

Mean curvature and compactification of surfaces in a negatively curved Cartan–Hadamard manifold

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We state and prove a Chern–Osserman-type inequality in terms of the volume growth for complete surfaces with controlled mean curvature properly immersed in a Cartan–Hadamard manifold N with sectional curvatures bounded from above by a negative quantity $K_N \leq b < 0$.

1. Introduction

In the articles [5, 6], Chen Qing and Cheng Yi proved the finiteness of the topology and the following Chern–Osserman-type inequality (see [3]) for complete and properly immersed minimal surfaces in $\mathbb{H}^n(b)$ with finite total extrinsic curvature $\int_S \|A^S\|^2 d\sigma < \infty$ (here $\|A^S\|$ denotes the Hilbert–Schmidt norm of the second fundamental form of S in $\mathbb{H}^n(b)$):

$$(1.1) \quad -\chi(S) \leq \frac{1}{4\pi} \int_S \|A^S\|^2 d\sigma - \text{Sup}_r \frac{\text{Vol}(S^2 \cap B_r^{b,n})}{\text{Vol}(B_r^{b,2})},$$

where $\chi(S)$ is the Euler characteristic of the surface, $B_r^{b,n}$ denotes the geodesic r -ball in $\mathbb{H}^n(b)$ and $\frac{\text{Vol}(S^2 \cap B_r^{b,n})}{\text{Vol}(B_r^{b,2})}$ is the volume growth of the domains $S^2 \cap B_r^{b,n}$.

A natural question arises in this context: can we prove the finiteness of the topology of a not necessarily minimal surface in a Cartan–Hadamard manifold and, moreover, establish a Chern–Osserman-type inequality for its Euler characteristic? (At this point we are referring to the work [19], where the finiteness of the topology and a Chern–Osserman inequality are proven for not necessarily minimal surfaces in the Euclidean spaces \mathbb{R}^n .)

In this paper, we provide a partial answer to this question. We consider a complete and connected surface S properly immersed in a Cartan–Hadamard manifold N with sectional curvatures K_N bounded from above by $b < 0$. As in [6], we assume that $\int_S \|A^S\|^2 d\sigma < \infty$ and that the sectional curvatures of the ambient manifold N satisfy $\int_S (b - K_N) d\sigma < \infty$. On the other hand,

we assume that the mean curvature of S in N , H_S , is controlled by a radial function $h(r)$ (which depends on the distance R to a fixed pole $o \in N$) and its total mean curvature $\int_S \|H_S\| d\sigma$ is finite. Then we obtain a Chern–Osserman-type inequality, thereby proving that the topology of such non-minimal surfaces is finite and generalizing the results directly in [5, 6].

The monotonicity and finiteness of the volume growth function $\frac{\text{Vol}(D_t)}{\cosh \sqrt{-bt}}$ (or a modified version of it) associated to the distinguished domains $D_t \subset S$ called *extrinsic balls* (see Definition 2.1) plays a fundamental role in the description of the topology of the surface. This monotonicity property is obtained from some isoperimetric inequalities satisfied by the extrinsic balls in S .

The isoperimetric inequalities are based, in turn, on the application of the divergence theorem and comparison with the Laplacian operator acting on radial functions which comes from the Hessian-Index analysis for manifolds with a pole that we can find in [8] (see also [12, 18]).

We basically follow the arguments set out in the works [5, 6]. However, several analytical and topological difficulties arising from the fact of considering an ambient space with variable curvature had to be overcome.

In particular, we present the following estimation of the Euler characteristic of an immersed surface:

$$-\chi(S) \leq \lim_{t \rightarrow \infty} (-\chi(D_t))$$

for an accurate exhaustion of S by *connected* extrinsic balls $\{D_t\}_{t>0}$ (see Theorem 5.8 in Section 4). The proof of this result is based on the proof of Huber’s classical theorem given by White in [19]. This is a key result which will allow us to argue in a similar way as in [5, 6], even though our ambient manifold has no constant curvature.

1.1. Outline of the paper

In Section 2, we present the basic tools we use (such as the co-area formula) and the definitions and facts about the rotationally symmetric spaces used as a model for comparison purposes. In Section 3, we state our main results: Theorem 3.2 and Corollary 3.4, and prove Corollary 3.4. Section 4 is divided into two parts: Section 4.1 is devoted to the basic results about the Hessian comparison theory of restricted distance function that we are going to use (see Proposition 4.1) and in Section 4.2 an estimate of the geodesic curvature of the boundary of the extrinsic balls covering the surface is obtained as a corollary, using the Gauss–Bonnet theorem, and from there an estimation of

the Euler characteristic of such extrinsic balls is also calculated. Section 5 is divided into two parts: in Section 5.1 the monotonicity property of the volume growth is studied in the non-minimal context and in Section 5.2 the estimation of the Euler characteristic of the surface is presented in terms of the Euler characteristics of the extrinsic balls. Section 6 is devoted to the proof of the main result (Theorem 3.2).

2. Preliminaries

We are now going to present the precise *controlled mean curvature* setting, where we can prove Chern–Osseman-type inequality by introducing the notion of *comparison constellations*.

We assume throughout the paper that $\varphi : S \rightarrow N$ is a complete and proper immersion of a non-compact surface S in a Cartan–Hadamard manifold N . Throughout the paper, we identify $\varphi(S) \equiv S$ and $\varphi(x) \equiv x$ for all $x \in S$. We also assume that the Cartan–Hadamard manifold N^n has sectional curvatures bounded from above by a negative bound $K_N \leq b < 0$. All the points in these manifolds are poles. Recall that a pole is a point o such that the exponential map $\exp_o : T_o N^n \rightarrow N^n$ is a diffeomorphism. For every $x \in N^n \setminus \{o\}$ we define $r(x) = \text{dist}_N(o, x)$, and this distance is realized by the length of a unique geodesic from o to x , which is the *radial geodesic from o* . We also denote by r the restriction $r|_S : S \rightarrow \mathbb{R}_+ \cup \{0\}$. This restriction is called the *extrinsic distance function* from o in S^m . The gradients of r in N and S are denoted by $\nabla^N r$ and $\nabla^S r$, respectively. Let us remark that $\nabla^S r(x)$ is just the tangential component in S of $\nabla^N r(x)$, for all $x \in S$. Then we have the following basic relation:

$$(2.1) \quad \nabla^N r = \nabla^S r + (\nabla^N r)^\perp,$$

where $(\nabla^N r)^\perp(x) = \nabla^\perp r(x)$ is perpendicular to $T_x S$ for all $x \in S$.

We are now going to define the *extrinsic balls*:

Definition 2.1. Given a connected and complete surface S^2 in a Cartan–Hadamard manifold N^n , we denote the *extrinsic metric balls* of radius R and center $o \in N$ by $D_R(o)$. They are defined as any connected component of the intersection:

$$B_R(o) \cap S = \{x \in S : r(x) < R\},$$

where $B_R(o)$ denotes the open geodesic ball of radius R centered at the pole o in N^n .

Remark 2.2. The restriction $r|_S$ is smooth in S and consequently, by Sard’s theorem and the regular level set theorem, the radii R that produce smooth boundaries $\partial D_R(o)$ are dense in \mathbb{R} .

Definition 2.3. Let o be a pole in the ambient Cartan–Hadamard manifold N and let $x \in M \setminus \{o\}$. The sectional curvature $K_N(\sigma_x)$ of the two-plane $\sigma_x \in T_x N$ is then called an o -radial sectional curvature of N at x if σ_x contains the tangent vector to a minimal geodesic from o to x . We denote these curvatures by $K_{o,N}(\sigma_x)$.

At this point we should remark that we assume that the o -radial sectional curvatures of N are bounded from above by the constant $b < 0$, which is the constant sectional curvature of the hyperbolic space $\mathbb{H}^n(b)$. This space can be viewed as a special kind of rotationally symmetric space called a *model space*.

Definition 2.4 (See [8, 9]). A ω -model M_ω^m is a smooth warped product with base $B^1 = [0, \Lambda[\subset \mathbb{R}$ (where $0 < \Lambda \leq \infty$), fiber $F^{m-1} = \mathbb{S}_1^{m-1}$ (i.e., the unit $(m - 1)$ -sphere with standard metric), and warping function $\omega: [0, \Lambda[\rightarrow \mathbb{R}_+ \cup \{0\}$, with $\omega(0) = 0$, $\omega'(0) = 1$ and $\omega(r) > 0$ for all $r > 0$. The point $o_\omega = \pi^{-1}(0)$, where π denotes the projection onto B^1 , is called the *center point* of the model space. If $\Lambda = \infty$, then o_ω is a pole of M_ω^m .

Remark 2.5. The hyperbolic space $\mathbb{H}^n(b)$ is a ω -model with warping function

$$\omega_b(r) = \frac{1}{\sqrt{-b}} \sinh(\sqrt{-b}r).$$

Proposition 2.6 (See [8, 9, 15]). Let M_ω^m be a ω -model with warping function $\omega(r)$ and center o_ω . The distance sphere of radius r and center o_ω in M_ω^m is the fiber $\pi^{-1}(r)$. This distance sphere has the constant mean curvature $\eta_\omega(r) = \frac{\omega'(r)}{\omega(r)}$. On the other hand, the o_ω -radial sectional curvatures of M_ω^m at every $x \in \pi^{-1}(r)$ (for $r > 0$) are all identical and determined by $K_{o_\omega, M_\omega}(\sigma_x) = -\frac{\omega''(r)}{\omega(r)}$.

Remark 2.7. The mean curvature of the geodesic spheres in the hyperbolic space $\mathbb{H}^n(b)$, “pointed inward,” is (see [17]):

$$\eta_{\omega_b}(t) = \frac{\omega'_b(t)}{\omega_b(t)} = \sqrt{-b} \coth \sqrt{-b}t.$$

Definition 2.8 (e.g., [14]). Given a function $h : \mathbb{R} \rightarrow \mathbb{R}$, the *isoperimetric comparison space* $C_{\omega_b, h}^2$ is the W -model space $[0, +\infty) \times_W S_1^{0,1}$ with base interval $B = [0, +\infty)$ and warping function $W(r)$ defined by:

$$(2.2) \quad W(t) = \frac{\sinh \sqrt{-b}t}{\sqrt{-b}e^{2 \int_0^t h(s) ds}}.$$

Note that

$$(2.3) \quad W(0) = 0, W'(0) = 1.$$

Remark 2.9. The warping function $W(r)$ is a/the solution of the differential equation:

$$(2.4) \quad \frac{W'}{W} = \frac{w'_b}{w_b} - 2h(t)$$

with the following boundary condition:

$$(2.5) \quad W'(0) = 1.$$

Remark 2.10. We observe that $C_{\omega_b, h}^2$ is indeed a model space $C_{\omega_b, h}^2 = M_{W}^2$ with a well-defined pole o_W at $r = 0$: $W(r) \geq 0$ for all r and $W(r)$ is only 0 at $r = 0$, where, additionally, because of equation (2.5), $W'(0) = 1$.

