_i$, $b_i > 0$, where:

(1) V_i is s-vertical and left-invariant.
(2) $V_i(0, \mathbf{e})$ belongs to some dense couple.

In the following, we will refer to this type of identification as *suitable*.

We also obtain that:

(a) Each $\pi^{-1}(i)$ is a connected component of the set of those points of M such that the closure of their X-trajectories is a torus of dimension ≥ 1.
(b) $\pi^{-1}(R_i)$ is the outset of $\pi^{-1}(i)$, that is the set of points of M whose X-trajectory has its α-limit included in $\pi^{-1}(i)$.
(c) For each $p \in P_i$ there exists one and only one open vector half-line E_p such that $E_p \times \mathbf{G}$ is the set of points of M whose X-trajectory has its α-limit included in $\pi^{-1}(i)$ and whose ω-limit is a zero of X of order $2\eta(p)$.

(Recall that $\pi^{-1}(R_i) = \left(\mathbb{R}^k \smallsetminus \bigcup_{p \in P_i}[p, \infty) \right) \times \mathbf{G}$ and $X = (\tau \circ \pi) \cdot (a_i \sum_{j=1}^k x_j \partial/\partial x_j + U)$ for some identification of $\pi^{-1}(S_i)$ with $\mathbb{R}^k \times \mathbf{G}$.)

Remark 4.4. From the preceding properties, it follows that if $F: M \to M$ is an automorphism of X, then $F(\pi^{-1}(i)) = \pi^{-1}(i)$ and, consequently, $F(\pi^{-1}(R_i)) = \pi^{-1}(R_i)$.

Moreover, if $F: \pi^{-1}(R_i) \to \pi^{-1}(R_i)$, when $\pi^{-1}(R_i)$ is suitably identified as $\mathbb{R}^k \times \mathbf{G}$, is given by $F(x, \mathbf{g}) = (\varphi_i(x), \lambda_i \cdot \mathbf{g})$ where $\varphi_i \in \mathrm{GL}(k, \mathbb{R})$ and $\lambda_i \in \mathbf{G}$ (refer to Proposition 3.3), then φ_i is a positive multiple of the identity. This is required to preserve the half-lines E_p, where $p \in P_i$, and these half-lines are in general position as a result of the construction of P_i.

Finally assume $k = 1$. In this case one cannot add additional zeros to construct the set P because in this scenario $\bigcup_{i \in I} R_i$ is no longer dense. Instead of that, we consider a proper and non-negative Morse function $\mu: B \to \mathbb{R}$, which has two or more maxima if B is not compact, and define P to be the set of sinks of Y. Then, given an injective set theoretical map $\eta: P \to \mathbb{N} \setminus \{0\}$, one can construct a bounded function $\tau: B \to \mathbb{R}$ that is positive on $B \setminus P$ and has a zero of order $2\eta(p)$ at every $p \in P$, as in the case $k \geq 2$. Therefore, if $p \in P$, then τY has a zero of order $2\eta(p) + 1$ at p.

The remaining arguments work in a similar way and one concludes that $F(\pi^{-1}(i)) = \pi^{-1}(i)$ and $F(\pi^{-1}(R_i)) = \pi^{-1}(R_i)$ where $R_i = S_i$. Obviously any $\varphi_i \in \mathrm{GL}(1, \mathbb{R})$ is a multiple of the identity.

Another way of dealing with the case $k = 1$ is to consider a Morse function μ on B with a single minimum. In this case $F(\pi^{-1}(R_i)) = \pi^{-1}(R_i)$ where $R_i = S_i$.

4.3. Construction of X_1

Let V_1 be a vertical vector field on M such that

(i) V_1 vanishes on $M \setminus \bigcup_{i \in I} \pi^{-1}(R_i)$.
(ii) On every $\pi^{-1}(R_i)$, suitably identified to $\mathbb{R}^k \times \mathbf{G}$, satisfies:
   (1) V_1 is left-invariant and its support is compact.
   (2) $V_i(0, \mathbf{e}), V_1(0, \mathbf{e})$ is a dense couple of $T_{\mathbf{e}} M \equiv \mathcal{G}$.

Let $h: M \to B$ be a function vanishing on $B \setminus \bigcup_{i \in I} R_i$ such that for every $i \in I$:

(a) $j_i^4 h = 0$ but $j_i^5 h \neq 0$.
(b) $h: R_i \to \mathbb{R}$ has compact support.

We define $X_1 = V_1 + (h \circ \pi) X$, which is obviously a complete and left-invariant vector field.

Then, for every $F \in \mathrm{Aut}(X, X_1)$, we have $F(\pi^{-1}(R_i)) = \pi^{-1}(R_i)$ for all $i \in I$, and $F: \pi^{-1}(R_i) \to \pi^{-1}(R_i)$ preserves both X and X_1. Making use of a suitable identification of $\pi^{-1}(R_i)$ and $\mathbb{R}^k \times \mathbf{G}$, our F can be seen as a self-diffeomorphism of $\mathbb{R}^k \times \mathbf{G}$ which preserves $X = b_i \sum_{j=1}^k x_j \partial / \partial x_j + V_i$, $b_i > 0$, and $X_1 = V_1 + (h \circ \pi_1) X$.

By Proposition 3.3 there exist $\varphi_i \in \mathrm{GL}(k, \mathbb{R})$ and $\lambda_i \in \mathbf{G}$ such that $F(x, \mathbf{g}) = (\varphi_i(x), \lambda_i \cdot \mathbf{g})$. From Remark 4.4 if $k \geq 2$, or directly if $k = 1$, it follows that φ_i is a multiple of the identity. Finally, Remark 3.4 (note that i is identified to $0 \in \mathbb{R}^k$) allows us to infer that $\varphi_i = \mathrm{Id}$, that is $F = L_{\lambda_i}$ on $\pi^{-1}(R_i)$.

