ON RECOVERY AND INTENSITY'S CORRELATION - A NEW CLASS OF CREDIT RISK MODELS*

Raquel M. Gaspar,
Advance Research Center,
ISEG, Technical University Lisbon
E-mail: Rmgaspar@iseg.utl.pt

Irina Slinko,
Group Risk Control
Swedbank AB
E-mail: irina.slinko@swedbank.se

WORKING PAPER N. 1/2007

July 2007

Abstract

There has been increasing support in the empirical literature that both the probability of default (PD) and the loss given default (LGD) are correlated and driven by macroeconomic variables. Paradoxically, there has been very little effort from the theoretical literature to develop credit risk models that would include this possibility.
The goals of this paper are: first, to develop the theoretical reduced-form framework needed to handle stochastic correlation of recovery and intensity, proposing a new class of models; and, second, to use concrete instance of our class to study the impact of this correlation in credit risk term structures. Our class of models is able to replicate and explain empirically observed features. For instance, we automatically get that periods of economic depression are periods of higher default intensity and where low recovery is more likely - the well-know credit risk business cycle effect.
Finally, we show how to calibrate this class of models to market data, and illustrate the technique using our concrete instance using US market data on corporate yields.

Key words: Credit Risk; Systematic Risk; Intensity Models; Recovery; Credit Spreads.

JEL Classification: C15, G12, G13, G33

____

* We gratefully acknowledge financial support from Jan Wallander and Tom Hedelius foundation. This research was also partially supported by the Austrian Science Foundation project P18022 at the Vienna University of Technology

Department of Management
Working paper Series
ISSN° 0874-8470

1 Introduction

Recent empirical studies show that there is a significant systematic risk component in defaultable credit spreads. See Altman, Resti, and Sironi (2004), Düllmann and Trapp (2000), Elton and Gruber (2004), Frye (2000a) or Frye (2003). The model underlying the Basel II internal ratings-based capital calculation – see Basel Committee (2003) and Wilde (2001) – measures credit portfolio losses only, that is, portfolio losses that are due to external influences and hence cannot be diversified away. This gives us an indication of what the main concerns are in practice and highlights the need for a realistic model of *systematic risk*.

Both the *probability of default* (PD) and the *loss given default* (LGD) are key in accessing expected capital losses and measuring the exposure of portfolios of defaultable instruments to credit risk. In accessing capital at risk, it is extremely important not to ignore the interdependence between PD and LGD, since this could lead to underestimation of the true risk borne by portfolio holders. In fact, there has been increasing support on the empirical literature agreeing on two observed facts: (i) PD and LGD are correlated and (ii) macroeconomic risks are likely to affect both these variables. See Allen and Saunders (2003), Altman, Braddy, Resti, and Sironi (2005), Frye (2000b), Giese (2005) or Hu and Perrandin (2002). Nonetheless, most of the theoretical literature considers models where only the default intensity, or equivalently the PD, is dependent on a state variable assuming that the LGD is either fixed or at least independent of default intensities. See Wilson (1997), Saunders (1999), JP Morgan (1997), Gordy (2000), or Schönbucher (2001).

The purpose of this study is to, based on the theoretical flexibility of DSMPP, present a reduced-form multiple default family of models that considers the influence of macroeconomic risks on *both* PD and LGD. This leads to the first theoretical family of models that is able to reproduce the empirical facts stated above, i.e., allows for correlated PD and LGD and considers the influence of systematic risks on both variables. The final goal is then to see if including these facts in the theoretical framework will help in explaining observed behavior of the term structure of credit spreads.

As a proxy for macroeconomic conditions we consider a market index. It is well known that market uncertainty and its level are negatively correlated. That is, periods of recession (low index level) also tend to be periods of high uncertainty (high index volatility) reflecting some sort of market panic, while periods of economic boom are perceived as safe periods and with low uncertainty. In setting up the dynamics of the market index, we incorporate this realistic feature by allowing the local volatility of the index to depend negatively on its level.

In terms of the PD and LGD, we assume that default intensity and the recovery given default depend on the market situation (the index level). With the PD dependence we try to account for the fact that during recessions it is reasonable to expect more defaults, while with the LGD dependence we try to account for the fact that if the entire market is down, the market value of any firm's assets should be lower, and debt holders should recover less if a default occurs.

The main contributions of this study can be summarized as follows.

- We suggest a multiple default reduced-form model where we use the flexibility of DSMPP to model the influence of systematic risks on both PD and LGD.
- We derive general qualitative impacts of the macroeconomic risks, on intensity of default, recovery given default and credit spreads, that should hold in realistic theoretical models.
- Using a concrete model, we are able quantify results and simulate realistic behaviors of

the term structure of credit spreads, showing that the correlation between PD and LGD (resulting from the influence of the systematic risk) must be considered.

- Finally, we show how to calibrate our model to market data.

The rest of the paper is organized as follows. In Section 2 we setup the framework and summarize the main theoretic results concerning the use of DSMPP in credit risk models. In Section 3 we propose a family of macroeconomic models, presenting the index dynamics and justifying the assumptions about the influence of such risks on the intensity and recovery processes using empirical facts. We derive qualitative results on the influence of the market index on default intensity, recovery and credit spreads. In Sections 4 we present a concrete instance of our class of models simulate it to access the impacts of our qualitative assumptions in terms credit spread term structures. We end the section discussion calibration issues of models with no closed-form solution and calibrate our own concrete model US market data. Section 5 concludes the paper, summarizing the main results and suggesting directions for future research.

2 The setup and fundamental theoretic results

We consider a financial market living on a filtered probability space $\left(\Omega, \mathcal{F}, \mathbb{Q}, (\mathcal{F}_t)_{0\leq t\leq T}\right)$ where $\mathbb{Q}$ is the risk-neutral probability measure. The probability space carries a multidimensional Wiener process W and, in addition, a doubly stochastic marked point process (DSMPP), $\mu(dt, dq)$, on a measurable mark space $(E, \mathcal{E})$ to model the default events. The filtration $(\mathcal{F}_t)_{0\leq t\leq T}$ is generated by W and μ, i.e. $\mathcal{F}_t = \mathcal{F}_t^W \vee \mathcal{F}_t^\mu$.

2.1 Default-free bond market

We assume the existence of a liquid market for default-free zero-coupon bonds, for every possible maturity T. We denote the price at time t of a default-free zero-coupon bond with maturity T by $p(t, T)$. The instantaneous forward rate with maturity T is denoted $f(t, T)$ and we recall one-to-one correspondence between zero-coupon bond prices and forward rates:

$$f(t,T) = -\frac{\partial \ln p(t,T)}{\partial T} \qquad \Leftrightarrow \qquad p(t,T) = \exp\left\{-\int_t^T f(t,s)ds\right\}. \tag{1}$$

The default-free short rate is denoted $r(t) = f(t, t)$.

2.2 Defaultable bond market

In addition to the risk-free bond market mentioned above, we consider a defaultable bond market. We assume that each company on the market issues a continuum of bonds with maturities T. Assumption 2.1 and Assumption 2.3 below characterize the default events and the dependence of both the default intensity and the recovery rate distribution on an abstract stochastic state variable X.

Assumption 2.1. *There exist an underlying stochastic state variable X, whose dynamics under the risk neutral measure $\mathbb{Q}$ is given by*

$$dX_t = \alpha_X(t, X_t)dt + \sigma_X(t, X_t)dW_t. \tag{2}$$

In our framework we will allow for multiple defaults. Behind this is the intuition that, given a distress situation for the obligator's business, the debt holders are willing to accept the renegotiation of their claims (accepting to lose some fraction q of the face value of the claims) in order to avoid a process of bankruptcy, which is typically costly, and allowing the firm to continue operating.[1] It is possible that a whole sequence of defaults is taking place, every time leading to reduction of the debt's face value and the bondholder accepting the conditions of the deal.

We now define the key notions for the defaultable bond market and formalize the assumptions concerning the occurrence of default events.

Definition 2.2. (Basic Definitions)

- The *loss quota* is the fraction by which the promised final payoff of the defaultable claim is reduced at each time of default. We denote the loss quota by q.
- The *remaining value*, after all reductions in the face value of the defaultable claim due to defaults in the time interval $[0,t]$, is denoted $V(t)$.
- $\bar{p}(t,T)$ is, at time t, the *price of a defaultable zero-coupon bond* with maturity T and the face value 1. The payoff at time T of the bond is, thus, $V(T)$ the remaining part of the face value of the bond after all reductions due to defaults in the time interval $[0,T]$, i.e., $\bar{p}(T,T) = V(T)$.
- We define the *instantaneous defaultable forward rate*, $\bar{f}(t,T)$, similarly to its risk-free equivalent, we have $\bar{p}(t,T) = V(t)\exp\left\{-\int_t^T \bar{f}(t,s)ds\right\}$ and $\bar{f}(t,T) = -\frac{\partial}{\partial T}\ln\bar{p}(t,T)$.
- The *defaultable short rate* is defined as $\bar{r}(t) = \bar{f}(t,t)$.
- The *short credit spread* $s(t)$ is defined as the difference between the defaultable and non-defaultable short rates $s(t) = \bar{r}(t) - r(t)$.
- The *forward credit spread* $s(t,T)$ is defined as the difference between the defaultable short rate and non-defaultable forward rates, $s(t,T) = \bar{f}(t,T) - f(t,T)$.

Assumption 2.3. (Default Events)

1. *We assume that default happens at the following sequence of the stopping times $\tau_1 < \tau_2 < \ldots$, where τ_i is the time of the i-th jump of a point process.*
2. *At each default time τ_i the loss quota q_i is drawn from the mark space $E = (0,1)$.*
3. *We assume that there is no total loss at default, i.e., the loss quota $q_i < 1$ for all $i = 1, 2, \cdots$.*
4. *We assume that both:*

   (i) the arrivals of default times $(\tau_i)_{i\geq 1}$;

   (ii) the distribution of the loss quotas given default $(q_i)_{i\geq 1}$

   depend upon the stochastic state process X.

[1]Multiple default models mimic the effect of a rescue plan as it is described in many bankruptcy codes. The old claimants have to give up some of their claims in order to allow for rescue capital to be invested in the defaulted firm. They are not paid out in cash (this would drain the defaulted firm of valuable liquidity) but in new defaultable bonds of the same maturity. [Schönbucher (2003)]

Given that at each default time τ_i the final claim amount is reduced by a loss quota q_i to $(1-q_i)$ times what it was before, we obtain

$$V(t) = \prod_{\tau_i \leq t} (1 - q_i), \tag{3}$$

where q_i is the stochastic marker to the default time τ_i. Obviously, in case of no default on the interval $[0,T]$, $V(t) = 1$.

2.2.1 The default process

We start by giving the abstract definition of a Marked *Poisson* Point Process and a *Doubly Stochastic* Marked Poisson Process (DSMPP). We introduce first the following filtrations.

Notation 1. (Filtrations)

- The filtration generated by $W(t)$, $\left(\mathcal{F}_t^W\right)_{t\geq 0}$, is the *background filtration*.[2]
- The filtration $\mathcal{G}^W = \bigvee_{t\geq 0} \mathcal{F}_t^W$ contains all future and past information on the background process W .
- The *full filtration* is reached by combining $\left(\mathcal{F}_t^W\right)_{t\geq 0}$ and the filtration $(\mathcal{F}_t^\mu)_{t\geq 0}$, is generated by MPP μ, $\mathcal{F}_t = \mathcal{F}_t^W \vee \mathcal{F}_t^\mu$.
- Finally, $\mathcal{G}_t^W = \mathcal{G}^W \vee \mathcal{F}_t^\mu$, is the filtration generated by *all* the information concerning the background process W, and only past information on the MPP μ.

Definition 2.4. (DSMPP)

- We call the Marked Point Process $\hat{\mu}$ an $\mathcal{F}_t^\mu$- Marked *Poisson* Process if there exists a deterministic measure $\hat{\nu}$ on $\mathbb{R}_+ \times E$ such that

$$\mathbb{P}\left(\hat{\mu}((s,t] \times B) = k \,|\mathcal{F}_s^\mu\right) = \frac{(\hat{\nu}((s,t] \times B))^k}{k!} e^{-\hat{\nu}((s,t]\times B)}, \quad \text{a.s.}, \quad B \in E.$$

- We call the Marked Point Process μ an $\mathcal{G}_t^W$- DSMPP if there exists a $\mathcal{G}^W$-measurable random measure ν on $\mathbb{R}_+ \times E$ such that

$$\mathbb{P}\left(\mu((s,t] \times B) = k \left|\mathcal{G}_s^W\right.\right) = \frac{(\nu((s,t] \times B))^k}{k!} e^{-\nu((s,t]\times B)}, \quad \text{a.s.}, \quad B \in E.$$

We note that the previous literature on credit risk have only used Doubly Stochastic Poisson process (also known as Cox Processes).[3] The focus has been in modeling the jump intensity λ and recovery has been assumed independent of λ. In factor models the state variable would influence only the λ.

