Título artículo / Títol article:	The Poincaré problem, algebraic integrability and dicritical divisors
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Revista:	*Journal of Differential Equations*, 2014, 256.11: 3614-3633
Versión / Versió:	PDF Preprint Autors
Cita bibliográfica / Cita bibliogràfica (ISO 690):	GALINDO, Carlos; MONSERRAT, Francisco. The Poincaré problem, algebraic integrability and dicritical divisors. *Journal of Differential Equations*, 2014, 256.11: 3614-3633. .
url Repositori UJI:	http://doi: 10.1016/j.jde.2014.02.015

THE POINCARÉ PROBLEM, ALGEBRAIC INTEGRABILITY AND DICRITICAL DIVISORS

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Abstract. We solve the Poincaré problem for plane foliations with only one dicritical divisor. Moreover, in this case, we give a simple algorithm that decides whether a foliation has a rational first integral and computes it in the affirmative case. We also provide an algorithm to compute a rational first integral of prefixed genus $g \neq 1$ of any type of plane foliation $\mathcal{F}$. When the number of dicritical divisors $\mathrm{dic}(\mathcal{F})$ is larger than two, this algorithm depends on suitable families of invariant curves. When $\mathrm{dic}(\mathcal{F}) = 2$, it proves that the degree of the rational first integral can be bounded only in terms of g, the degree of $\mathcal{F}$ and the local analytic type of the dicritical singularities of $\mathcal{F}$.

1. Introduction and Results

Denote by $\mathbb{P}^2$ the projective plane over the field of complex numbers. Poincaré, in [36], observed that "to find out whether a differential equation of the first order and of the first degree is algebraically integrable, it is enough to get an upper bound on the degree of the integral. Afterwards, one only needs to perform purely algebraic computations". The motivation for this observation, expressed in modern terminology, was the problem of deciding whether a singular algebraic foliation $\mathcal{F}$ on $\mathbb{P}^2$ (plane foliation) has a rational first integral and, when the answer is positive, to compute it. The so-called Poincaré problem consists of obtaining an upper bound of the degree of the first integral depending only on the degree of the foliation. Although it is well-known that such a bound does not exist in general, in the clause a) of the forthcoming Theorem 1, we shall give a bound of this type under the assumption that the minimal resolution of the singularities of $\mathcal{F}$ (which exists by a result of Seidenberg [38, 4]) has only one dicritical (i.e., non-invariant by $\mathcal{F}$) exceptional divisor.

Set $\pi_{\mathcal{F}} : Z_{\mathcal{F}} \to \mathbb{P}^2$ the composition of point blow-ups providing a minimal resolution of the dicritical singularities of $\mathcal{F}$ (see Definition 1). Clause b) of Theorem 1 states that, under the same assumption and when $\mathcal{F}$ is algebraically integrable, a rational first integral is given by the push-forward by $\pi_{\mathcal{F}}$ of the complete linear system given by a suitable multiple of a specific class in the Picard group of $Z_{\mathcal{F}}$. This gives a very simple procedure, Algorithm 2, to decide, from the resolution $\pi_{\mathcal{F}}$, whether $\mathcal{F}$ has a rational first integral and to compute it in the affirmative case. Alternative algorithms to do this are discussed in Section 4.

The natural extended version of the Poincaré problem consists of bounding the degree of the algebraic integral (reduced and irreducible) invariant curves of a foliation $\mathcal{F}$ (without assuming algebraic integrability) in terms of data obtained from the foliation and/or invariants related with the invariant curves themselves. There has been (and there is) a

Supported by Spain Ministry of Education MTM2012-36917-C03-03 and Univ. Jaume I P1-1B2012-04. The authors would like to thank the anonymous referee for his/her careful reading of the manuscript and for constructive suggestions that have improved it.

lot of activity concerning this or related problems, some of the main results (including higher dimension) being [10, 8, 6, 43, 40, 41, 44, 34, 16, 17, 9, 18].

The above mentioned problem was stated at the end of the 19th century as the problem of deciding whether a complex polynomial differential equation on the complex plane is algebraically integrable. The usefulness of nonlinear ordinary differential equations in practically any science turns this problem into a very attractive one, especially because when a differential equation admits a first integral, its study can be reduced in one dimension and because it is related with other interesting challenges. For example, it is related with the second part of the XVI Hilbert problem which tries to bound the number of limit cycles for a real polynomial vector field [30, 31], with the solutions of Einstein's field equations in general relativity [22] and with the center problem for vector fields [37, 15].

Algebraic integrability problem has a long history. In the 19th century, the main contributors were Darboux [13], Poincaré [35, 36], Painlevé [32] and Autonne [1]. They laid the foundations of a theory that has inspired a large quantity of papers, many of them published in the last twenty years. It was Darboux who gave a bound on the number of invariant integral algebraic curves of a polynomial differential equation that, when it is exceeded, implies the existence of a first integral. A close result was proved by Jouanolou [24] to guarantee that a foliation $\mathcal{F}$ as above has a rational first integral and that if one has enough reduced invariant curves; then the rational first integral can be computed. The existence of a first integral of that type is also equivalent to the fact that every invariant curve by $\mathcal{F}$ is algebraic and to the fact that there exist infinitely many invariant integral curves. These results have been adapted and extended to foliations on other varieties [23, 24, 5, 19, 12]. In [17], the authors gave an algorithm to decide about the existence of a rational first integral (and to compute it in the affirmative case) assuming that one has a well-suited set of $\mathrm{dic}(\mathcal{F})$ reduced invariant curves, where we stand $\mathrm{dic}(\mathcal{F})$ for the number of dicritical divisors appearing in the resolution of $\mathcal{F}$. In the same paper, it was also shown how to get sets of invariant curves as above for foliations such that the cone of curves of the surface obtained by the resolution of the dicritical singularities is polyhedral.

Painlevé in [32] posed the problem of recognizing the genus of the general invariant algebraic curve of a foliation admitting a rational first integral. Mixing the ideas of Painlevé and Poincaré, one can try to bound the degree of the rational first integral using also its genus. When $\mathcal{F}$ is non-degenerated, Poincaré himself provided a bound proving that $d(r-4) \leq 4(g-1)$, where d (respectively, r) is the degree of the first integral (respectively, $\mathcal{F}$) and g the mentioned genus. In the same sense, for foliations $\mathcal{F}$ as above with Kodaira dimension equal to 2, there exists a bound on the degree of the rational first integral which only depends on its genus, the degree of $\mathcal{F}$ and the sequence $\{\mathrm{h}^0(\mathbb{P}^2, \mathcal{K}_{\mathcal{F}}^{\otimes m})\}_{m>0}$, $\mathcal{K}_{\mathcal{F}}$ being the canonical sheaf of the foliation $\mathcal{F}$ (see [34] for a proof). With this philosophy, we shall show in clause a) of Theorem 2 that, for a foliation $\mathcal{F}$ on $\mathbb{P}^2$ having a rational first integral of genus $g \neq 1$ and such that $\mathrm{dic}(\mathcal{F}) \leq 2$, there exists a bound on the degree of the first integral which only depends on g, the degree of $\mathcal{F}$ and the local analytic type of the dicritical singularities of $\mathcal{F}$. It is worthwhile to add that, in [29], Lins Neto showed that, in general, such a bound does not exist. Clause b) of Theorem 2 states that, for foliations $\mathcal{F}$ of $\mathbb{P}^2$ satisfying also $\mathrm{dic}(\mathcal{F}) \leq 2$, there exists an algorithm that decides whether $\mathcal{F}$ has a rational first integral of fixed genus $g \neq 1$ (and computes it in the affirmative case). This algorithm assumes the knowledge of the resolution of dicritical singularities of $\mathcal{F}$ and only involves simple integer arithmetic and resolution of systems of linear equations.

Theorem 3 extends the results of Theorem 2 to the case when $\mathrm{dic}(\mathcal{F}) \geq 3$. Here it is required, as an additional hypothesis, the existence (and the knowledge) of a set of $\mathrm{dic}(\mathcal{F}) - 2$ independent algebraic solutions of $\mathcal{F}$ (see Definition 3). In a sense, this theorem is related with the above mentioned Darboux and Jouanolou's results because the knowledge of enough invariant curves allows us to obtain information concerning the rational first integral.

We finish this introduction stating the main results and summarizing briefly the aim of each section of the paper.

We need some notation for our first theorem (see details in sections 2 and 3). Set $\mathrm{Pic}_{\mathbb{Q}}(Z_{\mathcal{F}}) := \mathrm{Pic}(Z_{\mathcal{F}}) \otimes_{\mathbb{Z}} \mathbb{Q}$, where $\mathrm{Pic}(Z_{\mathcal{F}})$ stands for the Picard group of the surface $Z_{\mathcal{F}}$. Assume $\mathrm{dic}(\mathcal{F}) = 1$ and consider the hyperplane (see formula (3)) $\mathcal{W}$ of $\mathrm{Pic}_{\mathbb{Q}}(Z_{\mathcal{F}})$ spanned by the classes of the non-dicritical exceptional divisors and the class of $K_{\tilde{\mathcal{F}}} - K_{Z_{\mathcal{F}}}$, where $K_{\tilde{\mathcal{F}}}$ (respectively, $K_{Z_{\mathcal{F}}}$) denotes a divisor such that $\mathcal{O}_{Z_{\mathcal{F}}}(K_{\tilde{\mathcal{F}}})$ is the canonical sheaf of the pull-back of $\mathcal{F}$ by $\pi_{\mathcal{F}}$ (respectively, a canonical divisor of $Z_{\mathcal{F}}$). Consider a generator $\Delta_{\mathcal{F}} \in \mathrm{Pic}(Z_{\mathcal{F}})$ of the orthogonal complement of $\mathcal{W}$ with respect to the bilinear pairing given by the intersection form and such that $\Delta_{\mathcal{F}} \cdot \pi_{\mathcal{F}}^* L$ is positive and minimal, L being a general line of $\mathbb{P}^2$. Setting $\mathbb{Z}_{>0}$ the set of positive integers, it holds:

Theorem 1. *Let $\mathcal{F}$ be an algebraically integrable singular algebraic foliation on $\mathbb{P}^2$ of degree r such that* $\mathrm{dic}(\mathcal{F}) = 1$. *Then*

- **a):** *The degree d of a general integral invariant curve is less than or equal to $\frac{(r+2)^2}{4}$. Therefore, the Poincaré problem is solved in this case.*
- **b):** *There exists a value $\lambda \in \mathbb{Z}_{>0}$ such that $\mathcal{P}_{\mathcal{F}} := \pi_{\mathcal{F}*}|\lambda \Delta_{\mathcal{F}}|$ is a pencil and the rational map $\mathbb{P}^2 \cdots \to \mathbb{P}^1$ that it defines is a rational first integral of $\mathcal{F}$. Moreover λ is the minimum of the set* $\{\alpha \in \mathbb{Z}_{>0} \mid \dim\, \pi_{\mathcal{F}*}|\alpha \Delta_{\mathcal{F}}| \geq 1\}$.