Consider a $\mathbf{G}$-invariant Riemannian metric on M. Then F is an isometry on $\bigcup_{i \in I} \pi^{-1}(R_i)$ and, by continuity, on M. Thus fixed $q \in I$ then $F = L_{\lambda_q}$ on the open set R_q, which is non-empty. Since M is connected and F, L_{λ_q} are isometries, it follows that $F = L_{\lambda_q}$ on the whole M.

In other words, Theorem 1.1 is proved in the free case ($k \geq 1$).

Remark 4.5. Consider two $\mathbf{G}$-invariant, positive and bounded function $\varphi, \varphi_1 : M \to \mathbb{R}$, and define $\widetilde{X} = \varphi X$ and $\widetilde{X}_1 = \varphi_1 X_1$ where X, X_1 are the vector fields constructed above. Then both $\widetilde{X}$ and $\widetilde{X}_1$ are $\mathbf{G}$-invariant and complete, and the couple $\widetilde{X}, \widetilde{X}_1$ describes $\mathbf{G}$ as X, X_1 does so. Indeed, X and $\widetilde{X}$ share the same singularities with the same order and have identical trajectories, albeit with different parametrizations but maintaining the same orientation. The same holds for X_1 and $\widetilde{X}_1$. Therefore, the reasoning involving $\widetilde{X}$ and $\widetilde{X}_1$ follows a similar line of argumentation as that for X and X_1.

5. Actions with a free point

In this section we conclude the proof of Theorem 1.1. Consider an action of a connected compact Lie group $\mathbf{G}$, with dimension n, on a connected m-manifold M. Assuming the existence of a free point, the type of the principal orbits of the action is $\mathbf{G}$ itself, and if S is the set of non-free points, then $M \setminus S$ is open, connected and dense. Furthermore, the $\mathbf{G}$-action on $M \setminus S$ is free (see [2, Theorem IV.3.1]).

We first assume $m > n$. Let X', X_1' be a couple of vector fields on $M \setminus S$ describing $\mathbf{G}$ and constructed like in the preceding section.

By [10, Proposition 7.2] there are two functions $\rho, \varphi_1 : M \to \mathbb{R}$ bounded, $\mathbf{G}$-invariant, vanishing on S and positive on $M \setminus S$, such that the vector fields $\widehat{X}, \widehat{X}_1$ on M defined by $\widehat{X} = \rho X', \widehat{X}_1 = \varphi_1 X_1'$ on $M \setminus S$ and $\widehat{X} = \widehat{X}_1 = 0$ on S, are $\mathbf{G}$-invariant and differentiable.

Set $X_1 = \widehat{X}_1$; note that $X_1 = \varphi_1 X_1'$ on $M \setminus S$.

Let $h \colon \mathbb{R} \to \mathbb{R}$ be the function given by $h(t) = 0$ if $t \leq 0$ and $h(t) = \exp(-1/t)$ if $t > 0$. Set $X = (h \circ \rho)\widehat{X}$. Then $X = \varphi X'$ on $M \smallsetminus S$ where $\varphi = (h \circ \rho) \cdot \rho$. Moreover, X, X_1 are **G**-invariant and complete, and X vanishes at order infinity at p if and only if $p \in S$. By Remark 4.5 the couple X, X_1 describes **G** on $M \smallsetminus S$.

Given $F \in \mathrm{Aut}(X, X_1)$, then $F(S) = S$ and $F \colon M \smallsetminus S \to M \smallsetminus S$ is an automorphism of X, X_1 regarded as vector fields on $M \smallsetminus S$. Therefore, there exists $\boldsymbol{\alpha} \in \mathbf{G}$ such that $F = L_{\boldsymbol{\alpha}}$ on $M \smallsetminus S$ and, by continuity, on the whole M.

Finally, we assume $m = n$, that is we assume that **G** acts on itself on the left. Consider a couple dense X, X_1 of $\mathcal{G}$. Let $F \colon \mathbf{G} \to \mathbf{G}$ be an automorphism of this couple. By replacing F by $L_\lambda \circ F$, where $\lambda = (F(\mathbf{e}))^{-1}$, one may suppose $F(\mathbf{e}) = \mathbf{e}$. In this case $F_{|\mathbf{H}} = \mathrm{Id}$, where **H** is the connected Lie subgroup of **G** whose Lie algebra is spanned by X, X_1. But $\overline{\mathbf{H}} = \mathbf{G}$, so $F = \mathrm{Id}$.

This concludes the proof of Theorem 1.1.

We end this section by providing an example of an action with free points.

Example 5.1. Endow $V = \prod_{j=1}^{n-1} \mathbb{R}^n$, $n \geq 3$, with the SO(n)-action given by $\mathbf{g} \cdot v = (\mathbf{g} \cdot v_1, \ldots, \mathbf{g} \cdot v_{n-1})$, where $v = (v_1, \ldots, v_{n-1})$. This induces an action of SO(n) on the sphere

$$S^{n(n-1)-1} = \{v \in V : \|v_1\|^2 + \cdots + \|v_{n-1}\|^2 = 1\},$$

such that a point v of $S^{n(n-1)-1}$ is a free for this SO(n)-action if and only if the vectors $v_1, \ldots, v_{n-1}$ are linearly independent. Thus the SO(n)-action on $S^{n(n-1)-1}$ has both free and non-free points.

For instance, let $n = 3$, so $V = \mathbb{R}^3 \times \mathbb{R}^3$ and the induced SO(3)-sphere is S^5. Let $F \colon S^5 \to \mathbb{R}^2$ be the map defined by $F(v) = (\langle v_1, v_1 \rangle, \langle v_1, v_2 \rangle)$. Then a computation shows that $F(S^5) = E$, where $E = \{x \in \mathbb{R}^2 : (x_1 - 1/2)^2 + x_2^2 \leq 1/4\}$, and $v \in S^5$ is free if and only if $F(v)$ belongs to the interior of E. Indeed, given real numbers $a, b \geq 0$, and c, there exist two vectors $v_1, v_2 \in \mathbb{R}^3$ such that $\|v_1\|^2 = a$, $\|v_2\|^2 = b$ and $<v_1, v_2> = c$ if and only if $c^2 \leq ab$, and v_1, v_2 are linearly independent if and only if $c^2 < ab$. Therefore, $x \in \mathbb{R}^2$ belongs to $F(S^5)$ if and only if $x_2^2 \leq x_1(1 - x_1)$ while x is the image of a free point if and only if $x_2^2 < x_1(1 - x_1)$. Finally observe that $x_2^2 \leq x_1(1 - x_1)$ if and only if $(x_1 - 1/2)^2 + x_2^2 \leq 1/4$.