On the other hand, given the warping function $W(r)$, we introduce the isoperimetric quotient function $q_W(r)$ for the corresponding W -model space $C_{\omega_b, h}^2$ as follows:

$$(2.6) \quad q_W(r) = \frac{\text{Vol}(B_r^W)}{\text{Vol}(S_r^W)} = \frac{\int_0^r W(t) dt}{W(r)},$$

where B_r^W denotes the polar centered geodesic r -ball of radius r in $C_{\omega_b, h}^2$ with boundary sphere S_r^W .

3. Main results

With these definitions in hand, we present the notion of *strongly balanced isoperimetric comparison space* and our main results:

Definition 3.1. The isoperimetric comparison space $C_{\omega_b, h}^2$ is *strongly balanced* on the interval $[0, \infty)$ if and only if the following inequality holds

for all $r \geq 0$:

$$(3.1) \quad |h(r)| \leq \frac{1}{2}(\eta_{w_b}(r) - \sqrt{-b}),$$

where $\eta_{w_b}(r) = \sqrt{-b} \coth \sqrt{-b}r$ is the mean curvature of the geodesic r -spheres in the hyperbolic spaces $\mathbb{H}^n(b)$.

With all these concepts and definitions in hand, we have our main result:

Theorem 3.2. *Let us consider N^n to be a Cartan–Hadamard manifold, and $o \in N$ a pole in N . Let us suppose that its radial sectional curvatures are bounded from above by a negative bound $K_{o,N}(\sigma_x) \leq b < 0$. Let S be a complete, connected and properly immersed surface in N such that there exists a radial function $h(r)$ satisfying*

$$(3.2) \quad C(x) = -\langle \nabla^N r(x), H_S(x) \rangle \leq h(r(x)) \quad \text{for all } x \in S,$$

where $H_S(x)$ denotes the mean curvature vector of S in N .

Let $C_{\omega_b, h}^2$ denote the W -model constructed via ω_b and h , and assume that $C_{\omega_b, h}^2$ is a strongly balanced isoperimetric comparison space on the interval $[0, \infty)$.

Let us also assume that

$$\begin{aligned} (I) \quad & \int_S (b - K_N|_S) d\sigma < +\infty, \\ (II) \quad & \int_S \|A^S\|^2 d\sigma < +\infty, \\ (III) \quad & \int_S \|H_S\| d\sigma < +\infty, \end{aligned}$$

where A^S denotes the second fundamental form of S in N , $K_N|_S$ denotes the sectional curvature of N restricted to the tangent plane $T_p S$, for all $p \in S$, and H_S denotes the mean curvature vector of S .

Then

$$(1) \quad \text{Sup}_{t>0} \frac{v(t)}{\cosh \sqrt{-b}t} < +\infty, \text{ where } v(t) = \text{vol}(D_t) \forall t > 0 \text{ and } D_t \text{ denotes the extrinsic ball on the surface } S.$$

(2) S^2 has finite topological type, and there exists $t_0 \geq 0$ such that

$$(3.3) \quad \begin{aligned} -\chi(S) \leq & \frac{1}{2\pi} \int_S (b - K_N|_S) d\sigma + \frac{1}{4\pi} \int_S \|A^S\|^2 d\sigma - C \operatorname{Sup}_{t>0} \frac{\operatorname{vol}(D_t)}{\operatorname{Vol}(B_t^{b,2})} \\ & + \frac{\sqrt{-b}}{\pi} \int_S \|H_S\| d\sigma - \frac{bv(t_0)}{2\pi}, \end{aligned}$$

where $C \in [0, 1]$ is the constant defined as

$$C := \operatorname{Inf}_{t>0} \left(\cosh \sqrt{-bt} - q_W(t) \sqrt{-b} \sinh \sqrt{-bt} \right).$$

Remark 3.3. Given the surface S^2 immersed in N , the quantity $b - K_N|_S$, (where $K_N|_S = K_N(p, T_pS)$ is the sectional curvature of N at $p \in S$ of the tangent plane T_pS), only depends on the points $p \in S$. Hence, the assumption $\int_S (b - K_N|_S) d\sigma < +\infty$ makes sense. Indeed, when we consider the sectional curvature of the ambient manifold restricted to the two-dimensional tangent plane $T_pS \subseteq T_pN$, we have, by virtue of the Gauss formula and given an orthonormal basis $\{e_1, e_2\}$ of T_pS :

$$(3.4) \quad \begin{aligned} & K_N(p, T_pS) - K_S(p, T_pS) \\ & = \langle A^S(e_1, e_2), A^S(e_1, e_2) \rangle - \langle A^S(e_1, e_1), A^S(e_2, e_2) \rangle \\ & = \frac{1}{2} (\|A^S\|^2 - 4\|H_S\|^2), \end{aligned}$$

where $K_S(p, T_pS) = K_S$ is the Gauss curvature of S . Hence, $b - K_N$ does not depend on the basis $\{e_1, e_2\}$ of T_pS . If there is no risk of confusion, we shall denote as $\int_S (b - K_N) d\sigma$ the integral $\int_S (b - K_N|_S) d\sigma$.

As a corollary of Theorem 3.2, we have the following result, which is a generalization of the main theorem in [6], when we consider connected and minimal surfaces in a Cartan–Hadamard manifold (see also [7]):

Corollary 3.4. *Let S^2 be a complete, connected and minimal surface properly immersed in a Cartan–Hadamard manifold N , with sectional curvatures bounded from above by a negative quantity $K_N \leq b < 0$.*

Let us suppose that

$$(3.5) \quad \int_S \|A^S\|^2 d\sigma < +\infty$$

and

$$(3.6) \quad \int_S (b - K_N|_S) d\sigma < +\infty,$$

where A^S denotes the second fundamental form of S in N and $K_N|_S$ denotes the sectional curvature of N restricted to the tangent plane T_pS , for all $p \in S$.

Then:

- (1) $\text{Sup}_{t>0} \frac{\text{Vol}(D_t)}{\text{Vol}(B_t^{b,2})} < +\infty,$
- (2) S^2 has finite topological type and
- (3) $-\chi(S) \leq \frac{1}{4\pi} \int_S \|A^S\|^2 d\sigma - \text{Sup}_{t>0} \frac{\text{Vol}(D_t)}{\text{Vol}(B_t^{b,2})} + \frac{1}{2\pi} \int_S (b - K_N) d\sigma,$

where D_t denotes the connected extrinsic ball on surface S .

Proof. As S is minimal, then $\mathcal{C}(x) = 0 \ \forall x \in S$, so we consider $h(r) = 0$ for all $r > 0$. Then, $W(r) = w_b(r)$ trivially and we have that $C_{w_b,h}^2 = \mathbb{H}^2(b)$ is a strongly balanced isoperimetric comparison space. In particular, $t_0 = 0$.

It is straightforward to check that $q_W(t) = q_{w_b}(t) = \frac{1}{\sqrt{-b}}(\coth(\sqrt{-bt}) - \frac{1}{\sinh(\sqrt{-bt})})$, so $f_W(t) = \cosh \sqrt{-bt} - q_W(t)\sqrt{-b} \sinh \sqrt{-bt} = 1 \ \forall t \geq 0$, and hence

$$C = \text{Inf}_{t>0} \left(\cosh \sqrt{-bt} - q_W(t)\sqrt{-b} \sinh \sqrt{-bt} \right) = 1$$

in this case. □

Remark 3.5. It is clear from inequality (3.3) that Theorem 3.2 is a good generalization of classic Chern–Osserman inequality as long as the constant C is not zero. This fact depends on the function $h(r)$, which bounds the radial mean curvature of the surface, as we try to explain with the following consideration.

Let us consider S a complete, connected and properly immersed surface in a Cartan–Hadamard manifold N with pole $o \in N$ and with radial sectional curvatures $K_{o,N}(\sigma_x) \leq b < 0$. Let us assume that hypotheses *I*, *II* and *III* in Theorem 3.2 are fulfilled.

Let us suppose that, for some fixed constant $L \geq 1$, and for all $x \in S$, the bound for the mean curvature of S is given by

$$(3.7) \quad C(x) = -\langle \nabla^N r(x), H_S(x) \rangle \leq h_L(r(x)) = \frac{\sqrt{-b}}{L} e^{-2\sqrt{-b}r(x)}.$$

Then, it is straightforward that C_{w_b, h_L}^2 is strongly balanced and so, by applying Theorem 3.2, S^2 has finite topological type, and

$$\begin{aligned}
 -\chi(S) \leq & \frac{1}{2\pi} \int_S (b - K_N) d\sigma + \frac{1}{4\pi} \int_S \|A^S\|^2 d\sigma - C_L \sup_{t>0} \frac{\text{Vol}(D_t)}{\text{Vol}(B_t^{b,2})} \\
 & + \frac{\sqrt{-b}}{\pi} \int_S \|H_S\| d\sigma - \frac{1}{\pi} \int_S \|H_S\|^2 d\sigma - \frac{bv(t_0)}{2\pi},
 \end{aligned}$$

where it is straightforward to check that

$$1 \geq C_L = \inf_{t>0} \left(\cosh \sqrt{-bt} - q_{W_L}(t) \sqrt{-b} \sinh \sqrt{-bt} \right) \geq 1 - \frac{2}{3L} \geq \frac{1}{3} > 0.$$

Hence,

$$\lim_{L \rightarrow \infty} C_L = 1.$$

But, on the other hand, when L goes to infinity, then $h_L(r)$ goes to the constant function 0.

In turn, it is straightforward to check that the value $t_0(b, h)$ (which ultimately depends on the model space $C_{w_b, h}^2$) satisfies in this case the following inequality:

$$(3.8) \quad t_0(b, h_L) \leq \frac{2}{\sqrt{-b}} \operatorname{arcsech} \frac{\sqrt{L}}{\sqrt{L+1}}$$

and hence

$$\lim_{L \rightarrow +\infty} t_0(b, h_L) \leq \lim_{L \rightarrow +\infty} \frac{2}{\sqrt{-b}} \operatorname{arcsech} \frac{\sqrt{L}}{\sqrt{L+1}} = 0.$$

Therefore, the minimal case can be considered not only a corollary but also a limit case, when $L \rightarrow \infty$, of the assumptions in Theorem 3.2 (given a suitable choice of the bounding function $h(r) = h_L(r)$).