The above clause b) supports a very simple algorithm, our forthcoming Algorithm 2, which decides about the existence of a rational first integral of $\mathcal{F}$ (and computes it in the positive case) whenever $\mathrm{dic}(\mathcal{F}) = 1$. Other alternative algorithms are treated in Section 4. Our remaining main results are:

Theorem 2. *Let $\mathcal{F}$ be a singular algebraic foliation on $\mathbb{P}^2$ such that* $\mathrm{dic}(\mathcal{F}) \leq 2$. *Let $g \neq 1$ be a non-negative integer.*

- **a):** *Assume that $\mathcal{F}$ has a rational first integral of genus g. Then, there exists a bound on the degree of the first integral depending only on the degree of $\mathcal{F}$, the genus g and the local analytic type of the dicritical singularities of $\mathcal{F}$.*
- **b):** *There exists an algorithm to decide whether $\mathcal{F}$ has a rational first integral of genus g (and to compute it, in the affirmative case) whose inputs are: g, a homogeneous 1-form defining $\mathcal{F}$ and the minimal resolution of the dicritical singularities of $\mathcal{F}$.*

Theorem 3. *Let $\mathcal{F}$ be a singular algebraic foliation on $\mathbb{P}^2$ such that* $\mathrm{dic}(\mathcal{F}) \geq 3$ *and assume the existence (and the knowledge) of a* $[\mathrm{dic}(\mathcal{F}) - 2]$*-set S of independent algebraic solutions of $\mathcal{F}$ (see Definition 3). Let $g \neq 1$ be a non-negative integer.*

- **a):** *Assume that $\mathcal{F}$ has a rational first integral of genus g. Then there exists a bound on the degree of the first integral which depends on the degree of $\mathcal{F}$, the genus g, the local analytic type of the dicritical singularities of $\mathcal{F}$ and the degrees of the curves in S and their multiplicities at the centers of the sequence of blow-ups $\pi_{\mathcal{F}}$ giving rise to the minimal resolution of the dicritical singularities of $\mathcal{F}$.*

b): *There exists an algorithm to decide whether $\mathcal{F}$ has a rational first integral of genus g (and to compute it, in the affirmative case). Its inputs are: g, a homogeneous* 1-*form defining $\mathcal{F}$, $\pi_{\mathcal{F}}$ and the degrees of the curves in S and their above mentioned multiplicities.*

Section 2 provides the notations and preliminary facts devoted to make easier the reading of the paper. Section 3 contains the mentioned study of rational first integrals with fixed genus; we describe the algorithm announced in clause b) of Theorem 3 (Algorithm 1), which is supported mainly in Lemma 1; the algorithm of clause b) of Theorem 2 is nothing but a particular case. Clause a) in both theorems is deduced as a consequence of the obtained algorithm. Section 4 is mainly devoted to prove Theorem 1. The proof is supported in three results, Lemma 2 and Propositions 2 and 3. The mentioned Algorithm 2 is also presented in this section together with an alternative algorithm described in Remark 5. Its correctness is showed in Proposition 4, which allows us to get the integral components of the non-reduced curves of the pencil defined by an algebraically integrable foliation $\mathcal{F}$ such that $\mathrm{dic}(\mathcal{F}) = 1$. Finally, in Section 5, we give several examples that show how our ideas and algorithms work.

2. Preliminaries

2.1. **Basic definitions.** Let Z be an algebraic smooth projective complex surface. A *singular algebraic foliation* $\mathcal{F}$ (or simply a *foliation* in the sequel) on Z is given by a set of pairs $\{(U_i, v_i)\}_{i\in I}$, where $\{U_i\}_{i\in I}$ is an open covering of Z, $v_i \in \mathcal{T}_Z(U_i)$ (where $\mathcal{T}_Z$ denotes the tangent sheaf of Z) and, if $i, j, k \in I$, there exist functions $g_{ij} \in \mathcal{O}^*_Z(U_i \cap U_j)$ such that $v_i = g_{ij}v_j$ on $U_i \cap U_j$ and $g_{ij}g_{jk} = g_{ik}$ on $U_i \cap U_j \cap U_k$. If $\mathcal{L}$ denotes the invertible sheaf defined by the multiplicative cocycle given by $\{g_{ij}\}$, we can regard $\mathcal{F}$ as a global section of the sheaf $\mathcal{L} \otimes \mathcal{T}_Z$ and, therefore, there is an induced morphism $\mathcal{L}^{-1} \to \mathcal{T}_Z$. The sheaf $\mathcal{L}$ is called the *canonical sheaf* of the foliation $\mathcal{F}$, and it will be denoted by $\mathcal{K}_{\mathcal{F}}$. Conversely, given an invertible sheaf $\mathcal{J}$ on Z and a morphism $\mathcal{J} \to \mathcal{T}_Z$, we can canonically associate with $\mathcal{J}$ a foliation such that $\mathcal{J}^{-1}$ is its canonical sheaf.

From the dual point of view, the natural product map $\Omega^1_Z \otimes \Omega^1_Z \to \Omega^2_Z$ gives rise to an isomorphism $\Omega^1_Z \to \mathcal{T}_Z \otimes \Omega^2_Z$. Under this isomorphism the map $\mathcal{K}^{-1}_{\mathcal{F}} \to \mathcal{T}_Z$ corresponds to a global section of $\Omega^1_Z \otimes \mathcal{K}_{\mathcal{F}} \otimes \mathcal{K}_Z^{-1}$, where $\mathcal{K}_Z$ denotes the canonical sheaf of Z.

Given a point $p \in Z$, take an open set U_i such that $p \in U_i$. The *algebraic multiplicity* of $\mathcal{F}$ at p, $\nu_p(\mathcal{F})$, is the order of v_i at p, that is, $\nu_p(\mathcal{F}) = s$ if and only if $(v_i)_p \in \mathrm{m}_p^s\mathcal{T}_{Z,p}$ and $(v_i)_p \notin \mathrm{m}_p^{s+1}\mathcal{T}_{Z,p}$, where m_p denotes the maximal ideal of the local ring $\mathcal{O}_{Z,p}$. The *singularities* of $\mathcal{F}$ are those points p in Z such that $\nu_p(\mathcal{F}) \geq 1$. We shall assume that all considered foliations are saturated, that is they have finitely many singularities. Notice that if $\mathcal{F} \in H^0(Z, \mathcal{K}_{\mathcal{F}} \otimes \mathcal{T}_Z)$ vanishes on a divisor H of Z, one can regard $\mathcal{F}$ as a global section of $\mathcal{K}_{\mathcal{F}} \otimes \mathcal{T}_Z \otimes \mathcal{O}_Z(-H)$ which defines a foliation $\mathcal{F}^s$, called saturation of $\mathcal{F}$, with isolated singularities such that $\mathcal{K}_{\mathcal{F}^s} = \mathcal{K}_{\mathcal{F}} \otimes \mathcal{O}_Z(-H)$.

Recall that an integral (i.e., reduced and irreducible) projective curve $C \subseteq Z$ is called to be invariant by $\mathcal{F}$ if the restriction map $\mathcal{K}^{-1}_{\mathcal{F}} \mid_C \to \mathcal{T}_Z \mid_C$ factorizes through the natural inclusion $\mathcal{T}_C \to \mathcal{T}_Z \mid_C$ and that a projective curve $C \subseteq Z$ is named *invariant* by $\mathcal{F}$ if all its integral components are invariant. Integral invariant curves of a foliation $\mathcal{F}$ are usually called *algebraic solutions* of $\mathcal{F}$. Locally, it means that for all closed point $p \in Z$, $v_p(f) \in I_{C,p}$, whenever $f \in I_{C,p}$, $I_{C,p}$ being the ideal of C and v_p a generator of $\mathcal{F}$ both at p; or, dually, that the local differential 2-form $\omega_p \wedge df$ is a multiple of f, ω_p being a local equation as a form of $\mathcal{F}$ and $f = 0$ a local equation of C at p.

Assume now that $\mathcal{F}$ is a foliation on $\mathbb{P}^2$ (the projective plane over the complex field) and let r be the non-negative integer such that $\mathcal{K}_{\mathcal{F}} = \mathcal{O}_{\mathbb{P}^2}(r-1)$; r is named the *degree* of the foliation. The Euler sequence $0 \to \Omega^1_{\mathbb{P}^2} \to \mathcal{O}_{\mathbb{P}^2}(-1)^3 \to \mathcal{O}_{\mathbb{P}^2} \to 0$, in fact the dual sequence $0 \to \mathcal{O}_{\mathbb{P}^2} \to \mathcal{O}_{\mathbb{P}^2}(1)^3 \to \mathcal{T}_{\mathbb{P}^2} \to 0$, allows us to regard $\mathcal{F}$ as induced by a homogeneous vector field

$$\mathbf{X} = U\partial/\partial X_0 + V\partial/\partial X_1 + W\partial/\partial X_2,$$

where U, V, W are homogeneous polynomials of degree r in homogeneous coordinates $(X_0 : X_1 : X_2)$ on $\mathbb{P}^2$; two vector fields define the same foliation if, and only if, they differ by a multiple of the radial vector field of the form $H(X_0, X_1, X_2)(X_0\partial/\partial X_0 + X_1\partial/\partial X_1 + X_2\partial/\partial X_2)$, where H is a homogeneous polynomial of degree $r-1$. A detailed description of this fact, using coordinates, can be seen in [21, Capítulo 1.3].