6. Invariant vector fields and homogeneous spaces

In this section, we demonstrate that the hypothesis of the existence of a free point in Theorem 1.1 cannot be omitted. This is shown through Corollary 6.2 and Theorem 6.3.

Now, consider an effective and transitive action $\mathbf{G} \times P \to P$ of a compact and connected Lie group **G**, of dimension n, on a compact and connected manifold P, of dimension $r \geq 1$.

Let $\mathcal{A} \subset \mathfrak{X}(P)$ be the Lie algebra of fundamental vector fields associated to the action of **G**, thus $\mathcal{A}$ is isomorphic, in a natural way, to the algebra of right invariant vector fields on **G** (see [6]). Additionally, let $\mathcal{B} \subset \mathfrak{X}(P)$ be the Lie algebra of those vector fields on P that commute with (the action of) **G**, i.e., they are **G**-invariant. It is evident that $[\mathcal{A}, \mathcal{B}] = 0$.

For each $p \in P$, let $\mathbf{G}_p$ denote its isotropy subgroup, and set $\mathcal{B}(p) = \{X(p) | X \in \mathcal{B}\}$. As **G** and $\mathcal{B}$ commute one has:

- Every element of $\mathcal{B}$ is completely determined by its value at a point; thus $\dim \mathcal{B}(p) = \dim \mathcal{B}$, $p \in P$, and $\mathcal{B}$ defines a foliation, say $\mathcal{F}$, on P.
- If p and q belong to the same leaf of $\mathcal{F}$ then $\mathbf{G}_p = \mathbf{G}_q$.

Therefore, if $\dim \mathcal{B} = r$, then $\mathbf{G}_p$ does not depend on p, and since the action is effective, $\mathbf{G}_p = \{\mathbf{e}\}$. In other words, P may be identified to **G**, and $\mathcal{B}$ to the algebra of left-invariant vector fields on **G**. *Thus* $\dim \mathcal{B} < r$ *if and only if the isotropy subgroups have two or more elements.*

On the other hand, observe that if **G** is a torus then necessarily each $\mathbf{G}_p = \{\mathbf{e}\}$ and $\dim \mathcal{B} = \dim \mathbf{G} = \dim P$.

Proposition 6.1. *Let* $\mathbf{G} \times P \to P$ *be an effective and transitive action of a compact and connected Lie group* **G**, *of dimension n, on a compact and connected manifold P, of dimension $r \geq 1$. Let* $\mathcal{B} \subset \mathfrak{X}(P)$ *be the Lie algebra vector fields on P that commute with (the action of)* **G**. *If* $\dim \mathcal{B} < r$, *then there exists a diffeomorphism* $\lambda \colon P \to P$ *which preserves every element of $\mathcal{B}$ and such that $\lambda \neq L_{\mathbf{g}}$ for all* $\mathbf{g} \in \mathbf{G}$.

Proof. We assume $\dim \mathcal{B} \geq 1$ since the case $\mathcal{B} = 0$ is trivial. First we show that the leaves of $\mathcal{F}$ are compact regular submanifolds. Endow P with a **G**-invariant Riemannian metric $\tilde{g}$. Given any $p \in P$ there are normal coordinates on an open neighborhood A of p, $\varepsilon > 0$, and an integer $1 \leq \ell \leq r - 1$ (see below) such that A is identified to $B_\varepsilon(0)$, p to the origin and $\mathbf{G}_p$ to a subgroup of the orthogonal group O(r) whose set of fixed points in $B_\varepsilon(0)$ equals $B_\varepsilon(0) \cap (\{0\} \times \mathbb{R}^\ell)$ when one sets $\mathbb{R}^r = \mathbb{R}^{r-\ell} \times \mathbb{R}^\ell$.

Indeed, first observe that one may assume that $\tilde{g}(p)$, $p \equiv 0$, equals the standard scalar product on $\mathbb{R}^r$. Take any $\mathbf{g} \in \mathbf{G}_p$. As **g** is an isometry of $(P, \tilde{g})$, it maps geodesics into geodesics, which implies that, in normal coordinates around p, the map **g** is homogeneous of degree one, hence linear (see Remark 3.2). Since $\tilde{g}(p)$ is the scalar product of $\mathbb{R}^r$, here linear means that "**g** belongs to O(r)". Finally note that the set of fixed points of a subgroup of O(r) is always a vector subspace of $\mathbb{R}^r$.

Note that given $Y \in \mathcal{B}$, then $Y(p) \in T_p P$ is invariant under the linear action of $\mathbf{G}_p$ on $T_p P$. Conversely, if $v \in T_p P$ is invariant under this linear action, then one can define $Y \in \mathcal{B}$, with $Y(p) = v$, by setting $Y(\mathbf{g} \cdot p) = (L_{\mathbf{g}})_* v$ (the linear

invariance of v implies that this construction is correct). Therefore, in normal coordinates $\mathcal{B}(p) = \{0\} \times \mathbb{R}^{\ell} \subset T_0 P$ and $\dim \mathcal{B} = \ell$ (for this reason $1 \leq \ell \leq r - 1$ as announced above).