---

[2] In our setup, all the default-free processes are adapted to $\left(\mathcal{F}_t^W\right)_{t\geq 0}$.

[3] We recall that a counting process $N = (T_n)$ adapted to right-continuous filtration is a $\mathcal{G}_t^W$ - Cox Process if there is an $\mathcal{G}^W$-measurable random measure ν satisfying

$$\mathbb{P}\left(N(s,t] = k \left|\mathcal{G}_s^W\right.\right) = \frac{(\nu((s,t]))^k}{k!} e^{-\nu((s,t])}, \quad \text{a.s} \quad k \in \mathbb{N} .$$

As far as we know this study is the first using doubly stochastic *marked* point processes in credit risk. Our goal is to consider that both intensity and recovery can be affected by a common state variable. For that we need to model the default events using a $\mathcal{G}_t^W$-DSMPP whose compensator depends the Wiener driven stochastic state variable X presented in (2).

An important result for what follows is the existence of such processes, that is DSMPP with a compensator equal to a given random measure, $M_t(dq, X_t)$, and that can be written as a deterministic function of our state variable. Due to its rather technical level we present the proof of the next Theorem in the appendix.

Theorem 2.5. *Assume that a random measure ν on $\mathbb{R}_+ \times E$ admits representation $M_t(dq, X_t)dt$, where $M_t(dq, x)$ is a deterministic measure on E for any fixed x and t.*

Let $\hat{\nu}(dt, dq) = m_t(dq)dt$ be a deterministic compensator for some Marked Poisson Process $\hat{\mu}$. Assume that:

(i) $M_t(dq, x)$ is measurable w.r.t. $\mathcal{G}^W$

(ii) $M_t(dq, x)$ is absolutely continuous w.r.t. $m(t, dq)$ on $\mathcal{E}$, that is,

$$M_t(dq, x) << m_t(dq)$$

Then, there exists a $\mathcal{G}_t^W$-DSMPP μ, such that its compensator is of the form

$$\nu(dt, dq, \omega) = \nu(dt, dq, X_t) = M_t(dq, X_t)dt, \qquad \mathbb{Q} - a.s. \tag{4}$$

Given existence of DSMPP, we can now focus the the compensator's construction. Ideally we would like to model the intensity of default $\lambda(t, X_t)$ and the instantaneous conditional loss quota distribution $K(t, dq, X_t)$, separately allowing both those quantities to depend on the state varibale X, and not to model the compensator $\nu(dt, dq, X_t)$ at once. We propose what we consider to be a good construction procedure and then justify why it works.

Remark 2.6. (Construction procedure) *We construct the DSMPP μ as follows.*

1. *We specify the Wiener driven stochastic state variable X.*
2. *We specify the intensity $\lambda(t, X_t)$ as a function of the state variable.*
3. *We specify the instantaneous conditional loss quota distribution as a function of the state variable $K(t, dq, X_t)$.* [4]
4. *Finally, we construct the stochastic compensator ν*

$$\nu(dt, dq, X_t) = K(t, dq, X_t)\lambda(t, X_t)dt \ . \tag{5}$$

This construction does work because the multiplication of the two quantities $K(t, dq, X_t)$ and $\lambda(t, X_t)$ is a random measure on $\mathbb{R}_+ \times E$ (see, for example, Last and Brandt (1995)). This together with the result of Theorem 2.5 assures us that, for any given distribution of the loss quota given default $K(t, dq, X_t)$ and given intensity $\lambda(t, X_t)$, there exist a $\mathcal{G}_t^W$-DSMPP, whose compensator can be written as in (5).

[4] In all the practical applications we suppose the instantaneous conditional loss quota distributions $K(t, dq, X_t)$ are absolutely continuous with respect to Lebesgue measure on $\mathcal{E}$, that is, we consider conditional loss quota distributions of the form $K(t, dq, X_t) = \tilde{K}(t, q, X_t)dq$, thus $M_t(dq, x) << dq$. This is enough to cover all the most common conditional distributions.

The possibility to construct the quantities $\lambda(t, X_t)$ and $K(t, dq, X_t)$ separately has the additional advantage of satisfying a consistency requirement needed in any credit risk model. That is, we can derive consistently prices, or other credit risk relevant values, that depend exclusively upon default intensity (like the implied survival probability, the price default digital payoffs or the price defaultable bonds with zero recovery) and that depend on both intensity and recovery (like credit default swaps, recovery swaps, or defaultable bonds with non zero recovery).

2.2.2 Fundamental theoretic results

In this section we state the main results on the short and forward credit spreads. The proof of the next Proposition can be found in the appendix.

Proposition 2.7. *Given Assumption 2.3, and under the martingale measure $\mathbb{Q}$.*

1. *The short credit spreads, $s(t)$, have the following functional form*

$$s(t) = \lambda(t, X_t)q^e(t, X_t) > 0 \tag{6}$$

   where

$$q^e(t, X_t) = \int_0^1 qK(t, dq, X_t)$$

   can be interpreted as the locally expected loss quota (which is positive for $q > 0$).

2. *Then the forward credit spread $s(t,T)$ takes the form*

$$s(t,T) = \frac{\mathbb{E}_t^{\mathbb{Q}}\left[\{r(T) + \lambda(T, X_T)q^e(T, X_T)\}\, e^{-\int_t^T \{r(s)+\lambda(s,X_s)q^e(s,X_s)\}ds}\right]}{\mathbb{E}_t^{\mathbb{Q}}\left[e^{-\int_t^T \{r(s)+\lambda(s,X_s)q^e(s,X_s)\}ds}\right]} - f(t,T) \tag{7}$$

3. *The bond prices can be written as*

$$\bar{p}(t,T) = V(t)\mathbb{E}_t^{\mathbb{Q}}\left[e^{-\int_t^T \bar{r}_s ds}\right] . \tag{8}$$

We end up this section with a small remark concerning the modeling choice of the risk-neutral measure $\mathbb{Q}$ and its consequences in term of the objective probability measure $\mathbb{P}$.

Remark 2.8. **($\mathbb{P}$ considerations)**

Our setup has been defined under the martingale measure $\mathbb{Q}$. It is possible to show that if the market price of jump risk ϕ is a deterministic function of time then:[5]

1. *The $\mathbb{Q}$-default intensity, λ, relates to the $\mathbb{P}$-default intensity λ^P, by*

$$\lambda^P(t, X) = \lambda(t, X)(1 + \phi(t)) \tag{9}$$

2. *The $\mathbb{Q}$-loss quota distribution, conditional on default, $K_t(dq, X)$, equals to the conditional on default loss quota distribution under $\mathbb{P}$, $K_t^P(dq, X)$.*

We see that, in this case, the conditional distribution of the loss quota remains unchanged while intensity changes according to (9), i.e. multiplied by a deterministic function of time. The assumption the market price of jump risk is deterministic allows us to use our objective intuitions in setting up the applied model for macroeconomic risks[6].

[5]This result is a direct application of the Girsanov Theorem. Upon request the authors will gladly provide the exact proof.

[6]The unhappy reader can also interpret the results as only $\mathbb{Q}$-results

3 The Macroeconomic Risks

In this section we discuss the main economic arguments behind our general assumptions. All assumptions that result from empirical facts and economics arguments will be referred to as *properties* [7] and numbered with roman letters (i), (ii), etc.

We start by modeling the systematic risk of the economy. For that we consider what we call a *market index*, I. At all times, we consider both the case when this is the price of an important traded asset (say, oil prices) and the case when it is not the price of any traded asset (e.g. a stock market index).

It has been shown that market index volatility tends to increase when the market as a whole is depressed (low values of the index) and, conversely, volatility decreases when the market index is high, i.e. periods when the market as a whole is depressed are periods of higher volatility, while booms are associated with low volatilities. For example, Jiang and Sluis (1995) show that S&P500 has stochastic volatility. Gaspar (2001) does a comparative study of US and Europe stock markets (using S&P500 and EuroStoxx 50 indices) and shows this feature persists across markets. Selcuk (2005) shows that the innovations to a stock market index and innovations to volatility are negatively related, especially in emerging markets. In order to account for this fact (Property (i)), we consider a local volatility model where index volatility dependent on the index level.

Assumption 3.1. (Market Index)
Under the martingale measure $\mathbb{Q}$, the market index I, satisfies the following stochastic differential equation (SDE)

$$dI_t = \zeta(t)I_t dt + \gamma(t, I_t)I_t dW(t),$$

where γ is a row-vector, W is a $\mathbb{Q}$-Wiener process, and we assume that I is not a price of a traded asset. If I is a price of a traded asset, we replace ζ by the short rate r.

Furthermore, for each entry γ_i, the following holds

$$\frac{\partial \gamma_i}{\partial I}(t, I) < 0 \tag{i}$$

It is also reasonable to assume that firms may have different sensitivities to the market index. We, thus, introduce a measure of sensitivity to systematic risk, ϵ, $\epsilon \in [0, 1]$.

We now discuss the impact one would expect the market index I to have at default intensities and recoveries of firms.

Assumption 3.2. (The default Intensity)
The intensity is a deterministic function of (t, I, ϵ). Furthermore, we have ($\bar{\lambda} \in \mathbb{R}_+$)

$$\lambda(t, I, 0) = \bar{\lambda} \tag{ii}$$

$$\frac{\partial \lambda(t, I, \epsilon)}{\partial \epsilon} > 0 \tag{iii}$$

$$\frac{\partial \lambda(t, I, \epsilon)}{\partial I} < 0 \tag{iv}$$

We start by noting that if the firm's financial situation is strong enough, it should not really matter if the economy is booming or if it is in recession. That is, firms that are financially solid

[7] The same properties (qualitative relations) hold under $\mathbb{P}$ and $\mathbb{Q}$ as long as the market price of jump risk is positive, which seems a reasonable. Recall Remark 2.8.

should be much less (or not at all) sensitive to the business cycles than those in a less solid financial position. From this point of, the parameter ϵ, can be also regarded as a measure of a firm's credit worthiness. Firms with high credit worthiness typically tend to be less sensitive to the business cycle influence than less credit worthy firms. If this is so, then it makes sense to have properties (ii) and (iii). Property (ii) tells us that the default probability of some firms may be independent of the market situation. Property (iii) says that firms that are more sensitive to the market (have lower credit worthiness) have higher probability of default.

Finally, from Property (iv), we see that, if companies are sensitive to business cycle, then there is higher probability of default during the recession periods (low I) than in booms (high I).

Assumption 3.3. (Loss Quota)
The conditional distribution of loss quota is a deterministic function of (t, I). K *is a stochastic kernel from* $R_+ \times R_+ \times R_+ \to [0,1]$ *for any realization of* (t, I) .
We denote the cumulative distribution function of loss quota conditional on default as $\tilde{K}$

$$\tilde{K}(t,I,x) = \int_0^x K(t,I,dq), \qquad \int_0^1 K(t,I,dq) = 1, \quad \forall t, I$$

with the following property, $\tilde{K}(t,I_1,x) \geq \tilde{K}(t,I_2,x)$ *if* $I_1 \geq I_2$, $\forall x \in R$, *i.e.,*

$$\frac{\partial \tilde{K}(t,r,I,x)}{\partial I} > 0 \tag{v}$$

$\forall t$, $\tilde{K}(t,I,x)$ stochastically dominates *all the conditional distributions* $\underline{I} \leq I$.

Property (v) assumes it is *more likely*, for the loss quota q, to be below some fixed x when index values are *low* (low I). Indeed, if a firm is in distress and going to restructure its debt during a recession, its assets are worth less and hence debt holders are more likely to accept a higher loss of their debt face value. Moreover, bankruptcy costs tend to be higher in periods of recession, due to the decreased value of firm's assets, emphasizing this effect. From the stochastic dominance assumption above, we can now infer the impacts in terms of expected loss quota.