4. Hessian analysis, Gauss–Bonnet theorem and estimates for the Euler characteristic of the extrinsic balls

4.1. Hessian and Laplacian comparison analysis

We now assume that S^2 is a complete, non-compact and properly immersed surface in a Riemannian manifold N^n which possesses a pole o .

The second-order analysis of the restricted distance function $r|_S$ is governed by the Hessian comparison Theorem A in [8]. A corollary of this result is the following proposition (see [11] or [18] for further details):

Proposition 4.1. *Let N^n be a manifold with a pole o , and let M_ω^m denote a ω -model with center o_ω . Let us suppose that every o -radial sectional curvature at $x \in N - \{o\}$ is bounded from above by the o_ω -radial sectional curvatures in M_ω^m as follows:*

$$(4.1) \quad \mathcal{K}(\sigma(x)) = K_{o,N}(\sigma_x) \leq -\frac{\omega''(r)}{\omega(r)}$$

for every radial two-plane $\sigma_x \subset T_x N$ at distance $r = r(x) = \text{dist}_N(o, x)$ from p in N .

For every smooth function $f(r)$ with $f'(r) \geq (\leq) 0$ for all r , and given $X \in TqS$ unitary:

$$(4.2) \quad \text{Hess}^S(f \circ r)(X, X) \geq (\leq) (f''(r) - f'(r)\eta_w(r)) \langle X, \nabla^N r \rangle^2 + f'(r) (\eta_w(r) + \langle \nabla^N r, A^S(X, X) \rangle).$$

Tracing inequality (4.2)

$$(4.3) \quad \Delta^S(f \circ r) \geq (\leq) (f''(r) - f'(r)\eta_w(r)) \|\nabla^S r\|^2 + mf'(r) (\eta_w(r) + \langle \nabla^N r, H_S \rangle).$$

Proposition 4.2 (See [6, 7]). *Let S^2 be a complete, non-compact and properly immersed surface in a Cartan–Hadamard manifold N^n . Let us consider $\{D_t\}_{t>0}$ an exhaustion of S by extrinsic balls. Let $f : S \rightarrow \mathbb{R}$ be a positive C^∞ function. Then*

$$\int_S e^{-\sqrt{-b}r(x)} f(x) d\sigma < +\infty \quad \text{if and only if} \\ \int_0^{+\infty} e^{-\sqrt{-b}t} \int_{D_t} f(x) d\sigma dt < +\infty.$$

4.2. An application of the Gauss–Bonnet theorem: geodesic curvature of the extrinsic curves in the surface S

Proposition 4.3. *Let S^2 be a properly immersed surface in a Cartan–Hadamard manifold N . Let us assume that, given a pole $o \in N$, the o -radial sectional curvatures of N are bounded from above by a negative quantity*

$K_{o,N}(\sigma_x) \leq b < 0$. Let D_t be an extrinsic ball in S centered on the pole $o \in N$. The geodesic curvature of the extrinsic sphere ∂D_t , denoted as k_g^t , is bounded from below as follows:

$$\begin{aligned}
 (4.4) \quad k_g^t &\geq \frac{1}{\|\nabla^S r\|} \left\{ \eta_{\omega_b}(t) + 2 \langle H_S, \nabla^N r \rangle - \left\langle A^S \left(\frac{\nabla^S r}{\|\nabla^S r\|}, \frac{\nabla^S r}{\|\nabla^S r\|} \right), \nabla^\perp r \right\rangle \right\} \\
 &\geq \frac{1}{\|\nabla^S r\|} \left\{ \eta_{\omega_b}(t) - 2\|H_S\| - \left\langle A^S \left(\frac{\nabla^S r}{\|\nabla^S r\|}, \frac{\nabla^S r}{\|\nabla^S r\|} \right), \nabla^\perp r \right\rangle \right\},
 \end{aligned}$$

where A^S denotes the second fundamental form of S in N .

Proof. As $K_N \leq b$ by applying (4.3) to the radial function $f(r) = \cosh \sqrt{-b}r$ and having into account that

$$(4.5) \quad -\|H_S\| \leq \langle \nabla^N r, H_S \rangle \leq \|H_S\|$$

then

$$(4.6) \quad \Delta^S \cosh \sqrt{-b}r \geq -2b \cosh \sqrt{-b}r - 2\sqrt{-b} \sinh \sqrt{-b}r \|H_S\|.$$

Now, we again apply Proposition 4.1 to $f(r) = 1 \ \forall r \geq 0$ to conclude that the geodesic curvature k_g^t satisfies the inequality

$$\begin{aligned}
 (4.7) \quad k_g^t &= \frac{1}{\|\nabla^S r\|} \text{Hess}^S r(e, e) \\
 &\geq \frac{1}{\|\nabla^S r\|} \left\{ -\eta_{\omega_b} \langle e, \nabla^N r \rangle^2 + \eta_{\omega_b} + \langle A^S(e, e), \nabla^N r \rangle \right\} \\
 &= \frac{1}{\|\nabla^S r\|} \left\{ \eta_{\omega_b} + \langle A^S(e, e), \nabla^N r \rangle \right\},
 \end{aligned}$$

where e is unitary and tangent to ∂D_r .

As

$$(4.8) \quad H_S = \frac{1}{2} \left[A^S(e, e) + A^S \left(\frac{\nabla^S r}{\|\nabla^S r\|}, \frac{\nabla^S r}{\|\nabla^S r\|} \right) \right],$$

we obtain:

$$\begin{aligned}
 (4.9) \quad k_g^t &\geq \frac{1}{\|\nabla^S r\|} \left\{ \eta_{\omega_b}(t) + 2 \langle H_S, \nabla^N r \rangle - \left\langle A^S \left(\frac{\nabla^S r}{\|\nabla^S r\|}, \frac{\nabla^S r}{\|\nabla^S r\|} \right), \nabla^{\perp} r \right\rangle \right\} \\
 &\geq \frac{1}{\|\nabla^S r\|} \left\{ \eta_{\omega_b}(t) - 2\|H_S\| - \left\langle A^S \left(\frac{\nabla^S r}{\|\nabla^S r\|}, \frac{\nabla^S r}{\|\nabla^S r\|} \right), \nabla^{\perp} r \right\rangle \right\}.
 \end{aligned}$$

□

Proposition 4.4. *Let S^2 be a properly immersed surface in a Cartan–Hadamard manifold N . Let us assume that, given a pole $o \in N$, the o -radial sectional curvatures of N are bounded from above by a negative quantity $K_{o,N}(\sigma_x) \leq b < 0$. Let D_t be an extrinsic ball in S centered on the pole $o \in N$. The volume $v(t) = \text{vol}(D_t)$ satisfies the inequality:*

$$\begin{aligned}
 (4.10) \quad 2\pi\chi(D_t) &\geq \int_{\partial D_t} \frac{1}{\|\nabla^S r\|} \left\{ \eta_{\omega_b}(t) - 2\|H_S\| \right. \\
 &\quad \left. - \left\langle A^S \left(\frac{\nabla^S r}{\|\nabla^S r\|}, \frac{\nabla^S r}{\|\nabla^S r\|} \right), \nabla^{\perp} r \right\rangle \right\} d\sigma + \int_{D_t} K_S d\sigma,
 \end{aligned}$$

where K_S denotes the Gaussian curvature of S .

Proof. By applying the Gauss–Bonnet theorem:

$$(4.11) \quad \int_{\partial D_t} k_g^t d\mu + \int_{D_t} K_S d\sigma = 2\pi\chi(D_t).$$

Now, using Proposition 4.3

$$\begin{aligned}
 (4.12) \quad 2\pi\chi(D_t) &\geq \int_{\partial D_t} \frac{1}{\|\nabla^S r\|} \left\{ \eta_{\omega_b}(t) - 2\|H_S\| \right. \\
 &\quad \left. - \left\langle A^S \left(\frac{\nabla^S r}{\|\nabla^S r\|}, \frac{\nabla^S r}{\|\nabla^S r\|} \right), \nabla^{\perp} r \right\rangle \right\} d\mu + \int_{D_t} K_S d\sigma.
 \end{aligned}$$

□

Proposition 4.5. *Let S^2 be a properly immersed surface in a Cartan–Hadamard manifold N . Let us assume that, given a pole $o \in N$, the o -radial sectional curvatures of N are bounded from above by a negative quantity $K_{o,N}(\sigma_x) \leq b < 0$. Let $D_s \subset D_t$ be extrinsic balls in S centered on the pole*

$o \in N$. Then

$$\begin{aligned}
 & \frac{\int_{D_t} \left(\cosh \sqrt{-br} - \|H_S\| \frac{\sinh \sqrt{-br}}{\sqrt{-b}} \right) d\sigma}{\cosh^2 \sqrt{-bt}} \\
 (4.13) \quad & - \frac{\int_{D_s} \left(\cosh \sqrt{-br} - \|H_S\| \frac{\sinh \sqrt{-br}}{\sqrt{-b}} \right) d\sigma}{\cosh^2 \sqrt{-bs}} \\
 & \geq \int_{D_t - D_s} \frac{1 + \sinh^2 \sqrt{-br} \|\nabla^\perp r\|^2 - \frac{\sinh \sqrt{-br} \cosh \sqrt{-br}}{\sqrt{-b}} \|H_S\|}{\cosh^3 \sqrt{-br}} d\sigma.
 \end{aligned}$$

Proof. We integrate inequality (4.6), and then we apply the divergence theorem to obtain

$$\begin{aligned}
 (4.14) \quad & \sqrt{-b} \sinh \sqrt{-bt} \int_{\partial D_t} \|\nabla^S r\| d\sigma \\
 & \geq -2b \int_{D_t} \cosh \sqrt{-bs} d\sigma - 2\sqrt{-b} \int_{D_t} \|H_S\| \sinh \sqrt{-bs} d\sigma.
 \end{aligned}$$

Therefore,

$$\begin{aligned}
 (4.15) \quad & \int_{D_t} \left(\cosh \sqrt{-br} - \frac{\|H_S\| \sinh \sqrt{-br}}{\sqrt{-b}} \right) d\sigma \\
 & \leq \frac{1}{2} \frac{\sinh \sqrt{-bt}}{\sqrt{-b}} \int_{\partial D_t} \|\nabla^S r\| d\sigma_t.
 \end{aligned}$$

Deriving and using the above inequality

$$\begin{aligned}
 & \frac{d}{dt} \left(\frac{\int_{D_r} \left(\cosh \sqrt{-br} - \|H_S\| \frac{\sinh \sqrt{-br}}{\sqrt{-b}} \right) d\sigma}{\cosh^2 \sqrt{-bt}} \right) \\
 & \geq \frac{1}{\cosh^3 \sqrt{-bt}} \left\{ \int_{\partial D_t} \frac{\cosh^2 \sqrt{-br} - \frac{\sinh \sqrt{-br} \cosh \sqrt{-br}}{\sqrt{-b}} \|H_S\| - \sinh^2 \sqrt{-br} \|\nabla^S r\|^2}{\|\nabla^S r\|} d\mu \right\} \\
 & = \int_{\partial D_t} \frac{1}{\|\nabla^S r\|} \left\{ \frac{1 + \sinh^2 \sqrt{-br} \|\nabla^\perp r\|^2 - \frac{\sinh \sqrt{-br} \cosh \sqrt{-br}}{\sqrt{-b}} \|H_S\|}{\cosh^3 \sqrt{-bt}} d\mu \right\}.
 \end{aligned}$$

Now, integrate the above inequality, applying the co-area formula. \square

As direct corollaries of the above propositions, we have the corresponding results for the minimal case (see [4, 6]), where $H_S = 0$ and hence $\|H_S\| = 0$.

