# Closed-form estimates of the domain of attraction for nonlinear systems via fuzzy-polynomial models

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# Closed-form estimates of the domain of attraction for nonlinear systems via fuzzy-polynomial models

José Luis Pitarch, Antonio Sala, *Member IEEE* and Carlos Vicente Ariño

***Abstract*—In this work, the domain of attraction of the origin of a nonlinear system is estimated in closed-form via level sets with polynomial boundary, iteratively computed. In particular, the domain of attraction is expanded from a previous estimate, such as, for instance, a classical Lyapunov level set. With the use of fuzzy-polynomial models, the domain-of-attraction analysis can be carried out via sum of squares optimization and an iterative algorithm. The result is a function wich bounds the domain of attraction, free from the usual restriction of being positive and decrescent in *all* the interior of its level sets.**

***Index Terms*—domain of attraction, fuzzy polynomial systems, local stability, Sum of Squares, nonlinear systems, robust stability**

## I. Introduction

A large class of nonlinear systems can be *exactly* expressed, locally in a *compact* region (denoted as *modelling region*, $\Omega$, in the sequel), as a fuzzy Takagi-Sugeno (TS) model, using the "sector nonlinearity" methodology [1]. It expresses the nonlinearity as a convex time-varying combination of "vertex" linear equations. The works [2], [3] extend the idea to fuzzy polynomial models, by using a Taylor-series approach which expresses non-polynomial nonlinearities (or high-degree polynomial ones) as a convex interpolation between polynomials of reduced degree.

Once *locally exact* fuzzy models are available, stability and control design for the original nonlinear system via convex optimization (particularly with Linear Matrix Inequalities (LMI)) has been deeply explored in literature. Indeed, *global* stability conditions for Takagi-Sugeno models in LMI form have been explored with quadratic Lyapunov functions [1], parameter-dependent [4], non-quadratic [5], polyhedric ones [6], or even nonconvex Bilinear Matrix Inequalities (BMI) settings such as in [7] where a combination of genetic algorithms plus convex optimization is used. Output feedback designs with measurable premise variables have also been developed [8], [9].

Sum of squares techniques (SOS) [10], are used to extend the above framework to fuzzy polynomial models in stability analysis [11], [12] as well as controller synthesis [8], [13], via polynomial Lyapunov functions.

However, one of the key issues in practical usefulness of many of the above results is the fact that, given the locality of most fuzzy models, proving *global* fuzzy-model stability translates, actually, to only *local* stability of the original nonlinear system being modelled in most cases.

The above issue is usually disregarded in literature, considering the problem as solved once a feasible "global" fuzzy LMI solution is found with a Lyapunov function $V(x)$. However, stability is proved only for the largest Lyapunov level set $\{x : V(x) < V_c\}$ in the modelling region $\Omega$: in quite a few cases, a very small subset of the region $\Omega$ may be actually proved. A slight variation allows for expanding the proved domain of attraction (DA) to $\{x : V(x) < V_c\} \cap \Omega$ for $\{x : V(x) < V_c\} \not\subset \Omega$ in some cases [14], [15].

Given the above shortcomings, the objective of this paper is presenting a methodology to expand the proved domain of attraction of nonlinear systems, using fuzzy-polynomial models. The methodology is discussed for both continuous and discrete cases. Any (possibly small) subset of the domain of attraction found with current LMI/SOS results (to be denoted as, say, $B_1$) can be used as a "seed" of an iterative algorithm that expands it.

For completeness, note that, apart from Lyapunov methods, domain of attraction estimation can be done numerically [16]. Also, relevant results which discuss, specifically, domain of attraction estimation for polynomial systems are reported in [17]–[19].

This paper shows that, once any seed set $B_1$ is available, there is no need of using actual Lyapunov functions any more, but only proving that there is a set $B_2$ such that trajectories starting in it fall into $B_1$, so $B_2$ belongs to the DA, too. A SOS approach provides a numerical tool to obtain such a set, and an iterative algorithm naturally ensues by using $B_2 \cup B_1$ as the new seed. If $B_1 \subset B_2$, SOS algorithms provide $B_2$ which is an estimate of the DA expresed in closed-form as a polynomial.

