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Generalized Dynamic Systems

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GENERALIZED DYNAMIC SYSTEMS

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Introduction. The theory of generalized dynamic systems (GDS)

The evolution of dynamic a system with time suffers frequently structural changes in the sense that the laws that ruled the system during a certain time interval change to new laws that will rule the system for another time interval.

This change in the laws of the evolution can be considered either depending or not depending of the time

When the change depends on time the classical theory of the dynamic systems considers a non-autonomous system either continuous or discrete

$dx/dt = f(t, x(t))$ t real

$x_{t+1} = f(t, x_t)$ $t = 0, 1, ...$

As it is well known non-autonomous systems may be reduced to autonomous ones that is, systems that do not depend explicitly on the variable t.

However, when the change in the laws of the system is caused by other factors that not simply the flow of time, the classical theory of dynamic systems has no answers.

In case of structural change we may consider three types of situation.

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[1] UECE, ISEG

a) The evolution of the system after the structural change forgets the previous dynamics

b) The evolution of the system is directly but partially conditioned by the previous dynamics

c) The present dynamics is totally conditioned by the previous dynamics.

An example of the first situation happens when we have till moment T a path $x^*(t)$ that is the solution of a system $dx/dt = f(x)$ and from T on another path $x^{**}(t)$ that is the solution of another system $dx/dt = g(x)$, with no relation between functions g e f. The dynamics beginning at T forgets the previous dynamics and it is only the initial value of the new dynamics $x^{**}(T)$ that reflects a vague echo of that previous dynamics.

Let us now jump over situation b) and consider situation c). In this case the new dynamics from T on can be expressed as $x^{**}(t) = F(x^*(t))$ where $x^*(t)$ is the path, for $t > T$ of the previous solution of the system. That is the path that the system now takes depends directly and exclusively on the previous path $x^*(t)$ and on the values that this path would take (it is a virtual path) if it was extended beyond T.

Situation b) may be considered as the antecedent situation but where we have $x^{**}(t) = F(t, x^*(t))$ so co-existing a determination from the previous dynamics (represented by $x^*(t)$) and the present dynamics (represented by t).

The theory of generalized dynamics systems is the theory of the situations b) and c) Situation a) mat be dealt with the classical theory since is just a sequence of classical systems

1. Formalization of the GDS

To formalize adequately the evolution with time of systems such that their laws change we can use the following concepts.

Let [0 T] be a finite interval of time[2] including m+1 moments

$\{t_k\}$ k= 1,…m+1

such that $t_1 = 0$ e $t_{m+1} = T$.

Let us consider the sequence of m intervals $A_k = [t_k\ t_{k+1})$, k = 1,...m where A_m is also closed from the right

Let **C** be the set of functions x(t) defined and continuous on the interval

I = [0 T][3].

Let **F*** be the set of continuous maps of **C** into itself.

That is, $f \in$ **F*** f: **C**→**C**.

However not all the maps of **F***can be considered. It is necessary to restrict somewhat the field of study. To see this consider a time interval I and x(t) and y(t) defined on I, with y = f(x).

The existence of f does not imply that it exists a function g such for each t, y(t) = g(x(t)). However the reciprocal is true.

We call atomizable the functions of **F*** such that

y(t) = f(x(t)).

Formally

Let f : **C** → **C** and let **W** ⊂ **C**.

f is **atomizable** on **W** if and only if for each $x \in$ **W** and for each t* we have

$f(x)(t^*) = f(z_{x(t^*)})(t^*)$ for all the $z_{x(t^*)}$, where $z_{x(t^*)}$ is any function of **W** such that

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[2] In this section we consider finite intervals. Later on we will work with infinite intervals.

[3] Since the time intervals are finite we can work with all the continuous functions. For infinite intervals we have to restrict the analysis to bounded functions.

$z_{x(t^*)}(t^*) = x(t^*)$.

It is straightforward to see that the sum and the product of atomizable functions are atomizable

We have the following theorem.

TEOREM 1. Let $\{f_n\}$ be a sequence of atomizable functions on **W** converging uniformly on the set **W***, **W** $\subset$ **W***, to the function f (that is , $\forall \varepsilon > 0$, $\exists$ N such that for each $x \in$ **W*** we have for all the $n > N$ $\| f_n(x) - f(x)\| < \varepsilon$ and the value of N is not dependent on x; the norm involved is the supremum norm). Then f is atomizable on **W**.

Proof.

For each x and each z of **W** we have

$\sup_t|f(x)(t) - f(z_{x(t)})(t)| \leq (\sup_t | f(x)(t) - f_n(x)(t)|) + (\sup_t|f_n(x)(t) - f(z_{x(t)})(t)|)$

But as all the f_n are atomizable we have

$| f_n(x)(t) - f(z_{x(t)})(t)| = | f_n(z_{x(t)})(t) - f(z_{x(t)})(t)|$

Then if n tends to infinity and having in mind the uniform convergence of the sequence $\{f_n\}$ on **W**, we have necessarily $f(x)(t) = f(z_{x(t)})(t)$ for any t and for all the $z_{x(t)}$, as we had to prove. ***

From now on we consider only the set **F** of atomizable functions which is a subset of **F***.

The concept of GDS is now formalized in the following way:

For each k=1, … m e and each A_k we have

1) $x_k(t) = f_k [x_{k-1}(t)]$

where $f_k \in$ **F**, $x_k \in$ **C** and $x_k(t)$ and $x_{k-1}(t)$ are the values of the functions x_k e x_{k-1} respectively for each moment of the interval A_k.

Note that as was previously mentioned, we focus our attention on non-autonomous systems. However to simplify the notations we will not consider explicitly (unless there is the danger of confusion) the variable t in $f_k [x_{k-1} (t)]$.

A particular case of GDS is one where $f_k = f_{k-1}$ that is one for which the function f is always the same.