5. Extrinsic isoperimetry, volume growth and topology of surfaces

5.1. Extrinsic isoperimetry and the monotonicity property of the volume growth for non-minimal surfaces

In this section we are going to see how it is possible to deduce a monotonicity property satisfied by the volume growth function in the strongly balanced setting defined in Section 3.

We start by studying how to obtain the classic monotonicity property of the volume growth function (see [2, 13]) using a slightly more general isoperimetric inequality than the one used in [13]. This isoperimetric comparison is based, in turn, on a balance condition that is more general than the one used in [13].

Theorem 5.1. *Let us consider N^n to be a Cartan–Hadamard manifold, and $o \in N$ a pole in N . Let us suppose that its radial sectional curvatures are bounded from above by a negative bound $K_{o,N}(\sigma_x) \leq b < 0$. Let S be a complete, connected and properly immersed surface in N such that there exists a radial function $h(r)$ satisfying:*

$$(5.1) \quad C(x) = -\langle \nabla^N r(x), H_S(x) \rangle \leq h(r(x)) \quad \text{for all } x \in S,$$

where $H_S(x)$ denotes the mean curvature vector of S in N .

Let $C_{\omega_b, h}^2$ denote the W -model constructed via ω_b and h , and assume that $C_{\omega_b, h}^2$ is a strongly balanced isoperimetric comparison space on the interval $[0, \infty)$. Then, there exists $t_0 \geq 0$ such that

$$(5.2) \quad \frac{\text{vol}(\partial D_R)}{\text{vol}(D_R) - \text{vol}(D_{t_0})} \geq \frac{\text{vol}(\partial B_R^W)}{\text{vol}(B_R^W)}, \quad \forall R \geq t_0.$$

Proof. We shall show the following two lemmas first:

Lemma 5.2. *If the isoperimetric comparison space $C_{\omega_b, h}^2$ is strongly balanced, then*

(1) *The function $h(t)$ satisfies*

$$(5.3) \quad \lim_{t \rightarrow +\infty} h(t) = 0.$$

(2) The function $q_W(t) = \frac{\int_0^t W(s)ds}{W(t)}$ satisfies

$$(5.4) \quad q_W(t) \leq \frac{1}{\sqrt{-b}} \quad \forall t > 0,$$

$$(5.5) \quad \begin{aligned} \lim_{t \rightarrow +\infty} W(t) &= +\infty, \\ \lim_{t \rightarrow +\infty} q_W(t) &= \frac{1}{\sqrt{-b}}, \\ \lim_{t \rightarrow 0^+} q_W(t) &= 0. \end{aligned}$$

Proof. As $0 \leq \lim_{t \rightarrow \infty} |h(t)| \leq \frac{1}{2} \lim_{t \rightarrow \infty} (\eta_{\omega_b}(t) - \sqrt{-b}) = 0$, we have that

$$\lim_{t \rightarrow \infty} h(t) = 0.$$

To see (5.4), we use the fact that $h(r) \leq |h(r)| \leq \frac{1}{2}(\eta_{\omega_b}(r) - \sqrt{-b})$ for all $r \geq 0$, and equation (2.4).

To show the limits in (5.5), we use the fact that $\lim_{t \rightarrow \infty} h(t) = 0$. Therefore, it is straightforward to check that $\lim_{t \rightarrow \infty} W(t) = +\infty$ and, hence, to apply L'Hospital's rule in order to obtain the other two limits. \square

Lemma 5.3. *Let us consider an isoperimetric comparison space $C_{\omega_b, h}^2$. If $C_{\omega_b, h}^2$ is strongly balanced on $[0, \infty)$, then there exists some $t_0 \geq 0$ such that the following inequality holds for all $r \in [t_0, \infty)$:*

$$(5.6) \quad q_W(r) (\eta_{\omega_b}(r) - h(r)) \geq \frac{1}{2},$$

where $q_W(r)$ is the isoperimetric quotient function introduced in equation (2.6).

Proof. Applying (5.5) in Lemma 5.2, we have

$$\lim_{t \rightarrow \infty} q_W(t) (\eta_{\omega_b}(t) - h(t)) = \lim_{t \rightarrow \infty} q_W(t) \lim_{t \rightarrow \infty} (\eta_{\omega_b}(t) - h(t)) = 1.$$

Hence, by applying the definition of limit when t goes to infinity with $\epsilon = 1/2$, we obtain that there exists $t_0 \geq 0$ such that $q_W(t) (\eta_{\omega_b}(t) - h(t)) \geq 1/2$. \square

To show Theorem 5.1, let us now consider a fixed $R > t_0$. For all $t \in [t_0, R]$, we define

$$\psi(t) = \int_t^R \frac{1}{W(u)} \left(\int_0^u W(s) ds \right) du, \quad \forall t \geq t_0.$$

Using this definition and (2.6) we have:

$$(5.7) \quad \begin{aligned} \psi'(t) &= -q_W(t) = -\frac{\text{vol}(B_t^W)}{\text{vol}(\partial B_t^W)} \leq 0, \\ \psi''(t) &= -1 + q_W(t)(\eta_{\omega_b}(t) - 2h(t)). \end{aligned}$$

We transplant ψ to S , defining $\bar{\psi} : D_R - D_{t_0} \rightarrow \mathbb{R}$ as $\bar{\psi}(x) = \psi(r(x))$
Applying (4.3) in Proposition 4.1:

$$(5.8) \quad \begin{aligned} \Delta^S \psi(r(x)) &\leq (\psi''(r(x)) - \psi'(r(x))\eta_{\omega_b}(r(x))) \|\nabla^S r\|^2 \\ &\quad + 2\psi'(r(x))(\eta_{\omega_b}(r(x)) - h(t)). \end{aligned}$$

As $r(x) \geq t_0$, by applying the inequality (5.6) in Lemma 5.3, which holds for $\forall t \geq t_0$, we obtain:

$$\psi''(r(x)) - \psi'(r(x))\eta_{\omega_b}(r(x)) \geq 0.$$

Hence, as $\|\nabla^S r\|^2 \leq 1$ and using equations (5.7) and again inequality (5.6) we have $\Delta^S \psi(r(x)) \leq -1$.

By integrating inequality (5.8) on the annulus $A_{t_0}^R = D_R - D_{t_0}$ and applying the Divergence theorem, we obtain:

$$(5.9) \quad \begin{aligned} \text{Vol}(A_{t_0}^R) &\leq \int_{A_{t_0}^R} -\Delta^S \psi(r(x)) d\mu \\ &= -\psi'(R) \int_{\partial D_R} \|\nabla^S r\| d\mu + \psi'(t_0) \int_{\partial D_{t_0}} \|\nabla^S r\| d\mu. \end{aligned}$$

As $-\psi'(t) = q_W(t) \geq 0 \quad \forall t \geq 0$, we have:

$$(5.10) \quad \text{vol}(D_R) - \text{vol}(D_{t_0}) \leq q_W(R)\text{vol}(\partial D_R)$$

and hence $\text{vol}(D_R) - \text{vol}(D_{t_0}) \leq \frac{\text{vol}(B_R^W)}{\text{vol}(S_R^W)} \text{vol}(\partial D_R)$. □

As a first corollary, we obtain the comparison between the volume of extrinsic balls in the surface and the volume of the geodesic balls in the model space.

Corollary 5.4 (General monotonicity). *Let us consider N^n to be a Cartan–Hadamard manifold, and $o \in N$ a pole in N . Let us suppose that its radial sectional curvatures are bounded from above by a negative bound $K_{o,N}(\sigma_x) \leq b < 0$. Let S be a complete, connected and properly immersed surface in N such that there exists a radial function $h(r)$ satisfying*

$$(5.11) \quad C(x) = -\langle \nabla^N r(x), H_S(x) \rangle \leq h(r(x)) \quad \text{for all } x \in S,$$

where $H_S(x)$ denotes the mean curvature vector of S in N .

Let $C_{\omega_b, h}^2$ denote the W -model constructed via ω_b and h , and assume that $C_{\omega_b, h}^2$ is a strongly balanced isoperimetric comparison space on the interval $[0, \infty)$. Then the function

$$\frac{v(t) - v_0}{\text{Vol}(B_t^W(o_W))}$$

is non-decreasing in $[t_0, +\infty)$, where t_0 is given in Lemma 5.3, being $v(t) = \text{Vol}(D_t)$ and $v_0 = v(t_0) = \text{Vol}(D_{t_0})$.

Proof. Let us consider the functions $f(t) = \frac{\text{vol}(D_t) - v_0}{\text{vol}(B_t^W)}$ and $G(t) = \ln f(t)$.