Returning to the dual point of view, the foliation $\mathcal{F}$ corresponds to a global section of the sheaf $\Omega^1_{\mathbb{P}^2} \otimes \mathcal{O}_{\mathbb{P}^2}(r+2)$. Taking into account the Euler sequence, this section corresponds to three homogeneous polynomials A, B and C of degree $r+1$, without common factors, such that $X_0A + X_1B + X_2C = 0$ (Euler condition); equivalently, the section can be seen as the homogeneous differential 1-form on $\mathbb{A}^3$:

$$\mathbf{\Omega} := AdX_0 + BdX_1 + CdX_2.$$

Notice that the equality

$$\det \begin{pmatrix} dX_0 & dX_1 & dX_2 \\ X_0 & X_1 & X_2 \\ U & V & W \end{pmatrix} = \mathbf{\Omega}$$

allows us to compute $\mathbf{\Omega}$ from $\mathbf{X}$ and that a curve on $\mathbb{P}^2$ defined by a homogeneous equation $F = 0$ is invariant by $\mathcal{F}$ if, and only if, the polynomial F divides the projective 2-form $\mathbf{\Omega} \wedge dF$.

2.2. **Resolution of singularities.** Throughout this paper, we shall consider sequences of morphisms

$$(1) \qquad X_{n+1} \xrightarrow{\pi_n} X_n \xrightarrow{\pi_{n-1}} \cdots \xrightarrow{\pi_2} X_2 \xrightarrow{\pi_1} X_1 := \mathbb{P}^2,$$

such that each π_i is the blow-up of the variety X_i at a closed point $p_i \in X_i$, $1 \le i \le n$. The set of closed points $\mathfrak{K} = \{p_1, p_2, \ldots, p_n\}$ given by a sequence as (1) will be called a *configuration* over $\mathbb{P}^2$ and the variety X_{n+1} the *sky* of $\mathfrak{K}$; we shall identify two configurations with $\mathbb{P}^2$-isomorphic skies. We shall denote by E_{p_i} (respectively, $\tilde{E}_{p_i}$, $E^*_{p_i}$) the exceptional divisor provided by the blow-up π_i (respectively, its strict transform, its total transform on X_{n+1}). Also, given two points p_i, p_j in $\mathfrak{K}$, we shall say that p_i is *infinitely near to* p_j (denoted $p_i \ge p_j$) if either $p_i = p_j$ or $i > j$ and $\pi_j \circ \pi_{j+1} \circ \cdots \circ \pi_{i-1}(p_i) = p_j$. The relation $\ge$ is a partial ordering among the points of the configuration $\mathfrak{K}$. Furthermore, a point p_i will be called *proximate* to other one p_j whenever p_i is in the strict transform of the exceptional divisor E_{p_j} on the surface containing p_i. To represent sequences as (1), we shall use a combinatorial invariant named the *proximity graph*. It is a graph whose vertices correspond to to the points p_i in $\mathfrak{K}$ and the edges join points p_i and p_j whenever p_i is proximate to p_j. This edge is dotted excepting the case when $p_i \in E_{p_j}$.

If $\mathcal{F}$ is a foliation on $\mathbb{P}^2$, a sequence of morphisms (1) induces, for each $i = 2, 3, \ldots, n+1$, a foliation $\mathcal{F}_i$ on X_i given by the pull-back of $\mathcal{F}$ (see [4], for instance). By a result of Seidenberg [38] there exists a *resolution of singularities* of $\mathcal{F}$, that is, a sequence of blow-ups as (1) such that the foliation $\mathcal{F}_{n+1}$ on the last obtained surface X_{n+1} has only simple singularities. A singularity $p \in U_i$ is *simple* (or *reduced*) if at least one of the eigenvalues α and β of the linear part of the vector field v_i (that are well defined since $v_i(p) = 0$) does not vanish and, assuming $\beta \neq 0$, the quotient α/β is not an strictly positive rational number. These singularities have the property that they cannot be removed by blowing-up.

In the sequel, we shall denote by $\mathcal{S}_{\mathcal{F}}$ the configuration $\{p_i\}_{i=1}^n$ given by the centers of the blow-ups involved in a *minimal* (with respect to the number of blow-ups) resolution of singularities of $\mathcal{F}$, however in our development we shall not use the whole resolution of singularities, but only the sequence of blow-ups concerning the so-called configuration of *dicritical* points that we define next.

Definition 1. An exceptional divisor E_{p_i} (respectively, a point $p_i \in \mathcal{S}_{\mathcal{F}}$) of a minimal resolution of singularities of a foliation $\mathcal{F}$ on $\mathbb{P}^2$ is called *non-dicritical* if it is invariant by the foliation $\mathcal{F}_{i+1}$ (respectively, all the exceptional divisors E_{p_j}, with $p_j \geq p_i$, are non-dicritical). Otherwise, E_{p_i} (respectively, p_i) is said to be *dicritical.* We shall denote by $\mathcal{B}_{\mathcal{F}}$ the configuration of dicritical points in $\mathcal{S}_{\mathcal{F}}$ and by $Z_{\mathcal{F}}$ the sky of $\mathcal{B}_{\mathcal{F}}$.

2.3. **Foliations having a rational first integral.** This paper is devoted to study algebraic integrability of certain type of foliations on the projective plane $\mathbb{P}^2$, so we start this brief section by defining this concept.

Definition 2. A *rational first integral* of a foliation $\mathcal{F}$ on $\mathbb{P}^2$ is a rational map $f : \mathbb{P}^2 \cdots \to \mathbb{P}^1$ such that the closures of its fibers are invariant curves by $\mathcal{F}$. Equivalently, and from an algebraic point of view, if f is given by a rational function R, f is a rational first integral if, and only if, $\mathbf{\Omega} \wedge dR = 0$. $\mathcal{F}$ is called to be *algebraically integrable* (or that it has a *rational first integral*) whenever there exists such a rational map.

Consider an algebraically integrable foliation $\mathcal{F}$ on $\mathbb{P}^2$. The second theorem of Bertini [26] shows that $\mathcal{F}$ admits a *primitive* rational first integral $f : \mathbb{P}^2 \cdots \to \mathbb{P}^1$ (that is, such that the closures of its general fibers are integral curves). Taking projective coordinates, if $F(X_0, X_1, X_2)$ and $G(X_0, X_1, X_2)$ are the two homogeneous polynomials of the same degree d which are the components of f, then the closures of the fibers of f are the elements of the irreducible pencil $\mathcal{P}_{\mathcal{F}}$ defined by $\langle F, G\rangle \subseteq H^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(d))$ and, moreover, any algebraic solution of $\mathcal{F}$ is a component of a curve in $\mathcal{P}_{\mathcal{F}}$. The *degree* (respectively, *genus*) of the first integral f will be the degree (respectively, geometric genus) of a general fiber of $\mathcal{P}_{\mathcal{F}}$.

Let $Z_{\mathcal{F}}$ be the sky of the configuration $\mathcal{B}_{\mathcal{F}}$ (see Definition 1). By comparing both processes, that of elimination of indeterminacies of the rational map f [3, Theorem II.7] and the resolution of the dicritical singularities of $\mathcal{F}$ through $\pi_{\mathcal{F}} : Z_{\mathcal{F}} \to \mathbb{P}^2$, it can be proved that $\pi_{\mathcal{F}}$ is also the minimal resolution of the indeterminacies of f. Indeed, if $p_i \in \mathfrak{K}$ and $\mathfrak{f}$ and $\mathfrak{g}$ are local equations at p_i of the strict transform of two general elements F and G generating the pencil $\mathcal{P}_{\mathcal{F}}$, then the local solutions of the foliations $\mathcal{F}_i$ at p_i are the irreducible components of the local pencil in the completion with respect to the maximal ideal of $\mathcal{O}_{X_i,p_i}$ generated by $\mathfrak{f}$ and $\mathfrak{g}$ (see [20] for complete details). As a consequence if f is a rational first integral of a foliation $\mathcal{F}$ as above, then the map $\tilde{f} := f \circ \pi_{\mathcal{F}} : Z_{\mathcal{F}} \to \mathbb{P}^1$ is a morphism.

3. Rational first integral with given genus

Along this section $\mathcal{F}$ will be a foliation on $\mathbb{P}^2$ of degree r and $Z_{\mathcal{F}}$ the sky of its configuration $\mathcal{B}_{\mathcal{F}}$ of dicritical points. Denote by $\mathrm{Pic}_{\mathbb{Q}}(Z_{\mathcal{F}})$ the vector space over $\mathbb{Q}$, $\mathrm{Pic}(Z_{\mathcal{F}}) \otimes_{\mathbb{Z}} \mathbb{Q}$, where $\mathrm{Pic}(Z_{\mathcal{F}})$ stands for the Picard group of the surface $Z_{\mathcal{F}}$. Intersection theory provides a $\mathbb{Z}$-bilinear form: $\mathrm{Pic}(Z_{\mathcal{F}}) \times \mathrm{Pic}(Z_{\mathcal{F}}) \to \mathbb{Z}$ which induces a non-degenerate bilinear form over $\mathbb{Q}$: $\mathrm{Pic}_{\mathbb{Q}}(Z_{\mathcal{F}}) \times \mathrm{Pic}_{\mathbb{Q}}(Z_{\mathcal{F}}) \to \mathbb{Q}$. The image by this form of a pair $(x, y) \in \mathrm{Pic}_{\mathbb{Q}}(Z_{\mathcal{F}}) \times \mathrm{Pic}_{\mathbb{Q}}(Z_{\mathcal{F}})$ will be denoted $x \cdot y$.

Given a divisor A on $Z_{\mathcal{F}}$, we shall denote by $[A]$ its class in the Picard group $\mathrm{Pic}(Z_{\mathcal{F}})$ and also its image into $\mathrm{Pic}_{\mathbb{Q}}(Z_{\mathcal{F}})$. If C is either a curve on $\mathbb{P}^2$ or an exceptional divisor, $\tilde{C}$ (respectively, C^*) will denote its strict (respectively, total) transform on the surface $Z_{\mathcal{F}}$ via the composition of blow-ups $\pi_{\mathcal{F}}$. It is well known that the set $\mathbf{B} := \{[L^*]\} \cup \{[E_p^*]\}_{p \in \mathcal{B}_{\mathcal{F}}}$ is a $\mathbb{Z}$-basis (respectively, $\mathbb{Q}$-basis) of $\mathrm{Pic}(Z_{\mathcal{F}})$ (respectively, $\mathrm{Pic}_{\mathbb{Q}}(Z_{\mathcal{F}})$), where L denotes a general line on $\mathbb{P}^2$.