It may seem at first view hard to correlate the concept of GDS with the concepts usually applied both in natural and social sciences. We are used to consider that what happens in period t (in discrete time) results exclusively (apart from some stochastic factor) from the situation that existed in period t-1.But now what we present is a formulation where x_{kt} é determined by $x_{(k-1)t}$ and not by $x_{k(t-1)}$. What is the sense of this? The following interpretation may perhaps make easier the interpretation of what is really the matter.

2 Emerging situations, strong determinism and kinds of causality

To make interpretation easier we start by defining an **emerging situation** in a continuous system.

Let **K** be a complex at moment t composed by a set S_t of p elements and a set R_t of the relevant relations established at that moment between the p elements of S_t.

We assume the existence of a complex that origins a dynamic system, that is a time evolution of a variable x(t) that in a certain way characterizes the time evolution of the elements of the set S_t.

Let $A_k = [t_k, t_{k+1})$ be a time interval.

We say that the complex **K** presents a **homogeneous situation** during A_k if $R_t = R$ $\forall\ t \in A_k$, that is if the relevant relations between the elements of S do not change with time in that interval A_k.

The **emerging situation** of the complex **K** in moment t_k for the interval A_k is the pair $E_{tk} = (S_{tk}, R_{tk})$ when complex **K** presents a homogeneous situation during A_k. That is

$E_{tk} = (S_{tk}, R_{tk}) = (S_{tk} , R)$.

We these definitions we can proceed to the strong determinism hypothesis

2.1 Strong determinism

Let A_k be a time interval and let x_k (t) be the variable that characterizes the state of the system for each moment t of A_k. Assume also that the complex which originates the system presents a homogeneous situation during A_k. The following is the first version of the hypothesis of strong determinism.

Strong determinism (first version). For each emerging E_{tk} at moment t_k corresponds one and only one path x_k for each moment t of interval A_k and reciprocally..

The acceptability of this hypothesis depends on the way that the emerging situation is defined and on the way the path is characterized for each time interval .On the other hand the correspondence of the definition can be split in two correspondences.

Let us start by the correspondence that goes from the path to the emerging situation. If we consider a sufficient number of characteristics of the evolution during A_k, then we can assume that to each path in time interval A_k corresponds one and only one initial emerging situation. Indeed , suppose that we characterize that path in the interval A_k using a certain set of characteristics. If we verify that using those characteristics for characterizing the evolution in A_k, two different emerging situations could have generated the path evolution in A_k then we can add more characteristics in order to differentiate the paths of those two emerging situations..

If this reasoning is valid we have an important methodological consequence which is the following : the fact that we find possible that a certain path in a time interval could have been originated in two different initial situations is only a mere consequence of the fact that the characterization of the evolution is incomplete because of an excessive abstraction...

Let us now consider the inverse correspondence. If we admit the Leibnizian principle of sufficient reason (1983, pag 211) that nothing happens without a sufficient reason why it happens then to each emerging situation corresponds one and only one path.

From the two correspondences follows the hypothesis of strong determinism and therefore a one to one function $x_k = G(E_{tk})$.

Pushing the argument further we can make stronger the hypothesis and enunciate the second version of the hypothesis

Strong determinism (second version). To each emerging situation E_{tk} corresponds a path x_k and reciprocally and for two different emerging situations that originate respectively two paths x_k e x^*_k,we have

$x_k(t) \neq x^*_k(t)$ for all the t of A_k.

It is clear that this stronger hypothesis has less applicability then the weaker one. However it may be useful for studying some particular systems.

2.2 Causality of the second kind

A second hypothesis apart from strong determinism is that in the space of emerging situations there exist functions h_k of the space into itself such that

$E_{tk} = h_k(E_{tk-1})$,

That is there is a deterministic evolution of emerging situations. This means that there are two kinds of causality. One, the causality of the first kind determines the path followed by the system from a given emerging situation, The other, causality of the second kind determines an emerging situation from previous emerging situations.

With these two hypothesis (strong determinism-first version and causality of the second kind) we can write

$E_{tk-1} = G^{-1} (x_{k-1})$

On time interval A_k, the path followed by the system will be

$x_k = G.h_k.G^{-1}(x_{k-1}) = f_k(x_{k-1})$[4]

which is the formulation of the GDS that we introduced above.

It is easy to see that if the hypothesis of strong determinism (second version) applies function f_k is atomizable.

Indeed let $x_{k-1}(t)$ for any t in the time interval. Then by strong determinism (second version) there is only one possible path x_{k-1} that has the value $x_{k-1}(t)$ at moment t. Therefore there exists only one E_{tk-1} corresponding to the value $x_{k-1}(t)$ and so by the second kind causality and again by strong determinism there is only one x_k that corresponds to $x_{k-1}(t)$ and therefore only one $x_k(t)$.

However some additional comments are needed.

2.3 Strong determinism and causality of the second kind

The vision of reality that is implicit in these two assumptions is that the behavior of a system with time is nothing more that the generation of emerging situations coupled with the evolution of these of situations

We define the **developmen**t of the path of a system the progressive actualization of the path x(t) starting from the emerging situation E_{tk}

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[4] C Of course x_{k-1} (t) at each t of A_k is a virtual path that is, is the path that the system would follow if it were not interrupted by the emerging of a new situation E_{tk}

We call **evolution** of the system the process that leads to a new emergent situation E_{t-1k} from the previous emerging situation E_{tk}.

For the systems for which there is no intervention of conscious human action it is usually difficult to find functions h_k that describe the evolution of emergent situations. When we deal with human process for instance when the evolution of the system is the change of a computer program for another one it is easier to know the functions h_k

However even in the special case of conscious intervention it is in general difficult to characterize completely an emergent situation. That is why our formulation go the GDS which is implicitly based on the evolution of emergent situations is useful since it eliminates the need to consider explicitly emergent situations, albeit at a price which is that we have to admit the two assumptions of strong determinism and second kind causality.

A final aspect needs to be clarified

When we write $x_k(t) = f_k [x_{k-1}(t)]$ for $t \in A_k$, this may cause some difficulty in accepting the interpretation described above since we are considering a virtual and not actual path $x_{k-1}(t)$. However a simple example may help to see that virtual paths do not introduce undue complications.

