The dynamics of growth and distribution in a spatially heterogeneous world

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2.12.2004

Abstract

This paper tries to reconcile growth and geographical economics by dealing directly with capital accumulation through time and space and by seeing growth convergence and spatial agglomeration as jointly generated by dynamic processes displaying pattern formation. It presents a centralized economy in which a Bergson-Samuelson-Millian central planner finds a flow of optimal distributions of consumption, subject to a spatial-temporal capital accumulation budget constraint. The main conclusions are: first, if the behavioral parameters are symmetric, but there is an asymmetric distribution of the capital stock, then the long run asymptotic distribution will be spatially homogeneous; second, if there is homogeneous distribution of the capital stock, but there is an asymmetric shock in any parameter, then the economy will converge towards a spatially heterogeneous asymptotic state; third, spatially heterogeneous asymptotic states will only emerge exogenously, not endogenously; fourth, the spatial propagation mechanism can give birth, when the production function is close to linear, to a Turing instability, which implies that for some parameter values, a conditionally stable space-time distribution should display spatial pattern formation.
Keywords: Optimal growth and distribution; Spatial growth; Optimal control of partial differential equations; Traveling waves; Fourier transforms; Turing instability.
JEL Classification: C6, D9, E1, R1.

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[0]This paper has been maturing for a long time, since my *Agregação* lecture, presented at the ISEG

1 Introduction

The interaction between the temporal and the spatial dimensions of capital accumulation, has been recently readdressed, both theoretically and empirically, by the new (endogenous) growth and the new geography strands of literature. They tend to reach conflicting conclusions about the dynamics of growth across space.

The existence of convergence towards a long run distribution, displaying a higher degree of homogeneity across economies, seems to emerge as the consensus view among growth theorists. This conclusion appears to be fairly robust across different data sets, not only including countries but also regions within countries. There is empirical evidence of both $\beta-$ convergence and $\sigma-$ convergence, or, at least, evidence in favor of the existence of an ergodic long run distribution of income across countries (see the surveys by Temple (1999) or in (Barro and Sala-I-Martin, 2004, ch.11-12)).

However, new economic geography observes that the world economic activity is highly concentrated in a relatively tiny proportion of the earth's surface, and that the agglomeration seem to dominate the dispersive forces. Geographical concentration is observed, not only for the world economy but also at many several other levels: on metropoli or coasts, within countries, or on particular locations, for many industries (see Fujita et al. (1999) and Fujita and Thisse (2002)).

How can we reconcile analytically those two observations ? We explore in this paper the following answer: First, those disparate conclusions may be related to the structure of

Universidade Técnica de Lisboa, back in 1999. Several previous versions have been presented in conferences (CEMAPRE 2000, Conference on complexity - Aix-en-Provence 2000, ASSET 2001), workshops (UECE 1999, Max Planck Institute, Rostock, 2001, Marseille 2001) and seminars (ISEG, Technical University of Lisbon, 2002, Faculty of Economics of the Universidade Nova de Lisboa, 1999 and 2003, Universidade Católica Portuguesa, 2001, Universidade do Minho, 2002, Tor Vergata, Rome, 2003 and Universita degli Studi, Milan, 2003). In addition to the participants in those events, I thank, without implicating, the comments, reactions, ideas or encouragement of the following colleagues, the late José João Marques da Silva and Raouf Boucekkine, Pierre Cartigny, Rui Dilão, Gustav Feichtinger, Sofia Castro Gothen, Omar Licandro, Rui Vilela Mendes, Pascal Mossay, Danny Quah, Gerhard Sorger, Vladimir Veliov, and Benteng Zou.

the benchmark models in both areas, in particular to the independence between the spatial and the temporal dynamics of capital accumulation. Second, if we consider their interaction, then heterogeneous local growth dynamics, associated to the emergence of pattern formation, may exist.

The founding papers on the new endogenous growth theory, Romer (1986) and Lucas (1988), and the voluminous subsequent literature deal mainly with closed economies [1]. The other core assumptions are: utility functions are homogeneous, marginal returns to capital are decreasing, at the firm level, and there is a mechanism allowing for unbounded growth (which can be aggregate constant returns to scale and accumulation of human capital or externalities implying the existence of aggregate increasing returns to scale). Decreasing marginal returns to capital at the firm level is the main mechanism generating convergence, both within and across different locations. Asymmetry in parameters affecting the endogenous growth rate explain differences in growth experiences, between different pairs of countries and locations (absolute or relative $\beta-$ convergence).

Included in this strand of literature, there are some recent contributions dealing with endogenous growth in an open economy context, and with the existence of a (bounded) world distribution of income. In particular, Acemoglu and Ventura (2002) prove that even if there are some (mild) increasing returns domestically, the existence of a world capital market, together with spatial frictions, will produce convergence towards an asymptotic homogeneous long run world distribution of income. Capital diffusion and the dynamics of the real exchange rate will tend to equalize real rates of return throughout space, thereby producing aggregate decreasing returns at the country level.

The new economic geography strand of literature has elected location and agglomeration as the main subjects of enquiry. Most of the contributions deal with the spatial dynamics without considering capital accumulation through time. Agglomeration is conceived as a result of the trade-off between transport costs and increasing returns, within a Dixit-Stiglitz

[1] In particular, Romer (1990), Rebelo (1991), Caballè and Santos (1993), Bond et al. (1996).

monopolist competition framework (see Fujita et al. (1999) and Fujita and Thisse (2002)). In general, the only factor that is mobile inter-regionally is labor and capital is immobile. There is some work addressing endogenous growth and agglomeration in this framework, where human capital externalities and/or Schumpeterian R&D activities are the engines of growth, but again, the only mobile factor is labor (see Palivos and Wang (1996), (Fujita and Thisse, 2002, ch 11) and Martin and Ottaviano (2002)).

Given the spatial-temporal framework which is chosen in both strands of literature, the mechanisms that generate growth and convergence exclude the existence of agglomeration, and vice-versa.

This paper tries to reconcile growth and geography by dealing directly with the dynamics of heterogeneity and by defining variables as jointly dependent on time and a state variable [2]. Three main assumptions define the setup of this framework.

The first assumption is related to the features of the support for heterogeneity. We assume that there is a one-to-one correspondence between heterogeneous (or asymmetric) agents and their location in a particular point in space. This space has a metric which is related to economic distance, not geographical distance; we assume that it is one-dimensional and unbounded and serves as an indexing device as in Hotelling (1929). This choice will have consequences on the acceptable spatial weighting schemes and on spatial and temporal boundary conditions.

The second is related to the specification of the spatial-temporal constraints. Though the equilibrium condition between savings and investment still holds, the intertemporal and interspatial allocation of capital are mutually consistent. Capital flows among regions is a function of the spatial gradient of its initial distribution: regions with higher capital intensity have smaller real rates of return and therefore export capital. Mathematically, a forward

[2]This approach is common in other fields of science. In economics by it is used by, among others, Beckmann (1970) and, recently, by Robert E. Lucas and Rossi-Hansberg (2002), Quah (2002) and Mossay (2003).

parabolic partial differential equation represents the instantaneous budget constraint.

The last assumption is related to the specification of the spatial-temporal arbitrage conditions. We present a centralized economy in which the planner decides on the spatial-temporal allocation of consumption that maximizes a social welfare Bergson-Samuelson utility functional [3]. In order to deal with the problems arising from the unbounded support for space, we assume a Millian average utility function. A generalized Euler condition is represented by a backward parabolic partial differential equation.

In addition, the behavioral functions, which characterize the representative agents in every point in space, are neo-classical: utility and production functions are concave. Therefore, we present a generalization of the Ramsey (1928), Cass (1965) and Koopmans (1965) model for a continuum of open and interacting economies.

There is some literature dealing with a similar economy: Chatterjee (1994), study a discrete time discrete space economy, Isard and Liossatos (1979), present a continuous-time continuous and bounded space economy, but do not attempt at solving the model, and Camacho et al. (2004), which is closely related to our paper, deal extensively, and rigorously, with existence results for a model with a bounded space and linear utility.

Our model boils down to a particular problem of optimal control of parabolic partial differential equations, for which we derive heuristically a maximum principle. We also study qualitatively both the asymptotic distribution and the local stability properties of the optimal solution. The most surprising feature of the local (distributional) dynamics is the potential occurrence of total instability, when the related a-spatial Ramsey-Cass-Koopmans model displays conditional stability. The type of instability is analogous to the Turing (1952) type of instability. (Fujita et al., 1999, ch. 6) also consider diffusion in a continuous space framework and report the emergence of spatial heterogeneity, as a result of the existence of Turing instability. They conclude that spatial interaction among symmetric regions may eventually

[3] In order to avoid the collective choice impossibility results (see Yaari (1981)), we assume that agents do not have preferences on distributions of consumption but rather on their own level of consumption.

lead to the emergence of agglomeration, through a mechanism of pattern formation. In our model, spatial pattern formation may also emerge, for particular values of the parameters. However, differently from the two previous cases, it may have to characterize the conditionally (distributionally) stable optimal consumption and capital accumulation trajectories.

Intuitively, the two diffusion mechanisms, related to spatial contact, seem to generate externality effects, both in consumption and in production that, in some cases, offset the stabilizing properties of the decreasing marginal productivity of capital. In those cases, a bounded long run distribution of consumption and capital will only exist if the central planner limits the workings of the diffusion mechanisms. Therefore, both convergence towards a asymptotic distribution of capital and agglomeration may occur.

The main conclusions of the paper are the following: first, if there is symmetry in the behavioral parameters, but an initial non-homogeneous distribution of the capital stock, then the long run asymptotic distribution will be spatially homogeneous; second, if there is initially a homogeneous distribution of the capital stock, but there is an asymmetric shock in any parameter, then the economy will converge towards a spatially heterogeneous asymptotic state; third, spatially heterogeneous asymptotic states will only emerge as a result of exogenously determined factors and will not be created endogenously; fourth, the spatial propagation mechanism can give birth, when the production function is close to linear, to an analogous of the Turing instability, which implies that for some parameter values, a conditionally stable space-time distribution will have to display spatial pattern formation.

The rest of the paper is organized as follows: section 2 presents the components of the model, section 3 presents and derives the optimality conditions according to a generalized Pontriyagin principle, section 4 uses results on traveling waves' literature to address the asymptotic states for the optimal solution, section 5 studies local (distributional) stability by using Fourier transform methods, section 6 derives the comparative (distributional) dynamics formulae for productivity shocks and applies them to both symmetric and asymmetric shocks, and section 7 concludes.

2 The model

Let there be a continuum of potentially heterogeneous and interacting households. We will identify the support of that continuum with space. All the variables are referred to the time-space coordinates, $(t, x) \in (\mathbb{X}, \mathbb{T}) = (\mathbb{R} \times \mathbb{R}_+)$. The reference location, at $x = 0$, may be labeled after the location with an initially higher capital intensity [4].

We also assume that population is evenly distributed across space, that there is no labor migration and that there is only one homogeneous good in the economy. For most of the paper, spatial heterogeneity refers to differences in the quantities of the good which is consumed and produced in different locations.

The representative household, in each location, performs production, consumption and investment activities.

2.1 Households' problem at location x

Let $c(x, t)$ be the rate of consumption at time t and $C(x)_t := \{c(x, \tau) : \ t \leq \tau < +\infty\}$ be the path of consumption starting at time $t \in \mathbb{T}$, for the representative household located at x. We assume that household preferences are symmetric across space: both the intertemporal and the instantaneous utility functions and the rate of time preference are identical and space-independent. The intertemporal utility function, for the household located at x, is assumed to be additively separable and discounted,

$$V(C(x)_0) := \int_0^{+\infty} u(c(x, t))e^{-\delta t}dt \tag{1}$$

where $\delta > 0$ is the rate of time preference and $u(.)$ is the instantaneous utility function, which is assumed to be is increasing, concave and Inada: $u'(.) > 0$, $u''(.) < 0$, $\lim_{c \to 0} u(c) = +\infty$ and $\lim_{c \to +\infty} u(c) = 0$.