Now, let us suppose that $\mathcal{F}$ admits a rational first integral f (which we assume to be primitive) and set $\tilde{\mathcal{F}}$ the foliation on $Z_{\mathcal{F}}$ given by the pull-back of $\mathcal{F}$ by $\pi_{\mathcal{F}}$. $\tilde{f} := f \circ \pi_{\mathcal{F}}$ is a first integral of $\tilde{\mathcal{F}}$ and the integral invariant curves of $\tilde{\mathcal{F}}$ (which coincide with the integral components of the fibers of $\tilde{f}$) are, on the one hand, the strict transforms on $Z_{\mathcal{F}}$ of the integral invariant curves of $\mathcal{F}$ and, on the other hand, the strict transforms of the exceptional divisors E_{p_i} (with $p_i \in \mathcal{B}_{\mathcal{F}}$) which are non-dicritical. Denote by $D_{\tilde{f}}$ a general fiber of $\tilde{f}$.

The proof of the next proposition follows from the irreducibility of the pencil $\mathcal{P}_{\mathcal{F}}$ and well-known results that can be found in [2] or [27].

Proposition 1. *Assume that $\mathcal{F}$ has a rational first integral. With the above notations, it holds:*

(a) $\mathcal{P}_{\mathcal{F}} = \pi_{\mathcal{F}*}|D_{\tilde{f}}|$.

(b) *A curve C on $Z_{\mathcal{F}}$ is invariant by $\tilde{\mathcal{F}}$ if and only if $D_{\tilde{f}} \cdot C = 0$.*

(c) *If C is a curve on $Z_{\mathcal{F}}$ which is invariant by $\tilde{\mathcal{F}}$ then $C^2 \leq 0$.*

Remark 1. Clause (a) of the above result shows that, to compute a primitive rational first integral f of $\mathcal{F}$, it is enough to know a divisor Δ linearly equivalent to the strict transform on $Z_{\mathcal{F}}$ of a general fiber of the pencil $\mathcal{P}_{\mathcal{F}}$ and two linearly independent global sections of $\pi_{\mathcal{F}*}\mathcal{O}_{Z_{\mathcal{F}}}(\Delta)$, which will be the components of f.

The morphism $\tilde{f} : Z_{\mathcal{F}} \to \mathbb{P}^1$ is a fibration of the surface $Z_{\mathcal{F}}$ by the curve $\mathbb{P}^1$ in the sense that $\tilde{f}$ is surjective and with connected fibers. Taking duals as $\mathcal{O}_{Z_{\mathcal{F}}}$-modules in the corresponding to $\tilde{f}$ sequence of differentials

$$0 \to \tilde{f}^*\Omega^1_{\mathbb{P}^1} \to \Omega^1_{Z_{\mathcal{F}}} \to \Omega_{Z_{\mathcal{F}}/\mathbb{P}^1} \to 0,$$

one gets

$$0 \to \mathcal{T}_{Z_{\mathcal{F}}/\mathbb{P}^1} \to \mathcal{T}_{Z_{\mathcal{F}}} \to \tilde{f}^*\mathcal{T}_{\mathbb{P}^1},$$

where $\mathcal{T}_{\mathbb{P}^1}$ and $\mathcal{T}_{Z_{\mathcal{F}}}$ denote the tangent sheaves of $\mathbb{P}^1$ and $Z_{\mathcal{F}}$, and $\mathcal{T}_{Z_{\mathcal{F}}/\mathbb{P}^1}$ the relative tangent sheaf of the fibration, which is an invertible sheaf [39, Section 1]. The morphism $\mathcal{T}_{Z_{\mathcal{F}}/\mathbb{P}^1} \to \mathcal{T}_{Z_{\mathcal{F}}}$ is given by the differential of $\tilde{f}$ and it defines the foliation $\tilde{\mathcal{F}}$, therefore we obtain the equality $\mathcal{K}_{\tilde{\mathcal{F}}} = \mathcal{T}^{-1}_{Z_{\mathcal{F}}/\mathbb{P}^1}$. From [39, Lemma 1.1], it follows that

$$\mathcal{K}_{\tilde{\mathcal{F}}} = \mathcal{T}^{-1}_{Z_{\mathcal{F}}/\mathbb{P}^1} = \mathcal{K}_{Z_{\mathcal{F}}} \otimes \tilde{f}^*\mathcal{K}^{-1}_{\mathbb{P}^1} \otimes \mathcal{O}_{Z_{\mathcal{F}}}\left(-\sum_{i \in I}(n_i - 1)G_i\right),$$

where $\mathcal{K}_{Z_{\mathcal{F}}}$ and $\mathcal{K}_{\mathbb{P}^1}$ are the canonical sheaves of $Z_{\mathcal{F}}$ and $\mathbb{P}^1$, respectively, and $\{G_i\}_{i \in I}$ the set of integral components of the singular fibers of $\tilde{f}$, n_i being the multiplicity of G_i in the fiber to which belongs to. If we take divisors $K_{\tilde{\mathcal{F}}}$ and $K_{Z_{\mathcal{F}}}$ such that $\mathcal{K}_{\tilde{\mathcal{F}}} = \mathcal{O}_{Z_{\mathcal{F}}}(K_{\tilde{\mathcal{F}}})$ and $\mathcal{K}_{Z_{\mathcal{F}}} = \mathcal{O}_{Z_{\mathcal{F}}}(K_{Z_{\mathcal{F}}})$, the above equality may be rewritten in the following form

$$K_{\tilde{\mathcal{F}}} - K_{Z_{\mathcal{F}}} \sim 2D_{\tilde{f}} - \sum_{i \in I}(n_i - 1)G_i, \tag{2}$$

where $\sim$ means linear equivalence.

The linear equivalence class of the divisor $K_{\tilde{\mathcal{F}}} - K_{Z_{\mathcal{F}}}$ can also be expressed in terms of the above basis $\mathbf{B}$ of $\mathrm{Pic}(Z_{\mathcal{F}})$. Indeed, denote, as above, the degree of $\mathcal{F}$ by r and by $\nu_p(\mathcal{F})$ the algebraic multiplicity at p of the foliation given by the pull-back of $\mathcal{F}$ on the surface to which p belongs. Set $\epsilon_p(\mathcal{F})$ the value 0 (respectively, 1) whenever the exceptional divisor E_p is non-dicritical (respectively, dicritical). Then, by [7, Proposition 1.1], it happens that

$$\pi^* K_{\mathcal{F}} - K_{Z_{\mathcal{F}}} \sim \sum_{p \in \mathcal{B}_{\mathcal{F}}} \left(\nu_p(\mathcal{F}) + \epsilon_p(\mathcal{F}) - 1\right) E_p^*,$$

which gives the following equivalence

$$K_{\tilde{\mathcal{F}}} - K_{Z_{\mathcal{F}}} \sim (r+2)L^* - \sum_{p \in \mathcal{B}_{\mathcal{F}}} (\nu_p(\mathcal{F}) + \epsilon_p(\mathcal{F}))E_p^*. \tag{3}$$

Next, we provide the concepts and results that will allow us to state the algorithms that prove theorems 2 and 3 in this paper. For a while, we shall assume that the foliation $\mathcal{F}$ need not to have a rational first integral. Stand $\mathrm{dic}(\mathcal{F})$ for the number of dicritical exceptional divisors appearing in the minimal resolution of $\mathcal{F}$. The existence of a set of invariant curves for $\mathcal{F}$ as we are going to define is an hypothesis in Theorem 3.

Definition 3. Let $\mathcal{F}$ be a foliation on $\mathbb{P}^2$ and suppose that $s := \mathrm{dic}(\mathcal{F}) \geq 3$. A $[\mathrm{dic}(\mathcal{F})-2]$-*set of independent algebraic solutions of* $\mathcal{F}$ is a set $S = \{C_1, C_2, \ldots, C_{s-2}\}$ of $s-2$ integral projective curves on $\mathbb{P}^2$, invariant by $\mathcal{F}$, and such that the family of classes in $\mathrm{Pic}_{\mathbb{Q}}(Z_{\mathcal{F}})$

$$V(S) := \left\{[\tilde{C}_1], [\tilde{C}_2], \ldots, [\tilde{C}_{s-2}], [K_{\tilde{\mathcal{F}}} - K_{Z_{\mathcal{F}}}]\right\} \cup \left\{[\tilde{E}_p] \mid p \in \mathcal{B}_{\mathcal{F}} \text{ and } E_p \text{ is non-dicritical}\right\}$$

is $\mathbb{Q}$-linearly independent.

Let us consider the projective space over the field $\mathbb{Q}$ associated with the $\mathbb{Q}$-vector space $\mathrm{Pic}_{\mathbb{Q}}(Z_{\mathcal{F}})$:

$$\mathbb{P}\mathrm{Pic}_{\mathbb{Q}}(Z_{\mathcal{F}}) := (\mathrm{Pic}_{\mathbb{Q}}(Z_{\mathcal{F}}) \setminus \{0\})/\mathbb{Q}^*,$$

and, for any $x \in \mathrm{Pic}_{\mathbb{Q}}(Z_{\mathcal{F}})$, denote by $\mathbb{Q}x$ the element in $\mathbb{P}\mathrm{Pic}_{\mathbb{Q}}(Z_{\mathcal{F}})$ defined by x.

Sets S as in Definition 3 determine the following subsets of $\mathbb{P}\mathrm{Pic}_{\mathbb{Q}}(Z_{\mathcal{F}})$, which will be useful in our algorithms:

$$\mathcal{R}_{\mathcal{F}}(S) := \left\{\mathbb{Q}x \in \mathbb{P}\mathrm{Pic}_{\mathbb{Q}}(Z_{\mathcal{F}}) \mid x^2 = 0 \text{ and } z \cdot x = 0 \text{ for all } z \in V(S)\right\}.$$

Notice that $x \cdot [L^*] \neq 0$ for any $\mathbb{Q}x \in \mathcal{R}_{\mathcal{F}}(S)$.