Lemma 3.4. *Given Assumption 3.3, the following relations hold for the expected value*

$$q^e(t,I) = \int_0^1 qK(t,I,dq), \qquad \frac{\partial q^e(t,I)}{\partial I} < 0 \tag{vi}$$

Proof. We obtain the result integrating by parts, differentiating w.r.t I and using (v). ■

Remark 3.5. (Tractability)
We note that besides the above mentioned properties, K, *conditional on the state variable information, must be the distribution of a random variable taking values in* $(0,1)$, *and the intensity* λ *must be always positive. It is, thus, extremely hard to find a treatable model where these two facts together with properties (i)-(vi) are satisfied. In particular, we have found that no model of affine or quadratic spreads*[8] *will verify all the above properties.*

Given these tractability difficulties we go on with the analysis and draw *qualitative* results of the influence of the market index on credit spreads.

[8] Outside the class of affine or quadratic spread models it is basically impossible to find closed-form solutions. See also discussion in Gaspar and Schmidt (2007) or Chen, Filipović, and Poor (2004).

Remark 3.6. (Short spread impact)
Given the results in Proposition 2.7, Assumption 3.2 and Lemma 3.4, the short credit spread can be rewritten as a function of $(t, , I, \epsilon)$ *and*

$$s(t, I, \epsilon) = \lambda(t, r, I, \epsilon) q^e(t, I) \ . \tag{10}$$

Furthermore we have $s(t, I, 0) = \bar{\lambda} q^e(t, I)$, $\dfrac{\partial s(t, I, \epsilon)}{\partial \epsilon} = \underbrace{\dfrac{\partial \lambda(t, I, \epsilon)}{\partial \epsilon}}_{>0} q^e(t, I) > 0$ *and*

$$\frac{\partial s(t, I, \epsilon)}{\partial I} = \underbrace{\frac{\partial \lambda(t, I, \epsilon)}{\partial I}}_{<0} q^e(t, I) + \lambda(t, I, \epsilon) \underbrace{\frac{\partial q^e(t, I)}{\partial I}}_{<0} < 0.$$

We note that, given a concrete functional form for the intensity λ and for the loss quota distribution the above effects on the short spread in (10) can actually be quantified. Unfortunately, this is not going to be the situation when dealing with forward credit spreads. For the forward credit spreads $s(t, T)$ we obtain expressions in terms of expectations (see Equation (7)) that must be simulated.

4 A concrete model

In this section we illustrate the previous theoretical results using a concrete instance of our class of models. We aim to highlight the importance of considering the dependence between recovery and intensity of default, showing impacts obtained are substantial. We are more concerned with showing the applicability of the results than constructing a extremely realistic (necessary complicated) model. For that reason, our concrete model is as simple as possible. This has the additional advantage of more tractable formulas and a better understanding of what drives the simulation results. The theoretical results apply, of course, more generally many more examples could have been considered.
In order to have a concrete model we need to:

- establish the dependence of the volatility of the index γ, on the index level;
- provide the intensity functional form for λ, in terms of (t, I, ϵ);
- decide on a distribution for the loss quota q for all possible (t, I).

In the model we take the risk-free rate, r, as constant and abstain from considerations about the term structure of risk-free interest rates. Although unrealistic, it is not harmful to our goal of understanding the impact on *spreads*. I is assumed to be the price of a traded asset. To consider a non-traded asset, we would need further considerations on the market price of index risk. For simplicity we also take all functions to be time homogenous; the extension to non-time homogeneous functions is straightforward. Given these simplifications, and to have a completely specified model to simulate, we need to define a function $\gamma(I)$ for the index volatility, a function $\lambda(I, \epsilon)$ for the intensity, and a distribution function $K(dq, I)$ for the loss quota.

We start by defining a ratio, m, which relates the current value of the index to its long-run trend value. Let us define

$$m(I) = \frac{\bar{I}}{I},$$

where $\bar{I}$ is the long-run trend value and a priori given.

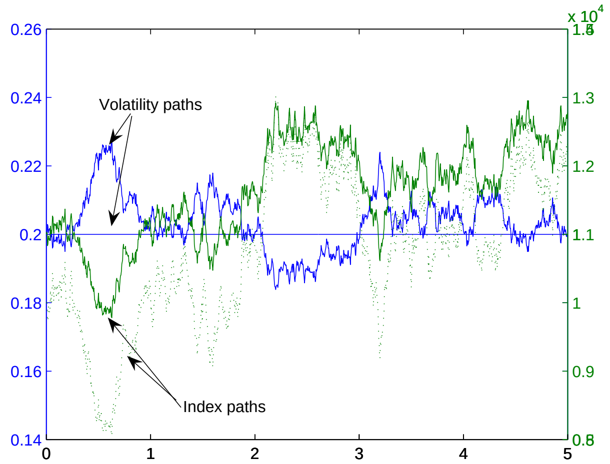

Figure 1: Two paths for the index level and volatility. The same noise was used for both cases, and we took $\bar{I} = 10000$ and $I_0 = 10000$. Case 1: constant volatility $\gamma = 0.2$, the index process is the full line. Case 2: stochastic volatility as in (11), the index process is the dotted line.

The ratio m measures how close or far away from the long-run trend value parameter, $\bar{I}$, the current value of the index I is. Intuitively, it seems reasonable to make all our functions dependent on some relative value of the index, instead on its absolute value. $\bar{I}$ will be assumed to grow at the risk-free rate over time. A reasonable range for $m(I)$ is the interval $[0.7, 1.3]$. We note that the higher the current level of the index the lower is $m(I)$, i.e. $\dfrac{\partial m}{\partial I} = -\dfrac{\bar{I}}{I^2} < 0$. That is, a value of, say, $m = 0.7$ refers to a bull market while $m = 1.3$ refers to a bear market.

4.1 The market index volatility γ

Based on the ratio m we define the *volatility of index*, $\gamma(I)$, in the following way,

$$\gamma(I) = \bar{\gamma}\,(m(I))^{\frac{1}{2}} \qquad \forall I, \bar{\gamma} \in \mathbb{R}_+ \ . \tag{11}$$

Agreeing with Assumption 3.1, the higher the current value of the index the lower is the index volatility γ,

$$\frac{\partial \gamma(I)}{\partial I} = \underbrace{\bar{\gamma}\frac{1}{2}}_{>0}\,\underbrace{[m(I)]^{-\frac{1}{2}}}_{>0}\,\underbrace{\frac{\partial m(I)}{\partial I}}_{<0} < 0 \quad \forall I > 0 \ .$$

Figure 1 shows us two possible paths for the index process, one assuming γ to be just a constant and the other where the index volatility depends on the index level as in (11).

4.2 The default intensity λ

Having defined the index volatility we now define the intensity function

$$\lambda(I, \epsilon) = \bar{\lambda}\,[m(I)]^{\epsilon} = \frac{\bar{\lambda}}{\bar{\gamma}}\,[m(I)]^{\epsilon - \frac{1}{2}}\,\gamma(I) \quad \text{for } \bar{\lambda} \in \mathbb{R}+ \ \text{ and } \epsilon \in [0, 1] \ .$$

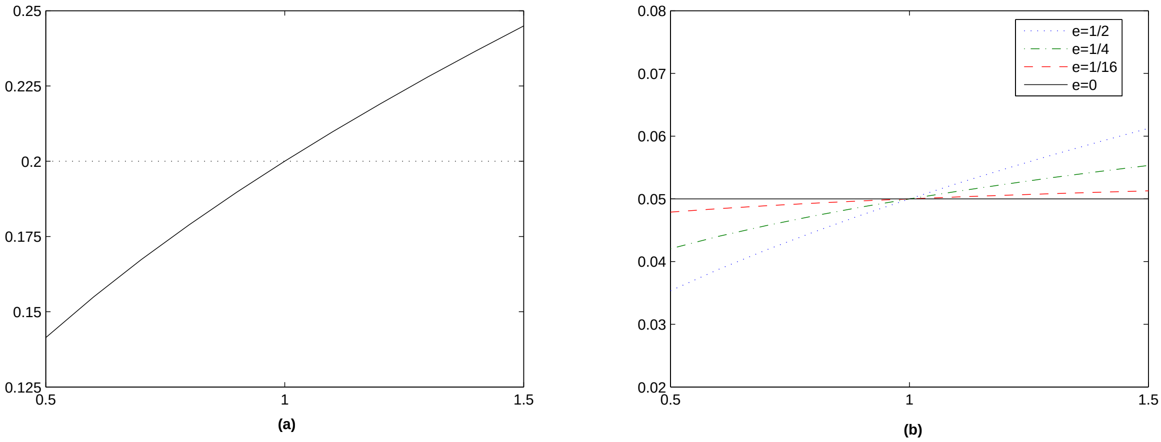

Figure 2: (a) : $\gamma(I)$ for different levels of $m(I)$ vs naive constant volatility $\bar{\gamma} = 0.2$. (b):$\lambda(I)$, for different levels of $m(I)$ and different $\epsilon = 0, 1/16, 1/4, 1/2$, $\bar{\lambda} = 0.05$.

We note that we can interpret the intensity function as a function of the index level or, if we prefer, as a function of the index volatility. One can argue that the intensity should not be affected by index level, but instead by its volatility since it is the volatility that represents the "risk". The above definition includes the two possibilities.

$$\frac{\partial \lambda}{\partial I} = \underbrace{\bar{\lambda}\epsilon}_{>0} \underbrace{(m(I))^{\epsilon-1}}_{>0} \underbrace{\frac{\partial m(I)}{\partial I}}_{<0} < 0 \quad , \quad \frac{\partial \lambda}{\partial \gamma} = \underbrace{\frac{\bar{\lambda}}{\bar{\gamma}}}_{>0} \underbrace{[m(I)]^{\epsilon - \frac{1}{2}}}_{>0} > 0.$$

Figure 2 show the functions $\lambda(I)$ and $\gamma(I)$ for different values of $m(I)$.

4.3 The loss quota q

Finally, we need to decide on the loss quota distribution. As before, we make use of the m ratio for defining the of loss process distribution on the market index I. We choose the Beta class of distributions. [9] We take

$$q \sim Beta\,(2m(I), 2) \quad \text{i.e.} \quad a = 2m(I) \quad \text{and} \quad b = 2\ , \tag{12}$$

which is consistent with the desired properties referred in Assumption 3.3. Thus,

$$\tilde{K}(q, I) = \frac{1}{B(2m(I), 2)} \int_0^q x^{2m(I)-1}(1-x)dx\ .$$

Figure 3 shows the loss quota density and its cumulative distribution function for three different values of m: $m = 0.7$ representing a bull market, $m = 1$ for the case where the market is at its long run level, and $m = 1.3$ representing a bear market. From the properties of the Beta

[9] Recall the beta density function is given by $f(x) = \frac{1}{B(a,b)} x^{a-1}(1-x)^{b-1}\mathbf{1}_{(0,1)}(x)$ where $a > 0$, $b > 0$ and $B(a,b) = \int_0^1 x^{a-1}(1-x)^{b-1}ds$ (beta function). Useful properties of the beta distribution are

$$\mathbb{E}\left[X\right] = \frac{a}{a+b} = \mu\ , \quad \text{Var}\, X = \frac{ab}{(a+b+1)(a+b)^2}\ , \quad \mathbb{E}\left[(X-\mu)^r\right] = \frac{B(r+a,b)}{B(a,b)}\ .$$

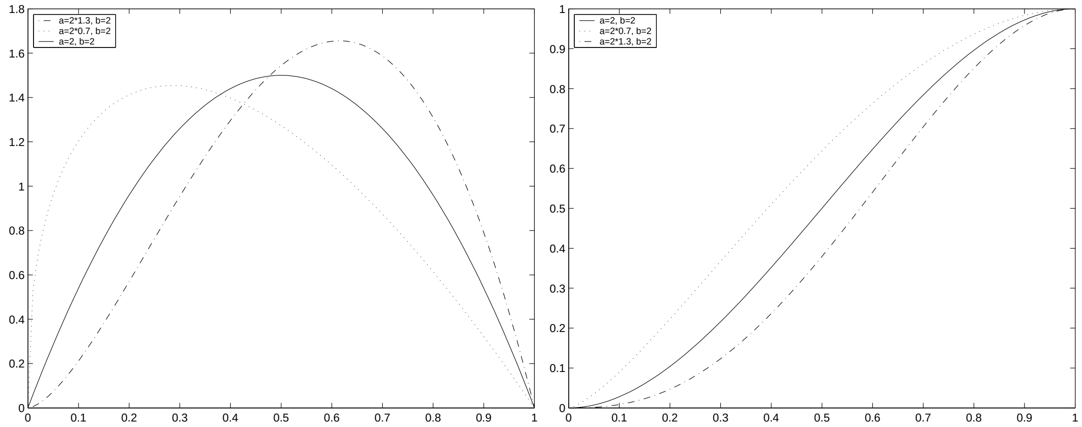

Figure 3: Density and Cumulative distribution functions of loss quota for $m = 1.3$,$m = 1$,$m = 0.7$

distribution we get that the expected loss is given by

$$q^e(I) = \mathbb{E}\left[q(I)\right] = \frac{m(I)}{1 + m(I)}, \qquad \text{and} \qquad \frac{\partial q^e(I)}{\partial I} = \frac{\frac{\partial m(I)}{\partial I}}{\left(1 + m(I)\right)^2} < 0\ . \tag{13}$$

Furthermore

- if default occurs exactly at the long-run level the loss expected quota is exactly 1/2;
- if default occurs when the index level is “high” ($m < 1$) one expects to recover more, expected loss quota decreases;
- if default occurs when the index level is “low” ($m > 1$) one expects to recover less, expected loss quota increases.