The function defining the boundary of the resulting DA estimates in this paper is free from the restriction of being decrescent and positive in its interior. That allows for improved estimates over previous literature.

The structure of the paper is as follows: next section presents notation and fuzzy polynomial modeling, known stability analysis results and motivation are discussed in Section III. Section IV presents the problem statement joint with auxiliary results and definitions. Section V discusses the expansion of the domain of attraction estimate for discrete systems and, based on it, proposes a computational low-cost iterative procedure using SOS techniques. Section VI discusses a similar procedure for continuous systems and section VII gives some examples in order to show the improving results. Finally Section VIII concludes the paper.

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## II. Preliminaries

This paper will consider either a continuous-time nonlinear system:

$$\dot{x}(t) = f(x(t)) \qquad x(t) \in \mathbb{R}^n,\ t \in \mathbb{R} \tag{1}$$

or a discrete-time one:

$$x_{k+1} = f(x_k) \qquad x_k \in \mathbb{R}^n,\ k \in \mathbb{N} \tag{2}$$

with sufficiently smooth $f$ (so it admits a Taylor series), being the origin a *stable* equilibrium point by assumption.

**Notation:** In this paper, the trajectory of system (1) –or (2) in the discrete developments–, starting in $x_0$ at time $t = 0$, will be denoted as $\psi(t, x_0)$. The set of polynomials in a variable $z$ will be denoted as $\mathcal{R}_z$, and the $n$-dimensional vectors of polynomials as $\mathcal{R}_z^n$. Polynomials in some variables $z$ which can be decomposed as a sum of squares of other polynomials will be denoted as $\Sigma_z$. SOS decompositions of polynomials can be found using well-known SDP software [20], [21]. Also, the notation used in [22] will be used in the rest of the paper: given polynomials $\{F_1, ..., F_{o_f}\}$, where $o_f$ denotes the number of them, $\mathcal{M}$ will denote the **multiplicative monoid**, $\wp$ denotes the **cone**, and $\Im$ the **ideal** generated by the set of $F_d$'s. In order to shorten notation, the ideal generated by a vector of polynomials $P \in \mathcal{R}_z^n$ will be defined as the ideal generated by its elements.

**Definition 1** ( [23])**.** *The set of initial conditions defined as*

$$\mathcal{D} = \left\{ x_0 \in \mathbb{R}^n : \lim_{t\to\infty} \psi(t, x_0) = 0 \right\} \tag{3}$$

*is denoted as the Domain of Attraction (DA) of the origin.*

In [2] a procedure is presented in which a system (1) or (2) can be equivalently transformed, locally when $x \in \Omega$, to a fuzzy-polynomial system

$$\dot{x}(t) = \sum_{i=1}^{r} \mu_i(z(t)) P_i(x(t)) \tag{4}$$

$$x_{k+1} = \sum_{i=1}^{r} \mu_i(z_k) P_i(x_k) \tag{5}$$

where $P_i \in \mathcal{R}_x^n$ are vertex polynomial models, $r$ is the number of fuzzy rules, $z$ are premise variables and the membership functions $\mu_i$ lie in the standard simplex $\Gamma = \{\mu_i \in \mathbb{R} \mid 0 \leq \mu_i,\ \sum_i \mu_i = 1\}$.

This paper will study the estimation of the DA of systems in the form (4) or (5), when $x$ fulfills some algebraic constraints (as a tool to estimate the DA of (1) or (2)). Indeed, proving global (or local) Lyapunov stability for the fuzzy-polynomial model (4) or (5), allows proving local stability for the nonlinear system (1) or (2) respectively.

First, a review on common approaches to DA estimation in literature will be outlined in next section, as well as a discussion on its limitations. Then, on section IV, additional definitions will allow to state in a more precise way the problem to be solved for polynomial fuzzy systems.

## III. Review of level-set approach to DA estimation

This section reviews DA estimation based on Lyapunov level sets in fuzzy systems literature.

First, simple global stability conditions which will form the basis of further developments will be reviewed.