When $\mathrm{dic}(\mathcal{F}) \leq 2$, *we shall say that* $S = \emptyset$ *is a* $[\mathrm{dic}(\mathcal{F})-2]$-*set of independent algebraic solutions of* $\mathcal{F}$ and $\mathcal{R}_{\mathcal{F}}(\emptyset)$ is defined as above, $V(\emptyset)$ being the set of classes

$$\left\{[K_{\tilde{\mathcal{F}}} - K_{Z_{\mathcal{F}}}]\right\} \cup \left\{[\tilde{E}_p] \mid p \in \mathcal{B}_{\mathcal{F}} \text{ and } E_p \text{ is non-dicritical}\right\}.$$

Now, assume again that $\mathcal{F}$ has a (primitive) rational first integral $f : \mathbb{P}^2 \cdots \to \mathbb{P}^1$. Then, the existence of a $[\mathrm{dic}(\mathcal{F})-2]$-set of independent algebraic solutions means that the classes in $\mathrm{Pic}_{\mathbb{Q}}(Z_{\mathcal{F}})$ of the irreducible components of the fibers of $\tilde{f}$ span a subspace of $\mathrm{Pic}_{\mathbb{Q}}(Z_{\mathcal{F}})$ of codimension less than or equal to 2. In the case $\mathrm{dic}(\mathcal{F}) = 1$, the mentioned subspace is a hyperplane.

Under the same condition ($\mathcal{F}$ algebraically integrable), let D be a general element of $\mathcal{P}_{\mathcal{F}}$ and suppose that $\pi_{\mathcal{F}}$ is the composition of a sequence as in (1). Notice that $[\tilde{D}] = [D_{\tilde{f}}]$, where $D_{\tilde{f}}$ is as in the beginning of this section. Let $\Delta_{\mathcal{F}}$ be the class in the Picard group $\mathrm{Pic}(Z_{\mathcal{F}})$ that satisfies $\mathbb{Q}[D_{\tilde{f}}] = \mathbb{Q}\Delta_{\mathcal{F}}$ and the value $\Delta_{\mathcal{F}} \cdot [L^*]$ is positive and minimal. The following result will be the key of our forthcoming Algorithm 1.

Lemma 1. *Let $\mathcal{F}$ be a foliation on $\mathbb{P}^2$ having a rational first integral such that it admits a* $[\mathrm{dic}(\mathcal{F})-2]$*-set of independent algebraic solutions S. Then $\mathbb{Q}\Delta_{\mathcal{F}} \in \mathcal{R}_{\mathcal{F}}(S)$ and the cardinality of $\mathcal{R}_{\mathcal{F}}(S)$ is either* 1 *or* 2*.*

Proof. The fact that $\mathbb{Q}\Delta_{\mathcal{F}} \in \mathcal{R}_{\mathcal{F}}(S)$ can be deduced from the definition of $\Delta_{\mathcal{F}}$ and Proposition 1.

With respect to our second assertion, notice that if $\mathbb{Q}x \in \mathcal{R}_{\mathcal{F}}(S)$ and $\Delta_{\mathcal{F}} \cdot x = 0$, then considering a class $px + q\Delta_{\mathcal{F}}$, such that $[H] \cdot (px + q\Delta_{\mathcal{F}}) = 0$ for some ample divisor H of $Z_{\mathcal{F}}$ and applying the Hodge index theorem it holds $\mathbb{Q}x = \mathbb{Q}\Delta_{\mathcal{F}}$. Thus, if some integer multiple of $\Delta_{\mathcal{F}}$ belongs to $V(S)$, then it is clear that $\mathbb{Q}\Delta_{\mathcal{F}}$ is the unique member of $\mathcal{R}_{\mathcal{F}}(S)$. To finish, set $\langle W \rangle$ the subspace of $\mathrm{Pic}_{\mathbb{Q}}(Z_{\mathcal{F}})$ generated by a subset $W \subseteq \mathrm{Pic}_{\mathbb{Q}}(Z_{\mathcal{F}})$ and assume that there exists an element $\mathbb{Q}x \in \mathcal{R}_{\mathcal{F}}(S)$ different from $\mathbb{Q}\Delta_{\mathcal{F}}$. Then, $\langle \Delta_{\mathcal{F}}, x \rangle$ is a hyperbolic plane of $\mathrm{Pic}_{\mathbb{Q}}(Z_{\mathcal{F}})$ [28, page 590] and the decomposition

$$\mathrm{Pic}_{\mathbb{Q}}(Z_{\mathcal{F}}) = \langle \Delta_{\mathcal{F}}, x \rangle \oplus \langle V(S) \rangle$$

is Witt's decomposition with respect to the intersection form [28, Corollary 10.7]. Finally, the condition $\Delta_{\mathcal{F}} \cdot x \neq 0$ implies that the only directions in $\langle \Delta_{\mathcal{F}}, x \rangle$ with zero self-intersection are precisely given by $\Delta_{\mathcal{F}}$ and x. This concludes the proof.

$\square$

Remark 2. If we do not assume that a foliation $\mathcal{F}$ has a rational first integral, then it also holds that the cardinality of the set $\mathcal{R}_{\mathcal{F}}(S)$ is less than or equal to 2; moreover, in this case, $\mathcal{R}_{\mathcal{F}}(S)$ may be empty. In addition, if $\mathrm{dic}(\mathcal{F}) = 1$, $\mathcal{R}_{\mathcal{F}}(S)$ is either empty or its unique element is $\mathbb{P}\langle V(\emptyset) \rangle^{\perp}$. Notice also that the set $\mathcal{R}_{\mathcal{F}}(S)$ can be easily computed from the classes in $V(S)$ expressed in terms of the basis $\mathbf{B}$. When $\mathrm{dic}(\mathcal{F}) \leq 2$, this can be done only using data obtained from the resolution of the singularities of $\mathcal{F}$. Indeed, with these data one is able to compute the class $[K_{\tilde{\mathcal{F}}} - K_{Z_{\mathcal{F}}}]$ (see formula (3)) and, for each $p \in \mathcal{B}_{\mathcal{F}}$, one has $[\tilde{E}_p] = [E_p^*] - \sum_q [E_q^*]$, where q runs over the set of points of $\mathcal{B}_{\mathcal{F}}$ which are proximate to p. Otherwise (when $\mathrm{dic}(\mathcal{F}) > 2$), one also needs the coordinates in the basis $\mathbf{B}$ of the set of classes of strict transforms on $Z_{\mathcal{F}}$ of the invariant by $\mathcal{F}$ curves in S, $\{[\tilde{C}_i]\}_{i=1}^{\mathrm{dic}(\mathcal{F})-2}$.

The following algorithm is also a *proof of clause b) of Theorems 2 and 3.* It can be applied to foliations $\mathcal{F}$ admitting a $[\mathrm{dic}(\mathcal{F})-2]$-set of independent algebraic solutions and it decides about existence of a rational first integral of $\mathcal{F}$ of a prefixed genus $g \neq 1$ (computing it in the affirmative case).

Algorithm 1.

Input: A projective differential 1-form $\mathbf{\Omega}$ defining $\mathcal{F}$, a non-negative integer $g \neq 1$, the configuration $\mathcal{B}_{\mathcal{F}}$ and a $[\mathrm{dic}(\mathcal{F})-2]$-set $S = \{C_1, C_2, \ldots, C_{s-2}\}$ of independent algebraic solutions of $\mathcal{F}$.

Output: Either a primitive rational first integral of $\mathcal{F}$ of genus g or "0" (which means that such a first integral does not exist).

1. If $\mathrm{dic}(\mathcal{F}) \leq 2$, define $S := \emptyset$.
2. Compute the set $\mathcal{R}_{\mathcal{F}}(S)$. If $\mathcal{R}_{\mathcal{F}}(S) = \emptyset$ then return "0". Else, let $\mathcal{L} := \mathcal{R}_{\mathcal{F}}(S)$.
3. While $\mathcal{L} \neq \emptyset$:

3.1. Choose $\ell = \mathbb{Q}q \in \mathcal{L}$ (with $q \in \mathrm{Pic}(Z_{\mathcal{F}})$ such that $q \cdot [L^*]$ is positive and minimal) and let $\mathcal{L} := \mathcal{L} \setminus \{\ell\}$.
3.2. Compute the expression of q in the basis $\mathbf{B}$: $q = d[L^*] - \sum_{p\in\mathcal{B}_{\mathcal{F}}} m_p[E_p^*]$. If $m_p < 0$ for some p, then go to Step 3.
3.3. Compute
$$\alpha := \frac{2(g-1)}{-3d + \sum_{p\in\mathcal{B}_{\mathcal{F}}} m_p}.$$
If α is not a positive integer then go to Step 3.
3.4. Compute the linear system $\pi_{\mathcal{F}*}|\alpha(dL^* - \sum_{p\in\mathcal{B}_{\mathcal{F}}} m_p E_p^*)|$.
3.5. If the above linear system is not a pencil then go to Step 3. Else, choose two homogeneous polynomials F and G defining a basis.
3.6. If $\mathbf{\Omega} \wedge (GdF - FdG) = 0$ then the rational map $\mathbb{P}^2 \cdots \to \mathbb{P}^1$ whose components are F and G is a primitive rational first integral of $\mathcal{F}$; return it. Else, go to Step 3.

4. Return "0". □

This algorithm is justified by the following facts: If $\mathcal{F}$ has a (primitive) rational first integral $f : \mathbb{P}^2 \cdots \to \mathbb{P}^1$ then, by Proposition 1, the linear system $\pi_{\mathcal{F}*}|D_{\tilde{f}}|$ is a pencil and any pair of homogeneous polynomials $F(X,Y,Z)$ and $G(X,Y,Z)$ defining a basis of $\pi_{\mathcal{F}*}|D_{\tilde{f}}|$ gives rise to a rational map $P \mapsto (F(P) : G(P))$ that is a primitive rational first integral of $\mathcal{F}$. Moreover $D_{\tilde{f}} = \alpha\Delta_{\mathcal{F}}$, where α is the greatest common divisor of the coefficients of $[D_{\tilde{f}}]$ with respect to the basis $\mathbf{B}$ and, in virtue of Lemma 1, $\mathbb{Q}\Delta_{\mathcal{F}}$ must belong to the set $\mathcal{R}_{\mathcal{F}}(S)$ (that has, at most, cardinality 2), where S is any $[\mathrm{dic}(\mathcal{F})-2]$-set of independent algebraic solutions of $\mathcal{F}$. Finally, notice that a general fiber of $\tilde{f}$ (whose class in $\mathrm{Pic}(Z_{\mathcal{F}})$ is $[D_{\tilde{f}}] = \alpha\Delta_{\mathcal{F}}$) is nonsingular; then, applying the adjunction formula and taking into account that $\Delta_{\mathcal{F}}^2 = 0$ we have that
$$\alpha = \frac{2(g-1)}{[K_{Z_{\mathcal{F}}}] \cdot \Delta_{\mathcal{F}}},$$
where g is the genus of a general fiber of $\tilde{f}$ (that is, the genus of a general invariant curve), that we assume to be different from 1 (and, therefore, $K_{Z_{\mathcal{F}}} \cdot D_{\tilde{f}} \neq 0$).