Let N be the leaf of $\mathcal{F}$ passing through p. As $\mathbf{G}_p = \mathbf{G}_q$ for all $q \in N$, then $N \cap A \equiv N \cap B_{\varepsilon}(0) \subset (\{0\} \times \mathbb{R}^{\ell}) \cap B_{\varepsilon}(0)$ and, if ε is small enough, $N \cap B_{\varepsilon}(0) = (\{0\} \times \mathbb{R}^{\ell}) \cap B_{\varepsilon}(0)$ since $T_p N = \mathcal{B}(p)$. Therefore, N is a regular submanifold (reasoning as before with every point of N).

Consider a leaf N' of $\mathcal{F}$ and a sequence $\{p_k\}_{k \in \mathbb{N}} \subset N'$ with limit p; then $N' = N$, that is $p \in N'$. Indeed, let $\mathbf{G}'$ be the isotropy subgroup of any point in N' (all the points in N' have the same isotropy subgroup). From $\{p_k\}_{k \in \mathbb{N}} \to p$ follows that $\mathbf{G}' \subset \mathbf{G}_p$. As $\mathbf{G}'$ and $\mathbf{G}_p$ are conjugated they have the same dimension and the same number of connected component, so $\mathbf{G}' = \mathbf{G}_p$.

If k is big enough and ε sufficiently small, since $\mathbf{G}_p \cdot p_k = p_k$, one has that $p_k \in (\{0\} \times \mathbb{R}^{\ell}) \cap B_{\varepsilon}(0) = N \cap B_{\varepsilon}(0)$; that is $p_k \in N$ and $N' = N$. This reasoning applied to the points of the closure of N' shows that the leaves of $\mathcal{F}$ are closed, so compact.

On the other hand, there are a simply connected Lie group $\mathbf{H}$ and an action $\mathbf{H} \times P \to P$ whose algebra of fundamental vector fields equals $\mathcal{B}$ (see [6]). Moreover, the actions of $\mathbf{G}$ and $\mathbf{H}$ commute because by construction $\mathbf{G}$ and $\mathcal{B}$ do so.

By the transitivity of the action of $\mathbf{G}$, all the isotropy subgroups of the action of $\mathbf{H}$ are equal. Denote $\mathbf{H}'$ this subgroup; then $\mathbf{H}'$ is the kernel of the morphism $\mathbf{h} \in \mathbf{H} \mapsto L_{\mathbf{h}} \in \text{Diff}(P)$. Therefore, the quotient group $\widetilde{\mathbf{H}} = \mathbf{H}/\mathbf{H}'$ acts freely on P with $\mathcal{B}$ as algebra of fundamental vector fields. Besides, the orbits of the action of $\widetilde{\mathbf{H}}$ are the leaves of $\mathcal{F}$, so $\widetilde{\mathbf{H}}$ is compact and connected.

Thus the action of $\widetilde{\mathbf{H}}$ gives rise to a $\widetilde{\mathbf{H}}$-principal fibre bundle $\pi : P \to Q$. Observe that $\dim Q = r - \ell$, so $1 \leq \dim Q < \dim P$ since $1 \leq \ell \leq r - 1$.

Consider any $q \in Q$ and an open neighborhood B of it such that $\pi : \pi^{-1}(B) \to B$ is identified, as $\widetilde{\mathbf{H}}$-principal fibre bundle, to $\pi_1 : B_a(0) \times \widetilde{\mathbf{H}} \to B_a(0)$ where $q \equiv 0$ and $a > 0$. Let $\mu : B_a(0) \to \widetilde{\mathbf{H}}$ be a map such that $\mu(0) \neq \tilde{\mathbf{e}}$ and $\mu(q') = \tilde{\mathbf{e}}$ outside of a compact $0 \in K \subset B_a(0)$, where $\tilde{\mathbf{e}}$ is the neutral element of $\widetilde{\mathbf{H}}$. One define $\lambda : P \to P$ by setting $\lambda = \text{Id}$ on $P \setminus \pi^{-1}(B)$ and $\lambda(x, \mathbf{h}) = (x, \mathbf{h} \cdot \mu(x))$ on $\pi^{-1}(B) \equiv B_a(0) \times \widetilde{\mathbf{H}}$.

With respect to the metric $\tilde{g}$, λ is not an isometry because equals the identity on a non-empty open set but $\lambda \neq \text{Id}$. Therefore, $\lambda \neq L_{\mathbf{g}}$ for all $\mathbf{g} \in \mathbf{G}$ since each $L_{\mathbf{g}}$ is an isometry.

Moreover, λ preserves each $Y \in \mathcal{B}$ because $Y_{|\pi^{-1}(B)}$, where $\pi^{-1}(B) \equiv B_a(0) \times \widetilde{\mathbf{H}}$, is a fundamental vector field of the action of $\widetilde{\mathbf{H}}$ on $B_a(0) \times \widetilde{\mathbf{H}}$ given by $\mathbf{h}' \cdot (x, \mathbf{h}) = (x, \mathbf{h}' \cdot \mathbf{h})$ (as we pointed out before, the fundamental vector fields of the left action of a Lie group on itself are the right invariant vector fields, see [6]). □

Finally the key result in this section is:

Corollary 6.2. *Let Q be a connected manifold. Under the hypotheses of Proposition 6.1, the action of $\mathbf{G}$ on $Q \times P$ given by $\mathbf{g} \cdot (q, p) = (q, \mathbf{g} \cdot p)$ is not determined by any family of $\mathbf{G}$-invariant vector fields.*

Proof. Consider any $\mathbf{G}$-invariant vector field X on $Q \times P$. Then X is foliated with respect to the foliation given by the second factor, that is the foliation given by the orbits of $\mathbf{G}$. Therefore, $X = Y + V$ where Y is s-horizontal, V vertical and each $V(q, \_)$, $q \in Q$, is $\mathbf{G}$-invariant, that is belongs to $\mathcal{B}$.

Define the diffeomorphism $\tilde{\lambda} : Q \times P \to Q \times P$ by setting $\tilde{\lambda}(q, p) = (q, \lambda(p))$ where λ is like in Proposition 6.1. Clearly $\tilde{\lambda}_* X = X$.