By SOS techniques, the following Lyapunov stability result is well-known:

**Lemma 1** (continuous-time, [24])**.** *Global asymptotic stability of a system* (4) *can be proved if a polynomial function* $V(x)$ *can be found verifying:*

$$V(x) - \epsilon(x) \in \Sigma_x \tag{6}$$

$$-\frac{\partial V}{\partial x} P_i(x) - \epsilon(x) \in \Sigma_x \quad \forall\ i = 1, \ldots, r \tag{7}$$

*where* $\epsilon(x)$ *is a radially unbounded positive polynomial, such as* $\|x\|_2^2$.

**Lemma 2** (discrete-time, [25], [26])**.** *Global asymptotic stability of a system* (5) *can be proved if* $V(x)$ *can be found such that* (6) *holds and*

$$V(x) - V(\sum_{i=1}^{r} \sigma_i^2 P_i(x)) - \epsilon(x) \in \Sigma_{\sigma,x} \tag{8}$$

In the above lemma, $\sigma_i = \sqrt{\mu_i}$ are auxiliary variables. Also, expression (8) needs to be modified for actual computations, making it *homogeneous* in the memberships (i.e., all the monomials must have the same degree in $\sigma_i$). It can be achieved by multiplying anything by $\sum_{i=1}^{r} \sigma_i^2$ (which is equal to one, anyway) as many times as needed (details omitted for brevity, as the procedure is well known).

If the above conditions are feasible, then the DA has been proved to be the whole $\mathbb{R}^n$. However, many nonlinear systems of interest are not globally stable[1]. Hence, refinements to the above conditions are needed in order to obtain a DA estimate (ideally, as large as possible).

### A. Local stability analysis: bounded DA estimates

If the above problem renders infeasible, a local stability condition can be posed, based on standard invariant-set arguments. Indeed, introducing the notation $V_\gamma = \{x : V(x) < \gamma\}$ to denote the level sets of $V(x)$, we have:

**Lemma 3** ( [23])**.** *If* $V(x) \geq 0$ *and* $\dot{V}(x) < 0$ *in* $\Omega$*, then* $V_\gamma \subset \Omega$ *implies* $V_\gamma \subset \mathcal{D}$*.*

Then, in many literature references, the estimated DA is given by $V_{\gamma^*}$ where $\gamma^*$ is the largest $\gamma$ such that $V_\gamma \subset \Omega$.

In order to apply the above lemma to polynomial systems, the *Positivstellensatz* theorem [22] enables checking local positiveness conditions with SOS programming (sufficient conditions). It will be used to modify conditions (6) and (7) or (6) and (8), in order to make them hold locally in $\Omega$, as follows:

[1] Even if they are, maybe a low-degree polynomial Lyapunov function might not be enough to prove such global stability.

**Lemma 4** ( [15]). *Assume that the modelling region can be defined as:*

$$\Omega = \{x : G_d(x) > 0, H_j(x) = 0, d = 1, \dots o_g, j = 1, \dots, o_h\} \quad (9)$$

*where $G_d$ and $H_j$ are, respectively, a collection of $o_g$ and $o_h$ polynomials defining the boundary of $\Omega$.*

*If polynomials $S_{i0}(x)$, $S_{id}(x) \in \Sigma_x$, $Z_{ij}(x) \in \mathcal{R}_x$, $i = 1, \dots, r$, can be found fullfilling*

$$-S_{i0}(x)(\frac{\partial V}{\partial x}P_i(x) + \epsilon(x)) - \sum_{d=1}^{o_g} S_{id}(x)G_d(x) + \sum_{j=1}^{o_h} Z_{ij}(x)H_j(x) \in \Sigma_x, \quad i = 1, ..., r \quad (10)$$

*then $\dot{V}(x)$ is locally negative in $\Omega$ except at the origin and, hence, its level sets belong to the DA of the origin if $V(x) \geq 0$ (Lemma 3).*

The above lemma is a simplified version of the original *Positivstellensatz* result in which (10) would be replaced by the less conservative expression:

$$F_{i,1}(x) + F_{i,2}(x) \in \Sigma_x \quad (11)$$

where $F_{i,1}$ belongs to the polynomial cone $\wp(-\frac{\partial V}{\partial x}P_i(x), G_1(x), ..., G_{o_g}(x))$ and $F_{i,2}$ belongs to the ideal $\Im(H_1(x), .., H_{o_h}(x))$, see [22] for details. However, computational complexity increases as more multipliers are added involving products of the $G_d(x)$ or $H_j(x)$ which are also members of the cones and ideals.