[4] If one would like to consider jointly heterogeneities related to both to economic distance and geographical distance we could add another dimension, as $x \in \mathbb{R}^2$.

In every location, production uses capital and labor with a space-independent neoclassical technology. We exclude the existence of explicit externalities leading to the existence of explicit agglomeration effects.

We work directly with per capita variables, by assuming that there is a fixed, constant and immobile labor supply, $L = 1$. Let (per capita) production and the stock of physical capital be $y(x,t)$ and $k(x,t)$, at location x at time t, respectively. The production function is $y(x,t) = Af(k(x,t))$ where $f(.)$ is increasing, concave and Inada: $f'(.) > 0$, $f''(.) < 0$, $\lim_{k\to 0} f(k) = +\infty$ and $\lim_{k\to +\infty} f(k) = 0$. A is an exogenous productivity parameter, not necessarily space-independent. The marginal productivity, or the real rate of return, of capital is $r(x,t) = Af'(k(x,t))$. The former assumptions exclude the existence of differentiated production technologies across space, implying that differences in the marginal productivity of capital will only be related to the distribution of the per-capita stock of capital. There is an inverse relationship between capital intensity and the real interest rate.

Households increase the scale of production by accumulating physical capital. If there are no (intertemporal) adjustment costs nor depreciation, then gross investment in location x at time t is $\frac{\partial k}{\partial t}$, and the x-household budget constraint is, for any pair $(x,t) \in (\mathbb{X}, \mathbb{T})$,

$$\frac{\partial k(x,t)}{\partial t} + c(x,t) + \tau(x,t) = Af(k(x,t)),$$

where $\tau(x,t)$ denotes both the households' excess of production over expenditures and its net lending capacity. The non-Ponzi game condition holds for every point in space, $x \in \mathbb{X}$,

$$\lim_{t\to\infty} e^{\int_0^t r(x,s)ds} k(x,t) \geq 0.$$

2.2 Regions

Identifying $\mathbb{R}$ with the world allows us to identify the $\sigma-$algebra of the Borel sets $(\mathbb{R}, \mathcal{B})$ with the sets of all regions. Let a particular region, i, be represented by $X_i = [x_i, x_i + \Delta x_i]$. Assume that the world is composed by the union of all the mutually disjoint regions, $\mathbb{X} = \bigcup_i X_i$, where $X_i \bigcap X_j = \emptyset$, for $i \neq j$.

Preferences, technology and endowments are symmetric and the capital market is perfect within regions. Therefore, we may consider a single representative agent located in each region and we may address the interactions among regions as interactions among potentially asymmetric representative agents.

As the world is a closed economy, meaning that

$$\int_{\mathbb{X}} \left(\frac{\partial k(x,t)}{\partial t} + c(x,t) - Af(k(x,t)) \right) dx = - \int_{\mathbb{X}} \tau(x,t) dx = 0 \;\; \forall t \in \mathbb{T},$$

then, spatial interactions will only take place among regions, if regions are open. When regions are closed, the aggregate distribution of capital may vary through time. But this will only occur as a result of independent intertemporal arbitrages within each region. If regions are open then there will be both (interdependent) intertemporal and interspatial arbitrages.

2.3 Autarkic regions

As a first approximation, assume that all regions are closed: there are no capital flows among regions. Then, real transfers of goods between regions cannot be financed, and, therefore trade balances must be instantaneously and permanently cleared, i.e., $\int_{X_i} \tau(x,t) = 0$ for all $(X_i, t) \in (\mathcal{B}, \mathbb{T})$. Therefore, in an autarkic economy the following regional balance equation should hold

$$\int_{X_i} \left(\frac{\partial k(x,t)}{\partial t} + c(x,t) - Af(k(x,t)) \right) dx = 0, \;\; \forall (X_i, t) \in (\mathcal{B}, \mathbb{T}).$$

From now on, let $\Delta x_i \to 0$ and denote the distribution of consumption and capital, at time t as $C(t) := \{c(x,t) : \; x \in \mathbb{X}\}$ and $K(t) := \{k(x,t) : \; x \in \mathbb{X}\}$.

A sufficient condition for regional balance is that

$$\frac{\partial k(x,t)}{\partial t} = Af(k(x,t)) - c(x,t), \;\; \forall (x,t) \in (\mathbb{X}, \mathbb{T}). \tag{2}$$

Independently from the existence of a centralized planner, or a distribution of planners for every region, the equilibrium for this economy will be Paretian. It can be seen as consisting of a continuum of spatially parameterized Ramsey (1928), Cass (1965), Koopmans (1965) models, such that given an initial distribution of capital $K(0)$, the representative agent determines an optimal trajectory for consumption $\hat{C}(x)_0$ and capital accumulation $\hat{K}(x)_0 := \{k(x,t) : t \in \mathbb{T}\}$ such that it maximizes the intertemporal utility function (1) subject to the budget constraint (2).

Applying the maximum principle to every autarkic location, x, there is a piecewise-continuous function, in time, $q(x,t)$ such that the first order conditions are, given by

$$\begin{aligned} u^{'}(c(x,t)) &= q(x,t) \\ \frac{dq(x,t)}{dt} &= q(x,t)(\delta - r(x,t)) \qquad (3) \\ \frac{dk(x,t)}{dt} &= Af(k(x,t)) - c(x,t) \qquad (4) \\ \lim_{t\to\infty} e^{-\delta t} q(x,t)k(x,t) &= 0. \end{aligned}$$

The standard neoclassical assumptions on preferences and technology imply that there is an equilibrium point for every location. If there is symmetry in the parameters, then there will be an asymptotic homogeneous distribution of c (monotonously related with q) and k, $(\overline{c}, \overline{k})$, which is continuously replicated in time and space. That is, $c(x,t) = \overline{c}$, and $k(x,t) = \overline{k}$ for every $(x,t) \in (\mathbb{X}, \mathbb{T})$, such that $\delta = Af^{'}(\overline{k}) = r(\overline{k})$ and $\overline{c}(\overline{q}) = Af(\overline{k})$.

If there is an initial, non-steady state, asymmetric distribution of capital, $K(0) \neq \overline{K} := \{k(x,t) = \overline{k} : \forall (x,t) \in (\mathbb{X}, \mathbb{T})\}$, then a distributional dynamics will follow. The evolution of $K(t)$ will only be generated by intertemporal arbitrage: first consumption will increase for regions in which capital is bellow their steady state level (because in these regions the rate on return of capital is higher than in the steady state) and net savings will also be positive, which will imply an increase in the region's stock of capital.

If, in a neighborhood of the homogeneous steady state, the determinant of jacobian of equations (3)-(4), is

$$D := -c^{'}(\overline{q})\overline{q}Af^{''}(\overline{k}) < 0 \qquad (5)$$

then there will be conditional saddle-path stability and β-convergence. As every region displays conditional stability, then the world distribution will also converge to an homogeneous distribution. If we measure the variance of the world's distribution of the capital stock by

$$\sigma^2(t;k) = \lim_{x\to\infty} \frac{1}{2x} \int_{-x}^{x} (k(x,t) - E(t;k))^2 dx$$

where the average is $E(t;k) = \lim_{x\to\infty} \frac{1}{2x} \int_{-x}^{x} k(x,t)dx$ then we will also have $\sigma-$convergence as $\sigma(0) > \sigma(\infty) = 0$.

Even in the case in which the steady state distribution is heterogeneous (because there is asymmetry in any parameter), two conclusions emerge: first, the dynamics of the aggregate distribution is only determined by the fact that, for each region, the initial capital endowment is different from the steady state capital stock, second, the dynamics of the capital distribution is independent from any spatial interaction, it will only change as a result of local intertemporal arbitrage.

2.4 Open regions

When capital and goods flow among regions, then the aggregate balance equation for region $X_i \in \mathcal{B}$ now becomes

$$\int_{X_i} \left(\frac{\partial k(x,t)}{\partial t} + c(x,t) + \tau(x,t) - Af(k(x,t)) \right) dx = 0, \;\; \forall (X_i, t) \in (\mathcal{B}, \mathbb{T}),$$

where $\tau(x,t) \neq 0$ is the net sales of goods produced in X_i to other regions. In a centralized economy, they will be matched by reallocations of capital among regions. In a decentralized equilibrium setting, we would assume that there would exist an interspatial capital market in which stocks would be traded. In the absence of adjustment, transactions or any other frictional costs, for moving across space, physical capital and its collateral will have the same value.

Then $\tau(x,t)$ represents both the trade balance of region x at time t and the symmetric of the capital account balance. Capital flows in order to eliminate inter-regional arbitrage opportunities, by flowing from regions with lower marginal productivity of capital towards regions with higher marginal productivity of capital. As there is symmetry in technology, and the production function displays diminishing marginal returns to capital, then capital flows from regions in which it is relatively abundant towards regions in which it is relatively scarce. If there are no institutional barriers to capital flows, and as regions are internally homogeneous, then the current account balance for region X_i is measured by the symmetric of the difference of capital intensities with the adjacent regions

$$\int_{X_i} \tau(x,t)dx = -\left[\frac{\partial k}{\partial x}(x_i+\Delta x_i,t) - \frac{\partial k}{\partial x}(x_i,t)\right].$$

As,

$$\frac{\partial k}{\partial x}(x_i+\Delta x_i,t) - \frac{\partial k}{\partial x}(x_i,t) = \int_{x_i}^{x_i+\Delta x_i} \frac{d}{dx}\left(\frac{\partial k}{\partial x}\right)dx = \int_{X_i} \frac{\partial^2 k}{\partial x^2}dx,$$

then the aggregate budget constraint, for region X_i is given by

$$\int_{X_i}\left[\frac{\partial k(x,t)}{\partial t} - \frac{\partial^2 k(x,t)}{\partial x^2} + c(x,t) - Af(k((x,t))\right]dx = 0, \quad \forall (X_i,t) \in (\mathcal{B},\mathbb{T}).$$

If $\Delta x_i \to 0$, then the instantaneous budget constraint is represented by the quasi-linear parabolic partial differential equation

$$\frac{\partial k(x,t)}{\partial t} = \frac{\partial^2 k(x,t)}{\partial x^2} + Af(k(x,t)) - c(x,t) \quad \forall (x,t) \in (\mathbb{X},\mathbb{T}).. \tag{6}$$

If the $x-$household has positive (negative) savings then this will imply both a temporal and a spatial change in the distribution of capital: the household may accumulate (deccumulate) capital by changing its capital stock, and therefore the level of its future production, and/or shift capital to other regions (or, in a decentralized setting, sell or buy equities).

The new economic geography theory highlights two main forces that operate through space: diffusion and agglomeration. Equation (8) presents capital accumulation through time and diffusion across space [5]. We will see that, when we consider the joint dynamics of

[5]See Isard and Liossatos (1979) for a more detailed exposition on the derivation of equation (8). Beckmann (1970) presents similar reasonings as regards the spatial diffusion of prices and innovation.

capital and consumption, agglomeration may emerge from their interplay, as a Turing instability. This means that by introducing spacial contact and diffusion we may get implicitly agglomeration dynamics, and do not need to introduce it explicitly [6].

2.5 Efficient consumption distribution

When goods and capital may be freely reallocated among heterogenous agents, we should have a single central planner that chooses not only the optimal intertemporal allocation of consumption, but also the optimal intratemporal distribution across different locations. The optimization criterium should consider not only aggregation of preferences across time but also across space.