Consider a system that follows the path $x_{k-1}(t)$ till moment t_k and such that we know that after that moment and caused by a new emergent situation E_{tk} the rate of growth of x will be one half of the rate of growth that would be the case if there was no evolution of the system, that is if there was no change from E_{tk-1} to E_{tk} . In this case we can obtain for all the moments of A_k the new path $x_k(t)$ using as reference a virtual path $x_{k-1}(t)$ that will never be actualized.

After this clarification is now time to proceed to the analysis of the GDS

3. The case $f_k = f_{k-1}$

In this case – the simpler one - the map f is the same for any moment of time
We can write

2) $x_k(t) = f[x_{k-1}(t)]$ $t \in A_k$ $x_0 \in \mathbf{C}$ for a given x_0

We start by defining solution of 2)

DEFINIÇÃO ***Solution*** of system 2) is a finite sequence $S = \{x^*_k\}$ of functions $x^*_k \in \mathbf{C}$ such that for all the $t \in A_k$

$x^*_k(t) = f[x^*_{k-1}(t)]$ given the values of $x^*_0(t)$ for all the t of A_k.

As it easily seen the solutions of a system are infinite in number. That is why we need to define a more operational concept, that we call the *elementary case* and that is the only case that is analyzed in this paper and that corresponds to the strong determinism.

DEFINITION The **elementary case** of system 2) is the solution S such that $x^*_k(t) = f[x^*_{k-1}(t)]$ for all the t of the interval [0 T]

Example

Given the system

$x_k(t) = e^{\lambda t} x_{k-1}(t)^{\beta}$ $x_0(t) = e^{\theta t}$

we have the elementary case solution

$e^{\theta t}$, $e^{(\lambda+\theta\beta)t}$, ... , $\exp[\lambda(1-\beta^k)/(1-\beta) + \theta\beta^k]$, ... for each t of the interval [0 T].

The path that the solution follows is composed by the values of the functions respectively for

$t \in A_1$ $t \in A_2$, ... , $t \in A_k$, ...

We can now proceed and define the *stationarity* of a solutions

DEFINITION. A solution $S=\{x^*_k\}$ is **stationary** if and only if $x^*_k = x^*$ for all the k

In the previous example $x(t) = \exp[\lambda t/(1-\beta)]$ are the values of a stationary solution provided that $\lambda/(1-\beta) = \theta$

Another important definition applies to some stationary solutions

DEFINITION. **Stability** of a stationary solution. Let $S =\{x^*\}$ be a stationary solution of a system 2). Then S is stable if and only if

$\forall\varepsilon> 0, \exists\ \delta(\varepsilon)> 0$ such that for any other solution $S'= \{y^*_k\}$ starting at y^* with $\| y^*- x^*\|<\delta$ we have $\| y^*_k - x^*\|< \varepsilon$ for all the k.

The norm is the supremum norm defined for all the space of functions **C** that is $\| x^*\| = \sup|x^*(t)|\ t \in[0\ T]$.

This definition introduces a **strong concept of stability** since it demands the proximity of y^*_k e x^* based on the norm calculated for all the values of t.

We can also define a weak stability.

Let $\|x^*\|_{Ak}$ be the restriction of the norm to the values of the supremum of the values $|x^*(t)|$ in the interval A_k.

DEFINITION. **Weak stability** of a stationary solution. In the same conditions of the previous definition S is stable if and only if

$\forall\varepsilon> 0\ \exists\delta\ (\varepsilon) > 0$ such that with com $\|y^* - x^*\|_{A1} < \delta$ we have

$\|y^*_k - x^*\|_{Ak} < \varepsilon$.

In what follows however the concept used is the strong one.

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To be useful this concept needs some more elaboration. We have the following theorem

THEOREM 2. Consider system 2) above and let f be Fréchet differentiable being Df the derivative. Let S= {x*} be a stable solution of 2) and let {y^*_k} , y^*_1 = y* be another solution.. Let, W be the set of the union of all the lines $\mathbf{R}_k$ = {z: z = x*+p(y^*_k - x*)} 0≤p≤1, k=0,…m .

Then if ||Df(z)|| ≤ 1 for all the z of W, the solution is stable.

Before proving the theorem some remarks are in order:

a) The functional space **C** of functions x(t) is a Banach space when we use the supremum norm ||x|| = sup|x(t)| t∈[0 T]

b)The Fréchet derivative is defined for all the maps of a Banach space into another Banach space so that we can use the Fréchet derivative for a map f of **C** into itself. f is Fréchet differentiable at x' if it exists the linear map Df(x) : **C** → **C** such that

$\lim_{x\to x'}$ ||f(x) - f(x') - Df(x-x')|| / || x-x'|| = 0

Df(x') is the Fréchet derivative of de f at x'[5].

we can now prove the theorem

Proof.

It is based on the theorem [6] that shows that if x e y ∈ **C** and if ||Df(z)|| ≤ M for all the z∈W we have || f(y) - f(x) || ≤ M|| y - x || . Then we may conclude that given the stationary solution S = {x*(t)} , and the solution S' ={y^*_k(t)}, and putting M=1, we have for all the k,

|| f(y^*_k) - x*|| ≤ || y^*_k - x*|| ,

since f(x*)=x* .

Proceeding successively for all k decreasingly till k=1 we have

|| y^*_k - x* || ≤ || y*-x*|| for all the k.

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[5] We consider only atomizable functions that have Fréchet derivatives atomizable.

then for any $\varepsilon > 0$, it is only necessary to choose $\delta < \varepsilon$ to obtain the condition for stability***

Example 1

Consider the system

$x_k(t)=(1-1/ax_1(0))\, e^{-mt}\, x_{k-1}(t) + 1/a$

on a given interval [0 T], the union of intervals A_k.

It is easy to see that $x^*(t) = 1/[a-(a-1/x_1(0))e^{-mt}]$ is a stationary solution (it is the logistic curve) if the initial condition is

$x_1(t) = 1/[a-(a-1/x_1(0))e^{-mt}]$.