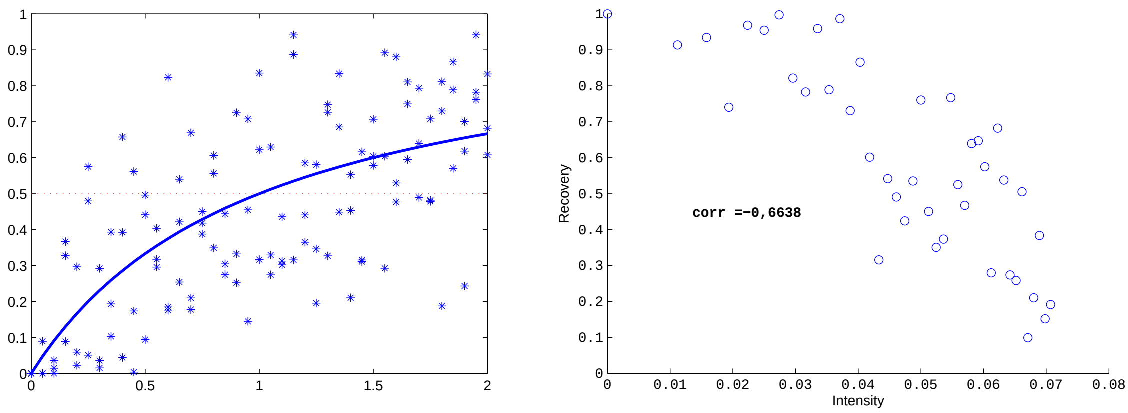

Figure 4: Left: Loss quota possible realizations and expected value for different values of m. Dotted line is the naive $q = \frac{1}{2}$. Right: Scatter plot of intensity versus a recovery realization for different values of m.

The figure above shows on the left-hand side possible realizations of the loss quota (drawn from the beta density with the appropriate mean for each m) (stars), the expected loss quota levels for different values of m (full line) and the naive approach of taking $\bar{q} = \frac{1}{2}$ (dotted line). On the right-hand side it shows a scatter plot of λ versus one possible recovery realization for different levels of the index.

4.4 Simulation Results

In simulations we use the Monte Carlo method where the step size is equivalent to one trading day (we do 250 steps per year) and all simulations concern 5,000 paths. The same noise matrix is used for all scenarios and cases so that the values obtained can actually be compared (discretization errors would be in the same direction for all scenarios). The spreads with zero maturity correspond to the short spread; all other maturities correspond to the forward spread. Table 1 tells us the reference parameters, while Table 2 characterizes all possible scenarios.

REFERENCE PARAMETERS

Maturities (T)	From days up to 5 years	
Risk-free interest rate	5%	
$m(I)$	Case A: bull market	0.7
	Case B: normal market	1.0
	Case C: bear market	1.3
Long-run index value	$10.000e^{0.5*T}$	
Fixed index volatility ($\bar{\gamma}$)	20%	
Fixed intensity value ($\bar{\lambda}$)	5%	
Fixed recovery value ($\bar{q} = \frac{1}{2}$)	50%	

Table 1: Reference values for the parameters in the model.

DIFFERENT SCENARIOS

Scenario	Index Volatility	Intensity	Recovery
(1)	F	F	F
(2)	S	F	F
(3)	F	F	S
(4)	S	F	S
(5)	F	S	F
(6)	S	S	F
(7)	F	S	S
(8)	S	S	S

Table 2: Basic reference scenarios for simulations. F= Fixed, S= Stochastic.

4.4.1 Impacts on the term structure of forward credit spreads

We start by looking into short spread dynamics. Figure 5 presents three possible paths for the short spread under each scenario. Obviously, three paths are not representative, still we believe the intuition is nice and we chose paths with different characteristics. In (a) the market index decreases over time, leading to an increase of the short spreads. In (b) we have a mixed path and in (c) the index value ends up increasing leading to a reduction in the short spreads. From the analysis of this figure, we can conclude that allowing for some stochasticity, either in the intensity process or in the expected loss quota, leads to similar short spread dynamics and that it is the combined effect that really makes the difference. In any of the presented paths, if just one of the effects would be considered, the short spreads do not oscillate more than 1% bellow or above the naive 2.5%, while for the combined effect the variation can be as large as 4% (in the case of path (a) and quite often above 2%. From Figure 6 we see that when we consider

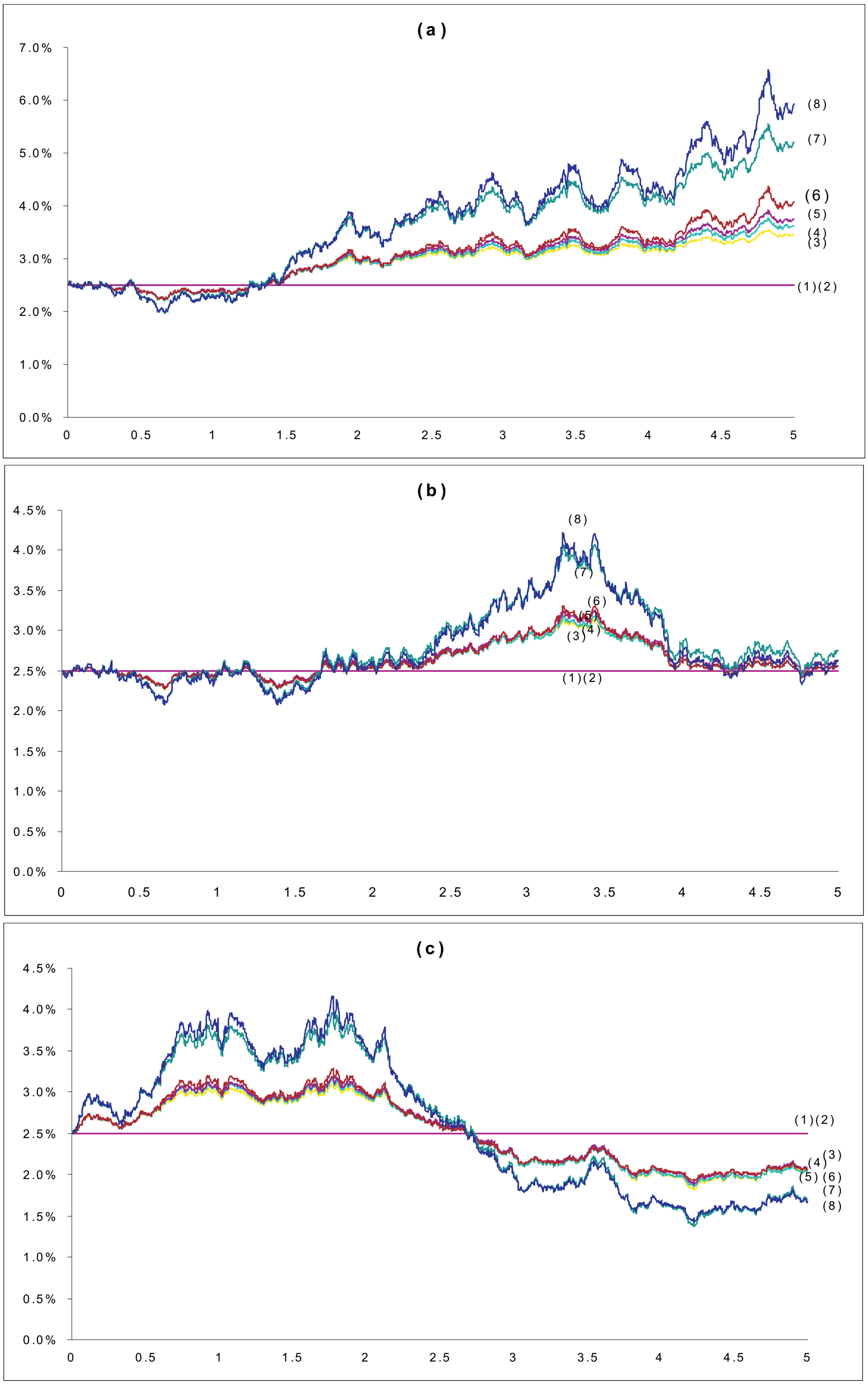

Figure 5: Three possible paths for the short spread, $s(t)$.

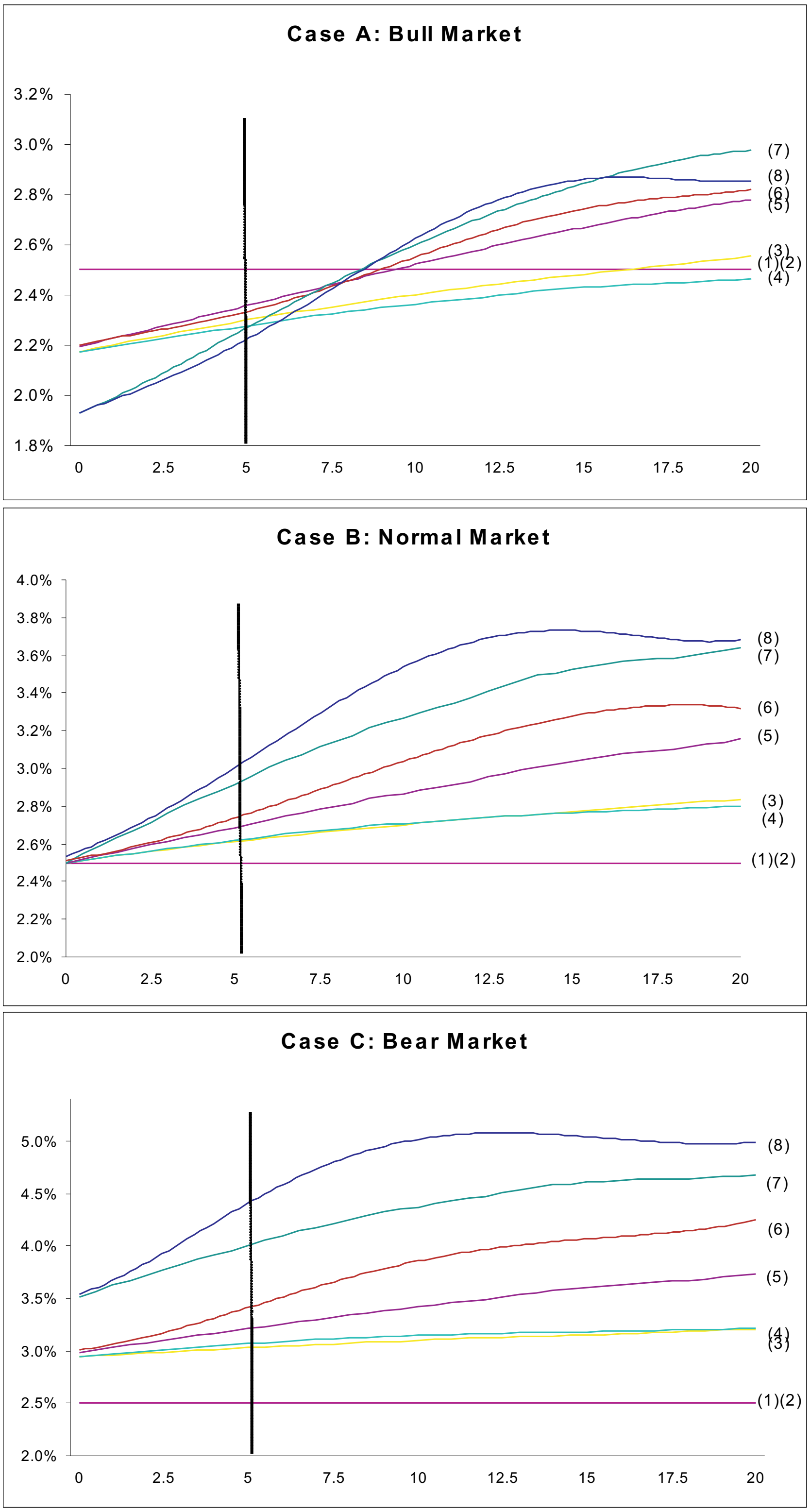

Figure 6: Term structure of forward spreads for all scenarios, under three possible market conditions.

the dependence of both PD and LGD and the negative relation the index level and volatility (scenario (8)), the TS seems to converge faster to its long-run level. In fact, for maturities higher than 15 years the TS of this scenario is relatively flat. Thus, the forward credit spreads are most sensitive to the influence of the market index at the relatively shorter maturities, and around the 15 years maturity, the credit spreads become relatively flatter and less sensitive to the market index, moving in fact closer to each other.