Thus if $\mathcal{L}$ is a family of $\mathbf{G}$-invariant vector fields, then $\tilde{\lambda} \in \text{Aut}(\mathcal{L})$ but $\tilde{\lambda} \notin \mathbf{G} \subset \text{Diff}(Q \times P)$. □

From Corollary 6.2 it immediately follows:

Theorem 6.3. *Consider an homogeneous space $\mathbf{G}/\mathbf{H}$ and the natural left action of $\mathbf{G}$ on it. Assume that:*

- *$\mathbf{G}$ is compact and connected.*
- *$\mathbf{H}$ has two or more elements and does not contain any normal subgroup of $\mathbf{G}$ but $\{\mathbf{e}\}$.*

Let Q be a connected manifold. Then the action of $\mathbf{G}$ on $Q \times (\mathbf{G}/\mathbf{H})$ given by $\mathbf{g} \cdot (q, [\mathbf{g}']) = (q, [\mathbf{gg}'])$ is effective and cannot be described by any family of $\mathbf{G}$-invariant vector fields.

We finish this section with an example that illustrates an effective $\mathbf{G}$-action on a manifold M satisfying the following conditions:

(1) The set of fixed point $M^{\mathbf{G}}$ is not trivial, indicating that the $\mathbf{G}$-action is not transitive,
(2) The $\mathbf{G}$-action on M is determined by a couple of vector field.
(3) However, the induced $\mathbf{G}$-action on $M \setminus M^{\mathbf{G}}$ cannot be determined by any family of $\mathbf{G}$-invariant vector fields.

Example 6.4. Consider the space $M = \mathbb{R}^{2n}$, where $n \geq 2$, equipped with coordinates $x = (x_1, \ldots, x_{2n})$. We introduce the complex structure $J = \sum_{j=1}^{n}(e_{2j} \otimes e_{2j-1}^* - e_{2j-1} \otimes e_{2j}^*)$, where $e_1, \ldots, e2n$ is the canonical basis of $\mathbb{R}^{2n}$. With respect to J, the space M becomes a complex vector space of complex dimension n, defined by $(a + bi)x = ax + bJx$. Therefore, M is equipped with an effective action of the unitary group $U(n)$, defined as the set of elements in $GL(n, \mathbb{C})$ that preserve the Hermitian inner product $\theta(x, y) = \langle x, y \rangle - i \langle Jx, y \rangle$. It is evident that $M^{U(n)} = \{0\}$.

We show that the $U(n)$-action on M can be described by a couple of vector field.

Let Y be the linear vector field associated to J, that is $Y(x) = Jx$, $x \in M$. Then $Y = \sum_{j=1}^{n}(-x_{2j}\partial/\partial x_{2j-1} + x_{2j-1}\partial/\partial x_{2j})$.

A real endomorphism A of M is $\mathbb{C}$-linear if and only if $A \circ J = J \circ A$, which is equivalent to say that A, as differentiable map, preserves the vector field Y.

Set $X = \xi = \sum_{\ell=1}^{2n} x_\ell \partial/\partial x_\ell$ and $X_1 = (\|x\|^2 - 1)Y$. Consider a diffeomorphism $f : M \to M$ which preserves X; then, as it is well known, $f \in GL(2n, \mathbb{R})$. Now suppose that f preserves X_1 too. Then f maps $X_1^{-1}(0)$ into itself, so $f(S^{2n-1}) = S^{2n-1}$. Thus if the length of x equals 1, then that of $f(x)$ equals 1 too, and from the linearity of f it follows that $\|f(x)\| = \|x\|$, $x \in M$. In other words, f is an isometry. This last fact implies that the function $\|x\|^2 - 1$ is f-invariant; therefore f preserves Y, which means that f is $\mathbb{C}$-linear. Thus $f \in O(2n) \cap GL(n, \mathbb{C}) = U(n)$.

Conversely, if $f \in U(n)$, then f preserves X and X_1. Hence $\text{Aut}(X, X_1) = U(n)$.

Note that in our case, the origin of $0 \in M$ plays a crucial role since the action of $U(n)$ on $M \smallsetminus \{0\} = M \smallsetminus M^{U(n)}$, $n \geq 2$, cannot be determined by means of $U(n)$-invariant vector fields.

Indeed, consider the diffeomorphism $\psi : M \smallsetminus \{0\} \to \mathbb{R}^+ \times S^{2n-1}$ given by $\psi(x) = (\|x\|, x/\|x\|)$. Then the action of $U(n)$ on $M \smallsetminus \{0\}$ becomes the action of $U(n)$ on $\mathbb{R}^+ \times S^{2n-1}$ defined by $\mathbf{g} \cdot (t, y) = (t, \mathbf{g} \cdot y)$ and Corollary 6.2 applies.

7. Some examples on compact connected linear groups

Both the natural action of $SO(2)$ on $\mathbb{R}^2$ and the natural action of $SU(2)$ on $\mathbb{C}^2 \equiv \mathbb{R}^4$ have a free point. Therefore, these actions are determinable by Theorem 1.1. However we shall see that the natural actions of $SO(n)$ on $\mathbb{R}^n$, $n \geq 3$, and of $SU(m)$ on $\mathbb{C}^m \equiv \mathbb{R}^{2m}$, $m \geq 3$, are not determinable.

Indeed, let us start with $SO(n)$, $n \geq 3$. First observe that at any $x \in \mathbb{R}^n \smallsetminus \{0\}$ the only direction that is invariant under the action of the isotropy group of x is given by $\xi(x)$, $\xi = \sum_{j=1}^{k} x_j \partial/\partial x_j$. Therefore, if X is an $SO(n)$-invariant vector field on $x \in \mathbb{R}^n \smallsetminus \{0\}$, then $X = f\xi$ where f is an $SO(n)$-invariant function, that is to say constant on each sphere centered at the origin.