The discounted and additively separable intertemporal utility function (1) presents a benchmark aggregation for utilities through time for the representative agent located in each point in space. Even when we consider intertemporally dependent preferences, the exponential time discounting would still present a natural weighting scheme.

Though we do not intend to dwell into the deep issues related to the definition of a collective preference relationship, we should observe that the choice of an aggregate utility function, when there is spatial asymmetry, is not as settled as for the case in which there is asymmetry. In order to stay close to a Pareto criterium, based upon the maximization of individual welfare, we will assume a Bergson-Samuelson social welfare function [7] and extend it to an intertemporal context.

Accepting an aggregate criterium, based upon a weighted sum of independent individual intertemporal utility functions, is only a first step. Next we have to address the problems of

[6]Explicit introduction of agglomeration may lead to a highly non-linear structure, that would have consequences on the characteristics of the asymptotic distribution of capital.

[7]Bergson (1938) and (Samuelson, 1947, p.219-229) presented an additive social welfare function as a sum of cardinal utility functions. Harsanyi (1955) showed that an aggregate social preference based upon ordinal utility functions and obeying some postulates (v.g, symmetry, independence, transitivity etc) would be represented as a weighted sum of individual cardinal (or Bernoullian) utility functions. These postulates also verify the two main Rawlsian criteria, impartiality and unanimity (see Mueller (2003)).

choosing a spatial weighting scheme and of dealing with the unboundedness of the spatial support.

Benthamian, Millian, von-Neumann-Morgenstern, egalitarian or Rawlsian utility functions [8] are based upon different weighting criteria, and verify reasonable ethical postulates. A Benthamian utility function would be defined, in our setup, as a simple, unweighted, sum of the individual intertemporal utility functions, for the representative households located in every point in space,

$$\int_{-\infty}^{+\infty}\int_{0}^{\infty} u(c(x,t))e^{-\delta t}dtdx.$$

This utility functional solves the aggregation problem but not the unboundedness problem: the intertemporal aggregate utility will be unbounded, even in the case in which all the admissible distributions would tend to a spatially homogeneous bounded steady state [9]

All the other collective utility functions introduce some type of spatial weighting. The simpler weighting schemes are based upon spatial discounting or averaging.

Spatial discounting introduces a symmetry between time and space, by penalizing dates and locations far away from $(x,t) = (0,0)$ [10]. For instance, space could be discounted in an exponential way, leading to the utility functional

$$\int_{-\infty}^{+\infty}\int_{0}^{\infty} u(c(x,t))e^{-(\delta t+\delta_x x^2)}dtdx,$$

where $\delta_w > 0$. However, spatial discounting has two unwelcome features: it introduces a preference relation over locations in space, which violates Harsanyis's symmetry postulate, and tends to force rejection of an homogeneous spatial distribution as an optimal distribution in the steady state (even in the case in which the other parameters of the model are spatially homogeneous).

[8] See Atkinson and Stiglitz (1980) for the related static counterpart.

[9] This is the spatial counterpart of the unboundedness problem arising in the undiscounted Ramsey intertemporal utility function, $\int_0^\infty u(c(t))dt$.

[10] Camacho et al. (2004) consider this case.

Spatial averaging, weights all the locations in space by the inverse of their relative distance to $x = 0$. As weights are spatially homogeneous, then there is not an implicit preference of the central planner for any particular location in space. The following Millian intertemporal utility function may be seen as a collective utility function based upon an averaging criteria [11]

$$V := \lim_{x\to\infty} \frac{1}{2x} \int_{-x}^{x} \int_{0}^{\infty} u(c(y,t))e^{-\delta t} dt dy.$$

This utility functional will be bounded for steady state spatially symmetric distributions of consumption, i.e., $V = \frac{u(\overline{c})}{\delta}$, allowing for the comparison of alternative optimal distributional strategies. We will assume a Millian central planner from now on.

2.6 Boundary conditions

The solutions of partial differential equations depend on the specification of the boundary conditions. Three types of alternative boundary conditions can be found in the applied mathematics literature and be adapted to our model: free boundaries if $\lim_{x\to\pm\infty} k(x,t)$ and $\lim_{x\to\pm\infty} \frac{\partial k(x,t)}{\partial x}$ are not specified, or Cauchy or Dirichlet boundaries if $\lim_{x\to+\infty} k(x,t) = \overline{k}(t)$ and $\lim_{x\to-\infty} k(x,t) = \underline{k}(t)$ or if $\lim_{x\to+\infty} \frac{\partial k(x,t)}{\partial x} = \frac{\partial \overline{k}}{\partial x}(t)$ and $\lim_{x\to-\infty} \frac{\partial k(x,t)}{\partial x} = \frac{\partial \underline{k}}{\partial x}(t)$ were given, respectively. Neumann (or no-flux) boundaries $\lim_{x\to+\infty} \frac{\partial k(x,t)}{\partial x} = \lim_{x\to-\infty} \frac{\partial k(x,t)}{\partial x} = 0$ are a popular special case.

The choice of the particular boundary condition depends on the type of heterogeneity, that we are considering. As, in our case, x is close to spatial location, the choice of the central place is arbitrary. A boundary condition that ties down the stock of capital of distant regions, would introduce incentives for the dispersion of capital and would imply that the choice of the optimal distribution of consumption would be determined exogenously. As our focus is related with the general properties of the dynamics of the capital distribution, it is irrelevant which geographical regions lay at a particular point in the distribution along time. Then

[11] It could also be seen as a von-Neumann-Morgenstern utility function for the case in which the particular location of a consumer is stochastic and equally probable.

Cauchy boundaries would be inappropriate.

Neumann boundaries would be a candidate for a state-space as $\mathbb{X} = [0, +\infty)$, where $x = 0$ could stand for the richest region and $x = +\infty$ for the poorest. The existence of a smooth distribution function would be possible if we would assume, tautologically, that capital could not move outside those boundaries. Neumann boundaries would allow for more flexibility than the Cauchy boundaries, because they would not eliminate the limit situation of homogeneity. However, from the application of Pontriyagin's maximum principle, the central planner would have a very strong incentive to allocate consumption to the extremes of the distribution. The dual boundary conditions would be $\lim_{k\to\pm\infty} \frac{\partial c(x,t)}{\partial x} = +\infty$.

We will assume, instead, the following boundary conditions,

$$\lim_{x\to\pm\infty} \frac{k(x,t)}{x} = 0, \quad \forall t \in \mathbb{T}.,$$

which is both weaker than the Neumann boundaries and is more realistic. It has the following property: the "tails" of the capital stock distribution are bounded functions of time and may be approximated by constant functions of space.

3 The optimal distributed growth dynamics

The central planner's problem is built by assembling the elements of the model presented in the last section. It consists in determining an optimal distributive strategies for consumption and capital, $C^* := \{c^*(x,t) : (x,t) \in \mathbb{X} \times \mathbb{T}\}$ and $K^* := \{k^*(x,t) : (x,t) \in \mathbb{X} \times \mathbb{T}\}$, respectively, such that

$$V := \max_{\{c(x,t):(x,t)\in(\mathbb{X},\mathbb{T})\}} \lim_{x\to\infty} \frac{1}{2x} \int_{-x}^{x} \int_{0}^{\infty} u(c(y,t)) e^{-\delta t} dt dy. \tag{7}$$

subject to

$$\frac{\partial k(x,t)}{\partial t} = \frac{\partial^2 k(x,t)}{\partial x^2} + Af(k(x,t)) - c(x,t) \quad \forall (x,t) \in (\mathbb{X}, \mathbb{T}), \tag{8}$$

$$\lim_{t\to\infty} e^{-\int_0^t r(x,s)ds} k(x,t) \geq 0, \quad \forall x \in \mathbb{X}, \tag{9}$$

$$\lim_{x\to\mp\infty} \frac{k(x,t)}{x} = 0, \quad \forall t \in \mathbb{T} \tag{10}$$

$$k(x,0) = k_0(x), \;\; \forall x \in \mathbb{X} \;\text{ given.} \tag{11}$$

The next proposition, assumes that an optimal solution exists, and presents an heuristic version of the necessary conditions according to the Pontryagin's maximum principle [12].

Proposition 1. *Let (C^*, K^*) be a solution of the centralized problem (7)-(11). Then there is a flow of distributions, $Q := \{q(x,t) : (x,t) \in \mathbb{X} \times \mathbb{T}\}$, where $q(x,t)$ is continuous in $\mathbb{X}$ and piecewise continuous in $\mathbb{T}$, such that the solution verifies equations (8), (9), (10), (11) and*

$$u'(c^*(x,t)) = q(x,t), \quad \forall (x,t) \in (\mathbb{X}, \mathbb{T}) \tag{12}$$

$$\frac{\partial q(x,t)}{\partial t} = -\frac{\partial^2 q(x,t)}{\partial x^2} + q(x,t)\left(\delta - A(x) f'(k^*(x,t))\right) \quad \forall (x,t) \in (\mathbb{X}, \mathbb{T}) \tag{13}$$

$$0 = \lim_{t\to+\infty} \lim_{x\to\infty} \frac{1}{2x} \int_{-x}^{x} e^{-\delta t} q(y,t) k(y,t) dy, \tag{14}$$

$$0 = \lim_{x\to\pm\infty} \frac{e^{-\delta t} q(x,t)}{x}. \quad \forall t \in \mathbb{T} \tag{15}$$

The planar system of partial parabolic equations, (8)-(13), together with the space-time limit conditions (10), (11), (14) and (15) and the optimality condition (12) present a direct generalization of the standard Ramsey-Cass-Koopmans model, for an optimal distribution policy.

Given an initial distribution of the stock of capital, $K(0)$, as in equation (11), an admissible solution should verify the open-economy budget constraint (8), for every date-location pair, the non-Ponzi game condition (9), as a terminal time constraint for every point in space, and the boundary conditions (10) for the "tails" of the distribution of capital in space, for every moment in time.

An optimal solution is an admissible solution that, in addition, has the following properties. First, a static arbitrage condition, given by equation (12), holds. It equates the measure

[12] All the proofs are presented in the Appendix.

of the marginal value of capital, represented by the co-state variable, $q(.)$, with the marginal utility of consumption. For any moment in time, and between any pair of locations, the ratios in q are directly related to the ratios in the true cost-of-living index, and inversely related to the ratios of consumption. Therefore, if we interpret q as a real exchange rate, locations in which consumption rises should face a real depreciation.

Second, an intertemporal-spatial arbitrage condition, given by equation (13), will also hold: for every date-location pair, the rate of return on capital should be equal to the rate of time preference. The rate of return on capital is equal to the sum of the real rate of interest with the change in value of the stock of capital, resulting from capital accumulation through time and space. In a decentralized economy setting, the temporal marginal income will be equal to the change in the value of capital resulting from the investment in one extra unit of equity, issued in the own region, and the spatial marginal income will be equal to the change in income obtained from investing in equities issued by neighboring locations. If we use the real exchange rate interpretation, then equation (13) states an arbitrage relationship between the rate of time preference and the sum of the rate of return on domestic assets plus the instantaneous rate of change in the real exchange rate (real appreciation) with neighboring locations.

Third, a generalized transversality condition, (14), should also be met: the asymptotic average value of capital should be dominated by an exponential discount factor, where the rate of time preference is the discount rate.

At last, equation (15) presents a dual counterpart of the boundary condition (10): the discounted shadow value of capital in the boundaries of the state space, should be close to constant.