4.4.2 Impacts on pricing and survival probabilities

Tables 3 and 4 presents forward spreads and defaultable bonds prices (non-zero recovery) for several maturities, low and high maturities respectively. The first point that should be highlighted is that even for low maturities there is a difference in the prices produces by naive scenario (scenarios (1) and (2)), scenarios where either the PD or LGD is dependent on the index level (scenarios (3),(4),(5),(6)), and scenarios where we consider the combined effect. For the bull and bear markets the pricing difference in clear already at $T = 0.1$. At 5 year maturities the under pricing of the naive model can be of up to 5% in a bull market, and up to 10% in a bear market. When we consider longer horizons, from 5 to 15 years, in the stochastic volatility scenario, the survival probability decreases by almost 40% for the bull market and up to 50% for the bear market. In our opinion, this is a realistic feature of the model since at the longer horizons when the market is in recession and firms are known to be sensitive to the fluctuations of the market, the probability of default is quite high. Moreover, it is interesting to note that in a bull market, although a stochastic volatility scenario yields higher survival probabilities at all the maturities, the difference in survival probabilities is much smaller at the higher maturities. In a bear market, on the other hand, survival probabilities are lower for the stochastic volatility case, and the difference in survival probabilities between the stochastic volatility and naive scenarios is more pronounced. In contrast to the bull market, the difference increases by approximately 5% when the investment horizon is extended from 5 to 15 years.

4.4.3 How to account for different ratings

We now take a closer look at the parameter ϵ which we recall (Assumption 3.2) is a measure of the sensitivity of a firms PD to the market situation. The intuition comes from the fact the PD of firms with high credit worthiness should depend very little (or not depend at all) on the market oscillation while less credit worthy firms are more sensitive to business cycles. In this sense, different ϵ parameters could represent the term structure of firms with different credit ratings. In the following we consider three different values for *epsilon*: high $\epsilon = 1/2$, medium $\epsilon = 1/4$ and low $\epsilon = 1/16$.[10] Figure 7 and Tables 5 show the simulation results for the different ϵ values under normal market conditions. The key feature is that the TS of less sensitive (higher ratings) firms have a smaller slope. This is particularly obvious for scenarios (7) and (8) when the index influences both PD and LGD, and less obvious when it affects only one of them. Thus, from a practical point of view, it is more important to take into account the correlation with the market index, especially when considering a portfolio of securities with low credit ratings. The effect will be even more pronounced when we have stochastic index volatility.

[10] The case of total insensitivity, or $\epsilon = 0$, is always considered since in scenarios (1)(2)(3)(4) $\lambda(I, \epsilon) = \bar{\lambda}$.

(a) SPREADS

Case A : Bull Market

T	(1)(2)	(3)	(4)	(5)	(6)	(7)	(8)
0	2.500%	2.174%	2.174%	2.192%	2.192%	1.906%	1.906%
0.1	2.500%	2.177%	2.176%	2.196%	2.195%	1.914%	1.912%
0.5	2.500%	2.188%	2.185%	2.209%	2.206%	1.945%	1.937%
1	2.500%	2.201%	2.196%	2.226%	2.220%	1.983%	1.968%
1.5	2.500%	2.215%	2.208%	2.245%	2.236%	2.025%	2.006%
2	2.500%	2.228%	2.219%	2.262%	2.251%	2.063%	2.040%
3	2.500%	2.253%	2.241%	2.296%	2.284%	2.140%	2.112%
5	2.500%	2.299%	2.275%	2.355%	2.334%	2.264%	2.217%

Case B : Normal Market

T	(1)(2)	(3)	(4)	(5)	(6)	(7)	(8)
0	2.5000%	2.5000%	2.5000%	2.5000%	2.5000%	2.5000%	2.5000%
0.1	2.5000%	2.5025%	2.5025%	2.5038%	2.5038%	2.5088%	2.5088%
0.5	2.5000%	2.5124%	2.5126%	2.5189%	2.5193%	2.5440%	2.5450%
1	2.5000%	2.5246%	2.5247%	2.5376%	2.5386%	2.5869%	2.5894%
1.5	2.5000%	2.5362%	2.5378%	2.5573%	2.5612%	2.6327%	2.6416%
2	2.5000%	2.5479%	2.5492%	2.5755%	2.5816%	2.6736%	2.6875%
3	2.5000%	2.5700%	2.5736%	2.6142%	2.6300%	2.7606%	2.7953%
5	2.5000%	2.6121%	2.6145%	2.6836%	2.7178%	2.9082%	2.9769%

Case C : Bear Market

T	(1)(2)	(3)	(4)	(5)	(6)	(7)	(8)
0	2.5000%	2.9413%	2.9411%	2.9881%	2.9880%	3.5157%	3.5155%
0.1	2.5000%	2.9433%	2.9439%	2.9925%	2.9941%	3.5258%	3.5296%
0.5	2.5000%	2.9512%	2.9551%	3.0103%	3.0199%	3.5672%	3.5897%
1	2.5000%	2.9608%	2.9687%	3.0324%	3.0533%	3.6182%	3.6666%
1.5	2.5000%	2.9704%	2.9822%	3.0544%	3.0882%	3.6686%	3.7458%
2	2.5000%	2.9803%	2.9940%	3.0742%	3.1167%	3.7121%	3.8084%
3	2.5000%	2.9982%	3.0201%	3.1189%	3.1985%	3.8114%	3.9857%
5	2.5000%	3.0303%	3.0688%	3.2088%	3.4289%	4.0014%	4.4276%

(b) ZERO-COUPON DEFAULTABLE BONDS W/ RECOVERY

Case A : Bull Market

T	(1)(2)	(3)	(4)	(5)	(6)	(7)	(8)
0.1	0.9925	0.9928	0.9928	0.9928	0.9928	0.9930	0.9930
0.5	0.9631	0.9647	0.9647	0.9646	0.9646	0.9659	0.9659
1	0.9277	0.9306	0.9306	0.9304	0.9304	0.9328	0.9329
1.5	0.8935	0.8976	0.8977	0.8973	0.8974	0.9007	0.9009
2	0.8606	0.8658	0.8659	0.8654	0.8655	0.8696	0.8698
3	0.7985	0.8053	0.8055	0.8046	0.8048	0.8100	0.8104
5	0.6872	0.6963	0.6966	0.6949	0.6953	0.7013	0.7021

Case B : Normal Market

T	(1)(2)	(3)	(4)	(5)	(6)	(7)	(8)
0.1	0.9924	0.9924	0.9924	0.9924	0.9924	0.9924	0.9924
0.5	0.9631	0.9631	0.9631	0.9631	0.9631	0.9630	0.9630
1	0.9277	0.9275	0.9275	0.9275	0.9275	0.9272	0.9272
1.5	0.8935	0.8933	0.8933	0.8931	0.8931	0.8926	0.8926
2	0.8606	0.8602	0.8602	0.8600	0.8599	0.8591	0.8590
3	0.7984	0.7976	0.7975	0.7971	0.7970	0.7953	0.7950
5	0.6872	0.6852	0.6852	0.6840	0.6836	0.6800	0.6792

Case C : Bear Market

T	(1)(2)	(3)	(4)	(5)	(6)	(7)	(8)
0.1	0.9924	0.9920	0.9920	0.9919	0.9919	0.9914	0.9914
0.5	0.9631	0.9609	0.9609	0.9607	0.9606	0.9581	0.9580
1	0.9276	0.9234	0.9234	0.9229	0.9228	0.9178	0.9176
1.5	0.8935	0.8874	0.8873	0.8865	0.8863	0.8789	0.8785
2	0.8606	0.8527	0.8526	0.8515	0.8511	0.8416	0.8407
3	0.7984	0.7872	0.7870	0.7853	0.7844	0.7710	0.7692
5	0.6872	0.6706	0.6700	0.6670	0.6644	0.6452	0.6400

Table 3: (a) Credit Spreads and (b)Price of defaultable bond with recovery, for several short maturities.

(a) SPREADS

Case A : Bull Market

T	(1)(2)	(3)	(4)	(5)	(6)	(7)	(8)
5	2.500%	2.299%	2.275%	2.355%	2.334%	2.264%	2.217%
6	2.5000%	2.3222%	2.2990%	2.3940%	2.3838%	2.3494%	2.3231%
7	2.5000%	2.3432%	2.3166%	2.4248%	2.4137%	2.4120%	2.3825%
8	2.5000%	2.3633%	2.3338%	2.4573%	2.4603%	2.4769%	2.4691%
10	2.5000%	2.4016%	2.3632%	2.5188%	2.5416%	2.5916%	2.6039%
12	2.5000%	2.4371%	2.3910%	2.5774%	2.6189%	2.6953%	2.7190%
15	2.5000%	2.4824%	2.4302%	2.6739%	2.7691%	2.8554%	2.8981%
20	2.5000%	2.5542%	2.4668%	2.7863%	2.8439%	2.9922%	2.8864%

Case B : Normal Market

T	(1)(2)	(3)	(4)	(5)	(6)	(7)	(8)
5	2.5000%	2.6121%	2.6145%	2.6836%	2.7178%	2.9082%	2.9769%
6	2.5000%	2.6310%	2.6404%	2.7281%	2.8012%	3.0045%	3.1441%
7	2.5000%	2.6495%	2.6595%	2.7625%	2.8530%	3.0731%	3.2376%
8	2.5000%	2.6671%	2.6768%	2.7988%	2.9226%	3.1432%	3.3545%
10	2.5000%	2.7011%	2.7054%	2.8669%	3.0235%	3.2643%	3.4930%
12	2.5000%	2.7326%	2.7315%	2.9314%	3.1352%	3.3706%	3.6233%
15	2.5000%	2.7696%	2.7640%	3.0359%	3.2997%	3.5253%	3.7635%
20	2.5000%	2.8352%	2.7992%	3.1547%	3.3834%	3.6428%	3.7094%

Case C : Bear Market

T	(1)(2)	(3)	(4)	(5)	(6)	(7)	(8)
5	2.5000%	3.0303%	3.0688%	3.2088%	3.4289%	4.0014%	4.4276%
6	2.5000%	3.0434%	3.0890%	3.2580%	3.5343%	4.0992%	4.6167%
7	2.5000%	3.0582%	3.1067%	3.2979%	3.6118%	4.1738%	4.7347%
8	2.5000%	3.0720%	3.1210%	3.3398%	3.7093%	4.2486%	4.8668%
10	2.5000%	3.0994%	3.1449%	3.4174%	3.8324%	4.3721%	4.9761%
12	2.5000%	3.1247%	3.1631%	3.4899%	3.9246%	4.4752%	5.0054%
15	2.5000%	3.1505%	3.1813%	3.6048%	4.1097%	4.6074%	5.0888%
20	2.5000%	3.2070%	3.2096%	3.7295%	4.2577%	4.6777%	4.9864%

(b) ZERO-COUPON DEFAULTABLE BONDS W/ RECOVERY

Case A : Bull Market

T	(1)(2)	(3)	(4)	(5)	(6)	(7)	(8)
5	0.6872	0.6963	0.6966	0.6949	0.6953	0.7013	0.7021
6	0.6374	0.6470	0.6475	0.6454	0.6459	0.6518	0.6529
7	0.5913	0.6012	0.6019	0.5993	0.5998	0.6054	0.6066
8	0.5486	0.5586	0.5594	0.5563	0.5568	0.5619	0.5632
10	0.4722	0.4819	0.4829	0.4789	0.4792	0.4832	0.4843
12	0.4064	0.4155	0.4167	0.4118	0.4118	0.4147	0.4155
15	0.3245	0.3321	0.3336	0.3275	0.3268	0.3283	0.3285
20	0.2231	0.2281	0.2299	0.2226	0.2215	0.2209	0.2217