From the co-area formula:

$$(5.12) \quad v'(t) = \int_{\partial D_t} \frac{1}{\|\nabla^S r\|} d\mu \geq \int_{\partial D_t} d\mu = \text{vol}(\partial D_t),$$

so

$$(5.13) \quad \frac{d}{dt} \text{vol}(D_t) \geq \text{vol}(\partial D_t).$$

On the other hand, in a rotationally symmetric space M_W we have that, (see [9]):

$$(5.14) \quad \text{vol}(B_t^W)' = \text{vol}(\partial B_t^W).$$

Hence, by applying Theorem 5.1, we obtain:

$$(5.15) \quad \begin{aligned} G'(t) &= \frac{v'(t)}{v(t) - v_0} - \frac{\text{vol}(\partial B_t^W)}{\text{vol}(B_t^W)} \\ &\geq \frac{\text{vol}(\partial D_t)}{v(t) - v_0} - \frac{\text{vol}(\partial B_t^W)}{\text{vol}(B_t^W)} \geq 0 \quad \forall t \geq t_0 \end{aligned}$$

so we have $f'(t) \geq 0 \quad \forall t \geq t_0$. □

Now, we are going to obtain two new monotonicity properties deduced from the isoperimetric inequality (5.2) in Theorem 5.1. The key difference with the generalized monotonicity property analyzed in Corollary 5.4 is that now we want to compare the volume of the extrinsic r -balls with the hyperbolic cosine (as in the minimal context given in [13]), and not with the volume of the geodesic r -balls in the model space (as is performed in Corollary 5.4).

Corollary 5.5 (Non-minimal monotonicity). *Let us consider N^n to be a Cartan–Hadamard manifold, and $o \in N$ a pole in N . Let us suppose that its radial sectional curvatures are bounded from above by a negative bound $K_{o,N}(\sigma_x) \leq b < 0$. Let S be a complete, connected and properly immersed surface in N such that there exists a radial function $h(r)$ satisfying*

$$(5.16) \quad C(x) = -\langle \nabla^N r(x), H_S(x) \rangle \leq h(r(x)) \quad \text{for all } x \in S,$$

where $H_S(x)$ denotes the mean curvature vector of S in N .

Let $C_{\omega_b, h}^2$ denote the W -model constructed via ω_b and h , and assume that $C_{\omega_b, h}^2$ is a strongly balanced isoperimetric comparison space on the interval $[0, \infty)$. Then, for some $t_0 \geq 0$, the function $\frac{v(t)-v_0}{\cosh \sqrt{-bt}-C}$ is non-decreasing in $[t_0, +\infty)$, where t_0 is given in Lemma 5.3 and the constant C is defined as (see Theorem 3.2)

$$(5.17) \quad C = \text{Inf}_{t>0} \left(\cosh \sqrt{-bt} - q_W(t) \sqrt{-b} \sinh \sqrt{-bt} \right).$$

As a consequence, the function $\frac{v(t)-v_0}{\cosh \sqrt{-bt}}$ is non-decreasing in $[t_0, +\infty)$, where $v_0 = \text{Vol}(D_{t_0})$.

Proof. We are going to study the constant C defined in the statement of the Theorem 3.2. To do so, we need the following consequence of Lemma 5.2:

Lemma 5.6. *Let us consider an isoperimetric comparison space $C_{\omega_b, h}^2$. Let us define the function $f(t) := \cosh \sqrt{-bt} - q_W(t) \sqrt{-b} \sinh \sqrt{-bt} \quad \forall t > 0$. Then $f(t) > 0 \quad \forall t > 0$ and $\lim_{t \rightarrow 0} f(t) = 1$.*

Proof. Applying (5.5) in Lemma 5.2 again, we have:

$$\lim_{t \rightarrow 0^+} f(t) = 1.$$

Finally, as $q_W(t) \leq \frac{1}{\sqrt{-b}} \forall t > 0$, then:

$$f(t) \geq \cosh \sqrt{-bt} - \sinh \sqrt{-bt} \geq 0 \quad \forall t > 0. \quad \square$$

Now, the proof of the theorem runs as follows: by applying Proposition 5.3 and Lemma 5.6, the function $f(t)$ is non-negative and $\lim_{t \rightarrow 0^+} f(t) = 1$. Hence, the infimum C exists, and we have

$$(5.18) \quad 0 \leq C \leq 1.$$

Note that C ultimately depends on the functions $h(r)$ and $\omega_b(r)$, namely $C = C_{h,b}$.

Now, we factor:

$$(5.19) \quad \frac{v(t) - v_0}{\cosh \sqrt{-bt} - C} = \frac{v(t) - v_0}{\int_0^t W(s) ds} \frac{\int_0^t W(s) ds}{\cosh \sqrt{-bt} - C}.$$

The function $\frac{\int_0^t W(s) ds}{\cosh \sqrt{-bt} - C} > 0$ is non-decreasing for all $t \geq 0$ if and only if, for all $t \geq 0$

$$(5.20) \quad W(t)(\cosh \sqrt{-bt} - C) - \sqrt{-b} \sinh \sqrt{-bt} \int_0^t W(s) ds \geq 0$$

which is in turn equivalent to inequality $C \leq \cosh \sqrt{-bt} - q_W(t)\sqrt{-b} \sinh \sqrt{-bt} \forall t \geq 0$, which is true by definition of C .

On the other hand, and as $C_{\omega_b, h}^2$ is strongly balanced, we apply Corollary 5.4 to conclude that the function $\frac{v(t) - v_0}{\text{Vol}(B_t^W(p_w))}$ is non-decreasing in $[t_0, +\infty)$, for some $t_0 \geq 0$.

Therefore, we have the product of two positive and non-decreasing functions in $[t_0, \infty)$, so the result is also non-decreasing in $[t_0, \infty)$, as we wanted to prove.

Finally, the function $\frac{v(t) - v_0}{\cosh \sqrt{-bt}}$ is non-decreasing in $[t_0, +\infty)$. It follows directly from the fact that, for all $0 \leq C \leq 1$, and for all $t \geq t_0$,

$$(5.21) \quad \begin{aligned} 0 &\leq v'(t)(\cosh \sqrt{-bt} - C) - (v(t) - v_0)\sqrt{-b} \sinh \sqrt{-bt} \\ &\leq v'(t) \cosh \sqrt{-bt} - (v(t) - v_0)\sqrt{-b} \sinh \sqrt{-bt}. \end{aligned} \quad \square$$

Remark 5.7. When the surface S is minimal, it is used the function $h(r) = 0$ as a radial controller for the mean curvature and the isoperimetric

comparison space $C_{w_b, h}^2$ becomes the hyperbolic space $\mathbb{H}^2(b)$. In this case $t_0 = 0$ and we have the isoperimetric inequality (see [14, 17])

$$(5.22) \quad \frac{\text{vol}(\partial D_R)}{\text{vol}(D_R)} \geq \frac{\text{vol}(\partial B_R^{b,2})}{\text{vol}(B_R^{b,2})}, \quad \forall R \geq 0.$$

As a corollary of inequality (5.22), we have the classic monotonicity property for properly immersed minimal surfaces in Cartan–Hadamard manifolds with strictly negative curvature (see [1, 13]). In this case, the volume of the extrinsic balls is compared with the volume of the geodesic balls in the model space, $\mathbb{H}^2(b)$, which is the hyperbolic cosine and we have that the function $\frac{v(t)}{\text{Vol}(B_t^{b,2})} = \frac{v(t)}{\cosh(\sqrt{-bt})-1}$ is non-decreasing in $[0, +\infty)$. This property also holds for minimal surfaces in the Euclidean spaces (see [13]).

5.2. Surfaces with finite topology

On the other hand, we have the following theorem, which provides an extrinsic version of the proof of Huber’s classical theorem (see [10]) given by White in [19]. As we have mentioned in the Introduction, this is a key result that will allow us to argue as in [5, 6] (where it is possible to conclude that $\chi(S) = \lim_{t \rightarrow \infty} \chi(D_t)$ for an exhaustion of S by extrinsic balls $\{D_t\}_{t>0}$).

Recall that an exhaustion of the surface S is a sequence of subsets $\{D_t \subseteq S\}_{t>0}$ such that:

- $D_t \subseteq D_s$ when $s \geq t$.
- $\cup_{t>0} D_t = S$.

Theorem 5.8. *Let S^2 be a complete, connected and oriented surface properly immersed in a Cartan–Hadamard manifold N . Let $\{D_{r_i}\}_{i=1}^\infty$ be an exhaustion of S by extrinsic balls centered at a pole $o \in N$, where $\{r_i\}_{i=1}^\infty$ is an increasing sequence of extrinsic radius such that $r_i \rightarrow \infty$ when $i \rightarrow \infty$. If we have:*

$$\liminf_{i \rightarrow \infty} (\{-\chi(D_{r_k})\}_{k=i}^\infty) < \infty.$$

Then,

- (1) S^2 has finite topology and
- (2) $-\chi(S) \leq \liminf_{i \rightarrow \infty} (\{-\chi(D_{r_k})\}_{k=i}^\infty)$.

Proof. As the extrinsic balls D_r in a properly immersed and connected submanifold S are precompact and connected sets, we have

$$-\chi(D_r) = 2g(r) + c(r) - 2,$$

where $g(r)$ and $c(r)$ are the genus (number of handles), and the number of boundary components of D_r , respectively.

Hence, if we consider $\{D_{r_i}\}_{i=1}^\infty$ to be the exhaustion of S by extrinsic balls (where $\{r_i\}_{i=1}^\infty$ is an increasing sequence of extrinsic radius such that $r_i \rightarrow \infty$ when $i \rightarrow \infty$) which satisfies $\lim_{i \rightarrow \infty} \inf(\{-\chi(D_{r_k})\}_{k=i}^\infty) < \infty$, we have, taking limits:

$$(5.23) \quad \liminf_{i \rightarrow \infty} (\{-\chi(D_{r_k})\}_{k=i}^\infty) = 2 \liminf_{i \rightarrow \infty} (\{g(r_k)\}_{k=i}^\infty) + \liminf_{i \rightarrow \infty} (\{c(r_k)\}_{k=i}^\infty) - 2 < \infty.$$

Therefore, as $\lim_{i \rightarrow \infty} \inf(\{g(r_k)\}_{k=i}^\infty) \geq 0$, $\lim_{i \rightarrow \infty} \inf(\{c(r_k)\}_{k=i}^\infty) \geq 0$ and $g(r)$ is a non-decreasing, integer-valued function of r ,

$$(5.24) \quad \liminf_{i \rightarrow \infty} (\{g(r_k)\}_{k=i}^\infty) = \lim_{i \rightarrow \infty} g(r_i) = g < \infty.$$

As $c(r)$ is also an integer-valued function of r ,

$$(5.25) \quad \liminf_{i \rightarrow \infty} (\{c(r_k)\}_{k=i}^\infty) < \infty.$$

On the other hand, as $\lim_{i \rightarrow \infty} \inf(\{c(r_k)\}_{k=i}^\infty) < \infty$ and $\{c(r_k)\}_{k=i}^\infty$ is a sequence of natural numbers, then for each $i \in \mathbb{N}$ there exists a natural number $l(i)$, $l(i) \geq i$, such that

$$\inf(\{c_k\}_{k=i}^\infty) = c_{l(i)}$$

and hence

$$\lim_{i \rightarrow \infty} c_{l(i)} = c < \infty.$$

Summarizing, there exists a natural number k_0 such that, for all $i \geq k_0$

$$\begin{aligned} g(r_i) &= g, \\ c(r_i) &= c \end{aligned}$$

and, therefore, such that any compact subset of $S \setminus D_{r_{k_0}}$ has a genus equal to zero.