Conditions (10) are not linear in decision variables if both $S$, $Z$ and $V$ have to be found. However, the problem becomes convex if either $V(x)$ is fixed (proposed in [19, chap. 4][2]) or $S_{i0}(x)$ is fixed, for instance to $S_{i0}(x) = 1$, as proposed in [12]. Once $V(x)$ is found, a bound for the maximum $\gamma$ fulfilling Lemma 3 can be also easily found via SOS techniques.

In order to avoid ill-shaped solutions, additional SOS constraints may be added to find the Lyapunov function level set containing the largest region with a particular predefined shape (circle, hypercube, ...) [14], [15], or maximising an approximation to the volume based on the maximum-volume formula for a quadratic form [19].

*Discrete systems:* Equivalent result can be proved for discrete-time systems using (8) instead of (7) in conditions (10). However, the result in Lemma 2 involves a polynomial whose degree is that of $P_i$ plus that of $V$ in the state variables as well as two plus the degree of $V$ in the auxiliary variables $\sigma_i$; it also needs the algebraic manipulations to make the inequality homogeneous in $\sigma_i$, see [2] for details. Hence, the degree of the polynomials and the number of decision variables may be high even for simple local stability problems. In order to allow for simpler conditions, if desired –at the expense of conservativeness–, an auxiliary lemma in next section (Lemma 5) will be useful.

[2] [19] uses a Lyapunov function from the linearized system, and take $\Omega \equiv V_\gamma$. If $\gamma$ is maximized, it can be recast as a quasi-convex problem (GEVP).

### B. Sources of conservativeness

Level-set based estimation of the DA with fuzzy models has some drawbacks.

*1.-Choice of modelling region:* If the modelling region $\Omega$ is chosen too large, the associated Lyapunov conditions may render infeasible (consequents separate too much). From classical Lyapunov theorems, if the linearised system is stable, a "small enough" modelling region will render a feasible problem[3]. The problem, then, is how to choose which is the right modelling region to obtain the largest DA estimate.

*2.-Conservative sign conditions:* One of the reasons for infeasibility is requiring $V$ and $-\dot{V}$ (increment of $V$ in the discrete case) to be positive in all $\Omega$. In fact, it would be need only inside a suitable level set. For instance, if there is more than one equilibrium point in the modelling region $\Omega$, all the above lemmas fail as there is one $x \neq 0$ where $\dot{V} = 0$ for any choice of $V$.

*3.-Existence of larger invariant sets in $\Omega$:* There exist invariant sets in $\Omega$ which are not level sets of a low-degree polynomial Lyapunov function (see later).

Some of these issues have been addressed in literature. For instance, the third one gives rise to piecewise Lyapunov functions in the form $V(x) = \min_i V_i(x)$, etc. [27].

Also, in the preliminary works [14], [15] by the authors, some considerations on DA estimation are discussed. In particular, the first drawback (iterations in the size of modelling region) and the third one (an *a posteriori* expansion of the DA estimate given a fixed Lyapunov function is proposed: indeed, in some cases there exists an invariant set in the form $V_{\gamma_2} \cap \Omega$ for $\gamma_2$ larger than $\gamma^*$ from Lemma 3).

This work presents a unified approach taking into account the three issues. The objective will be obtaining a DA estimate when starting from the following situation:

- a low-degree solution to the "global" stability problem (Lemmas 1 or 2) cannot be found,
- there is a small enough region around the origin where the "local" stability problem (Lemma 3) is strictly feasible and an initial level-set $B_1 = \{V_1(x) < 1\}$ is proven to belong to the DA.

In this paper, the goal is to obtain a "local" estimate of the DA as large as possible. An iterative approach is used in order to avoid as much as possible the above discussed sources of conservativeness. The main ideas are:

- setting "regions of interest" smaller than $\Omega$ in local stability conditions;
- lift the restriction of the DA estimate being a Lyapunov level set;
- allowing for more than one equilibrium point in the modelling region $\Omega$.

## IV. PROBLEM STATEMENT, AUXILIARY DEFINITIONS AND LEMMAS

In order to fulfill the objectives of the paper, the following definition is useful to refine the kind of DA to be obtained in later sections.