Let us confirm that it is a stable solution when $ax_1(0) > 1$.

It is easy to see that for every z(t),

$Df(z) = Df =(1-1/ax_1(0))e^{-mt}(.)$, since f is the sum of a linear operator with a constant (that is with a null derivative). So that Df does not vary with z. Being Df a continuous and linear operator we have, by definition of a norm of such an operator

$||Df|| = \sup ||Df(z)||$ for all the z such that $||z|| = 1$. As we have for every z

$||Df(z)|| = ||(1-1/ax_1(0))e^{-mt}(z)|| \leq ||(1-1/ax_1(0))e^{-mt}||.|| z ||$

we get $|| Df || \leq || (1-1/ax_1(0)e^{-mt}||$ since the norm is calculated for all the z such that

$||z|| = 1$. As $||e^{-mt}|| = 1$ (the supremum norm in [0 T]), we get

$||Df|| \leq |1- 1/ax_1(0)|$ and it is sufficient that $ax_1(0) > 1$ to have the condition of stability verified

if $ax_1(0) > 1$ then $||Df|| < 1$.

Example 2

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[6] Jost (1998) pag 103.

Consider now the more general case $f \equiv L(.) + a$

L is a limited linear transformation and a(t) is given, so that it is a constant in the space of the functions of t.

If $||L|| < 1$, then $(I - L)^{-1}$ exists and the stationary solution is given by

$x^* (t) = (I - L)^{-1} a(t)$.

Since L is linear, the Fréchet derivative is DL = L. As the norm of L is less than one, the stationary solution is stable.

It is also easy to see that in this case we have

$x_{k+1} = L^k x_1 + (I - L^k) (I - L)^{-1} a$

and

$x_{k+1} - x^* = L^k (x_1 - x^*)$,

so that

$|| x_{k+1} - x^*|| \leq || L^k|| .||x_1 - x^*|| \leq ||L||^k.|| x_1 - x^*||$.

In the Appendix is described another example, an application to economic growth theory.

4. A different representation of the GDS for the discrete case. A measure of variability

The systems analyzed so far are written in the form

$x_k (t) = f[x_{k-1} (t)]$ with x_1 given by $x_1(t) = f[x_0(t)]$ $t \in A_1$

Then for ever k = 1,...we have the paths

$x_2(t)=f[x_1(t)]$ $t\in A_2$

$x_3 (t)= f[x_2(t)]= f^2[x_1(t)]$ $t\in A_3$

$x_k(t) = f^{k-1}[x_1(t)]\ t \in A_k$

Let us consider the discrete case where the t are integers in the intervals A_k.

If $x_1(t)$ is also the solution of a simple discrete dynamic system that is if $x_1(t)= g^t(x_1(0))$, we can write

$x_k(t)= f^{k-1}[g^t(x_1(0))]$ where $t \in [t_k\ t_{k+1})$ and k goes from 1 to m.

This means that each actual value of the variable x(t) is defined by the ordered pair (k t).

Let $\mathbf{P} \subset N^2$ be the set of all the ordered pairs (k t), k from 1 to m, t from $t_1 = 0$ to $t_m = T-1$ that correspond to values of $x_k(t)$. We can define a square matrix A, of T rows /columns, such that its elements are :

$a_{ij} = 0$ se $(i\ j) \notin \mathbf{P}$

$a_{ij} = 1$ se $(i\ j) \in \mathbf{P}$

Obviously matrix A is a stochastic matrix in terms of columns since each column has only one non-null element and that element is equal to one. We call A the **characteristic matrix** of the solution of the system.

When $x_k = x_{k-1}$, that is, when we have a path corresponding to a stationary path of the system $x_k = f[x_{k-1}(t)]$ but not necessarily of the system $x_1(t)=g^t[x_1(0)]$, the characteristic matrix has the first line all of unitary elements and all the other elements are null..

For this case the variability of the solution of the system is minimum and reduces just to the variation associated with the simple dynamic system $x_1(t)=g^t[x_1(0)]$.

On the other hand if as time flows the solution is such that x_k changes from a period t to the next period then the variability is at its maximum and the matrix A is the identity matrix.

Based on these examples we can define a measure of the variability of the solution of a the system.

Let

$b_i \equiv \sum^{T}_{j=1} a_{ij}$

and define vector c de with components

$c_i = b_i/T$.

Obviously, $\sum_i c_i = 1$

We define the **degree of variability** of the solution S of the system, G(S) as

$G(S) = - \sum c_i \log c_i$

As we can easily see G is formally equivalent to the entropy of a system T of states i such that the probability of occurrence of each state i is c_i.

As it is well known from information theory the entropy has its maximum value G=log T when all the c_i are identical and has its minimum value G=0, that is when one of the c_i is 1 and all the others 0, which corresponds respectively to the cases of maximum and minimum variability of the system as we have seen above

We can now leave the case $f_k = f_{k-1}$ and proceed to the more complex case

5. The case $f_k \neq f_{k-1}$

We have studied so far the case where all the changes in the structure of the system occur as time flows but always in an identical manner However we can study now a more general case where we have

$f_k = g(f_{k-1})$

where g is a map into itself of the space **F** of the maps of **C** into itself.

However In order to make progress in the analysis we have to restrict the space **F**, and that is why we consider only the space of linear maps.

It is well known[7] that the space **L** $\subset$ **F** of the linear mappings of **C** into itself is a Banach space equipped with the norm

$||L|| = \sup \{ ||L(x)||: ||x|| = 1 \}$ where $x \in$ **C** and the norm L(x) and x is the norm of the supremum that we used for the $x \in$ **C**.

Choosing this norm, **L** is a Banach space and we can use all the theorems that apply to Banach spaces.

It is easy to define the concept of a solution $f_k = g(f_{k-1})$. It is the series $\{f^*_k\}$, $k = 1,\ldots m$ of maps f^*_k such that $f^*_k = g(f^*_{k-1})$and $f^*_0 = f^*$.