Two spatial diffusive forces are at work in our model. The first is related to the incentives which drive the redistribution of consumption, seeking to equalize the marginal productivity of capital with the rate of time preference through space and time. The second is related to the dynamics of the open economy budget constraint. While the latter acts as a forward diffusive force, the first acts as a backward diffusive force. That is, if at a particular time-space location there is an expectation that the rate of time preference is higher than the marginal productivity of capital in any point in the future, which means that there is too much capital, then consumption will not only increase in that location, but it also increases more than in neighboring locations. There are, immediately, net imports of goods from neighboring regions. If we consider the two effects together, then the increase in consumption will imply both a reduction in net investment, in the own location, and in the investment in assets on firms located in other regions. In the non-spatial Ramsey model, the decreasing returns to capital act as a dampening force which generates conditional stability. In our case, as we will see, the presence of both diffusion mechanisms generate instability, which may more than offset the stabilizing effect of the decreasing returns.

Next, we will characterize the solutions of the system (8)-(13) by using qualitative methods [13]. First, we will address the existence, uniqueness and characterization of an asymptotic state, and, next, we will study the local dynamics associated to changes in productivity.

4 Asymptotic states

An (optimal) asymptotic state is defined by the distributions of the dual price and of capital, $\tilde{Q} := \{q(x), x \in \mathbb{X}\}$ and $\tilde{K} := \{k(x), x \in \mathbb{X}\}$, respectively, that solve the system of ordinary differential equations on space,

$$\frac{\partial^2 q(x)}{\partial x^2} = q(x)(\delta - Af'(k(x))), \tag{16}$$

$$\frac{\partial^2 k(x)}{\partial x^2} = -Af(k(x))) + c(q(x)), \tag{17}$$

given the boundary conditions (10).

[13] There are several methods and a related huge literature in other fields of science on solving partial differential equations. However, the negative dependence of the diffusion term in equation (13) and the related terminal condition (14) is not common in the applied mathematics literature.

The main issue to address is related to the existence of a spatially heterogeneous asymptotic state, i.e., to the existence of particular solutions to the system (16)-(17) which vary with the independent variable. As the behavioral functions are defined implicitly, we may only supply a qualitative answer.

Heterogeneous asymptotic states may occur exogenously or endogenously. If any parameter is spatially dependent, v.g., $A = A(x)$, then the system becomes non-autonomous and the solution will be space dependent. In this case we say that a spatially heterogeneous asymptotic state is generated exogenously. If all the parameters are homogenous across space, then two types of solutions may be obtained: a constant or a spatially variable solution. In the first case we will have a spatially homogeneous asymptotic state, and, in the second, an endogenously generated heterogeneous asymptotic state.

The traveling wave literature (see Volpert et al. (1994)) offers some useful results to tackle this issue. Traveling waves are solutions of the system (8)-(13) such that the unknown functions, $q(.)$ and $k(.)$, can be written as $q(x,t) = y(\xi)$ and $k(x,t) = w(\xi)$ where $\xi := x - \eta t$. The constant speed of wave propagation is denoted by η. The solutions of system (16)-(17) are a particular case for $\eta = 0$.

Let $y_1(\xi) = \frac{dy(\xi)}{d\xi}$ and $w_1(\xi) = \frac{dw(\xi)}{d\xi}$, then the PDE system (8)-(13) is equivalent to the following four dimensional ODE system

$$\dot{y}(\xi) = y_1(\xi) \tag{18}$$

$$\dot{y}_1(\xi) = \eta y_1(\xi) + y(\xi)\left(\delta - Af'(w(\xi))\right) \tag{19}$$

$$\dot{w}(\xi) = w_1(\xi) \tag{20}$$

$$\dot{w}_1(\xi) = -\eta w_1(\xi) - Af(w(\xi)) + c(y(\xi)). \tag{21}$$

Again, two types of stationary solutions may exist:

Lemma 1. *Let the parameters δ, A and η be independent from ξ. Then the system (18)-(21) has only two possible stationary solutions: a stationary state or a stationary traveling wave of the pulse type.*

The second case may occur because the equilibrium point of the system (18)-(21) is unique and if an homoclinic orbit, such that $\lim_{\xi\pm\infty} z(\xi) = \overline{z}$, for $z := (y, y_1, w, w_1)$, exists. Observe that if it exists for $\eta = 0$ then system (16)-(17) will have an homoclinic orbit as well, and its existence is completely consistent with the boundary conditions (10).

Proposition 2. *Let $\eta = 0$ and let the other parameters be constant. Then the only asymptotic equilibrium point is the homogeneous distribution.*

This proposition is an implication of the fact that there is not enough curvature in the production function for allowing to the existence of endogenously generated asymptotic heterogeneous distributions of capital and consumption [14].

Summing up, we will only have a spatially heterogeneous asymptotic state if at least one parameter is spatially asymmetric. When the parameters are spatially symmetric, then the only asymptotic state for the system (8)-(13) is the Ramsey-Cass-Koopmans equilibrium, replicated across space, $\overline{K} = \{k(x) = \overline{k} : \quad \delta = Af'(\overline{k}), \quad \forall x \in \mathbb{X}\}$, and $\overline{Q} = \{q(x) = \overline{q} : c(\overline{q}) = Af(\overline{k}), \ \forall x \in \mathbb{X}\}$. Given the neo-classical assumptions on preferences and technology, that spatially homogeneous steady state will exist and be unique, for any point in space.

5 Local dynamics in the neighborhood of an homogeneous asymptotic state

A solution to the optimal distribution problem exists if, for any given initial distribution of capital, $K(0)$, not necessarily spatially homogeneous, there is an initial distribution of prices, $Q(0)$, (or a monotonously initial distribution of consumption, $C(0)$) and flow distributions through time, $Q(t)$ and $K(t)$, for $t > 0$, that will converge to a spatially homogeneous steady state, $\lim_{t\to\infty} Q(t) = \overline{Q}$, $\lim_{t\to\infty} K(t) = \overline{K}$.

Any unbounded trajectory violates both the boundary and the transversality conditions. As the forward-backward system of partial differential equations has not an explicit solution,

[14] The introduction of explicit externalities may produce enough non-linearities for allowing to the existence of heterogenous steady states.

we will have to rely again on qualitative methods in order to determine the existence and characterize the (generalized) stable manifold based upon linearization [15].

5.1 Linearization

Let $u_q(x,t) := q(x,t) - \overline{q}$ and $u_k(x,t) := k(x,t) - \overline{k}$ denote the space-time local variations of the marginal utility of consumption and of the capital stock, in the neighborhood of a spatially homogeneous asymptotic state. Assume that at time $t = 0$ the deviation from the steady state is given by $(u_q(x,0), u_k(x,0)) \neq (0,0)$, for any $x \in \mathbb{X}$. If the integrability and regularity conditions are fulfilled, then the local modified hamiltonian PDE system (8)-(13), can be approximated, by a system of two homogeneous semi-linear parabolic PDE's, $\forall (x,t) \in (\mathbb{X} \times \mathbb{R}_{++})$,

$$\frac{\partial u_q(x,t)}{\partial t} = -\frac{\partial^2 u_q(x,t)}{\partial x^2} - \overline{q} A f''(\overline{k}) u_k(x,t), \tag{22}$$

$$\frac{\partial u_k(x,t)}{\partial t} = \frac{\partial^2 u_k(x,t)}{\partial x^2} - c'(\overline{q}) u_q(x,t) + \delta u_k(x,t). \tag{23}$$

As the spatial support is unbounded, $-\infty < x < +\infty$, a Fourier transformation may be applied [16] as

$$u_j(x,t) = \int_{-\infty}^{+\infty} \mathcal{U}_j(\upsilon,t) e^{i\upsilon x} d\upsilon, \quad j = q, k, \quad \forall (x,t) \in (\mathbb{X} \times \mathbb{R}_+)$$

where $i = \sqrt{-1}$ and $\upsilon \in \Upsilon = \mathbb{R}$ and

$$\mathcal{U}_j(\upsilon,t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} u_j(x,t) e^{-i\upsilon x} dx, \quad j = q, k, \quad \forall (\upsilon,t) \in (\Upsilon \times \mathbb{R}_+).$$

That transformation changes the functional dependence of the variables from space, x, to frequency υ, which represents the speed of spatial propagation of any shock located at $x = 0$. This new representation has important analytical consequences. By using the properties of

[15] We will mainly develop a heuristic approach. See Henry (1981) and Evans (1998) for a rigorous approach on the existence of solutions. See (Smoller, 1994, ch. 11) on the linearization of parabolic PDE's.

[16] See (Evans, 1998, p. 182-190).

the Fourier transforms, then system (22)-(23) becomes

$$\begin{aligned}
0 &= \int_{-\infty}^{+\infty} \left[\frac{\partial \mathcal{U}_q(\upsilon,t)}{\partial t} + (i\upsilon)^2 \mathcal{U}_q(\upsilon,t) + \overline{q} A f''(\overline{k}) \mathcal{U}_k(\upsilon,t) \right] e^{i\upsilon x} d\upsilon, \\
0 &= \int_{-\infty}^{+\infty} \left[\frac{\partial \mathcal{U}_k(\upsilon,t)}{\partial t} - (i\upsilon)^2 \mathcal{U}_k(\upsilon,t) + c'(\overline{q}) \mathcal{U}_q(\upsilon,t) - \delta \mathcal{U}_k(\upsilon,t) \right] e^{i\upsilon x} d\upsilon.
\end{aligned}$$

A sufficient condition for the annihilation of the integrals is that integrand be equal to zero. Then an equivalent spectral representation of (22)-(23) is the linear planar ODE, parameterized by υ,

$$\frac{\partial \mathcal{U}_q(\upsilon,t)}{\partial t} = \upsilon^2 \mathcal{U}_q(\upsilon,t) - \overline{q} A f''(\overline{k}) \mathcal{U}_k(\upsilon,t), \tag{24}$$

$$\frac{\partial \mathcal{U}_k(\upsilon,t)}{\partial t} = -c'(\overline{q}) \mathcal{U}_q(\upsilon,t) + (\delta - \upsilon^2) \mathcal{U}_k(\upsilon,t), \tag{25}$$

$\forall (\upsilon,t) \in (\Upsilon \times \mathbb{R}_{++})$, given $\mathcal{U}_q(\upsilon,0)$ and $\mathcal{U}_k(\upsilon,0)$, at $t=0$. This system can be solved explicitly and its solution gives the local dynamics, through time, associated with every spatial frequency.

5.2 Local conditional spectral stability and Turing instability

Recall, from equation (5), that $D<0$ as an implication of the concavity of the production and utility functions. Additionally, let

$$\Upsilon_s = \begin{cases} \mathbb{R} & \text{if } \left(\frac{\delta}{2}\right)^2 + D < 0 \\ (-\infty, \upsilon_1) \cup (\upsilon_2, \upsilon_3) \cup (\upsilon_4, +\infty) & \text{if } \left(\frac{\delta}{2}\right)^2 + D > 0, \end{cases}$$

where, $\upsilon_1 < \upsilon_2 < 0 < \upsilon_3 < \upsilon_4$, because

$$\begin{aligned}
\upsilon_{1,2} &= -\left[\frac{\delta}{2} \pm \left[\left(\frac{\delta}{2} \right)^2 + D \right]^{\frac{1}{2}} \right]^{\frac{1}{2}} < 0 \\
\upsilon_{3,4} &= \left[\frac{\delta}{2} \mp \left[\left(\frac{\delta}{2} \right)^2 + D \right]^{\frac{1}{2}} \right]^{\frac{1}{2}} > 0.
\end{aligned}$$

We say that there is *conditional spectral stability* if the trajectory associated with any frequency is saddle-point stable.