Remark 3. The points of the configuration $\mathcal{B}_{\mathcal{F}}$ are used in the last steps (from 3.4 to 3.6) of Algorithm 1 because, there, it is required to compute and use linear systems of $\mathbb{P}^2$ obtained by pushing forward complete linear systems on $Z_{\mathcal{F}}$. To perform the remaining steps the algorithm only requires the following data: the degree of $\mathcal{F}$, the genus g of a general invariant curve, the proximity relations among the points of the configuration $\mathcal{B}_{\mathcal{F}}$, the above defined numbers $\nu_p(\mathcal{F})$ and $\epsilon_p(\mathcal{F})$ for each point $p \in \mathcal{B}_{\mathcal{F}}$ and (only when $\mathrm{dic}(\mathcal{F}) \geq 3$) the degrees of the curves in S and their multiplicities at the points of $\mathcal{B}_{\mathcal{F}}$.

To end this section *we shall prove clause a) of Theorems 2 and 3.* In both cases, this clause is an easy consequence of Algorithm 1. Indeed, if g is the genus of the rational first integral of a foliation $\mathcal{F}$ (assuming that it is different from 1), then the degree of the first integral can be bounded by the maximum of the numbers $2\,(g-1)\,d/\left(\sum_{p\in\mathcal{B}_{\mathcal{F}}} m_p - 3d\right)$, where $d[L^*] - \sum_{p\in\mathcal{B}_{\mathcal{F}}} m_p[E_p^*]$ are generators of the elements in $\mathcal{R}_{\mathcal{F}}(S)$ determined by a $[\mathrm{dic}(\mathcal{F})-2]$-set S of algebraic solutions (notice that, by Lemma 1, there are, at most, two possibilities for these elements). To compute this bound, one needs the following data: the genus g, the degree of $\mathcal{F}$, the degrees of the curves in S and their multiplicities at the points of $\mathcal{B}_{\mathcal{F}}$ (only when $\mathrm{dic}(\mathcal{F}) \geq 3$), the proximity relations among the points of the configuration $\mathcal{B}_{\mathcal{F}}$ and the above defined numbers $\nu_p(\mathcal{F})$ and $\epsilon_p(\mathcal{F})$ for each point $p \in \mathcal{B}_{\mathcal{F}}$. Since the last two data only depend on the local analytic type of the dicritical singularities of $\mathcal{F}$, we conclude clause a) of the mentioned theorems. □

4. Foliations with only one dicritical divisor

This section is mainly devoted to prove Theorem 1 which in its clause a) solves the Poincaré problem for foliations $\mathcal{F}$ on $\mathbb{P}^2$ such that $\mathrm{dic}(\mathcal{F}) = 1$. Moreover if we assume the knowledge of the resolution of the dicritical singularities of $\mathcal{F}$, clause b) of that theorem put the foundations to decide by means of Algorithm 2 whether $\mathcal{F}$ is algebraically integrable and to compute a rational first integral (if it exists). This is done by using only tools from linear algebra. Our arguments are supported in the facts that the degrees of the integral components of the non-reduced curves in $\mathcal{P}_{\mathcal{F}}$ are bounded by $\deg(\mathcal{F}) + 2$ (Proposition 2) and that they determine the degree of a primitive rational first integral of $\mathcal{F}$ (Proposition 3). Alternative procedures to Algorithm 2 are also discussed in this section.

For a start, fix a foliation $\mathcal{F}$ of degree r, having a rational first integral and such that $\mathrm{dic}(\mathcal{F}) = 1$. To avoid trivialities, we also assume that the cardinality of $\mathcal{B}_{\mathcal{F}}$ is greater than 1 (note that otherwise the foliation is defined by a pencil of lines). We shall need the following results:

Lemma 2. *Let $\mathcal{F}$ be as above. All the curves in the irreducible pencil $\mathcal{P}_{\mathcal{F}}$ defined in Section 2 are irreducible and, at most two of them, are non-reduced.*

Proof. From [25, Corollary 2] and the subsequent remark, it can be deduced that the cardinality of the set of dicritical exceptional divisors $\mathrm{dic}(\mathcal{F})$ attached to a foliation $\mathcal{F}$ satisfies the following inequality

$$1 + \sum (e_R - 1) \leq \mathrm{dic}(\mathcal{F}),$$

where the sum is taken over the set of curves R in the pencil $\mathcal{P}_{\mathcal{F}}$ and e_R stands for the number of different integral components of R. As a consequence any curve in $\mathcal{P}_{\mathcal{F}}$ is irreducible because $\mathrm{dic}(\mathcal{F}) = 1$. The second part of the statement follows from a result of Poincaré in [36, page 187 of I] (see [43, Proposition 3.1] for a different recent proof).

□

Proposition 2. *Let $\mathcal{F}$ and $\mathcal{P}_{\mathcal{F}}$ be as in Lemma 2. Let $\mathcal{A}$ be the set of integral components of the non-reduced curves in $\mathcal{P}_{\mathcal{F}}$. Then $\deg(A) < \deg(\mathcal{F}) + 2$ for all $A \in \mathcal{A}$. Moreover, if $\mathcal{A}$ has two elements (say A_1 and A_2) then $\deg(A_1) + \deg(A_2) = \deg(\mathcal{F}) + 2$.*

Proof. Let δ be the degree of a primitive rational first integral of $\mathcal{F}$ and let $r = \deg(\mathcal{F})$. Set θ the number of non-reduced curves in the pencil $\mathcal{P}_{\mathcal{F}}$ and χ the sum of the degrees of the integral components of these curves. Notice that $\theta \leq 2$ by Lemma 2.

Taking, in (2), intersection products with the total transform of a general line of $\mathbb{P}^2$, one has that

$$2\delta - r - 2 = \sum (e_R - 1) \deg(R), \tag{4}$$

where the sum is taken over the set of integral components R of the curves in $\mathcal{P}_{\mathcal{F}}$ and e_R denotes the multiplicity of R as a component of such curves. Therefore

$$2\delta - r - 2 = \theta\delta - \chi.$$

This concludes the proof of the first assertion because $\theta = 1$ implies $\delta < r + 2$ and $\theta = 2$ shows $\chi = r + 2$, and in both cases $r + 2$ is a strict upper bound of the degrees of the mentioned integral components. The last assertion holds because, for $\theta = 2$, we have the equality $\chi = r + 2$.

□

Proposition 3. *Let $\mathcal{F}$ be a foliation on $\mathbb{P}^2$ of degree r having a primitive rational first integral f and such that* $\mathrm{dic}(\mathcal{F}) = 1$. *Let $D_{\tilde{f}}$ be a general fiber of $\tilde{f} = f \circ \pi_{\mathcal{F}}$ and γ the positive integer such that $[D_{\tilde{f}}] = \gamma \Delta_{\mathcal{F}}$. Let $\mathcal{A}$ be the set of integral components of the non-reduced curves in $\mathcal{P}_{\mathcal{F}}$. Then, the following statements hold:*

(a) *If $\mathcal{A} = \emptyset$, then $\gamma = \frac{r+2}{2s_0}$, where $s_0 = \Delta_{\mathcal{F}} \cdot [L^*]$.*
(b) *If $\mathcal{A} = \{A_1\}$, then $\gamma = \frac{r+2-\deg(A_1)}{s_0}$.*
(c) *Otherwise $\mathcal{A} = \{A_1, A_2\}$ and then $\gamma = \frac{\mathrm{lcm}(\deg(A_1),\deg(A_2))}{s_0}$.*

Proof. (a) and (b) are direct consequences of equality (4). To prove (c) observe first that, by Lemma 2, there exist positive integers n_1, n_2 such that the pencil $\mathcal{P}_{\mathcal{F}}$ is spanned by homogeneous polynomials giving equations of $n_1 A_1$ and $n_2 A_2$. Moreover n_1 and n_2 are relatively primes because the pencil is irreducible. Since $n_1 \deg(A_1) = n_2 \deg(A_2)$ we have that $n_1 = \frac{\deg(A_2)}{\gcd(\deg(A_1),\deg(A_2))}$ and, therefore, the degree of a general integral invariant curve is

$$\frac{\deg(A_1)\deg(A_2)}{\gcd(\deg(A_1),\deg(A_2))} = \mathrm{lcm}(\deg(A_1),\deg(A_2)).$$

□

Now we are able to prove Theorem 1:

Proof of Theorem 1. As before assume that $\mathcal{F}$ is a foliation $\mathcal{F}$ on $\mathbb{P}^2$ of degree r having a (primitive) rational first integral $f : \mathbb{P}^2 \cdots \to \mathbb{P}^1$ of degree d and such that $\mathrm{dic}(\mathcal{F}) = 1$. Let $\mathcal{A}$ be as in Proposition 3. If either this set is empty or its cardinality is 1, the inequality

$$d \leq \frac{(r+2)^2}{4},$$

is trivially satisfied by clauses (a) and (b) of Proposition 3. Therefore, let us take $\mathcal{A} = \{A_1, A_2\}$. Applying Proposition 2 and clause (c) of Proposition 3 one has that

$$d = \mathrm{lcm}(\deg(A_1), r+2-\deg(A_1)) \leq \deg(A_1)(r+2-\deg(A_1)) \leq \frac{(r+2)^2}{4},$$

completing the proof of clause a).

Clause b) follows taking into account Proposition 1 and the fact that $[D_{\tilde{f}}]$ is a multiple of $\Delta_{\mathcal{F}}$. □

Next, we state the mentioned very simple algorithm supported in clause b) of Theorem 1 to decide about algebraic integrability. Notice that, when $\mathrm{dic}(\mathcal{F}) = 1$, the classes of strict transforms of the non-dicritical exceptional divisors and the class $[K_{\tilde{\mathcal{F}}} - K_{Z_{\mathcal{F}}}]$ span a hyperplane $\mathcal{W}$ in $\mathrm{Pic}_{\mathbb{Q}}(Z_{\mathcal{F}})$. Therefore, independently of the algebraic integrability of $\mathcal{F}$ and, as we said in the paragraph before the statement of Theorem 1, we can define the class $\Delta_{\mathcal{F}}$ as the element of $\mathrm{Pic}(Z_{\mathcal{F}})$ that is orthogonal to $\mathcal{W}$ and such that $\Delta_{\mathcal{F}} \cdot [L^*]$ is positive and minimal.