Now, given the sequence $\{r_i\}_{i=k_0}^\infty$ and for each r_i , let A_i be the union of D_{r_i} with those connected components of $S \setminus D_{r_i}$ which are compact (if there are none, then $A_i = D_{r_i}$). Let $g(A_i)$ and $c(A_i)$ denote the number of handles and boundary components, respectively, of A_i . As A_i is precompact itself, then, provided $j \geq i$ is large enough

$$D_{r_i} \subseteq A_i \subseteq D_{r_j}.$$

Hence, as $j > i \geq k_0$, $g = g(r_i) \leq g(A_i) \leq g(r_j) = g$, so:

$$(5.26) \quad g(A_i) = g \quad \forall i \geq k_0$$

and, by construction of A_i , we also have that $c(A_i) \leq c(r_i) \quad \forall i \geq k_0$, so additionally we can conclude that

$$(5.27) \quad c(A_i) \leq c \quad \forall i \geq k_0.$$

As a consequence of (5.26) and (5.27), we have that the A_i , ($i \geq k_0$), are homeomorphic, with A_{i+1} obtained from A_i by attaching annuli.

Therefore, S has finite topology, because $S = A_{k_0} \cup S \setminus A_{k_0}$, and A_{k_0} is compact and $S \setminus A_{k_0}$ is homeomorphic to a finite union of cylinders. Moreover:

$$(5.28) \quad \chi(S) = \chi((S \setminus A_{k_0}) \cup A_{k_0}) = \chi(S \setminus A_{k_0}) + \chi(A_{k_0}) = \chi(A_{k_0})$$

so, as $g(A_k) = g(r_k) = g$ and $c(A_k) \leq c(r_k) \leq c$,

$$\chi(S) = \chi(A_k) \geq 2 - 2g - c$$

and therefore:

$$(5.29) \quad \begin{aligned} & \liminf_{i \rightarrow \infty} (\{-\chi(D_{r_k})\}_{k=i}^\infty) \\ &= 2 \liminf_{i \rightarrow \infty} (\{g(r_k)\}_{k=i}^\infty) + \liminf_{i \rightarrow \infty} (\{c(r_k)\}_{k=i}^\infty) - 2 \\ &= 2g + c - 2 \geq -\chi(S). \end{aligned} \quad \square$$

6. Proof of main theorem

Let us consider $\{D_t\}_{t>0}$ to be an exhaustion of S by extrinsic balls, centered at a pole $o \in N$.

Let us denote

$$(6.1) \quad I(t) = \int_{\partial D_t} \left\langle A^S \left(\frac{\nabla^S r}{\|\nabla^S r\|}, \frac{\nabla^S r}{\|\nabla^S r\|} \right), \frac{\nabla^\perp r}{\|\nabla^S r\|} \right\rangle d\mu.$$

Then we have, by applying Proposition 4.4, the co-area formula, and adding and subtracting $b \cdot v(t)$ in inequality (4.10),

$$(6.2) \quad \begin{aligned} 2\pi\chi(D_t) &\geq \int_{D_t} (K_S - b) d\sigma + \eta_{\omega_b} v'(t) + b \cdot v(t) \\ &\quad - 2 \int_{\partial D_t} \frac{\|H_S\|}{\|\nabla^S r\|} d\sigma_t - I(t). \end{aligned}$$

As, for any $b < 0$ and for all $t > 0$,

$$\eta_{\omega_b}(t)v'(t) + b v(t) = \sqrt{-b} \frac{\cosh^2(\sqrt{-bt})}{\sinh(\sqrt{-bt})} \frac{d}{dt} \frac{v(t)}{\cosh(\sqrt{-bt})}$$

then

$$(6.3) \quad \begin{aligned} \frac{d}{dt} \left(\frac{v(t)}{\cosh \sqrt{-bt}} \right) &\leq \frac{\sinh \sqrt{-bt}}{\sqrt{-b} \cosh^2 \sqrt{-bt}} \left\{ 2\pi\chi(D_t) + \int_{D_t} (b - K_S) d\sigma \right. \\ &\quad \left. + 2 \int_{\partial D_t} \frac{\|H_S\|}{\|\nabla^S r\|} d\mu + I(t) \right\}. \end{aligned}$$

Now, using that, for all $t \geq 0$,

$$(6.4) \quad \frac{\sinh(\sqrt{-bt})}{\cosh^2(\sqrt{-bt})} \leq 2e^{-\sqrt{-bt}}$$

so we obtain

$$(6.5) \quad \begin{aligned} \frac{d}{dt} \left(\frac{v(t)}{\cosh \sqrt{-bt}} \right) &\leq \frac{1}{\sqrt{-b}} \left\{ 2e^{-\sqrt{-bt}} \int_{D_t} (b - K_S) d\sigma + \frac{\sinh \sqrt{-bt}}{\cosh^2 \sqrt{-bt}} I(t) \right. \\ &\quad \left. + 4e^{-\sqrt{-bt}} \int_{\partial D_t} \frac{\|H_S\|}{\|\nabla^S r\|} d\mu + 4e^{-\sqrt{-bt}} \pi\chi(D_t) \right\}. \end{aligned}$$

As we have observed before, the extrinsic balls D_t in a properly immersed and connected surface S are connected, precompact domains. Hence, for all

$t \geq 0$, we have:

$$(6.6) \quad \chi(D_t) = 2 - 2g(t) - c(t) \leq 1,$$

where $g(t)$ and $c(t)$ are the genus (number of handles), and the number of boundary components of D_r , respectively (see [16], p. 43).

Then, we integrate both sides of inequality (6.5) between 0 and a fixed $t > 0$, having into account that $\frac{v(0)}{\cosh(0)} = 0$ and applying the co-area formula:

$$(6.7) \quad \begin{aligned} \frac{v(t)}{\cosh(\sqrt{-bt})} &\leq \frac{1}{\sqrt{-b}} \left\{ 2 \int_0^t e^{-\sqrt{-bs}} \int_{D_s} (b - K_S) d\sigma ds \right. \\ &\quad + \int_0^t \frac{\sinh(\sqrt{-bs})}{\cosh^2(\sqrt{-bs})} I(s) ds + 4 \int_0^t \int_{\partial D_s} e^{-\sqrt{-bs}} \frac{\|H_S\|}{\|\nabla^S r\|} d\sigma_s ds \\ &\quad \left. + 4\pi \int_0^t \chi(D_s) e^{-\sqrt{-bs}} ds \right\} \\ &\leq \frac{1}{\sqrt{-b}} \left\{ 2 \int_0^t e^{-\sqrt{-bs}} \int_{D_s} (b - K_S) d\sigma ds \right. \\ &\quad + \int_0^t \frac{\sinh(\sqrt{-bs})}{\cosh^2(\sqrt{-bs})} I(s) ds \\ &\quad \left. + 4 \int_0^t \int_{\partial D_s} e^{-\sqrt{-bs}} \frac{\|H_S\|}{\|\nabla^S r\|} d\sigma_s ds + C(0) \right\}, \end{aligned}$$

where

$$(6.8) \quad C(0) = 4\pi \int_0^\infty e^{-\sqrt{-bs}} ds = \frac{4\pi}{\sqrt{-b}} < \infty$$

because, as D_s is (pre)compact for all finite radii s , then $\chi(D_s) < \infty \quad \forall s$.

We are going to estimate $\text{Sup}_{t>0} \frac{v(t)}{\cosh(\sqrt{-bt})}$ using the above inequality. To do so, we proceed as follows.

As $\int_S \|A^S\|^2 d\sigma < +\infty$, then $\int_S e^{-\sqrt{-br}} \|A^S\|^2 d\sigma < +\infty$, and similarly, using hypotheses (I) and (III), we have that $\int_S e^{-\sqrt{-br}} (b - K_N) d\sigma < +\infty$ and $\int_S e^{-\sqrt{-br}} \|H_S\| d\sigma < +\infty$.

Now, by applying Proposition 4.2 to the non-negative functions $\|A^S\|^2$, $b - K_N(x)$, and $\|H_S\|$ we have:

$$(6.9) \quad \begin{aligned} \int_0^{+\infty} e^{-\sqrt{-bt}} R(t) dt &< +\infty; \int_0^{+\infty} e^{-\sqrt{-bt}} \int_{D_t} (b - K_N) d\sigma dt < +\infty \\ \int_S e^{-\sqrt{-br(x)}} \|H_S\| d\sigma &< +\infty. \end{aligned}$$

With these estimates we can conclude, using equality (3.4) in Remark 3.3 and the co-area formula, and taking into account that, for all $t \geq 0$, the quantity $-\frac{4}{\sqrt{-b}} \int_0^t e^{-\sqrt{-bs}} \int_{D_s} \|H_S\|^2 d\sigma ds$ is negative:

$$\begin{aligned}
 \frac{v(t)}{\cosh(\sqrt{-bt})} &\leq \frac{2}{\sqrt{-b}} \int_0^t e^{-\sqrt{-bs}} \int_{D_s} (b - K_N) d\sigma ds \\
 &\quad + \frac{1}{\sqrt{-b}} \int_0^t e^{-\sqrt{-bs}} \int_{D_s} \|A^S\|^2 d\sigma ds \\
 &\quad - \frac{4}{\sqrt{-b}} \int_0^t e^{-\sqrt{-bs}} \int_{D_s} \|H_S\|^2 d\sigma ds \\
 (6.10) \quad &\quad + \frac{1}{\sqrt{-b}} \int_0^t \frac{\sinh(\sqrt{-bs})}{\cosh^2(\sqrt{-bs})} I(s) ds \\
 &\quad + \frac{4}{\sqrt{-b}} \int_{D_s} e^{-\sqrt{-br}} \|H_S\| d\sigma + \frac{C(0)}{\sqrt{-b}} \\
 &\leq C_1(0) + \frac{1}{\sqrt{-b}} \int_0^t \frac{\sinh(\sqrt{-bs})}{\cosh^2(\sqrt{-bs})} I(s) ds,
 \end{aligned}$$

where

$$\begin{aligned}
 C_1(0) &= \frac{1}{\sqrt{-b}} C(0) + \frac{2}{\sqrt{-b}} \int_0^{+\infty} e^{-\sqrt{-bt}} \int_{D_t} (b - K_N) d\sigma dt \\
 &\quad + \frac{1}{\sqrt{-b}} \int_0^{+\infty} e^{-\sqrt{-bt}} R(t) dt + \frac{4}{\sqrt{-b}} \int_S e^{-\sqrt{-br}} \|H_S\| d\sigma
 \end{aligned}$$

is a positive and finite constant.