This last fact implies that f is $O(n)$-invariant too and, as a consequence, that X is $O(n)$-invariant too. Since the origin is a fixed point, necessarily both the natural actions of $O(n)$ and $SO(n)$ have the same set of invariant vector fields on $\mathbb{R}^n$. Therefore, the natural action of $SO(n)$ on $\mathbb{R}^n$, $n \geq 3$, cannot be determined by means of $SO(n)$-invariant vector fields.

Now consider the case of $SU(m)$, $m \geq 3$. As in Example 6.4, set $Y = \sum_{j=1}^{m}(-x_{2j}\partial/\partial x_{2j-1} + x_{2j-1}\partial/\partial x_{2j})$. Then the isotropy group of each $x \neq 0$ has just one invariant vector complex line that, as real plane, has $\{\xi(x), Y(x)\}$ as a basis. Since Y is invariant under the action of $U(m)$, every $SU(m)$-invariant vector field X on $\mathbb{R}^{2m} \smallsetminus \{0\}$ writes $X = f\xi + gY$ where f and g are $SU(m)$-invariant functions (and therefore $U(m)$-invariant). Finally the same reasoning as before shows that $SU(m)$ cannot be described by means of $SU(m)$-invariant vector fields, although the natural action of $U(m)$ can be determined by two invariant vector fields (Example 6.4).

Our next example shows that the natural action of the symplectic group can be described by a couple of invariant vector field.

Example 7.1. On $\mathbb{R}^{4r}$, $r \geq 1$, let consider the following complex structures

$$J = \sum_{\ell=1}^{r}\left(e_{4\ell-2} \otimes e_{4\ell-3}^* - e_{4\ell-3} \otimes e_{4\ell-2}^* + e_{4\ell} \otimes e_{4\ell-1}^* - e_{4\ell-1} \otimes e_{4\ell}^*\right)$$

$$K = \sum_{\ell=1}^{r}\left(e_{4\ell-1} \otimes e_{4\ell-3}^* - e_{4\ell} \otimes e_{4\ell-2}^* - e_{4\ell-3} \otimes e_{4\ell-1}^* + e_{4\ell-2} \otimes e_{4\ell}^*\right)$$

$$L = \sum_{\ell=1}^{r}\left(e_{4\ell} \otimes e_{4\ell-3}^* + e_{4\ell-1} \otimes e_{4\ell-2}^* - e_{4\ell-2} \otimes e_{4\ell-1}^* - e_{4\ell-3} \otimes e_{4\ell}^*\right)$$

where $\{e_1, \ldots, e_{4r}\}$ is the canonical base of $\mathbb{R}^{4r}$.

With respect to J, K and L the space $\mathbb{R}^{4r}$ becomes a quaternionic (left) vector space of dimension r by setting $(a + bi + cj + dk)x = ax + bJx + cKx + dLx$.

Let Y, Z, U be the linear vector fields on $\mathbb{R}^{4r}$ given by $Y(x) = Jx$, $Z(x) = Kx$ and $U(x) = Lx$, $x \in \mathbb{R}^{4r}$, respectively. A real endomorphism A of $\mathbb{R}^{4r}$ is $\mathbb{H}$-linear if and only if $A \circ J = J \circ A$, $A \circ K = K \circ A$ and $A \circ L = L \circ A$, which is equivalent to say that A, as differentiable map, preserves Y, Z and U. Thus the symplectic group $Sp(r)$, $r \geq 1$, is the set of those $A \in O(4r)$ that preserves Y, Z and U.

Let consider the open intervals $I_1 = (4, 9)$, $I_2 = (16, 25)$ and $I_3 = (36, 49)$, and let $\varphi_1, \varphi_2, \varphi_3 : \mathbb{R} \to \mathbb{R}$ be functions such that:

(1) $\varphi_1^2 + \varphi_2^2 + \varphi_3^2 > 0$ on $\mathbb{R} \setminus \{1\}$ and $\varphi_1(1) = \varphi_2(1) = \varphi_3(1) = 0$.
(2) $\varphi_a(I_b) = \delta_{ab}$ for every $a = 1, 2, 3$ and every $b = 1, 2, 3$.

Now set $X = \xi$ and $X_1(x) = \varphi_1(\|x\|^2)Y(x) + \varphi_2(\|x\|^2)Z(x) + \varphi_3(\|x\|^2)U(x)$, $x \in \mathbb{R}^{4r}$, which defines a vector field X_1 on $\mathbb{R}^{4r}$. Observe that $X_1^{-1}(0) = \{0\} \cup S^{4r-1}$.

Then $\mathrm{Sp}(r)$ equals $\mathrm{Aut}(X, X_1)$. Indeed, obviously $\mathrm{Sp}(r) \subset \mathrm{Aut}(X, X_1)$. Conversely, consider any $f \in \mathrm{Aut}(X, X_1)$. As f preserves X, necessarily it is $\mathbb{R}$-linear. On the other hand as f preserves X_1 then $f(X_1^{-1}(0)) = X_1^{-1}(0)$, so $f(S^{4r-1}) = S^{4r-1}$ and f is an isometry (see Example 6.4).

Observe that $X_1 = Y$ on $B_3(0) \setminus \overline{B}_2(0)$. Therefore, f preserves Y on this open set and, by the analyticity of f and Y, on the whole $\mathbb{R}^{4r}$. Analogously f preserves Z and U, hence $f \in \mathrm{Sp}(r)$.

We now consider a generic manifold M of dimension m, and let $\mathrm{Act}(M)$ be the set of all the effective actions of compact connected Lie groups on M. Thinking of these actions as subgroups of $\mathrm{Diff}(M)$ gives rise, by inclusion, to a partial order on $\mathrm{Act}(M)$. Finally, let $\mathrm{Act}_0(M)$ be the set of those elements of $\mathrm{Act}(M)$ that are determined by some family of invariant vector fields.