Case B : Normal Market

T	(1)(2)	(3)	(4)	(5)	(6)	(7)	(8)
5	0.6872	0.6852	0.6852	0.6840	0.6836	0.6800	0.6792
6	0.6373	0.6347	0.6346	0.6330	0.6322	0.6276	0.6261
7	0.5913	0.5880	0.5878	0.5858	0.5846	0.5791	0.5768
8	0.5485	0.5446	0.5444	0.5419	0.5402	0.5340	0.5308
10	0.4721	0.4670	0.4668	0.4633	0.4605	0.4531	0.4483
12	0.4064	0.4003	0.4000	0.3956	0.3917	0.3836	0.3776
15	0.3245	0.3172	0.3170	0.3113	0.3060	0.2976	0.2907
20	0.2230	0.2147	0.2148	0.2077	0.2020	0.1938	0.1882

Case C : Bear Market

T	(1)(2)	(3)	(4)	(5)	(6)	(7)	(8)
5	0.6872	0.6706	0.6700	0.6670	0.6644	0.6452	0.6400
6	0.6372	0.6185	0.6176	0.6138	0.6098	0.5888	0.5811
7	0.5911	0.5707	0.5696	0.5650	0.5597	0.5373	0.5274
8	0.5484	0.5264	0.5252	0.5199	0.5132	0.4900	0.4781
10	0.4720	0.4478	0.4463	0.4397	0.4305	0.4067	0.3918
12	0.4063	0.3808	0.3792	0.3713	0.3604	0.3368	0.3207
15	0.3244	0.2983	0.2967	0.2872	0.2748	0.2529	0.2369
20	0.2230	0.1982	0.1969	0.1862	0.1741	0.1561	0.1438

Table 4: (a) Credit Spreads and (b)Price of defaultable bond with recovery, for several long maturities.

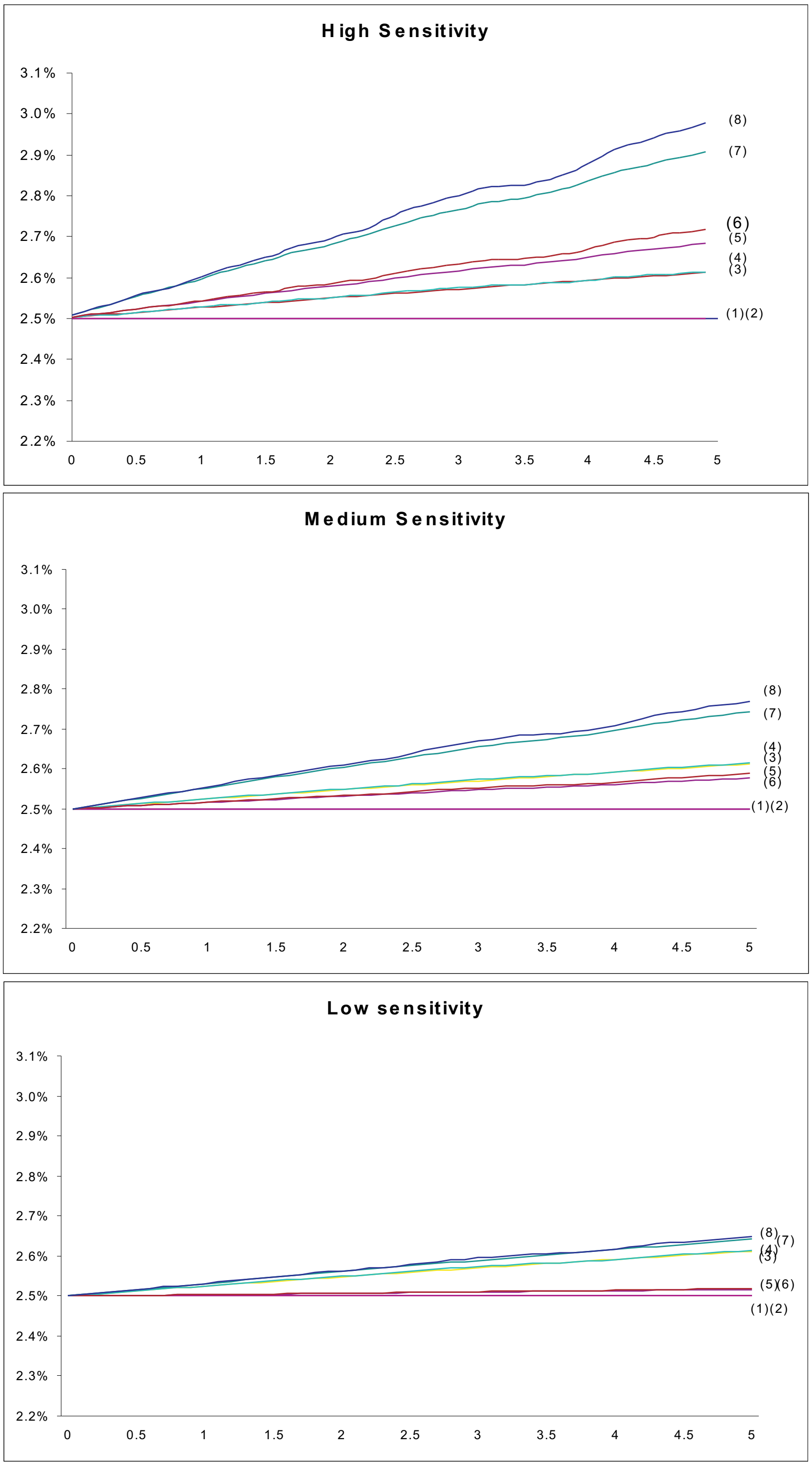

Figure 7: Term structure of forward spreads for all scenarios, under normal market conditions, and for three different values of ϵ: high $\epsilon = \frac{1}{2}$, medium $\epsilon = \frac{1}{4}$ and low $\epsilon = \frac{1}{16}$.

(a) SPREADS

High

T	(1)(2)	(3)	(4)	(5)	(6)	(7)	(8)
0	2.5000%	2.5000%	2.5000%	2.5000%	2.5000%	2.5000%	2.5000%
0.1	2.5000%	2.5025%	2.5025%	2.5038%	2.5038%	2.5088%	2.5088%
0.5	2.5000%	2.5124%	2.5126%	2.5189%	2.5193%	2.5440%	2.5450%
1	2.5000%	2.5246%	2.5247%	2.5376%	2.5386%	2.5869%	2.5894%
1.5	2.5000%	2.5362%	2.5378%	2.5573%	2.5612%	2.6327%	2.6416%
2	2.5000%	2.5479%	2.5492%	2.5755%	2.5816%	2.6736%	2.6875%
3	2.5000%	2.5700%	2.5736%	2.6142%	2.6300%	2.7606%	2.7953%
5	2.5000%	2.6121%	2.6145%	2.6836%	2.7178%	2.9082%	2.9769%

Medium

T	(1)(2)	(3)	(4)	(5)	(6)	(7)	(8)
0	2.5000%	2.5000%	2.5000%	2.5000%	2.5000%	2.5000%	2.5000%
0.1	2.5000%	2.5025%	2.5025%	2.5016%	2.5016%	2.5054%	2.5053%
0.5	2.5000%	2.5124%	2.5126%	2.5079%	2.5080%	2.5266%	2.5271%
1	2.5000%	2.5246%	2.5247%	2.5157%	2.5160%	2.5525%	2.5535%
1.5	2.5000%	2.5362%	2.5378%	2.5236%	2.5252%	2.5792%	2.5837%
2	2.5000%	2.5479%	2.5492%	2.5313%	2.5335%	2.6038%	2.6099%
3	2.5000%	2.5700%	2.5736%	2.5471%	2.5527%	2.6544%	2.6697%
5	2.5000%	2.6121%	2.6145%	2.5768%	2.5879%	2.7433%	2.7701%

Low

T	(1)(2)	(3)	(4)	(5)	(6)	(7)	(8)
0	2.5000%	2.5000%	2.5000%	2.5000%	2.5000%	2.5000%	2.5000%
0.1	2.5000%	2.5025%	2.5025%	2.5003%	2.5003%	2.5032%	2.5032%
0.5	2.5000%	2.5124%	2.5126%	2.5017%	2.5017%	2.5156%	2.5159%
1	2.5000%	2.5246%	2.5247%	2.5033%	2.5034%	2.5310%	2.5313%
1.5	2.5000%	2.5362%	2.5378%	2.5050%	2.5053%	2.5460%	2.5482%
2	2.5000%	2.5479%	2.5492%	2.5067%	2.5070%	2.5606%	2.5629%
3	2.5000%	2.5700%	2.5736%	2.5100%	2.5110%	2.5891%	2.5951%
5	2.5000%	2.6121%	2.6145%	2.5165%	2.5184%	2.6419%	2.6491%

(b) ZERO-COUPON DEFAULTABLE BONDS W/ RECOVERY

High

T	(1)(2)	(3)	(4)	(5)	(6)	(7)	(8)
0.1	0.9924	0.9924	0.9924	0.9924	0.9924	0.9924	0.9924
0.5	0.9631	0.9631	0.9631	0.9631	0.9631	0.9630	0.9630
1	0.9277	0.9275	0.9275	0.9275	0.9275	0.9272	0.9272
1.5	0.8935	0.8933	0.8933	0.8931	0.8931	0.8926	0.8926
2	0.8606	0.8602	0.8602	0.8600	0.8599	0.8591	0.8590
3	0.7984	0.7976	0.7975	0.7971	0.7970	0.7953	0.7950
5	0.6872	0.6852	0.6852	0.6840	0.6836	0.6800	0.6792

Medium

T	(1)(2)	(3)	(4)	(5)	(6)	(7)	(8)
0.1	0.9924	0.9924	0.9924	0.9924	0.9924	0.9924	0.9924
0.5	0.9631	0.9631	0.9631	0.9631	0.9631	0.9630	0.9630
1	0.9277	0.9275	0.9275	0.9276	0.9276	0.9274	0.9274
1.5	0.8935	0.8933	0.8933	0.8934	0.8933	0.8930	0.8930
2	0.8606	0.8602	0.8602	0.8604	0.8603	0.8597	0.8597
3	0.7984	0.7976	0.7975	0.7979	0.7978	0.7966	0.7964
5	0.6872	0.6852	0.6852	0.6859	0.6858	0.6829	0.6826

Low

T	(1)(2)	(3)	(4)	(5)	(6)	(7)	(8)
0.1	0.9924	0.9924	0.9924	0.9924	0.9924	0.9924	0.9924
0.5	0.9631	0.9631	0.9631	0.9631	0.9631	0.9631	0.9631
1	0.9277	0.9275	0.9275	0.9276	0.9276	0.9275	0.9275
1.5	0.8935	0.8933	0.8933	0.8935	0.8935	0.8932	0.8932
2	0.8606	0.8602	0.8602	0.8606	0.8606	0.8601	0.8601
3	0.7984	0.7976	0.7975	0.7983	0.7983	0.7974	0.7973
5	0.6872	0.6852	0.6852	0.6869	0.6869	0.6847	0.6846

Table 5: (a) Credit Spreads and (b)Price of defaultable bond with recovery for several maturities and three different values of ϵ: high $\epsilon = \frac{1}{2}$, medium $\epsilon = \frac{1}{4}$ and low $\epsilon = \frac{1}{16}$.

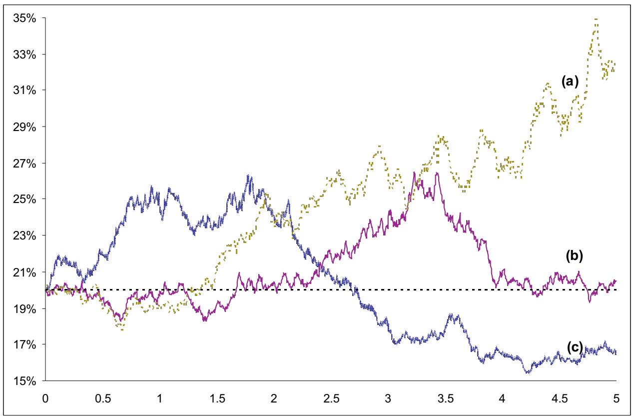

Figure 8: Volatility paths corresponding to the spot spread paths.