We now have the following result.

Lemma 6.1. *There exist two non-negative constants C_2 and C_3 such that*

$$(6.11) \quad \int_0^t \frac{\sinh \sqrt{-bs}}{\cosh^2 \sqrt{-bs}} I(s) ds \leq C_2 \sqrt{C_3 + \frac{v(t)}{\cosh \sqrt{-bt}}}.$$

Proof. Let us consider $\{e_1, e_2\}$ to be an orthonormal basis of $T_p S$, ($p \in S$), being $e_1 = \frac{\nabla^S r}{\|\nabla^S r\|}$. Then

$$(6.12) \quad \left\| A^S \left(\frac{\nabla^S r}{\|\nabla^S r\|}, \frac{\nabla^S r}{\|\nabla^S r\|} \right) \right\|^2 \leq \|A^S\|^2$$

so

$$(6.13) \quad \left\langle A^S \left(\frac{\nabla^S r}{\|\nabla^S r\|}, \frac{\nabla^S r}{\|\nabla^S r\|} \right), \nabla^\perp r \right\rangle \leq \|A^S\| \|\nabla^\perp r\|.$$

Applying Cauchy-Schwartz inequality to the functions $\frac{\|A^S\|}{(\cosh(\sqrt{-br}))^{1/2}}$ and $\frac{\sinh(\sqrt{-br})\|\nabla^\perp r\|}{(\cosh(\sqrt{-br}))^{3/2}}$, we obtain:

$$(6.14) \quad \begin{aligned} & \int_{D_t} \frac{\sinh(\sqrt{-br})}{\cosh^2(\sqrt{-br})} \left\langle A^S \left(\frac{\nabla^S r}{\|\nabla^S r\|}, \frac{\nabla^S r}{\|\nabla^S r\|} \right), \nabla^\perp r \right\rangle d\sigma \\ & \leq \int_{D_t} \sinh(\sqrt{-br}) \|A^S\| \frac{\|\nabla^\perp r\|}{\cosh^2(\sqrt{-br})} d\sigma. \end{aligned}$$

On the other hand, if we consider $s_0 = 0$ and $t_0 = t$ in Proposition 4.5, as $\cosh \sqrt{-br}$ is non-decreasing, we have the following inequalities:

$$(6.15) \quad \begin{aligned} & \frac{\int_{D_t} \left(\cosh \sqrt{-br} - \|H_S\| \frac{\sinh \sqrt{-br}}{\sqrt{-b}} \right) d\sigma}{\cosh^2 \sqrt{-bt}} \\ & \geq \int_{D_t} \frac{\sinh^2 \sqrt{-br} \|\nabla^\perp r\|^2}{\cosh^3 \sqrt{-br}} d\sigma - \frac{1}{\sqrt{-b}} \int_{D_t} \frac{\sinh \sqrt{-br} \|H_S\|}{\cosh^2 \sqrt{-br}} d\sigma. \end{aligned}$$

But as $\cosh \sqrt{-br}$ is non-decreasing and $\frac{\int_{D_t} \sinh \sqrt{-br} \|H_S\| d\sigma}{\cosh^2 \sqrt{-bt}} \geq 0$, we have

$$(6.16) \quad \begin{aligned} & \frac{\int_{D_t} \left(\cosh \sqrt{-br} - \|H_S\| \frac{\sinh \sqrt{-br}}{\sqrt{-b}} \right) d\sigma}{\cosh^2 \sqrt{-bt}} \\ & \leq \frac{\int_{D_t} (\cosh \sqrt{-br}) d\sigma}{\cosh^2 \sqrt{-bt}} \leq \frac{v(t)}{\cosh \sqrt{-bt}} \end{aligned}$$

and therefore

$$(6.17) \quad \int_{D_t} \frac{\sinh^2 \sqrt{-br} \|\nabla^\perp r\|^2}{\cosh^3 \sqrt{-br}} \leq \frac{v(t)}{\cosh \sqrt{-bt}} + \frac{1}{\sqrt{-b}} \int_{D_t} \frac{\sinh \sqrt{-br} \|H_S\|}{\cosh^2 \sqrt{-br}} d\sigma.$$

Returning to the main computation in the Lemma, taking into account that $\frac{1}{\cosh \sqrt{-br}} \leq 2e^{-\sqrt{-br}}$ and $\frac{\sinh \sqrt{-br}}{\cosh^2 \sqrt{-br}} \leq 2e^{-\sqrt{-br}}$, we have:

$$\begin{aligned}
 & \int_0^t \frac{\sinh \sqrt{-bs}}{\cosh^2 \sqrt{-bs}} I(s) ds \\
 & \leq \sqrt{\int_{D_t} \frac{\|A^S\|^2}{\cosh \sqrt{-br}} d\sigma} \sqrt{\int_{D_t} \frac{\sinh^2 \sqrt{-br} \|\nabla^\perp r\|^2}{\cosh^3 \sqrt{-br}} d\sigma} \\
 (6.18) \quad & \leq \sqrt{\int_{D_t} 2e^{-\sqrt{-br}} \|A^S\|^2 d\sigma} \sqrt{\frac{v(t)}{\cosh \sqrt{-bt}} + \frac{1}{\sqrt{-b}} \int_{D_t} e^{-\sqrt{-br}} \|H_S\| d\sigma}.
 \end{aligned}$$

Applying hypotheses (II) and (III):

$$(6.19) \quad \int_0^t \frac{\sinh \sqrt{-bs}}{\cosh^2 \sqrt{-bs}} I(s) ds \leq C_2 \sqrt{\frac{v(t)}{\cosh \sqrt{-bt}}} + C_3,$$

where

$$\begin{aligned}
 0 \leq C_2 &= \sqrt{\int_{D_t} 2e^{-\sqrt{-br}} \|A^S\|^2 d\sigma} < \infty, \\
 0 \leq C_3 &= \frac{1}{\sqrt{-b}} \int_{D_t} e^{-\sqrt{-br}} \|H_S\| d\sigma < \infty
 \end{aligned}$$

and the Lemma is proven. □

By applying Lemma 6.1 to inequality (6.10), we obtain

$$(6.20) \quad \frac{v(t)}{\cosh \sqrt{-bt}} \leq C_1(0) + C_2 \sqrt{\frac{v(t)}{\cosh \sqrt{-bt}}} + C_3.$$

By putting $f(t) = \sqrt{\frac{v(t)}{\cosh \sqrt{-bt}}} + C_3$, the above inequality becomes

$$f(t)^2 - C_2 f(t) - (C_3 + C_1(0)) \leq 0 \forall t$$

and hence the values of $f(t)$ lie between the zeros of the function $g(x) = x^2 - C_2 x - (C_3 + C_1(0))$, which are real and distinct numbers (because $C_1(0) > 0$, $C_2, C_3 \geq 0$). Hence, $f(t)$ (and also $f^2(t) = \frac{v(t)}{\cosh \sqrt{-bt}} + C_3$) are bounded,

so therefore:

$$\text{Sup}_{t \geq 0} \frac{v(t)}{\cosh \sqrt{-bt}} = M < +\infty.$$

On the other hand, from Corollary 5.5 we know that $\frac{v(t)-v_0}{\cosh \sqrt{-bt}}$ is a non-decreasing function, so because for all $t \geq t_0$, $\frac{v(t)-v_0}{\cosh \sqrt{-bt}} \leq \frac{v(t)}{\cosh \sqrt{-bt}} \leq M$, we have that the limit $\lim_{t \rightarrow \infty} \frac{v(t)}{\cosh \sqrt{-bt}}$ exists and

$$(6.21) \quad \lim_{t \rightarrow \infty} \frac{v(t)}{\cosh \sqrt{-bt}} = \lim_{t \rightarrow \infty} \frac{v(t) - v_0}{\cosh \sqrt{-bt}} \leq M < \infty$$

and hence

$$(6.22) \quad \lim_{t \rightarrow \infty} \frac{v(t)}{\cosh \sqrt{-bt} - 1} = \lim_{t \rightarrow \infty} \frac{v(t)}{\cosh \sqrt{-bt}} \leq M < \infty$$

and

$$(6.23) \quad \lim_{t \rightarrow \infty} \frac{v(t)}{\sinh \sqrt{-bt}} = \lim_{t \rightarrow \infty} \frac{v(t)}{\cosh \sqrt{-bt} - 1} \leq M < \infty$$

To prove assertion (2) of the theorem, we need the following:

Lemma 6.2. *Let us consider C to be the constant defined in Theorem 3.2. Then*

$$(6.24) \quad \int_0^t \cosh \sqrt{-bt} v'(s) ds \geq \frac{\cosh \sqrt{-bt} + C}{2} v(t) - \frac{v_0(\cosh \sqrt{-bt} - 1)}{2}.$$

Proof. By Corollary 5.5 we have that $\frac{v(t)-v_0}{\cosh \sqrt{-bt}-C}$ is non-decreasing in $]t_0, +\infty[$ and $v(t)$ is non-decreasing, so we have:

$$(6.25) \quad (\cosh \sqrt{-bt} - C)v'(t) \geq (v(t) - v_0) \sqrt{-b} \sinh \sqrt{-bt}, \forall t \geq 0.$$

Integrating by parts:

$$(6.26) \quad \int_0^t \cosh \sqrt{-bt} v'(s) ds \geq v(t) \cosh \sqrt{-bt} - v_0 (\cosh \sqrt{-bt} - 1) + Cv(t) - \int_0^t \cosh \sqrt{-bt} v'(s) ds$$

and we obtain the result by isolating $\int_0^t \cosh \sqrt{-bt} v'(s) ds$. □

Once we have proven Lemma 6.2, we proceed as follows.