The poset $\mathrm{Act}_0(M)$ has nice structural properties:

Lemma 7.2. *The ascending chain condition holds in* $\mathrm{Act}(M)$*. In particular, every element in* $\mathrm{Act}(M)$ *is included in some maximal element.*

Proof. Recall that elements in $\mathrm{Act}(M)$ can be regarded as connected compact groups of isometries for a suitable Riemannian metric on M. Consequently, the dimension of any $\mathbf{G}$ in $\mathrm{Act}(M)$ is bounded above by $m(m+1)/2$ by Theorems 3.3 and 3.4 in [5, Chapter VI].

Now, consider any strictly ascending sequence

$$\mathbf{G}_1 \subset \mathbf{G}_2 \subset \ldots \subset \mathbf{G}_r \subset$$

in $\mathrm{Act}(M)$. Since every group is connected, we have $0 < \dim \mathbf{G}_i < \dim \mathbf{G}_{i+1} \leq m(m+1)/2$, and this sequence must eventually stop. Moreover, if the sequence cannot be extended, it must conclude with a maximal element. $\square$

A direct consequence of Lemma 7.2 is that the ascending chain condition also holds in $\mathrm{Act}_0(M)$, and we obtain the following stability property for the maximal elements in $\mathrm{Act}_0(M)$:

Proposition 7.3. *Let* $\mathbf{G}$ *be a maximal element in* $\mathrm{Act}_0(M)$*, and let* $\mathcal{W}$ *be a family of* $\mathbf{G}$*-invariant vector fields on* M*. Then either* $\mathcal{W}$ *does not determine any element of* $\mathrm{Act}_0(M)$ *or* $\mathbf{G} = \mathrm{Aut}(\mathcal{W})$*.*

Remark 7.4. A particular case of the above proposition is as follows. Suppose that $\mathbf{G}$ is a maximal element in $\mathrm{Act}_0(M)$ and is determined by two $\mathbf{G}$-invariant vector fields X, X_1. If X', X_1' is a new pair of $\mathbf{G}$-invariant vector fields which is" close" to the original pair X, X_1 in some Whitney C^k topology [3, p. 43], then one of two scenarios arises: either X', X_1' do not determine any element in $\mathrm{Act}_0(M)$, or we have $\mathbf{G} = \mathrm{Aut}(X', X_1')$.

Proposition 7.5. *The natural action of* $\mathrm{U}(m)$ *on* $\mathbb{R}^{2m}$*,* $m \geq 1$*, is maximal in* $\mathrm{Act}_0(\mathbb{R}^{2m})$*.*

Proof. Let us start recalling a result needed later on. Consider two connected compact Lie subgroup $\mathbf{H}, \mathbf{H}'$ of $\mathrm{SO}(2m)$, ≥ 1, the first one isomorphic to $\mathrm{U}(m)$. If $\mathbf{H} \subset \mathbf{H}'$ then either $\mathbf{H} = \mathbf{H}'$ or $\mathbf{H} = \mathrm{SO}(2m)$ [1].

Let $\mathbf{G} \in \mathrm{Act}_0(\mathbb{R}^{2m})$ such that $\mathrm{U}(m) \subset \mathbf{G}$. Then every $\mathbf{G}$-orbit has dimension strictly smaller than $2m$, as otherwise $\mathbb{R}^{2m}$ would be compact. Furthermore, since $\mathrm{U}(m) \subset \mathbf{G}$, every $\mathbf{G}$-orbit contains the induced $\mathrm{U}(m)$-orbit. This induced orbit is either a $(2m-1)$-sphere centered at the origin or the origin, which we identify as the degenerate sphere with radius zero. Combining both arguments, we conclude that the $\mathbf{G}$-orbits consist of all the spheres centered at the origin, including the degenerate one, and the origin is a fix point of the $\mathbf{G}$-action.

Endow $\mathbb{R}^{2m}$ with a $\mathbf{G}$-invariant Riemannian metric $\tilde{g}$. For this metric consider normal coordinates $(u_1, \ldots, u_{2m})$ in an open neighborhood A of the origin, which is identified to some open ball $B_\varepsilon(0)$ (origin to origin). As usual we suppose that $\tilde{g}(p)$, $p \equiv 0$, equals the scalar product of $\mathbb{R}^{2m}$. Then both $\mathrm{U}(m)$ and $\mathbf{G}$ are identified to subgroups of $\mathrm{SO}(2m)$ (see the second paragraph of the proof of Proposition 6.1).

If $m = 1$ then $\dim \mathrm{U}(1) = \dim \mathrm{SO}(2) = 1$ and necessarily $\mathbf{G} = \mathrm{U}(1)$. Therefore, assume $m \geq 2$ and $\mathbf{G} \neq \mathrm{U}(m)$, which implies that $\mathbf{G}$ is isomorphic to $\mathrm{SO}(2m)$ and $\dim \mathbf{G} = m(2m-1)$.

Moreover the action of $\mathbf{G}$ on every sphere of positive radius is effective. Indeed, if not the Lie algebra of $\mathbf{G}$, and so that of $\mathrm{SO}(2m)$, includes a proper ideal, which excludes the case $m \geq 3$. If $m = 2$, as the action of $\mathrm{U}(2)$ on these spheres is effective, this ideal ought to have codimension ≥ 4, but there is no such ideal.

Set $Y = \sum_{j=1}^{m}(-x_{2j}\partial/\partial x_{2j-1} + x_{2j-1}\partial/\partial x_{2j})$ (see Example 6.4 again). At each $x \neq 0$, and for the action of U(m), the isotropy group of this point has just a vector real plane of invariant vectors one of whose basis is $\{\xi(x), Y(x)\}$. Therefore, each U(m)-invariant vector field X on $\mathbb{R}^{2m} \setminus \{0\}$ writes $X = f\xi + gY$ where f and g are functions of $\|x\|^2$.