4.4.4 Using implied ATM volatilities as credit spread trackers

An interesting side effect of our concrete model is that, when we take the index volatility to be stochastic and negatively related to the index level, the short spread dynamics can be tracked quite well by observing the index volatility. See Figure 8 with three possible volatility paths and compare with the short spread evolution in Figure 5. An interesting conjecture arises: *since the (spot) volatility seems to be a good tracker of the short spread, then implied volatilities of options with longer maturities may be good trackers of the forward spread TS.* This is all due to the negative correlation between the index level and its volatility. Still, it provides a fundamental reason for using implied volatilities of options on indices as predictors of forward credit spread term structures, which seems to be common practice among traders (who use ATM volatility term structures as predictors). Collin-Dufresne, Goldstein, and Martin (2001) also investigated the determinants of credit spread and showed that credit spreads are mostly driven by a single common factor and that implied volatilities of index options contain important information for credit spreads[11].

4.5 Calibration of the model to market data

Lack of the closed form solutions complicates the calibration procedure. In order to calibrate our concrete model to the observed credit spreads, we specify its intensity and expected loss quota as a perturbation of the naive model. We then *linearize* the observed bond prices in the perturbation parameter around the value $k = 0$. Concretely, we specify the intensity and the loss quota as follows

$$\lambda(m(I)) = \lambda \{1 + km(I)\}^{\epsilon} \qquad q^e(m(I)) = \frac{a(1 + km(I))}{a(1 + km(I)) + b}, \tag{14}$$

[11] Recent papers (see e.g. Cremers, Dreissen, and Weinbaum (2004)) start using measures of volatility and skewness based also on individual stock options to explain credit spreads on corporate bonds. Implied volatilities of individual options are shown to contain important information for credit spreads. They showed that those implied volatilities improve on both implied volatilities of index options. However, in the suggested framework we cannot model this feature since the reduced models do not allow us to model stock and corporate bonds together.

Estimation results

Default intensity λ	0.0105	(0.0039)
Sensitivity ϵ_{Aaa}	0.0737	(1.0555)
Sensitivity ϵ_{Bbb}	0.9001	(3.7663)
Volatility of index γ	0.35	(0.0017)
Beta-distribution param. a	1.8587	(0.0984)
Beta-distribution param. b	11.5947	(0.2286)
Perturbation param. k	1.6738	(0.1594)

Table 6: Least-square estimates of the model parameters (standard deviations in brackets).

where we assume that the Beta distribution, for the loss quota, has parameters $a(1 + km(I))$ and b.

We observe that if $k = 0$ the model reduces to the naive model (constant intensity λ and constant expected loss quota $q^e = \frac{a}{a+b}$).

Our data consists of daily data from august 2004 to march 2007, on benchmark yields of Moody's Aaa and Bbb rated long term US corporate bonds. We use long term US government yields as a proxy to the risk-free short rate the S&P 500 as our market index. Since the data comes in *yields* we note that the yield spread between the defaultable and non-defaultable bonds can be computed as follows

$$\bar{y}(t,T) - y(t,T) = -\frac{1}{T-t}\int_t^T s(t,u,k)du, \tag{15}$$

where $s(t,T,k)$ can be found as in (7).

In order to be able to calibrate our model to the observed credit yield spreads we linearize $s(t,T,k)$ around the value $k = 0$, that is we find the linear perturbation of the benchmark model in the parameter k.

$$\bar{y}(t,T) - y(t,T) = -\frac{1}{T-t}\int_t^T \left\{ s(t,u,0) + \frac{\partial s(t,u,0)}{\partial k}k \right\} du$$

$$= \frac{\lambda a}{a+b} + \frac{\lambda a}{(a+b)^2}\frac{(1+\epsilon)(a+b)-a}{T-t}\frac{1}{\frac{\gamma^2}{2}-r}\left\{ e^{(T-t)(\frac{\gamma^2}{2}-r)} - 1 \right\}\frac{\bar{I}}{I_t}\Delta k \tag{16}$$

We use our model to create model-consistent time series of yields of corporate bond and find the set of parameters which minimizes the difference (in the least square sense) to the observed term structure. We present the estimation results in Table 6. Figure 9 shows the *estimated* loss quota density function for three different values of $m(I)$: $m = 0.7$ representing a bull market, $m = 1$ for the case where the market is at its long run level, and $m = 1.3$ representing a bear market.

5 Conclusions and future research

In terms of general contributions, we introduce DSMPP intro credit risk modeling and propose a class of reduced-form models where both the PD and LGD are dependent on a macroeconomic index. We explain the economics intuition behind the fact that during recessions both the PD and the LGD increase (the reverse happens during economic booms) and relate that to

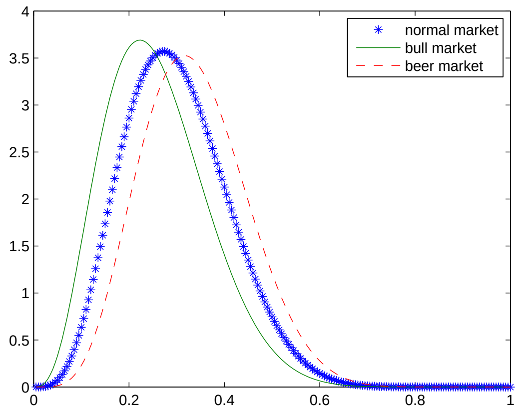

Figure 9: Estimated beta-distribution for loss quotas

qualitative properties of both the intensity and the loss quota distribution. Finally, we discuss (in) existence of tractable models that would that into account all the required properties.

We then use a concrete (simple) instance of our class of model and use simulations to compute survival probabilities, defaultable bond prices, and forward spread term structures and show that it is possible to account for many empirically observed features, such as:

- the difference between short spreads in bull versus bear markets can be up to three times more than the difference produced by models that consider the market influence in the PD or LGD only;
- the convergence to long-run levels is faster, originating flat TS for maturities higher than 15 years;
- market volatility tracks the short credit spread dynamics quite well, suggesting that the TS of ATM implied volatilities of index options may do the same for forward credit spreads.

Given the simplicity of the proposed concrete model we found this extremely encouraging. We also show how we can calibrate models with no closed-form solutions to market data and use US market data to calibrate our own concrete model.

For future research, and since cannot hope for closed-form solutions in our class of models (maybe this is the right price to pay, given all we can account for) we suggest that new concrete models should be analyzed where both the intensity and the distribution of the loss quota should be modeled as realistically as possible (this may involve different functional forms and a different market price of jump risk assumption). Also, a study of the credit TS shapes observed in the market can help to define such functional forms. In addition to the single firm setup, this framework could also be extended to several firms and help deal with portfolio credit risk issues. For portfolio credit risk, the relation between PD and LGD, is likely to be even more important than when considered at the firm level. In fact, portfolio losses depend upon both quantities, and the fact that periods when default is more likely may also be periods when recovery is lower suggests caution in using naive models to establish bank reserves and related precautionary measures. Finally, our last comment on volatility trackers may help to construct a bridge between equity and credit markets, and also deserves further investigation.

A Appendix

Proof of Theorem 2.5:

Proof. We fix $\left(\Omega, \mathcal{F}, \mathbb{P}, (\mathcal{F}_t)_{0\leq t\leq T}\right)$ and a Marked Point Process $\hat{\mu}$ with the compensator

$$\hat{\nu}(dt, dq) = m_t(dq)dt$$

and, as before, $\mathcal{G}_t^W = \mathcal{G}^W \vee \mathcal{F}_t^\mu$. Since $M_t(x, dq)$ is absolutely continuous w.r.t $m_t(dq)$ on $\mathcal{E}$, then according to the Radon-Nikodym Theorem for every t there exists a $\mathcal{E} \times \mathcal{G}^W$-measurable nonnegative function $\varphi_t(q, x)$, $\varphi : E \times R_+ \to R_+$, such that

$$M(t, Ax) = \int_A \varphi(t, q, x) m(t, dq), \quad \text{for all} \quad A \in \mathcal{E}$$

or

$$M(t, dq, x) = \varphi(t, q, x) m(t, dq).$$

We define the process L_t as

$$\begin{cases} dL_t &= L_{t-} \int_E \{\varphi(t, q, X_t) - 1\} \{\hat{\mu}(dt, dq) - m_t(dq)dt\} \\ L_0 &= 1. \end{cases}$$

We notice that $\varphi(t, q, X_t) \in \mathcal{G}_0^W$. Define the new measure on $\mathcal{G}_t^W$, $0 \leq t \leq T$ as $d\mathbb{Q} = L_t d\mathbb{P}$. According to the Girsanov transformation the $\mathbb{Q}$-compensator of the new process is exactly

$$\nu(dt, dq) = \hat{\nu}(dt, dq)(1 + \varphi_t(q, X_t) - 1) = \varphi_t(q, X_t) m_t(dq)dt = M_t(dq, X_t)dt.$$

First, we would like to show that the $\mathbb{Q}$-distribution of ν is the same as the $\mathbb{P}$-distribution. We note that $\mathcal{G}_0^W = \mathcal{G}^W$ and that

$$\frac{d\mathbb{Q}}{d\mathbb{P}}\Big|_{\mathcal{G}_0^W} = L_0 = 1 \ ,$$

thus, $\mathbb{P} = \mathbb{Q}$ on $\mathcal{G}_0^W$. Second, we would like to show that

$$\mathbb{P}\left(\mu((s,t] \times B) = k \,\middle|\, \mathcal{G}_s^W\right) = \frac{(\nu((s,t] \times B))^k}{k!} e^{-\nu((s,t]\times B)}, \quad \text{a.s.,} \quad B \in E. \tag{17}$$

We prove (17) using characteristic functions. Define the stochastic process

$$Y_t = \int_0^t \int_E q \hat{\mu}(ds, dq).$$

Changing the measure we obtain that

$$\mathbb{E}^{\mathbb{Q}}\left[e^{iuY_t} \middle| \mathcal{G}_0^W\right] = \mathbb{E}^{\mathbb{P}}\left[L_t e^{iuY_t} \middle| \mathcal{G}_0^W\right].$$

Define $Z_t = L_t e^{iuY_t}$, then the dynamics of Z_t is

$$\begin{aligned} dZ_t &= L_t \int_E \left\{e^{iu(Y_{t-}+q)} - e^{iuY_{t-}}\right\} \mu(dt, dq) \\ &\quad + L_{t-} e^{iuY_t} \int_E (\varphi(t, q, X_t) - 1) \{\hat{\mu}(dt, dq) - m_t(dq)dt\} \\ &\quad + \int_E L_{t-}(\varphi(t, q, X_t) - 1) \left\{e^{iu(Y_{t-}+q)} - e^{iuY_{t-}}\right\} \hat{\mu}(dt, dq) \\ &= \int_E Z_{t-} \varphi(t, q, X_t) m_t(dq)(e^{iuq} - 1)dt + \int_E Z_{t-}(e^{iuq} - 1)\varphi(t, q, X_t)\tilde{\mu}(dt, dq) \\ &\quad + \int_E Z_{t-}(\varphi(t, q, X_t) - 1)\tilde{\mu}(dt, dq) \end{aligned}$$

where $\tilde{\mu}(dt,dq) = \hat{\mu}(dt,dq) - m_t(dq)$. We notice also that $Z_0 = 1$, then

$$\begin{aligned} Z_t &= 1 + \int_0^t \int_E Z_{s-}\varphi(s,q,X_s) m_t(dq)(e^{iuq}-1)ds + \int_0^t \dots \tilde{\mu}(ds,dq) \\ &= 1 + \int_0^t \int_E Z_{s-}(e^{iuq}-1) M_s(dq,X_s)ds + \int_0^t \dots \tilde{\mu}(ds,dq). \end{aligned}$$

Denote $\xi_t = \mathbb{E}^{\mathbb{P}}\left[Z_t | \mathcal{G}_0^W \right]$, then

$$\xi_t = 1 + \int_0^t \int_E \xi_s (e^{iuq}-1) M_s(dq,X_s)ds.$$

thus since ξ_t does not depend on q and $M_s(dq,X_s)$ is $\mathcal{G}_0^W$-measurable

$$\xi_t = e^{\int_0^t \int_E (e^{iuq}-1) M_s(dq,X_s)ds}.$$

Note that $\nu(dt,dq,X_t) = M_t(dq,X_t)dt$ is $\mathcal{G}^W$ measurable. The final result follows from the fact that the characteristic function of the process

$$\bar{Y}_t = \int_0^t \int_E q\bar{\mu}(ds,dq)$$

where $\bar{\mu}$ is a Marked Poisson Process with compensator $\bar{\nu}(t,dq)$ is given by

$$\mathbb{E}\left[e^{iu\bar{Y}_t} \right] = \exp\left\{ \int_0^t \int_E (e^{iuq}-1)\bar{\nu}(s,dq) \right\}.$$