Lemma 2. *If $\upsilon \in \Upsilon_s$ then there is conditional spectral stability.*

Therefore two cases may occur: if $\left(\frac{\delta}{2}\right)^2 + D < 0$ then there is conditional spectral stability for any frequency of spatial propagation; but, if $\left(\frac{\delta}{2}\right)^2 + D > 0$ then there will only be conditional spectral stability for some frequencies [17].

A similar result occurs in the presence of the *Turing instability*, which may exist in planar forward parabolic partial differential equations. In those equations, Turing (1952) observed that the diffusive terms may reduce the dimension of the stable manifold associated to the kinetic part, for some parameter values. In initial value well posed problems, associated to forward equations, it implies the emergence of pattern formation, i.e., emergence of a non-homogenous asymptotic distributions. In our case, we have a mixed initial-terminal condition in time, associated to a system composed by a forward and a backward parabolic partial differential equation. Then, a *transient pattern formation* results, for the case in which $\left(\frac{\delta}{2}\right)^2 + D > 0$, when we are forced to eliminate the frequencies associated with total instability, in order to get a non-empty stable manifold [18].

Formally, the conditions enabling the case $\left(\frac{\delta}{2}\right)^2 + D > 0$ are the following:

Lemma 3. *The likely occurrence of a Turing instability is higher when the rate of time preference and the concavity of the utility function are higher and the concavity of the production function is lower.*

5.3 The tangent to the generalized local stable manifold

Proposition 3. *Assume that the system is at a given initial distribution $K(0) \neq \overline{K}$, at time $t = 0$, and let*

$$\lambda_s = \left\{ \frac{\delta}{2} - \left[\left(\upsilon^2 - \frac{\delta}{2} \right)^2 - D \right]^{\frac{1}{2}} : \upsilon \in \Upsilon_s \right\} < 0. \tag{26}$$

[17] There will be mode interaction, associated to a fold bifurcation, if $\left(\frac{\delta}{2}\right)^2 + D = 0$. See Mei (2000).

[18] This solution is stronger than necessary. As the transversality condition is computed as an average and as the sum of the eigenvalues is independent from frequencies, because $\lambda_u + \lambda_s = \delta$, if we could compute an explicit solution, then it is possible than the pattern formation may emerge not only locally but also globally.

Then, the solutions along the stable space, which is tangent to the stable manifold, are

$$\begin{aligned}\tilde{u}_q(x,t) &= \int_{-\infty}^{+\infty} h_s(w)\tilde{u}_k(x-w,t)dw, &(27)\\ \tilde{u}_k(x,t) &= \int_{-\infty}^{+\infty} u_k(z,0)\phi(x-z,t)dz,, &(28)\end{aligned}$$

for any pair $(x,t) \in (\mathbb{X}, \mathbb{R}_{++})$*, where the Green's function for the stock of capital is*

$$\phi(y,t) = \frac{1}{2\pi}\int_{\Upsilon_s} e^{\lambda_s(\upsilon)t + i\upsilon y} d\upsilon, \quad \forall (y,t) \in (\mathbb{X}, \mathbb{R}_{++}), \tag{29}$$

and the slope of the tangent to the stable manifold in the (q,k)*-space is*

$$h_s(y) = \frac{1}{2\pi}\int_{\Upsilon_s} \frac{\lambda_u(\upsilon) - \upsilon^2}{c'(\overline{q})} e^{i\upsilon y} d\upsilon, \quad \forall y \in \mathbb{X}. \tag{30}$$

The economy will conditionally converge to a spatially homogeneous steady state $(\overline{Q}, \overline{K})$, through a time-varying distribution which is tangent to (27)-(28). Though there is not an explicit expression for the integrals, it is easy to see that

$$\lim_{t\to\infty} \tilde{u}_q(x,t) = \lim_{t\to\infty} \tilde{u}_k(x,t) = 0, \quad \forall x \in \mathbb{X}$$

and

$$\lim_{t\downarrow 0} \tilde{u}_q(x,t) \neq u_q(x,0) \quad \lim_{t\downarrow 0} \tilde{u}_k(x,t) = u_k(x,0), \quad \forall x \in \mathbb{X}.$$

This result generalizes the well know result for the "jump" to the saddle manifold in the Ramsey-Cass-Koopmans model, to the distributional "jump" $\tilde{u}_q(x,0) - u_q(x,0)$.

When we have both intertemporal and inter-spatial arbitrage mechanisms, their interaction gives birth to an analogous to Turing instability, which is not present in the homogenous agent model. In deriving the spectral conditional stability and the related (tangent to the) stable manifold, we have to eliminate that source of instability. The intuition is the following: if the economy is not in an initial homogeneous asymptotic state, and the spatial asymmetry generate two spatial diffusion forces, with a frequency belonging to Υ_s, then a particular generalized saddle path is followed, and the economy will converge to a spatially homogeneous and bounded asymptotic state. If $\left(\frac{\delta}{2}\right)^2 - c^{'}(\overline{q})\overline{q}Af^{''}(\overline{k}) < 0$ then the optimal consumption-investment policy will generate a spatially monotonous distribution reallocation of consumption and capital. However, if the $\left(\frac{\delta}{2}\right)^2 - c^{'}(\overline{q})\overline{q}Af^{''}(\overline{k}) > 0$ then the spatial reallocation of capital and consumption will tend to generate an unbounded asymptotic distribution if that reallocation does not "discriminate" against particular locations. That is, the optimal consumption and investment strategies may have to be non-monotonous across space [19]. Quah (2002) also found, in a spatial aggregation model, that some sinusoidal components have to be omitted for the existence of convergence to a uniform steady-state equilibrium. Looking at the expression for λ_s, in equation (26), it appears that the effect of some medium-ranged spatial frequencies has to be eliminated. The explanation seems to be the following: while the low frequencies tend to counter the effect of the decreasing marginal productivity of capital, which will work for instability, they seem to inhibit the destabilizing effect of the spatial re-distribution of consumption by the planner.

Even if we assume concave utility and production functions, as in the Ramsey-Cass-Koopmans model, the spatial propagation mechanism may originate instability by the way of an implicit externality. Therefore, spatial agglomeration is a property of the optimal spatial capital allocation policy in the presence of a potential Turing instability. According to lemma 3, an optimal agglomeration policy is likelier if the production function is close to linear and the degree of relative risk aversion is high. This means that both spatial differences among real interest rates and changes in the real exchange rates are not high. In this case, there is a complementarity between stronger intertemporal mechanisms (as opposed to inter-spatial) and the externality effects generated by diffusion. The optimal solution will only verify the transversality condition if the working of that externality is hampered by the presence of spatial agglomeration.

[19] The formal reason for the emergence of a transient pattern formation can be seen in the expressions for the Green function for the stock of capital and for the slope of the stable manifold ($\phi(.)$ and $h_s(.)$).

5.4 Numerical illustrations

To illustrate the results in Lemma 3 and Proposition 3, consider the examples in table 1. Both assume an isoelastic utility function,$u(c) = \frac{c^{1-\theta}}{1-\theta}$, where $\theta > 0$, and a Cobb-Douglas production function, $y = Ak^{\alpha}$, where $0 < \alpha < 1$. The two sets of parameters are chosen in order to allow for $\left(\frac{\delta}{2}\right)^2 + D < 0$, in case 1, and $\left(\frac{\delta}{2}\right)^2 + D > 0$, in case 2. As $D = -\frac{\bar{c}}{\theta}\alpha(\alpha-1)\bar{k}^{\alpha-2}$, then case 2 occurs for higher δ and θ are an α close to one, as stated in Lemma 3. We consider a slightly more general case with capital depreciation at a constant rate ρ.

Table 1: Numerical examples

case	δ	α	θ	ρ	A	a_{12}	a_{21}	$\left(\frac{\delta}{2}\right)^2 + D$	$b_{1,A}$	$b_{2,A}$
1	0.03	0.3	0.5	0.1	0.704	0.051	1.431	-0.073	-0.207	1.421
2	0.2	0.9	5	0.1	0.357	0.046	0.052	0.008	-2.563	2.799

We also assume, in both cases, that there is an initial asymmetric deviation from a homogeneous steady state, given by $u_k(x, 0) \sim N(0, 1)$. Figures 1 and 2 present the space tangent to the saddle manifold for case 1 and case 2, respectively. We observe that the qualitative dynamics is similar, in both cases. Initially, the locations with more initial capital have lower relative prices. Therefore, they have higher consumption and a depreciated real exchange rate. When time passes, those regions export capital and face a process of real appreciation towards a long run homogeneous distribution of (optimal) capital, prices and consumption. This positive correlation between the increase in capital endowments real depreciation is consistent with the findings of Acemoglu and Ventura (2002), who models a world with imperfectly substitutable goods.

Two other observations are worth mentioning: first, there is both absolute β- and σ-convergence, and, second, the transition is slower through time and non-monotonous across space in case 2. This is, again, a consequence of the effects of the potential Turing instability and of the their elimination through a transient pattern formation.

6 Comparative distributional dynamics

In the last section we considered an initial heterogeneous distribution, $K(0)$, and symmetric parameters. Now, we will assume that the economy is at a stationary homogeneous distribution, $(\overline{Q}, \overline{K})$, and that there is a, possibly asymmetric, productivity shock $dA(x)$.

The next proposition presents comparative dynamics formulae.

Proposition 4. *Assume that $\upsilon \in \Upsilon_s$ and that there is a permanent time-invariant and possibly spatially asymmetric productivity shock, $dA(x)$. Then the short run multipliers, along the saddle manifold are given by equations*

$$u_{q,A}(x,t) = \overline{u}_{q,A}(x) - \int_{-\infty}^{+\infty} \overline{u}_{k,A}(w) \int_{-\infty}^{+\infty} h_s(y)\phi(x-y-w,t)dydw, \tag{31}$$

$$u_{k,A}(x,t) = \overline{u}_{k,A}(x) - \int_{-\infty}^{+\infty} \overline{u}_{k,A}(z)\phi(x-z,t)dz, \tag{32}$$

for $(x,t) \in (\mathbb{X}, \mathbb{R}_{++})$, where $\phi(.)$ and $h_s(.)$ are given in equations (29) and (30), respectively, and the asymptotic multipliers are

$$\overline{u}_{q,A}(x) = \int_{\Upsilon_s} \overline{\mathcal{U}}_{q,A}(\upsilon)e^{i\upsilon x}d\upsilon \tag{33}$$

$$\overline{u}_{k,A}(x) = \int_{\Upsilon_s} \overline{\mathcal{U}}_{k,A}(\upsilon)e^{i\upsilon x}d\upsilon, \tag{34}$$

where

$$\overline{\mathcal{U}}_{q,A}(\upsilon) = \frac{(\delta-\upsilon^2)\mathcal{B}_{1,A}(\upsilon) + \overline{q}Af''(\overline{k})\mathcal{B}_{2,A}(\upsilon)}{\upsilon^2(\delta-\upsilon^2)+D} \tag{35}$$

$$\overline{\mathcal{U}}_{k,A}(\upsilon) = \frac{c'(\overline{q})\mathcal{B}_{1,A}(\upsilon) + \upsilon^2\mathcal{B}_{2,A}(\upsilon)}{\upsilon^2(\delta-\upsilon^2)+D}. \tag{36}$$

and

$$\mathcal{B}_{1,A}(\upsilon) = -\frac{1}{2\pi}\int_{-\infty}^{+\infty} \overline{q}f'(\overline{k})dA(x)e^{-i\upsilon x}dx \tag{37}$$

$$\mathcal{B}_{2,A}(\upsilon) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} f(\overline{k})dA(x)e^{-i\upsilon x}dx. \tag{38}$$