Let us start by defining the stationary point of type I

DEFINITION **Stationary point of type I** [8]is the map f* such that

$f^* = g(f^*)$

All the analysis we made in the above sections has to do precisely with a map f that may be considered as a stationary point of the system

3) $f_k = g(f_{k-1})$

In the same way we can define the stability of a stationary point

DEFINITION $f^* \in$ **L** be a stationary point of $f_k = g(f_{k-1})$, f* is **stable** if and only if $\forall \varepsilon > 0\ \exists \delta(\varepsilon) > 0$ such that for any other sequence $\{f^{**}_k\}$ of solutions of 3) beginning at f** and with $||f^{**} - f^*|| < \delta$ we have $||f^{**}_k - f^*|| < \varepsilon$ $k = 1,\ldots m$ (the involved norms are of course those of the space **L**)

It is easy to prove as was done for the previous case, that if g is Fréchet differentiable and if $||Dg(f)|| \leq 1$ for all the $f \in$ **L** then the solution $f_k = f^*$ is stable.

We can now give an example that is a generalization of the logistic curve

[7] Céa (1971).

[8] We designate by Stationarity II the possible stationarity of the functions x(t).

Example1

Consider the system

$x_k(t) = b_k(1-1/ax_1(0))e^{-mt}x_{k-1}(t)+c_k/a$

with b_k and c_k real numbers such that

$b_k=\alpha b_{k-1} + \beta$

$c_k=\alpha c_{k-1}+s$

We have then

$x_k(t)=\alpha b_{k-1}(1-1/ax_1(0))e^{-mt}x_{k-1}(t) + \alpha c_{k-1}/a + \beta(1-1/ax_1(0))e^{-mt}x_{k-1}(t)+s/a$

that is

$x_k(t)=\alpha f_{k-1} [x_{k-1}(t)] +\varphi[x_{k-1}(t)]$

where

$\varphi \equiv \beta(1-1/ax_1(0))e^{-mt} (.)+s/a$

does not depend of any f belonging to **L,** so that it may be considered a constant in this space

We have obviously

$f_k =\alpha f_{k-1} + \varphi$

$||Dg(f)|| = |\alpha|$

so that the system will be stable I if $|\alpha| \leq 1$

The stationary solution is

$f^*=\varphi /(1-\alpha)= \beta(1-1/ax_1(0))e^{-mt} (.)/(1- \alpha) + s/a(1-\alpha)$

and we have

$b^*=\beta/(1-\alpha)$ e $c^*=s/(1-\alpha)$

so that

$f^*=b^*(1-1/ax_1(0))e^{-mt}(.) + c^*/a$

We can now proceed to stationarity II that is stationarity relative to the functions x(t).

Applying f to the function $x(t)\in\mathbf{C}$ we have

$x_k(t)=b^*(1-1/ax_1(0))e^{-mt}x_{k-1}(t)+c^*/a$

The stationary solution is

$x^*(t)= c^*/a[1-b^*(1-1/ax_1(0))e^{-mt}]$

which is stable if $|b^*(1-1/ax_1(0))|\leq 1$

The logistic case with the scale parameter c^*, is found when $b^* =1$, that is when $\alpha+\beta =1$.

Therefore the logistic curve can be interpreted as a solution that is double stationary of a generalized system when $b^*=1$. It is a stable solution when $ax_1(0) > 1$ and $|\alpha| \leq 1$.

Example 2

Consider now a more general case (we continue to assume as in the previous example that the moments k when changes of behavior of x_k are the same of the changes in f_k).

$$x_k = L_k x_{k-1} + a$$

a) $L_k = ML_{k-1} + B$

Where the L_k and B are bounded linear maps on the space of the functions of t and M is a bounded linear map in the (Banach) space $\mathbf{L}$ of the linear transformations on the space of the functions of t

Assuming $||M|| \leq c < 1$

we may write (as in the case $f_k = f_{k-1}$)

$L_k = M^{k-1} [L_1 - (I -M)^{-1}B] + (I - M)^{-1}B$

So that

$x_k =\{ M^{k-1}[L_1 - (I - M)^{-1}B] + (I-M)^{-1}B\} x_{k-1} + a \equiv N_k x_{k-1} + a$

A double stationary solution exists whenever

$||B|| < 1-c$.

Indeed, if M is a bounded linear transformation such that $||M|| \leq c < 1$ we have $|| (I - M)^{-1}|| \leq 1/(1-c)$ (Saaty, 1981 page34).

So that as the stationary solution of a) is $L^* = (I - M)^{-1}B$, we have

$||L^*|| \leq ||(I - M)^{-1}||.||B|| \leq ||B|| /(1-c)$ and since $L^* \equiv N$, $||N||$ will be lower than 1 if $||B|| < 1-c$.

Note that this condition implies obviously

$||M|| + ||B|| < 1$.

Note also that in this case of coincidence of moments k of changes in x_k and f_k, to obtain the double stationarity it is sufficient to impose conditions to the mappings M e B relative to f_k, being unnecessary to impose them at the level of x_k.

6. Cycles

The formulation of the GDS offers the possibility of defining a new concept of cycle that goes beyond the traditional definition.

According to the theory of the simple dynamic systems given a solution $x^*(t)$ there exists a s-cycle if $x^*(t+s) = x^*(t)$ for some s and for all the t.

For a GDS and for the case $x_k(t) = f[x_{k-1}(t)]$ where the solution is $x^*(t)$, there exists a s-cycle, with s integer if

$x^*_{k+s}(t) = x^*_k(t)$ for every k and for all the $t \in A_{k+s}$.

The recurrence therefore is not the recurrence of *values* that the solution takes at different moments t but it is a recurrence of *behaviors*. History does have similitude of eras but does not repeat itself. .

Consider the following example for a linear GDS

$x_k = L_k x_{k-1} + a$

where all the operators L_k have an inverse.