■

Lemma A.1. *Consider a T-defaultable claim $\mathcal{X}$. For the purpose of computing expectations, and in particular its price at time $t \le T$*

$$\mathbb{E}_t^{\mathbb{Q}}\left[e^{\int_t^T r_s ds} V(T)\mathcal{X} \right],$$

it is equivalent to use the following two dynamics for the remaining value process

$$\frac{dV(t)}{V(t-)} = -\int_0^1 q\mu(dt,dq), \quad V(t) = v \tag{18}$$

and

$$\frac{dV(t)}{V(t-)} = -q^e(t-,X_{t-})dN_t, \quad V(t) = v \tag{19}$$

where μ is a DSMPP with compensator $\nu(t,X_t) = \lambda(t,X_t)K(t,dq,X_t)dt$, N is a Cox process with intensity $\lambda(t,X_t)$, and we define

$$q^e(t,X_t) = \int_0^1 K(t,dq,X_t) \ .$$

Proof. Using the V dynamics in (18) we get,

$$\mathbb{E}^{\mathbb{Q}}\left[e^{-\int_t^T r_s ds}V(T)\mathcal{X}\middle|\mathcal{F}_t\right] =$$

$$\begin{aligned}
&= V(t)\underbrace{\mathbb{E}^{\mathbb{Q}}\left[e^{-\int_t^T r_s ds}\mathcal{X}\middle|\mathcal{F}_t\right]}_{\pi(t,\mathcal{X})} - \mathbb{E}^{\mathbb{Q}}\left[e^{-\int_t^T r_s ds}\int_t^T\int_0^1 qV_{s-}\mu(dq,ds)\mathcal{X}\middle|\mathcal{F}_t\right] \\
&= V(t)\pi(t,\mathcal{X}) - \mathbb{E}^{\mathbb{Q}}\left[\mathbb{E}^{\mathbb{Q}}\left[e^{-\int_t^T r_s ds}\int_t^T\int_0^1 qV_{s-}\mu(dq,ds)\mathcal{X}\middle|\mathcal{G}_t^W\right]\middle|\mathcal{F}_t\right] \\
&= V(t)\pi(t,\mathcal{X}) - \mathbb{E}^{\mathbb{Q}}\left[e^{-\int_t^T r_s ds}\int_t^T V_{s-}\left\{\int_0^1 qK(s,dq,X_s)\right\}\lambda(s,X_s)ds\mathcal{X}\middle|\mathcal{F}_t\right] \\
&= V(t)\pi(t,\mathcal{X}) - \mathbb{E}^{\mathbb{Q}}\left[e^{-\int_t^T r_s ds}\int_t^T V_{s-}q^e(s,X_s)\lambda(s,X_s)ds\mathcal{X}\middle|\mathcal{F}_t\right]
\end{aligned}$$

Using the V dynamics in (19) we get,

$$\mathbb{E}^{\mathbb{Q}}\left[e^{-\int_t^T r_s ds}V(T)\mathcal{X}\middle|\mathcal{F}_t\right] =$$

$$\begin{aligned}
&= V(t)\underbrace{\mathbb{E}^{\mathbb{Q}}\left[e^{-\int_t^T r_s ds}\mathcal{X}\middle|\mathcal{F}_t\right]}_{\pi(t,\mathcal{X})} - \mathbb{E}^{\mathbb{Q}}\left[e^{-\int_t^T r_s ds}\int_t^T V_{s-}q^e(s,X_s)dN(s)\mathcal{X}\middle|\mathcal{F}_t\right] \\
&= V(t)\pi(t,\mathcal{X}) - \mathbb{E}^{\mathbb{Q}}\left[\mathbb{E}^{\mathbb{Q}}\left[e^{-\int_t^T r_s ds}\int_t^T V_{s-}q^e(s,X_s)dN(s)\mathcal{X}\middle|\mathcal{G}_t^W\right]\middle|\mathcal{F}_t\right] \\
&= V(t)\pi(t,\mathcal{X}) - \mathbb{E}^{\mathbb{Q}}\left[e^{-\int_t^T r_s ds}\mathbb{E}^{\mathbb{Q}}\left[\int_t^T V_{s-}q^e(s,X_s)dN(s)\mathcal{X}\middle|\mathcal{G}_t^W\right]\middle|\mathcal{F}_t\right] \\
&= V(t)\pi(t,\mathcal{X}) - \mathbb{E}^{\mathbb{Q}}\left[e^{-\int_t^T r_s ds}\int_t^T V_{s-}q^e(s,X_s)\lambda(s,X_s)ds\mathcal{X}\middle|\mathcal{F}_t\right]
\end{aligned}$$

The results follow from comparing the final expressions on both cases. ■

Proof of Proposition 2.7:

Proof. The time t price of the defaultable zero-coupon bond with maturity T is equal to

$$\bar{p}(t,T) = \mathbb{E}^{\mathbb{Q}}\left[e^{-\int_t^T r_s ds}V(T)\middle|\mathcal{F}_t\right], \tag{20}$$

where $V(T)$ is the residual of the face value after multiple defaults up to time T. Making use of Lemma A.1, instead of $\frac{dV(t)}{V(t-)} = -\int_0^1 q\mu(dt,dq)$ with our DSMPP μ (these dynamics follow directly from (3)), we use

$$\frac{dV(t)}{V(t-)} = -q^e(t-,X_{t-})dN_t$$

where N is the Cox process with intensity $\lambda(t,X_t)$. For every fixed t, define $Z(u)$ as follows

$$Z(u) = e^{\int_t^u q^e(s,X_s)\lambda(s,X_s)ds}V(u).$$

We note that then the dynamics of $Z(u)$ take the form

$$dZ(u) = -Z_{u-} q^e(u-, X_{u-}) \{dN_u - \lambda(u, X_u) du\} \ , \qquad u \geq t, \quad t\text{-fixed}$$

and $Z(u)$ is a $\mathbb{Q}$-martingale conditional on the filtration $\mathcal{F}_t^W$. Thus,

$$\mathbb{E}^{\mathbb{Q}}\left[Z(T) | \mathcal{F}_t^W \right] = Z(t).$$

The price of a defaultable bond is then can be found as

$$\begin{aligned}
\bar{p}(t,T) &= \mathbb{E}^{\mathbb{Q}}\left[e^{-\int_t^T r_s ds} V(T) \Big| \mathcal{F}_t \right] = \mathbb{E}^{\mathbb{Q}}\left[e^{-\int_t^T r_s ds} e^{-\int_t^T q^e(s,X_s)\lambda(s,X_s)ds} Z(T) \Big| \mathcal{F}_t \right] \\
&= \mathbb{E}^{\mathbb{Q}}\left[\mathbb{E}^{\mathbb{Q}}\left[e^{-\int_t^T r_s ds} e^{-\int_t^T q^e(s,X_s)\lambda(s,X_s)ds} Z(T) \Big| \mathcal{G}_t^W \right] \Big| \mathcal{F}_t \right] \\
&= \mathbb{E}^{\mathbb{Q}}\left[e^{-\int_t^T r_s ds} e^{-\int_t^T q^e(s,X_s)\lambda(s,X_s)ds} \mathbb{E}^{\mathbb{Q}}\left[Z(T) | \mathcal{G}_t^W \right] \Big| \mathcal{F}_t \right] \\
&= \mathbb{E}^{\mathbb{Q}}\left[e^{-\int_t^T r_s ds} e^{-\int_t^T q^e(s,X_s)\lambda(s,X_s)ds} Z(t) \Big| \mathcal{F}_t \right] \\
&= V(t)\mathbb{E}^{\mathbb{Q}}\left[e^{-\int_t^T \{r(s) + q^e(s,X_s)\lambda(s,X_s)\}ds} \Big| \mathcal{F}_t \right].
\end{aligned}$$

Using the basic relations between defaultable bond prices and defaultable forward rates in (**??**) we can, thus, write

$$\bar{f}(t,T) = \frac{\mathbb{E}_t^{\mathbb{Q}}\left[\{r(T) + \lambda(T,X_T) q^e(T,X_T)\} e^{-\int_t^T \{r(s) + \lambda(s,X_s) q^e(s,X_s)\}ds} \right]}{\mathbb{E}_t^{\mathbb{Q}}\left[e^{-\int_t^T \{r(s) + \lambda(s,X_s) q^e(s,X_s)\}ds} \right]}. \tag{21}$$

Finally using that $\bar{f}(t,t) = \bar{r}(t)$ in the above expression we obtain

$$\bar{r}(t, r_t, X_t) = r(t) + q^e(t, X_t)\lambda(t, X_t).$$

The result follow from $s(t) = \bar{r}(t) - r(t)$ and $s(t,T) = \bar{f}(t,T) - f(t,T)$. ∎

References

Allen, L. and A. Saunders (2003). A survey of cyclical effects in credit risk measurement models. BIS Working papers N.126.

Altman, E., B. Braddy, A. Resti, and A. Sironi (2005). The link between default and recovery rates: theory, empirical evidence and implications. *Journal of Bussiness 78*, 2203–2228.

Altman, E. B., A. Resti, and A. Sironi (2004). Default recovery rates in credit risk modelling: a review of the literature and empirical evidence. *Economic Notes by Banca Monte dei Paschi di Siena SpA 33*(2), 183–308.

Basel Committee (2003). *The new Basel capital Accord*. Basel Comittee on Banking Supervision.

Chen, L., D. Filipović, and H. V. Poor (2004). Quadratic term structure models for risk-free and defaultable rates. *Mathematical Finance 14*(4), 515–536.

Collin-Dufresne, P., R. S. Goldstein, and J. S. Martin (2001). Individual stock-option prices and credit spreads. *Journal of Finance 56*(6), 2177–2207.

Cremers, M., J. Dreissen, and D. Weinbaum (2004). The determinants of credit spread changes. Working paper.

Düllmann, K. and M. Trapp (2000). Systematic risk in recovery rates - an empirical analysis of us corporate credit expousures. Discussion Paper Serie 2: Banking and Financial Supervision.

Elton, E. and M. J. Gruber (2004). Factors affecting the valuation of corporate bonds. *Journal of Business and Finance 28*(11), 31–53.

Frye, J. (2000a). Collateral damage. *Risk 13*(4), 91–94.

Frye, J. (2000b). Depressing recoveries. *Risk Magazine November*, 108–111.

Frye, J. (2003). A false sense of security. *Risk 16*(8), 63–67.

Gaspar, R. M. (2001). *Sobre o efeito da correlação entre rendibilidade e volatilidade do activo subjacente na valorização de opções.* Serie Moderna Finança n.25, Euronext.

Gaspar, R. M. and T. Schmidt (2007). Shot-noise quadratic term structure models. *submitted*.

Giese, G. (2005). The impact of PD/LGD correlations on credit risk capital. *Risk Magazine 8*(4), 79–84.

Gordy, M. (2000). A comparative anatomy of credit risk models. *Journal of Banking and Finance 24*, 119–149.

Hu, Y. and W. Perrandin (2002). The dependence of recovery rates and default. Birbeck College and Bank of England working paper.

Jiang, G. and P. Sluis (1995). Index option pricing models with stochastic volatility and stochastic interest rates. *Review of Finance 3*, 273–310.

JP Morgan (1997). Creditmetrics. Technical Document.

Last, G. and A. Brandt (1995). *Probability and its applications-Marked point processes on the real line.* Spring verlag.

Saunders, A. (1999). *Credit Risk measurement.* John Wiley & Sons.

Schönbucher, P. (2001). Factors models: portfolio credit risk when defaults are correlated. *Journal of Risk Finance 3*(1), 31–53.

Schönbucher, P. (2003). *Credit derivatives pricing models - models, pricing and implementation.* JWS.

Selcuk, F. (2005). Asymmetric stochastic volatility models in emerging stock markets. *Applied Financial Economics 15*, 867–874.

Wilde, T. (2001). IRB approach explained. *Risk 14*(5), 87–90.

Wilson, T. (1997). Portfolio credit risk. *Risk 10*(9-10), 111–117, 56–61.