Figure 1: Local stable manifold: case 1

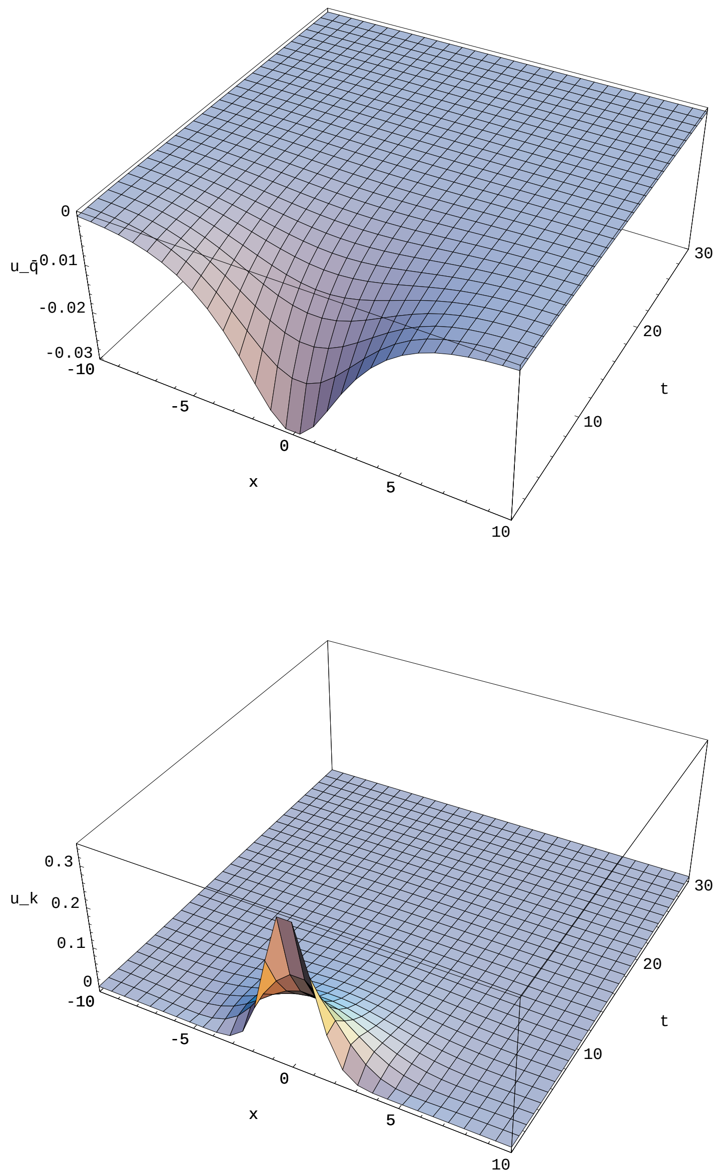
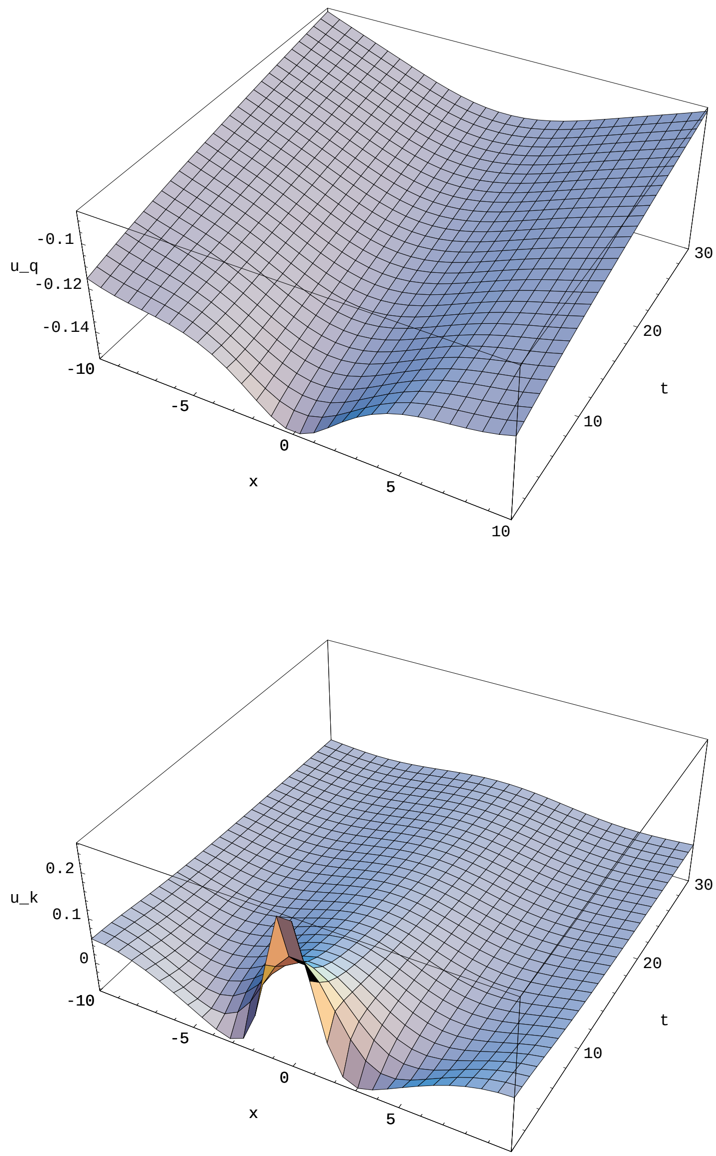


Figure 2: Local stable manifold: case 2, transient spatial pattern formation



Now, on impact, the changes in both variables are $\lim_{t\downarrow 0} \tilde{u}_{q,A}(x,0) \neq 0$ and $\lim_{t\downarrow 0} \tilde{u}_{k,A}(x,0) = 0$, for any $x \in \mathbb{X}$.

Next, we determine the long run multipliers for two particular cases: for a symmetric shock and for an asymmetric shock in location $x = 0$.

Corollary 1. *Assume that there is a symmetric, constant, and permanent unit shock, $dA(x) = dA = 1$ for any $x \in \mathbb{X}$. Then, the long run multipliers are also symmetric,*

$$\begin{aligned} \overline{u}_{q,A} &= \frac{(f'(\overline{k}))^2 - f(\overline{k})f''(\overline{k})}{c'(\overline{q})f''(\overline{k})} < 0, \quad \forall x \in \mathbb{X} \\ \overline{u}_{k,A} &= -\frac{f'(\overline{k})}{Af''(\overline{k})} > 0, \quad \forall x \in \mathbb{X}. \end{aligned}$$

where $(\overline{q}, \overline{k})$ are the space-wise pre-shock values.

Corollary 2. *Assume that there is a Dirac's delta permanent productivity shock at location $x = 0$, i.e., $dA(x) = \delta(x)$. Then the asymptotic multipliers are asymmetric across space,*

$$\begin{aligned} \overline{u}_{q,A}(x) &= \int_{\Upsilon_s} \frac{\overline{q}}{2\pi} \left(\frac{Af(\overline{k})f''(\overline{k}) - (\delta - \upsilon^2)f'(\overline{k})}{\upsilon^2(\delta - \upsilon^2) + D} \right) e^{i\upsilon x} d\upsilon \\ \overline{u}_{k,A}(x) &= -\frac{1}{2\pi} \int_{\Upsilon_s} \left(\frac{c'(\overline{q})\overline{q}f'(\overline{k}) + \upsilon^2 f(\overline{k})}{\upsilon^2(\delta - \upsilon^2) + D} \right) e^{i\upsilon x} d\upsilon \end{aligned}$$

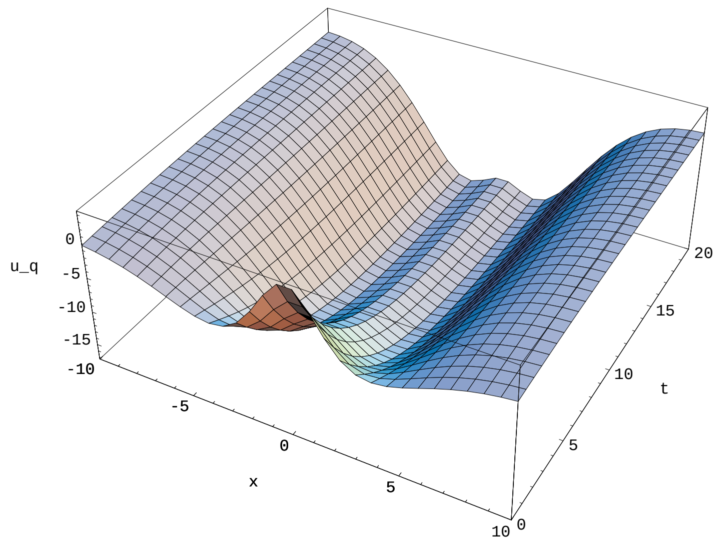
where $(\overline{q}, \overline{k})$ are the space-wise pre-shock values.

Given an initial symmetric state, a symmetric shock will generate an asymptotic symmetric shift in the capital stock and an asymmetric shock will generate an asymmetric one. Also, the long run multipliers are formally the same, independently of the structure of Υ_s.

We get a geometrical representation by going back to the examples presented in the last section and introducing a permanent asymmetric productivity shock $dA(x) \sim \frac{N(0,1)}{A_0} \times 100$ (see Figures 3 and 4). This case is a weighted sum of the Dirac's delta shocks dealt in Corollary 2.

They show, firstly, that there is both a short run and a long run increase in the asymmetry of the distribution of capital and consumption, and there will be a permanent real

Figure 3: Dynamics for a shock in A, percentage changes: case 1



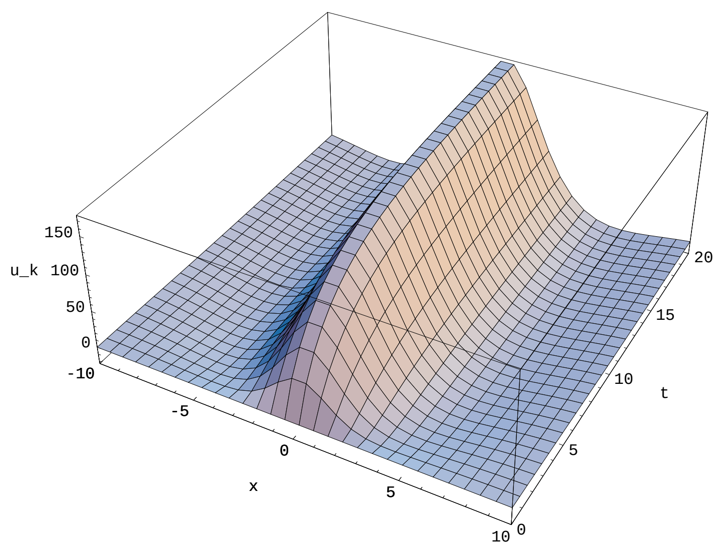
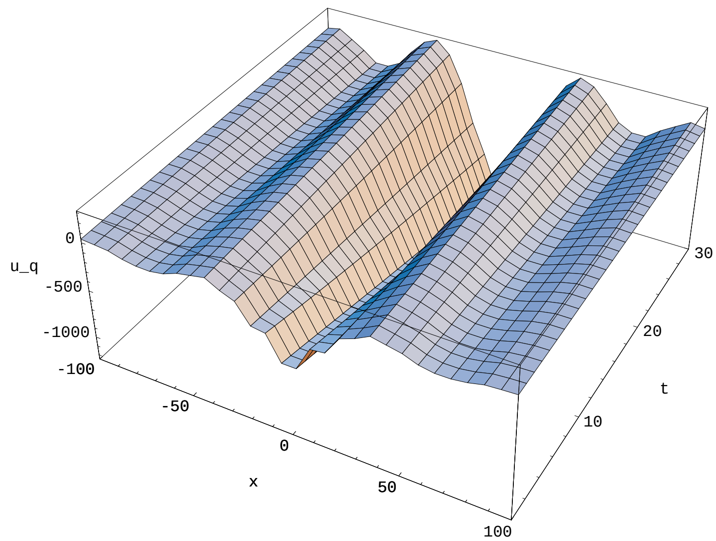
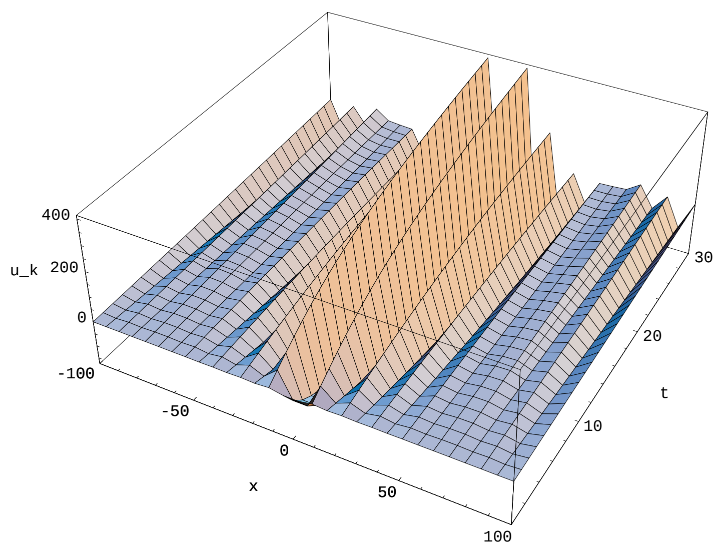