Suppose that $L_k = (L_{k-1})^{-1}$ e $La = -a$. We have the 2-cycle:

$x_k = L_2x_1 + a$

$x_{k+1} = x_1$ for every even k.

7. Structural changes

So far we have studied GDS supposing that at some moments t_k some structural change happen. However we have not studied how these changes happen.

The task of this section is too study this question. However and contrary to the previous analysis we consider now an infinite interval that is the interval $[0 +\infty)$, which means that the sequences of intervals A_k may be infinite

There are two possibilities of emergence of structural changes: stochastic and deterministic. Let us begin by the stochastic ones.

7.1 Stochastic structural changes

We limit ourselves to the case where x(t) is discrete and as always to what we previously called the elementary case.

Let p be the probability of the occurrence of a structural change at each moment t (integer). We assume that this probability remains constant all the time, which means that it does not change in consequence of the possible existence of previous changes.

Let x*(t) be the solution of the system. Then the probability of x*(t) being equal to x_k for $t\in A_k = [t_k\, t_{k+1})$ that is of having happened k-1 changes before t_k, is given by

$$Q_{k-1tk} = ({}^{tk-1}{}_{k-1})\, p^{k-1}(1-p)^{tk-k}$$

And the expected value of x*(t) for each t integer is given by

$$E[x^*(t)] = \sum_{k=1}^{tk} Q_{k-1tk}\, f_{k-1}\, [x_1(t)]$$

This is of course a simple but perhaps useful model for the study of stochastic changes. However interesting as it is the stochastic case is not the more adequate one to explain structural changes, namely in what concerns both natural and social sciences. So let us now look at the deterministic case.

7.2 Deterministic structural changes .Structural jumps. Hypercomplex dynamic systems [9]

Consider a GDS corresponding to the elementary case. The solution is given by . $x^*(t) = f^k [x_0(t)]$ where t takes values on the interval $[0 + \infty)$ and x_1 (t) is continuous and bounded and f a map f : **Cl** →**Cl** in the space **Cl** of the bounded and continuous functions of t.

DEFINITION A **structural jump** at moment t with a time-lag $h > 0$ exists if and only if

$x^* (t+j) = f^k [x_0 (t+j)]$ for $0 \leq j < h$

$x^* (t+h) = f^{k+1} [x_0 (t+h)]$

There is a structural jump with time lag $h = 0$ when there is a $\varepsilon > 0$ and a $\delta \geq 0$ such that

$x^* (t-\varepsilon) = f^k [x_0 (t-\varepsilon)]$ e $x^*(t+\delta) = f^{k+1}[x_0(t+\delta)]$

REMARK. We assume that if there is a structural jump at t with time lag $h > 0$ no more structural jumps will exist before t+h

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[9] The term hypercomplex system was used by Professor Almeida Costa sixty years ago when studying algebraic matters that have nothing in common with the present study. To avoid confusion we call the models hypercomplex *dynamic* systems.

In this section we study a simple case of structural jump that may be used to explain several evolutions in social and natural sciences

Suppose that there is a function y(t) such that whenever x*(t) reaches the value y(t) a structural jump of time lag $h \geq 0$ happens.

We can think for instance of a society where the intensity of conflicts reaches such a point that the social agents change their behaviors. Or of an individual subject to such a psychological tension that he changes his regular behavior and so one.

DEFINITION A system is **structurally stable** if and only if

$\exists$ T real and R integer and positive such that

$x^* (t) = f^R[x_0 (t)]$ for all the t > T

That is after a certain moment T there are no more structural jumps[10].

Consider the following example :

The GDS is given by

$x_k (t) = e^{-.08t} x_{k-1}(t) + e^{0.05t} \qquad x_0(t) = e^{.08t}$

and suppose that there is a structural jump of lag 0 whenever x*(t) = y(t) with

$y(t) = -0.0253t^2 + 0.6525t - 1.108$

We obtain the evolution $x^*(t) = x_0(t) = e^{0.08t}$ for $4.9 > t \geq 0$

$x^*(t) = 1+e^{0.05t}$ for $8 > t \geq 4.9$

$x^*(t) = e^{-008t} + e^{-0.03t} + e^{0.05t}$ for $14.5 > t \geq 8$

$x^* (t) = e^{-0.16t} + e^{-0.11t} + e^{-0.03t} + e^{0.05t}$ for all the t such that $t \geq 14.5$

[10] This concept is used by Prigogine et al (1984 pag 189) with a meaning that is quite similar to our own., since it means for those authors that a system is able to maintain unchanged as time goes by its laws of functioning

The system is structurally stable since there is a value of t (t =14.5) such that for posterior moments there are no more structural jumps.

For this case, that is the case where there is a structural jump whenever $x^*(t) = y(t)$ we have the following theorem

THEOREM 3. If the sequence of maps $f^m(x)$ converges uniformly to a certain map $a(x)$ when m increases and for all the $x \in W \subset \mathbf{CI}$, where $a(x)$ is such that

$\inf_t |a[x(t)] - y(t)| > 0$, for $t > T^*$

then the system is structurally stable when the initial path x_0 belongs to W.

Proof..

Due to the uniform convergence of the sequence f^m we have for each function $x(t)$, with $x \in W$

$\forall \varepsilon > 0 \ \exists \ n \in N$ such that $\forall m \geq n$ we have $\|f^m(x) - a(x)\| < \varepsilon$

That is we have for $m \geq n$

$\sup_t | f^m(x)-a(x)| < \varepsilon$

Then for all the t

$|f^m[x(t)]-a[x(t)]| < \varepsilon$

Let $\delta = \inf_t|a[x(t)]- y(t)|$

Then for all the o $t > T^*$ $|a[x(t)]-y(t)| \geq \delta$ holds

Taking $\varepsilon = \delta/2$, there exists a certain n^* such hat for $m \geq n^*$ we have for all the t t

$|f^m[x(t)]- a[x(t)]| < \delta/2$. Then combining with condition

$|a[x(t)]- y(t)| \geq \delta$

We have for all the x de **W**, aa the $t > T^*$ and all the $m \geq n^*$

$|f^m[x(t)] - y(t)| > \delta/2$ and no more structural jumps will happen after max (t_{n*}, T*) where t_{n*} is the left limit of interval A_{n*}.***

An issue that may be of great importance for the analysis of the GDS, namely in the case that we are considering is the question of knowing how the solution x* (t) behaves when t goes to infinity.