Let S_x, $x \neq 0$, be the sphere centered at the origin and passing through x. Since the action of **G** is effective on all its orbits but $\{0\}$ and dim **G** $=$ dim SO($2m$), a vector $v \in T_x\mathbb{R}^{2m}$, $x \neq 0$, is invariant under the action of the isotropy group of x if an only if it is $\tilde{g}$-orthogonal to $T_x S_x$. Therefore, there exists a **G**-invariant vector field V on $\mathbb{R}^{2m} \setminus \{0\}$ such that:

(1) $\tilde{g}(V, V) = 1$.
(2) V is $\tilde{g}$-orthogonal to every S_x, $x \neq 0$, and points outward.

Obviously V is U(m)-invariant; therefore, it writes $V = a\xi + bY$ where a and b are functions of $\|x\|^2$ and $a > 0$ everywhere.

Observe that any **G**-invariant vector field defined on a punctured open ball $B_\rho(0) \setminus \{0\}$ is the product of a function of $\|x\|^2$ and V. Let R be the radial vector field of the normal coordinates $(u_1, \ldots, u_{2m})$, that is to say $R = \sum_{j=1}^{2m} u_j\partial/\partial u_j$. On a punctured open ball $B_\tau(0) \setminus \{0\}$ included in A one has $R = fV$ where f is a positive function of $\|x\|^2$.

Consider a positive function φ of $\|x\|^2$ defined on $\mathbb{R}^{2m} \setminus \{0\}$ such that φV is complete and $\varphi = f$ on $B_{\tau/2}(0) \setminus \{0\}$. Define a new vector field W on $\mathbb{R}^{2m}$ by setting $W = R$ on $B_{\tau/2}(0)$ and $W = \varphi V$ on $\mathbb{R}^{2m} \setminus \{0\}$. Then W is complete at its linear part at the origin equals the identity. Moreover, W is **G**-invariant.

Now by Sternberg linearization theorem (see Theorem 3.7 and page 319 of [9]) there exists a diffeomorphism $F: \mathbb{R}^{2m} \to \mathbb{R}^{2m}$ such that $F_*W = \xi$.

Consider the conjugate action on $\mathbb{R}^{2m}$, which will be called $\mathbf{G}^*$, given by $\mathbf{g} \cdot x = F(\mathbf{g} \cdot F^{-1}(x))$. Clearly $\mathbf{G}^*$ belongs to Act$_0(\mathbb{R}^{2m})$ and ξ is $\mathbf{G}^*$-invariant; therefore, $\mathbf{G}^*$ is linear. As this group is compact, there always exists a $\mathbf{G}^*$-invariant scalar product. Thus via a linear automorphism of $\mathbb{R}^{2m}$, the action of $\mathbf{G}^*$ can be assimilated to the natural action of SO($2m$) on $\mathbb{R}^{2m}$. But this last one does not belong to Act$_0(\mathbb{R}^{2m})$, *contradiction*. $\quad\square$

8. Some open questions

In this last section we collect a list of open questions that naturally arise from our work.

First, we observe that our main result, Theorem 1.1, can be expressed in terms of centralizers of elements within the full group of diffeomorphisms.

Indeed, let us Z be a complete vector field Z on a manifold P, and let Φ_t be its corresponding flow. Consider two rationally independent real numbers, denoted as a and b. Now, given a diffeomorphism $f: P \to P$, it commutes with every Φ_t, where $t \in \mathbb{R}$, if and only if it commutes with both Φ_a and Φ_b.

Additionally, we observe that f commutes with the flow Φ_t, that is $f \in C_{\text{Diff}(M)}(\Phi_t)$, if and only if it preserves the vector field Z. Therefore, Theorem 1.1 leads us to the following conclusion:

Corollary 8.1. *Consider an action of a connected compact Lie group **G** on a connected manifold M. If the action of **G** possess a free point, then there exist $f_i \in \text{Diff}(M)$, $i = 1, \ldots, 4$, such that each f_i is diffeotopic to the identity map, and*

$$\mathbf{G} = \bigcap_{i=1}^{4} C_{\text{Diff}(M)}(f_i).$$

Then, it is natural to ask whether group actions of connected Lie groups on connected manifolds can be described as centralizer of diffeomorphisms:

Question 8.2. *Consider an action of a connected compact Lie group **G** on a connected manifold M.*

(a) *Under which hypothesis is it possible to find a family of diffeomorphisms $\{f_i \in \text{Diff}(M) : i \in I\}$ such that*

$$\mathbf{G} = \bigcap_{i \in I} C_{\text{Diff}(M)}(f_i)?$$

(b) *If such a family exists, what is the minimum number of diffeomorphisms it must contain?*

In Section 7 it is shown that there exist connected linear compact groups whose natural action is not determinable by any family of invariant vector fields. Lemma 7.2 and Proposition 7.3 suggest that an inductive argument via maximal subgroups of connected compact Lie groups, as classified in [1], may allow to tackle the following open question:

Question 8.3. *For every integer $n > 0$, what is the complete description of all connected compact subgroups $\mathbf{G} \subset \text{GL}(n, \mathbb{R})$ such that the natural **G**-action on $\mathbb{R}^n$ is determinable?*

In a completely general setting, solving the problem equivalent to Question 8.3, which seeks the complete description of elements in $\mathrm{Act}_0(M)$, may seem unattainable for a generic M. Nevertheless, Proposition 7.3 and Remark 7.4 indicate that maximal elements in $\mathrm{Act}_0(M)$ could be characterized using invariant vector fields and a suitable definition of action stability. This leads to the following inquiry:

Question 8.4. *Consider a connected m-manifold M and a family of vector fields $\mathcal{F}$ on M. Is it possible to devise a suitable definition of stability for the Aut($\mathcal{F}$)-action in terms of the elements in $\mathcal{F}$ and characterize those actions that are stable?*

Finally, our methods are not applicable in the C^0 class because they rely on Lemma 3.4 in [9], which does not hold for continuous maps. Similarly, they cannot be applied in the analytic case due to the use of plateau functions. Therefore, a natural question arises:

Question 8.5. *Do our results hold true in the continuous or analytic classes?*

Data availability

No data was used for the research described in the article.

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