Algorithm 2.

Input: A projective differential 1-form $\mathbf{\Omega}$ defining a foliation $\mathcal{F}$ on $\mathbb{P}^2$ such that $\mathrm{dic}(\mathcal{F}) = 1$

and the class $\Delta_{\mathcal{F}}$ (computed from the resolution of the dicritical singularities).

Output: Either a primitive rational first integral of $\mathcal{F}$ or "0" (which means that $\mathcal{F}$ is not algebraically integrable).

1. Compute the linear systems $\pi_{\mathcal{F}*}|\alpha\Delta_{\mathcal{F}}|$ for successive positive integer values $\alpha \leq \frac{(r+2)^2}{4(\Delta_{\mathcal{F}}\cdot[L^*])}$. If none of them has projective dimension 1 then return "0".
2. Else, take two homogeneous polynomials $F(X,Y,Z)$ and $G(X,Y,Z)$ defining curves that provide a basis of the linear system $\pi_{\mathcal{F}*}|\alpha\Delta_{\mathcal{F}}|$ with projective dimension 1. If $\mathbf{\Omega} \wedge (GdF - FdG) = 0$ then the rational map $\mathbb{P}^2 \cdots \to \mathbb{P}^1$ whose components are F and G is a rational first integral of $\mathcal{F}$; return it. Otherwise return "0".

Remark 4. The first step of this algorithm uses only linear algebra (as in Step 3.4 of Algorithm 1, one needs to compute the push-forward of complete linear systems and it consists of the resolution of systems of linear equations) and the second step involves to check easy conditions.

The bound given in clause a) of Theorem 1 allows us to use Algorithm 1 in [17]. However, Algorithm 2 is simpler because we take advantage of the knowledge of $\Delta_{\mathcal{F}}$, which is easily computed from data obtained in the resolution of the dicritical singularities of $\mathcal{F}$ (see (3)).

It could also be used the Poincaré idea of considering undetermined coefficients, but in this case one is forced to solve a system of quadratic equations in several variables.

Furthermore, when the coefficients of a differential 1-form providing $\mathcal{F}$ are integer numbers, a different alternative algorithm is that described in [11], which relies on the factorization of the extactic curves studied in [33] (see also [14]). Nevertheless, to check whether $\mathrm{dic}(\mathcal{F}) = 1$, a resolution of the singularities of $\mathcal{F}$ is needed.

Let $\mathcal{F}$ be as in the above paragraphs. To finish this section, we shall show a new alternative procedure to get a rational first integral that is useful when $\mathcal{F}$ has invariant curves of low degree. First, we define a set of divisors on $Z_{\mathcal{F}}$. For any positive integer x, we denote by $\Gamma_{\mathcal{F}}(x)$ the (finite) set of divisors $C = xL^* - \sum_{p\in\mathcal{B}_{\mathcal{F}}} y_p E_p^*$ satisfying the following conditions:

(a) $0 \leq y_p \leq x$ for all $p \in \mathcal{B}_{\mathcal{F}}$ and $\Delta_{\mathcal{F}} \cdot [C] = 0$.
(b) $C \cdot \tilde{E}_p \geq 0$ for all $p \in \mathcal{B}_{\mathcal{F}}$.
(c) Either $C^2 = K_{Z_{\mathcal{F}}} \cdot C = -1$, or $C^2 \leq 0$, $K_{Z_{\mathcal{F}}} \cdot C \geq 0$ and $C^2 + K_{Z_{\mathcal{F}}} \cdot C \geq -2$.
(d) The complete linear system $|C|$ has (projective) dimension 0 and $\pi_{\mathcal{F}*}|C|$ is invariant by $\mathcal{F}$.

It is clear that to get $\Gamma_{\mathcal{F}}(x)$, one only needs to check easy numerical relations and to solve systems of linear equations. These sets will be useful for our mentioned procedure. It is also worthwhile to mention that Proposition 2 and the following result show that, when $\mathcal{F}$ is algebraically integrable, the integral components of the non-reduced curves of $\mathcal{P}_{\mathcal{F}}$ can be computed from these sets $\Gamma_{\mathcal{F}}(x)$ (see the forthcoming examples 2 and 3).

Proposition 4. *Let $\mathcal{F}$ be an algebraically integrable foliation of $\mathbb{P}^2$ of degree r such that $\mathrm{dic}(\mathcal{F}) = 1$. Any integral component of a non-reduced curve in $\mathcal{P}_{\mathcal{F}}$ is the push-forward $\pi_{\mathcal{F}*}|C|$ for some divisor C on $Z_{\mathcal{F}}$ which belongs to $\bigcup_{x<r+2} \Gamma_{\mathcal{F}}(x)$.*

Proof. Let H be an integral component of a non-reduced curve of the pencil $\mathcal{P}_{\mathcal{F}}$. Let x be the degree of H and, for each $p \in \mathcal{B}_{\mathcal{F}}$, denote by y_p the multiplicity at p of the strict transform of H on the surface to which p belongs. Then, it holds the following linear equivalence between divisors on $Z_{\mathcal{F}}$:

$$\tilde{H} \sim C := xL^* - \sum_{p \in \mathcal{B}_{\mathcal{F}}} y_p E_p^*,$$

and it happens that $H = \pi_{\mathcal{F}*}|C|$. Let us see that C belongs to $\Gamma_{\mathcal{F}}(x)$. Indeed, condition (a) of the definition of $\Gamma_{\mathcal{F}}(x)$ is clear by Proposition 1, (b) is true because $\tilde{H}$ is irreducible and non-exceptional, (c) follows from statement (c) in Proposition 1 and the adjunction formula and (d) holds because the integral components of the curves in $\pi_{\mathcal{F}*}|C|$ are also integral components of the curves in the pencil $\mathcal{P}_{\mathcal{F}}$. This concludes the proof because $x < r+2$ by Proposition 2. □

Remark 5. Propositions 3 and 4 support the following alternative algorithm to Algorithm 2. Its input is a projective differential 1-form $\mathbf{\Omega}$ defining a foliation $\mathcal{F}$ on $\mathbb{P}^2$ of degree r such that $\mathrm{dic}(\mathcal{F}) = 1$, and its output is either a rational first integral of $\mathcal{F}$ or "0"(which means that $\mathcal{F}$ has no such a first integral):

1. Compute $\Delta_{\mathcal{F}}$ (in terms of the basis $\mathbf{B}$) and $s_0 := \Delta_{\mathcal{F}} \cdot [L^*]$.
2. If $2s_0$ divides $r+2$ and $\pi_{\mathcal{F}*}|\frac{r+2}{2s_0}\Delta_{\mathcal{F}}|$ is a pencil defined by two homogeneous polynomials F and G such that $\mathbf{\Omega} \wedge (GdF - FdG) = 0$, then return the rational map $\mathbb{P}^2 \cdots \to \mathbb{P}^1$ whose components are F and G; otherwise go to Step 3.
3. If the set $\bigcup_{x<r+2} \Gamma_{\mathcal{F}}(x)$ is empty then return "0".
4. Else, take a divisor C in $\bigcup_{x<r+2} \Gamma_{\mathcal{F}}(x)$ with minimal $x = C \cdot L^*$.
5. If s_0 divides $r+2-x$ and $\pi_{\mathcal{F}*}|\frac{r+2-x}{s_0}\Delta_{\mathcal{F}}|$ is a pencil defined by two homogeneous polynomials F and G such that $\mathbf{\Omega} \wedge (GdF - FdG) = 0$, then return the rational map $\mathbb{P}^2 \cdots \to \mathbb{P}^1$ whose components are F and G; otherwise go to Step 6.
6. If s_0 divides $\mathrm{lcm}(x, r+2-x)$ and $\pi_{\mathcal{F}*}|\frac{\mathrm{lcm}(x,r+2-x)}{s_0}\Delta_{\mathcal{F}}|$ is a pencil defined by two homogeneous polynomials F and G such that $\mathbf{\Omega} \wedge (GdF - FdG) = 0$, then return the rational map $\mathbb{P}^2 \cdots \to \mathbb{P}^1$ whose components are F and G; otherwise return"0".

5. Examples

This last section is devoted to provide some examples that show how our ideas and algorithms work. We have used the version 7 of the software system *Mathematica* [42] to get the resolution and apply our algorithms to the involved foliations. For a start, we shall use Algorithm 1 to get a rational first integral of a foliation of degree 4.

Example 1. Consider the singular algebraic foliation $\mathcal{F}$ given by the differential 1-form

$$\mathbf{\Omega} = (2X_1X_2^5)\, dX_0 + (-7X_1^5X_2 - 3X_0X_2^5 + X_1X_2^5)\, dX_1 + (7X_1^6 + X_0X_1X_2^4 - X_1^2X_2^4)\, dX_2.$$