Figure 4: Dynamics for a shock in A: case 2, spatial pattern formation





depreciation affecting the locations that benefit from the shock. Secondly, some partial spillovers to neighboring locations occur, although the optimal growth and distribution of consumption will be achieved by keeping capital concentrated in the locations in which it is more productive. We have relative β-convergence but possibly not σ-convergence. At last, in case 2, the optimal distributive policy is clearly non-monotonous across space and there will be a clear local pattern formation: for a shock that varies monotonously across space, there will be long run local agglomerations in the distribution of capital. In this case, the pattern formation has both a transient and a long run nature.

7 Final remarks

This paper presented an attempt at integrating both spatial and temporal dynamics, by dealing directly with distributions and with the dynamics of distributions. The main extensions and questions worth addressing with our framework are the following, in our opinion. First, under which conditions can we get an endogenous long run distribution of capital, v.g. consistent with Quah (1996) bi-modal distribution of per capita income: will explicit agglomeration mechanisms do the job ? Second, the Turing instability may supply another avenue for generating endogenous unbounded growth: will spatial spillovers generate unbounded growth even when there are locally decreasing returns to capital ? At last, in modeling a decentralized economy: is there a structure of markets and institutions which would generate an equilibrium equivalent, in a Pareto sense, to the solutions of our model ?

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A Appendix: Proofs

Proof of Proposition 1. We have an optimal control problem of partial differential equations or an optimal distributed control problem [20]. Let us assume that there is a solution (C^*, K^*), for the problem, and define the value function as

$$V(C^*, K^*) = \lim_{x\to\infty} \frac{1}{2x} \int_{-x}^{x} \int_{0}^{\infty} u(c^*(y,t))e^{-\delta t} dtdy.$$

Consider a small continuous perturbation $(C(\epsilon), K(\epsilon)) = \{(c(x,t), k(x,t)) : (x,t) \in \mathbb{X} \times \mathbb{T}\}$, where ϵ is any positive constant, such that $c(x,t) = c^*(x,t) + \epsilon h_c(x,t)$ and $k(x,t) = k^*(x,t) + \epsilon h_k(x,t)$, for $t > 0$, and $h_c(x,0) = h_k(x,0) = 0$, for every $x \in \mathbb{X}$. The value of this strategy is

$$V(\epsilon) = \lim_{x\to\infty} \frac{1}{2x} \int_{-x}^{x} \int_{0}^{\infty} u(c(y,t))e^{-\delta t} dtdy.$$

But,

$$\begin{aligned} V(\epsilon) \quad := \quad & \lim_{x\to\infty} \frac{1}{2x} \int_{-x}^{x} \int_{0}^{\infty} u(c(y,t))e^{-\delta t} dtdy - \\ & - \lim_{x\to\infty} \frac{1}{2x} \int_{-x}^{x} \int_{0}^{\infty} \lambda(y,t) \left[\frac{\partial k(y,t)}{\partial t} - \frac{\partial^2 k(y,t)}{\partial y^2} - Af(k(y,t)) + c(y,t) \right] dtdy + \\ & + \lim_{t\to\infty} \lim_{x\to\infty} \frac{1}{2x} \int_{-x}^{x} e^{-r(y,t)} \mu(y,t) k(y,t) dy \end{aligned}$$

where $\lambda(.)$ is the co-state variable and $\mu(.)$ is a Lagrange multiplier associated with the solvability condition. In the optimum, the Kuhn-Tucker condition should hold

$$\lim_{t\to\infty} \lim_{x\to\infty} \frac{1}{2x} \int_{-x}^{x} e^{-r(y,t)} \mu(y,t) k(y,t) dy = 0.$$

By using integration by parts we find that

$$\int_{0}^{\infty} \lambda(x,t) \frac{\partial k(x,t)}{\partial t} dt = \lambda(x,t) k(x,t)|_{t=0}^{\infty} - \int_{0}^{\infty} \frac{\partial \lambda(x,t)}{\partial t} k(x,t) dt$$

[20] Butkovskiy (1969), Lions (1971), Derzko et al. (1984) or Neittaanmaki and Tiba (1994) present optimality results with varying generality. We draw mainly upon the last two references. See also, for applications in economics Carlson et al. (1991, chap.9). Camacho et al. (2004) study the existence of solutions in a related problem with a linear utility function.

and that

$$\begin{aligned}\int_{-x}^{x}\int_{0}^{\infty}\lambda(y,t)\frac{\partial^2 k(y,t)}{\partial y^2}dtdy &= \int_{0}^{\infty}\lambda(y,t)\frac{\partial k(y,t)}{\partial y}\bigg|_{y=-x}^{x} - k(y,t)\frac{\partial \lambda(y,t)}{\partial y}\bigg|_{y=-x}^{x}dt + \\ &+\int_{-x}^{x}\int_{0}^{\infty}\frac{\partial^2 \lambda(y,t)}{\partial y^2}k(y,t)dtdy,\end{aligned}$$

where the second term is canceled by the boundary conditions (10). Then

$$\begin{aligned}V(\epsilon) &= \lim_{x\to\infty}\frac{1}{2x}\int_{-x}^{x}\int_{0}^{\infty}\Big(u(c(y,t))e^{-\delta t}+ \\ &+\frac{\partial\lambda(y,t)}{\partial t}k(y,t)+\frac{\partial^2\lambda(y,t)}{\partial y^2}k(y,t)+\lambda(y,t)[Af(k(y,t))-c(y,t)]\Big)dtdy - \\ &-\lim_{x\to\infty}\frac{1}{2x}\left(\int_{-x}^{x}\lambda(y,t)k(y,t)|_{t=0}^{\infty}dy+\int_{0}^{\infty}\lambda(y,t)\frac{\partial k(y,t)}{\partial y}\bigg|_{y=-x}^{x}dt\right)\end{aligned}$$

If an optimal solution exists, then we may characterize it by applying the variational principle,

$$\frac{\partial V(C^*,K^*)}{\partial \epsilon} = \lim_{\epsilon\to 0}\frac{V(C(\epsilon),K(\epsilon))-V(C^*,K^*)}{\epsilon} = 0.$$

But

$$\begin{aligned}\frac{\partial V}{\partial \epsilon} &= \lim_{x\to\infty}\frac{1}{2x}\left\{\int_{-x}^{x}\int_{0}^{\infty}\left[\left(u^{'}(c^*(y,t))e^{-\delta t}-\lambda(y,t)\right)h_c(y,t)+\right.\right. \\ &+\left.\left(\frac{\partial\lambda(y,t)}{\partial t}+\frac{\partial^2\lambda(y,t)}{\partial y^2}+\lambda(y,t)Af^{'}(k^*(y,t))\right)h_k(y,t)\right] \\ &-\int_{-x}^{x}\lambda(y,t)h_k(y,t)\ |_{t=0}^{\infty}\ dy+\int_{0}^{\infty}\lambda(y,t)\frac{\partial h_k(y,t)}{\partial y}\ |_{y=-x}^{x}\ dt - \\ &\left.-\lim_{t\to\infty}\int_{-x}^{x}\mu(y,t)e^{-r(y,t)}h_k(y,t)dy\right\}.\end{aligned}$$

The last and the third to last expressions are canceled if $\lim_{t\to\infty}[\mu(x,t)e^{-r(x,t)}-\lambda(t,x)]=0$, and by the fact that $h_k(x,0)=0$, for any x. Then, substituting in the Kuhn-Tucker condition we get a generalized transversality condition. We get the first order conditions by equating to zero all the remaining components of $\frac{\partial V}{\partial \epsilon}$. Equations (12)-(15) are obtained by simply making $q(x,t)=e^{\delta t}\lambda(x,t)$. □

Proof of Lemma 1. The steady state of system (18)-(21) exists and is unique. It is given by $\overline{y}_1 = \overline{w}_1 = \dot{k} = \dot{w} = 0$, $\overline{k} = \overline{w} = \{w > 0 : Af'(w) = \delta\}$ and $\overline{q} = \overline{y} = \{y > 0 : Af(\overline{w}) = c(y)\}$. and is a function of the (constant) parameters. In this case, it has been proved in the traveling wave literature (see (Volpert et al., 1994, p.5)) that, in addition to the equilibrium point, seen as a degenerate wave, the only stationary traveling wave that may exist is of the pulse type. This is a stationary wave, which has a bell-like shape, that traverses across a space-time frame with a constant speed and shape. If it exists, for a speed of propagation as $\eta = 0$, then there will be an permanent spatially non-homogeneous asymptotic state, for our original system. □

Proof of Proposition 2. Pulses exist if two conditions are met: first, the steady state of system (18)-(21) is locally a saddle point, second, an homoclinic orbit, passing through $(\overline{y}_1, \overline{y}, \overline{w}_1, \overline{w})$ exists. In order to check if those conditions are met, we start by computing the Jacobian,

$$\frac{\partial(\dot{y}, \dot{y}_1, \dot{w}, \dot{w}_1)}{\partial(y, y_1, w, w_1)} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & \eta & -\overline{y}Af''(\overline{w}) & 0 \\ 0 & 0 & 0 & 1 \\ c'(\overline{y}) & 0 & -\delta & -\eta \end{bmatrix}.$$

Its characteristic polynomial is $c(\lambda) = \lambda^4 + (\delta - \eta^2)\lambda^2 - \eta\delta\lambda - D$, where the determinant, $\det = -D := -c'(\overline{y})\overline{y}Af''(\overline{w})$, is strictly positive. The eigenvalues are

$$\begin{aligned} \lambda_{1,2} &= \frac{1}{2}\left[\sqrt{\zeta_1} \pm \left(\sqrt{\zeta_2} + \sqrt{\zeta_3}\right)\right] \\ \lambda_{1,2} &= -\frac{1}{2}\left[\sqrt{\zeta_1} \pm \left(\sqrt{\zeta_2} - \sqrt{\zeta_3}\right)\right] \end{aligned}$$

where $\zeta_1 = s_1 + s_2 - \frac{2}{3}(\delta - \eta^2)$, $\zeta_{2,3} = -\frac{1}{2}(s_1 + s_2) - \frac{2}{3}(\delta - \eta^2) \pm \frac{\sqrt{3}}{2}(s_1 - s_2)i$, $s_{1,2} = \left(r \pm \sqrt{q^3 + r^2}\right)^{\frac{1}{3}}$, $r = \frac{1}{3}\left(\delta - \eta^2\right)^3 + \frac{4}{3}\left(\delta - \eta^2\right)D + \frac{\eta^2\delta^2}{2}$ and $q = -\frac{1}{9}\left[\left(\delta - \eta^2\right)^2 - 12D\right] < 0$

Therefore: (1) if $\eta = 0$, then the four eigenvalues will be complex with zero real parts; (2) if $\eta \neq 0$, then the fixed point is hyperbolic, for any values of the parameters, and the local stable manifold is two-dimensional. In the last case, the fixed point can be a saddle, or a saddle focus or a focus-focus, that is, we may have two real negative eigenvalues or two complex conjugate eigenvalues with negative real parts.