By definition of $I(t)$, inequality (6.13), and the arithmetic–geometric mean inequality $xy \leq \frac{x^2+y^2}{2}$, we have:

$$\begin{aligned}
 (6.27) \quad I(t) &\leq \int_{\partial D_t} \|A^S\| \frac{\|\nabla^\perp r\|}{\|\nabla^S r\|} d\sigma_t \\
 &= \int_{\partial D_t} \frac{\|A^S\|}{\sqrt{\eta_{\omega_b}(t)}\sqrt{\|\nabla^S r\|}} \frac{\sqrt{\eta_{\omega_b}(t)}\|\nabla^\perp r\|}{\sqrt{\eta_{\omega_b}(t)}\sqrt{\|\nabla^S r\|}} d\sigma_t \\
 &\leq \frac{1}{2} \int_{\partial D_t} \left(\frac{\|A^S\|^2}{\eta_{\omega_b}(t)\|\nabla^S r\|} + \frac{\eta_{\omega_b}(t)\|\nabla^\perp r\|^2}{\|\nabla^S r\|} \right) d\sigma_t \\
 &\leq \frac{1}{\eta_{\omega_b}(t)} \int_{\partial D_t} \frac{\|A^S\|^2}{\|\nabla^S r\|} + \eta_{\omega_b}(t) \int_{\partial D_t} \frac{\|\nabla^\perp r\|^2}{\|\nabla^S r\|} d\sigma_t.
 \end{aligned}$$

But, on applying the co-area formula:

$$\frac{1}{\eta_{\omega_b}(t)} R'(t) = \frac{1}{\eta_{\omega_b}(t)} \int_{\partial D_t} \frac{\|A^S\|^2}{\|\nabla^S r\|} d\sigma_t,$$

so we have:

$$(6.28) \quad I(t) \leq \frac{R'(t)}{\eta_{\omega_b}(t)} + \eta_{\omega_b}(t) \int_{\partial D_t} \frac{\|\nabla^\perp r\|^2}{\|\nabla^S r\|} d\sigma_t.$$

We are now going to analyze the integral $\int_{\partial D_t} \frac{\|\nabla^\perp r\|^2}{\|\nabla^S r\|} d\sigma_t$.

By integrating inequality (4.6) and applying the divergence theorem ($\sinh \sqrt{-b}s$ is increasing) and the co-area formula:

$$(6.29) \quad \int_{\partial D_t} \|\nabla^S r\| d\mu \geq \frac{2\sqrt{-b}}{\sinh \sqrt{-bt}} \int_0^t \cosh \sqrt{-bt} v'(s) d\sigma - 2 \int_{D_t} \|H_S\| d\sigma.$$

If we apply this inequality and the co-area formula:

$$\begin{aligned}
 (6.30) \quad &\eta_{\omega_b}(t) \int_{\partial D_t} \frac{\|\nabla^\perp r\|^2}{\|\nabla^S r\|} d\sigma_t \\
 &\leq \eta_{\omega_b}(t) v'(t) - \frac{2\sqrt{-b}\eta_{\omega_b}(t)}{\sinh \sqrt{-bt}} \int_0^t \cosh \sqrt{-bt} v'(s) ds \\
 &\quad + 2\eta_{\omega_b}(t) \int_{D_t} \|H_S\| d\sigma.
 \end{aligned}$$

Hence, on now applying Lemma 6.2, we have:

$$\begin{aligned} & \eta_{\omega_b}(t) \int_{\partial D_t} \frac{\|\nabla^\perp r\|^2}{\|\nabla^S r\|} d\sigma_t \\ & \leq \eta_{\omega_b}(t)v'(t) - \eta_{\omega_b}(t)^2v(t) - C\sqrt{-b}\eta_{\omega_b}(t)\frac{v(t)}{\sinh \sqrt{-bt}} \\ & \quad + \eta_{\omega_b}(t)\frac{\sqrt{-b}(\cosh \sqrt{-bt} - 1)}{\sinh \sqrt{-bt}}v_0 + 2\eta_{\omega_b}(t) \int_{D_t} \|H_S\|d\sigma. \end{aligned}$$

On the other hand, as $\frac{\sqrt{-b}(\cosh \sqrt{-bt}-1)}{\sinh \sqrt{-bt}} \leq \eta_{\omega_b}(t)$ we obtain:

$$\begin{aligned} (6.31) \quad & \eta_{\omega_b}(t) \int_{\partial D_t} \frac{\|\nabla^\perp r\|^2}{\|\nabla^S r\|} d\sigma_t \\ & \leq \eta_{\omega_b}(t)v'(t) - \eta_{\omega_b}(t)^2v(t) - C\sqrt{-b}\eta_{\omega_b}(t)\frac{v(t)}{\sinh \sqrt{-bt}} \\ & \quad + \eta_{\omega_b}(t)^2v_0 + 2\eta_{\omega_b}(t) \int_{D_t} \|H_S\|d\sigma. \end{aligned}$$

Therefore, by replacing the last inequality in (6.28)

$$\begin{aligned} (6.32) \quad I(t) & \leq \frac{1}{\eta_{\omega_b}(t)}R'(t) + \eta_{\omega_b}(t)v'(t) + bv(t) \\ & \quad + Cb\frac{v(t)}{\sinh \sqrt{-bt}} + \eta_{\omega_b}(t)^2v_0 + 2\eta_{\omega_b}(t) \int_{D_t} \|H_S\|d\sigma. \end{aligned}$$

From inequality (6.2) and equality (3.4) in Remark 3.3:

$$\begin{aligned} (6.33) \quad \eta_{\omega_b}(t)v'(t) + bv(t) & \leq \int_{D_t} (b - K_N)d\sigma + \frac{1}{2}R(t) - 2 \int_{D_t} \|H_S\|^2d\sigma + I(t) \\ & \quad + 2\pi\chi(D_t) + 2 \int_{\partial D_t} \frac{\|H_S\|}{\|\nabla^S r\|} d\sigma_t \\ & \leq \int_{D_t} (b - K_N)d\sigma + \frac{1}{2}R(t) + I(t) + 2\pi\chi(D_t) \\ & \quad + 2 \int_{\partial D_t} \frac{\|H_S\|}{\|\nabla^S r\|} d\sigma_t. \end{aligned}$$

So, on applying (6.32) we obtain:

$$\begin{aligned}
 (6.34) \quad -2\pi\chi(D_t) &\leq \int_{D_t} (b - K_N)d\sigma + \frac{1}{2}R(t) + \frac{1}{\eta_{\omega_b}(t)}R'(t) \\
 &\quad + Cb\frac{v(t)}{\sinh\sqrt{-bt}} + \eta_{\omega_b}(t)^2v_0 + 2\eta_{\omega_b}(t) \int_{D_t} \|H_S\|d\sigma \\
 &\quad + 2 \int_{\partial D_t} \frac{\|H_S\|}{\|\nabla^S r\|}d\sigma_t.
 \end{aligned}$$

On the other hand, as

$$\begin{aligned}
 (6.35) \quad &\int_S \left(\frac{\|A^S\|^2}{\eta_{\omega_b}(t)} + \|H_S\| \right) d\sigma \\
 &= \lim_{t \rightarrow +\infty} \left(\frac{1}{\eta_{\omega_b}(t)} \int_{D_t} \|A^S\|^2 d\sigma + \int_{D_t} \|H_S\| d\sigma \right) \\
 &= \frac{1}{\sqrt{-b}} \int_S \|A^S\|^2 d\sigma + \int_S \|H_S\| d\sigma < +\infty
 \end{aligned}$$

we have that there exists a monotone increasing (sub)sequence $\{t_i\}_{i=1}^\infty$ tending to infinity (namely, $t_i \rightarrow \infty$ when $i \rightarrow \infty$), such that:

$$(6.36) \quad \lim_{i \rightarrow \infty} \int_{\partial D_{t_i}} \left(\frac{\|A^S\|^2}{\eta_{\omega_b}(t)\|\nabla^S r\|} + \frac{\|H_S\|}{\|\nabla^S r\|} \right) d\sigma_{t_i} = 0.$$

So, as both addenda are non-negative, we have:

$$\begin{aligned}
 (6.37) \quad \lim_{i \rightarrow \infty} \frac{1}{\eta_{\omega_b}(t_i)}R'(t_i) &= \lim_{i \rightarrow \infty} \int_{\partial D_{t_i}} \frac{\|A^S\|^2}{\eta_{\omega_b}(t_i)\|\nabla^S r\|} d\sigma_{t_i} = 0 \\
 \lim_{i \rightarrow \infty} \int_{\partial D_{t_i}} \frac{\|H_S\|}{\|\nabla^S r\|} d\sigma_{t_i} &= 0.
 \end{aligned}$$

Let us consider the exhaustion of S by these extrinsic balls, namely, $\{D_{t_i}\}_{i=1}^\infty$. Since $\{D_{t_i}\}_{i=1}^\infty$ is a family of connected and precompact open sets which exhaust S , then $\{-\chi(D_{r_i})\}_{i=1}^\infty$ is monotone non-decreasing. Then, on replacing t for t_i and taking limits when $i \rightarrow \infty$ in inequality (6.34), we have that:

$$\begin{aligned}
 (6.38) \quad &2\pi \liminf_{i \rightarrow \infty} (\{-\chi(D_{r_k})\}_{k=i}^\infty) \\
 &\leq \int_S (b - K_N)d\sigma + \frac{1}{2} \int_S \|A^S\|^2 d\sigma \\
 &\quad + bC \lim_{t \rightarrow +\infty} \frac{v(t)}{\cosh\sqrt{-bt} - 1} + 2\sqrt{-b} \int_S \|H_S\|d\sigma - bv_0,
 \end{aligned}$$

since we have the equality between the limits

$$(6.39) \quad \lim_{t \rightarrow \infty} \frac{v(t)}{\sinh \sqrt{-bt}} = \lim_{t \rightarrow \infty} \frac{v(t)}{\cosh \sqrt{-bt} - 1} < \infty.$$

Hence, by applying Theorem 5.8, S^2 has finite topology and

$$(6.40) \quad -2\pi\chi(S) \leq \int_S (b - K_N) d\sigma + \frac{1}{2} \int_S \|A^S\|^2 d\sigma \\ + bC \lim_{t \rightarrow +\infty} \frac{v(t)}{\cosh \sqrt{-bt} - 1} + 2\sqrt{-b} \int_S \|H_S\| d\sigma - bv_0.$$

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