From the minimal resolution of singularities we compute the configuration of dicritical points $\mathcal{B}_{\mathcal{F}}$. It has 13 points, $\{p_i\}_{i=1}^{13}$, and its proximity graph is displayed in Figure 1. We recall that the vertices of this graph represent the points in $\mathcal{B}_{\mathcal{F}}$, and two vertices, $p_i, p_j \in \mathcal{B}_{\mathcal{F}}$, are joined by an edge if p_i belongs to the strict transform of the exceptional divisor E_{p_j}. This edge is curved-dotted except when p_i belongs to the first infinitesimal neighborhood of p_j (here the edge is straight-continuous). For simplicity's sake, we delete those edges which can be deduced from others (for instance, we have deleted the curved-dotted edges joining p_7 and p_6 with p_4 since there is an edge joining p_8 and p_4). From the local differential 1-forms defining the transformed foliations in the resolution process, it can be easily deduced that the unique dicritical divisors are E_{p_3} and $E_{p_{13}}$. Therefore $\mathrm{dic}(\mathcal{F}) = 2$. Then we can use Algorithm 1 to check whether $\mathcal{F}$ has a rational first integral of genus $g = 0$ because $g \neq 1$ and $\mathcal{F}$ admits an empty $[\mathrm{dic}(\mathcal{F}) - 2]$-set of independent algebraic solutions. From the minimal resolution, we can obtain the divisor class

$$[K_{\tilde{\mathcal{F}}} - K_{Z_{\mathcal{F}}}] = 7[L^*] - [E^*_{p_1}] - [E^*_{p_2}] - 2[E^*_{p_3}] - 5[E^*_{p_4}] - 2\sum_{i=5}^{8}[E^*_{p_i}] - \sum_{i=9}^{12}[E^*_{p_i}] - 2[E^*_{p_{13}}]$$

and the set $\mathcal{R}_{\mathcal{F}}(\emptyset)$:

$$\begin{aligned}\mathcal{R}_{\mathcal{F}}(\emptyset) = \{&(10 : -2 : -1 : -1 : -8 : -2 : -2 : -2 : -2 : -2 : -2 : -2 : -1 : -1),\\ &(2770 : -762 : -381 : -381 : -2152 : -538 :\\ &-538 : -538 : -538 : -538 : -538 : -538 : -269 : -269)\},\end{aligned}$$

where we have taken projective coordinates with respect to the basis $\mathbf{B}$. Following Algorithm 1, we must consider the first element in $\mathcal{R}_{\mathcal{F}}(\emptyset)$ given by q and compute the value α in Step 3.3. Here $\alpha = 1$,

$$q = 10[L^*] - 2[E^*_{p_1}] - [E^*_{p_2}] - [E^*_{p_3}] - 8[E^*_{p_4}] - 2\sum_{i=5}^{11}[E^*_{p_i}] - [E^*_{p_{12}}] - [E^*_{p_{13}}]$$

and the linear system $\pi_{\mathcal{F}*}|q|$ is a pencil, being $F = X_1^3X_2^7$ and $G = X_1^{10} - 2X_0X_1^5X_2^4 + 2X_1^6X_2^4 + X_0^2X_2^8 - 2X_0X_1X_2^8 + X_1^2X_2^8$ a basis. Finally, $\mathcal{F}$ has a rational first integral given by F and G because $\mathbf{\Omega} \wedge (GdF - FdG) = 0$. Notice that this example is [17, Example 2], where we proved the same result with a different procedure.

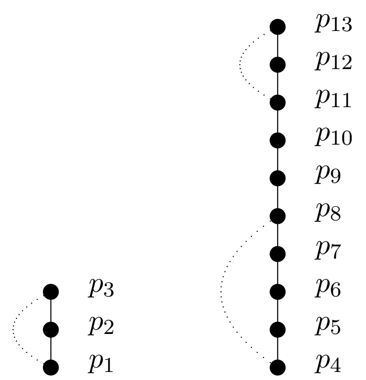

Figure 1. Proximity graph of $\mathcal{B}_{\mathcal{F}}$ in Example 1

Example 2. Set $\mathcal{F}$ the foliation attached to the differential 1-form

$$\mathbf{\Omega} = (3X_0^2X_2^3)\,dX_0 - (5X_1^4X_2)\,dX_1 + (5X_1^5 - 3X_0^3X_2^2)\,dX_2.$$

The configuration $\mathcal{B}_{\mathcal{F}}$ has 19 points $\{p_i\}_{i=1}^{19}$ and only one dicritical divisor: $E_{p_{19}}$. We show the corresponding proximity graph in Figure 2. From the resolution of singularities it is deduced that

$$[K_{\tilde{\mathcal{F}}} - K_{Z_{\mathcal{F}}}] = 6[L^*] - 2[E^*_{p_1}] - 3[E^*_{p_2}] - 2[E^*_{p_3}] - 2[E^*_{p_4}] - \sum_{i=5}^{18}[E^*_{p_i}] - 2[E^*_{p_{19}}]$$

and, moreover, it can be checked that

$$\Delta_{\mathcal{F}} = 5[L^*] - 2[E^*_{p_1}] - 2[E^*_{p_2}] - \sum_{i=3}^{19}[E^*_{p_i}].$$

Applying Algorithm 2, we get that the linear system $\pi_{\mathcal{F}*}|\Delta_{\mathcal{F}}|$ is the pencil spanned by the curves defined by the homogeneous polynomials $F = X_1^5 - X_0^3X_2^2$ and $G = X_2^5$. Furthermore $\mathbf{\Omega} \wedge (GdF - FdG) = 0$. So, we deduce that $\mathcal{F}$ is algebraically integrable and that $P \mapsto (F(P) : G(P))$ defines a rational first integral of $\mathcal{F}$.

The same conclusion can be obtained applying the algorithm described in Remark 5. Indeed, the divisor $C = L^* - \sum_{i=1}^3 E_{p_i}^*$ belongs to $\Gamma_{\mathcal{F}}(1)$ and this implies that the line $\pi_{\mathcal{F}*}|C|$, defined by $X_2 = 0$, is invariant by $\mathcal{F}$ (notice that this is also evident from $\mathbf{\Omega}$). Therefore, if $\mathcal{F}$ is algebraically integrable, the pencil $\mathcal{P}_{\mathcal{F}}$ should be $\pi_{\mathcal{F}*}|\frac{4+2-1}{5}\Delta_{\mathcal{F}}| = \pi_{\mathcal{F}*}|\Delta_{\mathcal{F}}|$.

After checking that $\Gamma_{\mathcal{F}}(5)$ is empty and, taking into account Propositions 2 and 4, we deduce that X_2^5 defines the unique non-reduced curve in $\mathcal{P}_{\mathcal{F}}$. Finally notice that $\Delta_{\mathcal{F}}$ is the class in $\mathrm{Pic}(Z_{\mathcal{F}})$ of a general curve of $\mathcal{P}_{\mathcal{F}}$, which is nonsingular. As a consequence, applying the adjunction formula it happens that the genus of the rational first integral of $\mathcal{F}$ is 4.

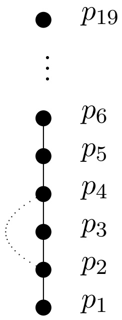

Figure 2. Proximity graph of $\mathcal{B}_{\mathcal{F}}$ in Examples 2 and 4

Example 3. Consider now the foliation $\mathcal{F}$ defined by the projective differential 1-form $\mathbf{\Omega} = AdX_0 + BdX_1 + CdX_2$, where

$$A = 8X_0^4X_1^2 + 10X_0X_1^5 + 2X_0^5X_2 - 4X_0^2X_1^3X_2 - 4X_0^3X_1X_2^2 - 4X_1^4X_2^2 + 2X_0X_1^2X_23,$$

$$B = -8X_0^5X_1 - 10X_0^2X_1^4 + 10X_0^3X_1^2X_2 + 5X_1^5X_2 - X_0^4X_2^2 - 2X_0X_1^3X_2^2 + 2X_0^2X_1X_2^3 - X_1^2X_2^4,$$

$$C = -2X_0^6 - 6X_0^3X_1^3 - 5X_1^6 + 5X_0^4X_1X_0 + 6X_0X_1^4X_2 - 4X_0^2X_1^2X_2^2 + X_1^3X_2^3.$$

$\mathcal{B}_{\mathcal{F}} = \{p_i\}_{i=1}^{10}$ and the reader can see its proximity graph in Figure 3. $E_{p_{10}}$ is the unique dicritical divisor and it can be checked that

$$\Delta_{\mathcal{F}} = 10[L^*] - 4\sum_{i=1}^{6}[E_{p_i}^*] - \sum_{i=7}^{10}[E_{p_i}^*].$$

Applying Algorithm 2, $\pi_{\mathcal{F}*}|\Delta_{\mathcal{F}}|$ is a pencil that defines a rational first integral of $\mathcal{F}$ (of genus 0, by the adjunction formula).

We can obtain the same conclusion applying the algorithm in Remark 5. Indeed, it holds that $C_1 := 2L^* - \sum_{i=1}^5 E_{p_i}^*$ belongs to $\Gamma_{\mathcal{F}}(2)$ and, then, $\pi_{\mathcal{F}*}|C_1|$ is a conic (defined by the equation $F_1 := X_1X_2 - X_0^2 = 0$) that is invariant by $\mathcal{F}$. Thus, if $\mathcal{F}$ is algebraically integrable, then the pencil $\mathcal{P}_{\mathcal{F}}$ should be $\pi_{\mathcal{F}*}|\frac{\mathrm{lcm}(2,5+2-2)}{10}\Delta_{\mathcal{F}}| = \pi_{\mathcal{F}*}|\Delta_{\mathcal{F}}|$.

Finally, it can be verified that the divisor $C_2 = 5L^* - 2\sum_{i=1}^6 E_{p_i}^* - E_{p_7}^* - E_{p_8}^*$ belongs to $\Gamma_{\mathcal{F}}(5)$. Therefore the curve $\pi_{\mathcal{F}*}|C_2|$, whose equation is $F_2 = 2X_0^3X_1^2 + X_1^5 + X_0^4X_2 - 2X_0X_1^3X_2 - 2X_0^2X_1X_2^2 + X_1^2X_2^3 = 0$, is an integral component of a non-reduced curve of

$\mathcal{P}_{\mathcal{F}}$. Hence the curves defined by F_1^5 and F_2^2 are the unique non-reduced curves of the pencil $\mathcal{P}_{\mathcal{F}}$ (and determine a rational first integral of $\mathcal{F}$).

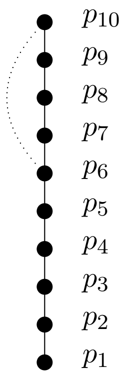

FIGURE 3. The proximity graph of $\mathcal{B}_{\mathcal{F}}$ in Example 3

Example 4. Let $\mathcal{F}$ be the foliation given by the differential 1-form

$$\mathbf{\Omega} = (3X_0^2X_2^3 - X_1^2X_2^3)\, dX_0 - (5X_1^4X_2 - X_0X_1X_2^3)\, dX_1 + (5X_1^5 - 3X_0^3X_2^2)\, dX_2.$$

The configuration $\mathcal{B}_{\mathcal{F}}$ has 19 points $\{p_i\}_{i=1}^{19}$, only one dicritical divisor, $E_{p_{19}}$, and its proximity graph is that of Figure 2. $\Delta_{\mathcal{F}}$ has the same expression than the one of Example 2. It can be checked that $\pi_{\mathcal{F}*}|\Delta_{\mathcal{F}}|$ is the pencil generated by the curves defined by $X_1X_2^4$ and X_2^5. It does not correspond to a rational first integral of $\mathcal{F}$ because it is not an irreducible pencil. As a consequence, $\mathcal{F}$ is not algebraically integrable.

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