An homoclinic orbit is associated to a global bifurcation, and therefore is related to the structure of the non-linear part of system (18)-(21). However, there are necessary conditions on the linear part, for its the existence, that can be checked (see (Kuznetsov, 1998, chap. 6)): the saddle quantity should be different from zero, $\sigma^s \neq 0$. The saddle quantity is the sum of the real parts of the two leading eigenvalues (i.e., the sum of the real parts of the eigenvalues with positive and negative real parts with the smallest absolute value) evaluated at the equilibrium point.

If $\delta - \eta^2 > 0$ then there will be two pairs of complex eigenvalues, with one pair having negative real parts. This is so, because the discriminant $\Delta := q^3 + r^2$ is negative and some other conditions are met. Then the eigenvalues may be represented as $\lambda_{1,2} = \alpha_1 \pm \psi_1 i$ and $\lambda_{3,4} = \alpha_2 \pm \psi_2 i$, where $\alpha_1 < 0 < \alpha_2$ and $\psi_{1,2} > 0$. Then $\sigma^s = \alpha_1 + \alpha_2 = 0$, and a homoclinic will not exist.

If $\delta - \eta^2 < 0$, then the discriminant $\Delta := q^3 + r^2$ may have any sign. In this case, it is not possible to prove unambiguously that the saddle quantity is equal to zero. If it is different from zero, then an homoclinic orbit may exist.

However, if $\eta = 0$ then $\delta - \eta^2 = \delta > 0$ and $\alpha_1 = \alpha_2 = \sigma^s = 0$. Then a homoclinic bifurcation will not exist. □

Proof of Lemma 2. The jacobian matrix associated to equations (24)-(25) is

$$J(\upsilon) = \begin{bmatrix} \upsilon^2 & a_{12} \\ a_{21} & \delta - \upsilon^2 \end{bmatrix},$$

where $a_{12} := -\overline{q}Af''(\overline{k})$ and $a_{21} := -c'(\overline{q})$ are positive constants, has $\text{tr}(J) = \delta$ and $\det(J) = \upsilon^2(\delta - \upsilon^2) + D$, where $D := -a_{12}a_{21} < 0$. Though the determinant of the kinetic part, D is always negative, the determinant of J may have any sign. Therefore, the spectrum of matrix J, $\sigma(\upsilon; \delta, D) := \{\lambda : \lambda^2 - \delta\lambda + \upsilon^2(\delta - \upsilon^2) + D = 0, \;\; \upsilon \in \Upsilon, \delta > 0, \;\; D < 0\}$ is dependent not only on the parameters of the model but also on the frequencies υ. The eigenfunctions are

$$\lambda_{u,s} = \frac{\delta}{2} \pm \left[\left(\upsilon^2 - \frac{\delta}{2} \right)^2 - D \right]^{\frac{1}{2}}.$$

Then, for any value of $\upsilon \in \Upsilon$, the eigenfunctions are real, as $D < 0$, λ_u is positive and $\lambda_s + \lambda_u = \delta$. The eigenfunction λ_s may take positive or negative values, depending on the value of υ. When it is positive we say that there is a Turing instability. Turing instability is ruled out for the values of υ such that $\det(J) < 0$. Let us define the subset of Υ, $\Upsilon_s(\delta, D) := \{\upsilon : \ \det(J) = \upsilon^2(\delta - \upsilon^2) + D < 0, \}$. Then,

$$\Upsilon_s = \begin{cases} \mathbb{R} & \text{if} \quad \left(\frac{\delta}{2}\right)^2 + D < 0 \\ (-\infty, \upsilon_1) \cup (\upsilon_2, \upsilon_3) \cup (\upsilon_4, +\infty) & \text{if} \quad \left(\frac{\delta}{2}\right)^2 + D > 0, \end{cases}$$

where $\upsilon_{1,2,3,4} = \mp \left[\frac{\delta}{2} \pm \left[\left(\frac{\delta}{2}\right)^2 + D \right]^{\frac{1}{2}} \right]^{\frac{1}{2}}$. Therefore: if $\upsilon \in \Upsilon_s$ then $\lambda_s(\upsilon) < 0$ and if $\upsilon \in \Upsilon / \Upsilon_s$ then $\lambda_s(\upsilon) > 0$. In the second case the stable manifold is empty. □

Proof of Proposition 3. The solution of the system (24)-(25), along the stable manifold is, using well-known methods,

$$\begin{aligned} \mathcal{U}_q(\upsilon, t) &= \mathcal{H}_s(\upsilon)\mathcal{U}_k(\upsilon, 0)e^{\lambda_s(\upsilon)t}, \quad \upsilon \in \Upsilon_s, t > 0, \\ \mathcal{U}_k(\upsilon, t) &= \mathcal{U}_k(\upsilon, 0)e^{\lambda_s(\upsilon)t}, \quad \upsilon \in \Upsilon_s, t > 0, \end{aligned}$$

where we restrain the frequencies to Υ_s, in order to get $\lambda_s(\upsilon) < 0$, and the slope of the stable manifold is

$$\mathcal{H}_s(\upsilon) := -\frac{a_{12}}{\upsilon^2 - \lambda_s(\upsilon)} = -\frac{\delta - \lambda_s(\upsilon) - \upsilon^2}{a_{21}} < 0, \quad \upsilon \in \Upsilon_s.$$

If we observe that $\mathcal{U}_q$ depends upon $\mathcal{U}_k$, and apply the convolution theorem for Fourier transforms and the inverse Fourier transform, then we get the solutions of the linearized system, along the stable manifold, (27)-(28). □

Proof of Proposition 4. In this case the variational system, differently of (22)-(23), will become non-homogeneous and possibly non-autonomous, if the shocks are asymmetric,

$$\frac{\partial u_q(x,t)}{\partial t} = -\frac{\partial^2 u_q(x,t)}{\partial x^2} + a_{12}u_k(x,t) + b_{1,A}(x), \quad \forall t > 0$$

$$\frac{\partial u_k(x,t)}{\partial t} = \frac{\partial^2 u_k(x,t)}{\partial x^2} + a_{21} u_q(x,t) + \delta u_k(x,t) + b_{2,A}(x), \quad \forall t > 0$$

where the perturbation terms are $b_{1,A}(x) = -\overline{q} f'(\overline{k}) dA(x)$ and $b_{2,A}(x) = f(\overline{k}) dA(x)$. The Fourier transforms of the perturbation terms, $\mathcal{B}_{1,A}$ and $\mathcal{B}_{2,A}$, are given in equations (37) (38) and the spectral representation of planar PDE is the following non-homogeneous planar ODE

$$\begin{aligned} \frac{\partial \mathcal{U}_{q,A}(\upsilon,t)}{\partial t} &= \upsilon^2 \mathcal{U}_{q,A}(\upsilon,t) + a_{12} \mathcal{U}_{k,A}(\upsilon,t) + \mathcal{B}_{1,A}(\upsilon), \quad \forall t > 0 \\ \frac{\partial \mathcal{U}_{k,A}(\upsilon,t)}{\partial t} &= a_{21} \mathcal{U}_{q,A}(\upsilon,t) + (\delta - \upsilon^2) \mathcal{U}_{k,A}(\upsilon,t) + \mathcal{B}_{2,A}(\upsilon), \quad \forall t > 0 \end{aligned}$$

for $\mathcal{U}_{q,A}(\upsilon,0) = \mathcal{U}_{k,A}(\upsilon,0) = 0$, if $t = 0$, for any $\upsilon \in \Upsilon_s$. As the dynamic properties are the same as in system (24)-(25), the spectral representation of the stable manifold becomes

$$\mathcal{U}_{q,A}(\upsilon,t) = \overline{\mathcal{U}}_{q,A}(\upsilon) - \mathcal{H}_s(\upsilon) \overline{\mathcal{U}}_{k,A}(\upsilon) e^{\lambda_s(\upsilon)t}, \quad \upsilon \in \Upsilon_s, t > 0, \tag{39}$$

$$\mathcal{U}_{k,A}(\upsilon,t) = \overline{\mathcal{U}}_{k,A}(\upsilon)(1 - e^{\lambda_s(\upsilon)t}), \quad \upsilon \in \Upsilon_s, t > 0,, \tag{40}$$

where $\overline{\mathcal{U}}_{q,A}$ and $\overline{\mathcal{U}}_{q,A}$ are given in equations (35) and (36). Then, equations (31) and (32) are obtained as the solutions in the neighborhood of the stable manifold, after applying the inverse Fourier transforms to (39) and (40). □

Proof of Corollary 1. The perturbation terms are constant, $b_{1,A} = -\overline{q} f'(\overline{k})$ and $b_{2,A} = f(\overline{k})$. Their Fourier transforms are $\mathcal{B}_{1,A} = b_{1,A}\delta(\upsilon)$ and $\mathcal{B}_{2,A} = b_{1,A}\delta(\upsilon)$, where $\delta(\upsilon)$ is Dirac's delta function centered at $\upsilon = 0$. Then,

$$\begin{aligned} \overline{\mathcal{U}}_{q,A}(\upsilon) &= \frac{(\delta - \upsilon^2) b_{1,A}\delta(\upsilon) - a_{12} b_{2,A}\delta(\upsilon)}{\upsilon^2(\delta - \upsilon^2) + D} \\ \overline{\mathcal{U}}_{k,A}(\upsilon) &= -\frac{a_{21} b_{1,A}\delta(\upsilon) - \upsilon^2 b_{2,A}\delta(\upsilon)}{\upsilon^2(\delta - \upsilon^2) + D}. \end{aligned}$$

and the long run multipliers, $\overline{u}_{q,A}$ and $\overline{u}_{k,A}$ are their inverse Fourier transforms, for $\upsilon \in \Upsilon_s$. □

Proof of Corollary 2. The perturbation terms are $b_{1,A}\delta(x)$ and $b_{2,A}\delta(x)$, where $\delta(x)$ is Dirac's delta function centered at $x = 0$. Their Fourier transforms are independent of υ,

$\mathcal{B}_{1,A} = \frac{b_{1,A}}{2\pi}$ and $\mathcal{B}_{2,A} = \frac{b_{2,A}}{2\pi}$. Then

$$\begin{aligned}\overline{\mathcal{U}}_{q,A}(\upsilon) &= \frac{(\delta - \upsilon^2)b_{1,A} - a_{12}b_{2,A}}{2\pi[\upsilon^2(\delta - \upsilon^2) + D]} \\ \overline{\mathcal{U}}_{k,A}(\upsilon) &= -\frac{a_{21}b_{1,A} - \upsilon^2 b_{2,A}}{2\pi[\upsilon^2(\delta - \upsilon^2) + D]}.\end{aligned}$$

□