For the analysis of this issue the following definition may prove helpful.

DEFINITION Let f^m be a sequence of maps of **CI** into itself. Then f^m **converges uniformly (weakly)** f of **CI** on W if and only if for every x of W

$\forall\varepsilon>0\ \exists$ n* integer such that sup $|f^m[x(t)]-f[x(t)]| < \varepsilon$ for all the m> n*

and for $t\in[t_{n*}, +\infty)$.

A sequence that is uniformly convergent is uniformly convergent in the weak sense .

Another useful definition is the following

DEFINITION The map f is **continuous in the strong sense** in x if and only if $\lim x(t)_{t\to\infty}$ exists, is finite and

$\lim f[t, x(t)]_{t\to\infty} = \lim_{t\to\infty} f[t, \lim x(t)_{t\to\infty}]$.[11]

Example

The map $f \equiv (t+1).x$ is not continuous in the strong sense for $x(t) = 1/(t+1)$ but it is continuous in the strong sense for $x(t) = 1/(t+1)^2$.

It is easy to see, that for $f \equiv m(t).x$ with a given m(t) and with bounded functions x the functions x for which f is not continuous in the strong sense are those such that $\lim x(t)_{t\to\infty} = 0$ and $\lim m(t)x(t)_{t\to\infty} \neq 0$. In particular, if m(t) is bounded, f is continuous in the strong sense for all the functions x bounded.

With this definition we have the following theorem

THEOREM 4. Let f^m be a sequence of maps converging uniformly in the weak sense to f^∞ em W. Then, if f^∞ is strongly continuous at $x_0 \in$ W, with $\lim x_0(t)_{t\to\infty} = r$, we have

$\lim x^*(t)_{t\to\infty} = \lim f^\infty [t, r]_{t\to\infty}$.

Proof..

By the definition of uniform convergence in the weak sense

$\forall\varepsilon > 0\ \exists n^*$ such that for $m \geq n^*$ we have

$|f^m[t, x_0(t)]- f^\infty[t, x_0(t)]| < \varepsilon/2$ for each $t \geq n^*$

On the other hand since f^∞ is continuous in the strong sense at x_0 (t),

$\forall\varepsilon/2\ \exists$ T such that for $t \geq T$ we have $|f^\infty[t, x_0(t)] - f^\infty[t, r]| < \varepsilon/2$

so that for $t \geq \max (t_{n^*}, T)$ we have

$|f^m[t, x_0(t)] - f^\infty[t, x_0(t)]| + |f^\infty[t, x_0(t)] - f^\infty[t, r]| < \varepsilon$

therefore $|f^m[t, x_0(t)]- f^\infty[t, r]| < \varepsilon$

That is with $x^* (t) = f^m [t, x_0(t)]$ for $m \geq n^*$and $t\in A_m$,

we have that for $t > \max (t_{n^*}, T)$

$|x^*(t) - f^\infty[t, r]| < \varepsilon$ so that $\lim x^*(t)_{t\to\infty} = \lim f^\infty[t, r]_{t\to\infty}$ as we had to prove .***

This theorem is important because it allows us to study what happens to the asymptotic evolution of the system following a small variation in the initial function $x_0(t)$.

Other important theorems for the study of the trajectory if the solution can be found. One of such theorem is the following that provides a sufficient condition for the continuity of the functions $x_k(t)$.

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[11] As there is some possibility of confusion and differently from what we have been writing till now we used the symbol f[t, x(t)] to emphasize that we are considering non autonomous systems.

THEOREM 5.Suppose a non-autonomous GDS. Let x_{k-1} be a continuous function and $x_{k-1}(a)$ the value of x_{k-1} for t = a. Let $x_{k-1}(a)^*$ be the constant function that for each t has the value $x_{k-1}(a)$.

Then if f [t, x(t)] is continuous on the space of functions x(t) defined for every interval (t_1, t_2) and if $f[t, x_{k-1}(a)^*]$ is continuous as function of t at the point t = a, then $x_k(t) = f[t, x_{k-1}(t)]$ is continuous as a function of t at the point t=a.

Proof

As $f[t,x_{k-1}(a)^*]$ is continuous we have

a) $\forall \delta > 0 \; \exists \varepsilon > 0$ such that $|t - a| < \varepsilon \Rightarrow |f[t, x_{k-1}(a)^*] - f[a, x_{k-1}(a)^*]| < \delta/2$

On the other hand since f is continuous on the space of the functions we have

b) $\forall \delta > 0 \; \exists \varepsilon^* > 0$ such that

$||x_{k-1} - x_{k-1}(a)^*|| < \varepsilon^* \Rightarrow || f[t, x_{k-1}(t)] - f[t, x_{k-1}(a)^*]|| < \delta/2$, where the supremum norm may be calculated for each interval $(t_1\ t_2)$.

As $x_{k-1}(t)$ is assumed continuous and as the norm is a supremum norm we can write for every $\varepsilon' < \varepsilon^*$

c) $\exists \varepsilon^{**}(\varepsilon')$ such that

$|t - a| < \varepsilon^{**} \Rightarrow |x_{k-1}(t) - x_{k-1}(a)| < \varepsilon' \Rightarrow ||x_{k-1} - x_{k-1}(a)^*|| < \varepsilon^*$, where the supremum of the norm is calculated in the neighborhood ε^{**} of a.

And so using a), b) e c) and noting that $x_k(t) \equiv f[t, x_{k-1}(t)]$ and

$x_k(a) \equiv f[a, x_{k-1}(a)^*]$ we have

$\forall \delta > 0 \; \exists \varepsilon^{***} = \min(\varepsilon, \varepsilon^{**})$ such that $|t - a| < \varepsilon^{***} \Rightarrow |x_k(t) - x_k(a)| < \delta$, as we had to prove.***

COROLLARY . If the GDS is autonomous the continuity of the mapping f and of $x_{k-1}(t)$ as a function of t are sufficient for obtaining the continuity of $x_k(t)$. (The proof is straightforward).

Another important formula has to do with the derivative $x_k(t)$ that we designate by $x'_k(t)$.

THEOREM 6.

We have for t=a

$$x'_k(a) = (df[t, x_{k-1}(a)^*]/dt)_a + (Df(x_{k-1}(a)^*)\bullet x'_{k-1}(t)^*)_a$$

Were the second summand of the right side represents the value for t =a of the function of t that is obtained applying the Fréchet derivative, calculated at $x_{k-1}(a)^*$, to the derivative of $x_{k-1}(t)$.

Proof

We have

$$x_k(a+h) - x_k(a) = f[a+h, x_{k-1}(a+h)^*] - f[a, x_{k-1}(a)^*] =$$

$$= f[a+h, x_{k-1}(a+h)^*] - f[a+h, x_{k-1}(a)^*] + f[a+h, x_{k-1}(a)^*] - f[a, x_{k-1}(a)^*].$$

On the other hand using a property resulting from the definition of the Fréchet derivative[12] and using as always the symbol "*"to designate the respective constant function we have ,

$$f[a+h, x_{k-1}(a+h)^*] = f[a+h, x_{k-1}(a)^* + x'_{k-1}(a)h^* + \upsilon(h)^*] =$$

$$= f[a+h, x_{k-1}(a)^*] + [Df(x_{k-1}(a)^*)\bullet[(x'_{k-1}(a)h]^*+\upsilon(h)^*)] +$$

$$+ \varphi(x'_{k-1}(a)h^*+\upsilon(h)^*)$$

where $\lim \upsilon(h)/h = 0$ when $h\to 0$ and $\lim \varphi /\|x'_{k-1}(a)h^*+\upsilon(h)^*\|=0$ when $h\to 0$.

so that $x_k(a+h) - x_k(a) = [Df(x_{k-1}(a)^*)\bullet[(x'_{k-1}(a)h]^* +\upsilon(h)^*)] + f[a+h, x_{k-1}(a)^*] -$

[12] $f(u+v) = f(u) + Df(u)\bullet v + \|v\|\varepsilon (u,v)$ where $\lim \varepsilon(u,v) = 0$ when $v\to 0$.

- $f[a, x_{k-1}(a)^*] + \varphi$

Dividing both sides by h, multiplying and dividing φ by $\|x'_{k-1}(a)h+\upsilon(h)\|$, calculating the limit as $h\to 0$ and recalling the linearity of $Df(x_{k-1}(a)^*)$ and the continuity of the functions involved we get immediately what we wanted to prove.***

COROLLARY. If the GDS is autonomous we have

$x'_k(a) = (Df(x_{k-1}(a)^*) \bullet x'_{k-1}(t))_a$.

Proof

It follows directly from the fact that the first summand of the previous formula is null.***.

These results show that it is possible to study as a GDS a complex system that suffer endogenous structural changes.

We call **Hypercomplex Dynamic System** a system where structural changes are at least partially a consequence of the working of the system and not totally due to exogenous causes. Hypercomplex systems may therefore be analyzed with the help of the theory of GDS.

Conclusion

As we have seen GDS are useful for studying hypercomplex systems and also for defining and studying cycles of a new type where the recurrence is one of paths and not values of a variable.

It is also apparent that the analysis may be extended to the degrees that we want for instance to systems where g is itself variable with a map

g_k=h (g_{k-1}).

The interest of this generalization is however limited. We have to restrict ourselves to the linear case. On the other hand the actual meaning of the systems is rapidly lost.

There is a crucial problem that is not solved by the previous analysis. When we deal with time series how can we distinguish in statistical terms an evolution that may be described by a simple dynamic system from an evolution that is the result of structural jumps?. The development of statistical tests to distinguish this two situations is of course crucial for the empirical applications of the theory of GDS .

APPENDIX: An Example: the Harrod-Domar model

Consider the moment t =1 at the beginning of period 1.

At this moment there is a given capital of knowledge (D_1) in a given economy. This capital permits that the production in period 1 is obtained by the production function

$K_1 = M(D_1)Y_1$

where K_1 is the physical capital existent at moment 1, Y_1 the GDP produced in period 1 and $M(D_1)$ the capital /output ratio in that period..

If we assume as in the Harrod-Domar model that $\Delta K_t = sY_t$ we have

$\Delta Y_t / Y_{t-1} = s/M(D_1)$

Now suppose that at moment T, at the beginning of period T, there is a structural jump that is an increase of the capital of knowledge from D_1 to D_2.

Then the new capital output ratio is $M(D_2)$ and the behavior of GDP after T is given by

$x_t \equiv \Delta Y_t/Y_{t-1} = s/M(D_2)$

Suppose that the knowledge capital increases as follows. After five years there is an increase of the coefficient M(D) at the same rate i, so that for each moment k k =1, 2, ... where a jump happens $M(D_k) = M(D_{k-1})(1+i)$

Then for t = k, k+1, ...k+4 com k = 1, 2,... we have

$x_{kt} = s/M(D_k) = s/ [M(D_{k-1}) (1+i)] = x_{k-1t}/(1+i)$ that is we our notation of the GDS

$x_{kt} = f(x_{k-1t}) \equiv 1/(1+i) (x_{k-1t})$

If i < 0, that is if the increase in the knowledge capital leads to new technologies that use less physical capital by unit of output the system has not a stationary path.

A more sophisticated model would be

$M(D_k) = \{s/M(D_{k-1}) / [bs/M(D_{k-1}) + a] \} M(D_{k-1})$

with b > 0and a < 0. In this case we have

$x_{kt} = bx_{kt-1} + a$

and the analysis could proceed using the GDS theory.

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