[3] Indeed, if the linearised system is stable, a quadratic $V(x)$ will suffice for small $\Omega$.

**Definition 2.** *The Local Robust Domain of Attraction (LRDA) of system* (4) *or* (5)*, referred to region* $\Omega$*, will be denoted by* $\mathcal{D}_\Omega$*. It is defined as the set of initial conditions fulfilling:*

$$\mathcal{D}_\Omega = \left\{ x_0 \in \Omega : \begin{array}{l} \psi(t, x_0, \mu) \in \Omega \forall t \geq 0, \forall \mu \in \Gamma \\ \lim_{t\to\infty} \psi(t, x_0, \mu) = 0, \forall \mu \in \Gamma \end{array} \right\} \quad (12)$$

Note that the condition for the trajectories not leaving $\Omega$ above is needed as (4) and (5) are, by assumption, not valid outside $\Omega$ to analyse (1) or (2) respectively.

The term "robust" in the definition is due to the fact that $\mathcal{D}_\Omega$ is defined considering "all" possible $\mu$ in the simplex $\Gamma$ and not the particular, possibly non-polynomial, $\mu(x)$ giving actually exact equivalence with the nonlinear system. That allows polynomial techniques to be used at the price of conservativeness. Indeed, based on the above, as (4) and (5) include (1) and (2), respectively, in $\Omega$ (plus many other systems), then obviously, $\mathcal{D}_\Omega \subset \mathcal{D}$: by finding the LRDA from a fuzzy polynomial model, we have found an inner estimate of the DA of (1) –or (2)–, as defined in [19, Chap. 6].

*Problem statement:* The above-defined LRDA may be a very complicated region and hardly characterizable. The goal of this paper is to "fit" the LRDA with a *closed-form* expression given by a low-degree polynomial boundary which gives larger results than Lyapunov literature. The polynomial degree will be chosen depending on the available computing resources.

In particular, consider a compact set defined by $o_q$ polynomial bounds $\Theta = \{x : Q_l(x) \leq 1\}\ l : 1, ..., o_q$, and an inner region $B = \{x \in \Theta : V(x) < 1\}$ containing the origin ($V(0) < 1$).

The following definition will be later taken in the rest of paper as the best low-degree fit of $\Theta$.

**Definition 3.** *Consider a decision-variable polynomial of predefined degree denoted as* $R(x)$*. The best fitting region* $\Theta_R = \{x \in \Theta : R(x) \leq 1\}$*, fulfilling* $B \subset \Theta_R \subset \Theta$*, is defined to be the solution of the following problem:*

minimize $\tau$ s.t.

$$1 + \tau \geq R(x)\ when\ x \in \Theta_m,\ m = 1, \ldots, o_q \quad (13)$$

$$R(x) \leq 1\ when\ x \in B \quad (14)$$

$$R(0) = 0 \quad (15)$$

*being* $\Theta_m$ *each one of the* $o_q$ *portions of the frontier of* $\Theta$*, defined as* $\Theta_m = \{x :\ V(x) \geq 1,\ Q_m(x) = 1\}$.

In this way, $\Theta_R$ will be an inner approximation to $\Theta$ with a *single* polynomial restriction.

Note that condition (15) is needed in order to avoid the trivial solution $\tau = 0$, $R = 1$, requiring that at least one point has a value different from one[4].

*Auxiliary lemma:* Let us last discuss an auxiliary result regarding lower-complexity SOS conditions for stability. As commented in the previous section, some developments (particularly in discrete-time) require a high-degree polynomial in both state and auxiliary membership variables $\sigma$. As an alternative, a dummy variable $\rho$ may be introduced jointly with the equality constraint $\rho = x_{k+1}$, i.e., $\rho - \sum_{i=1}^r \mu_i P_i = 0$. In this way, equation (8) may be changed, following a Positivstellensatz argumentation.

**Lemma 5.** *The system* (5) *is globally stable if there exist functions* $V(x)$ *and* $G_1(\rho, x)$ *such that:*

$$V(x) - V(\rho) - \epsilon(x) + G_1(\rho, x) > 0 \quad (16)$$

*with*[5] $G_1 \in \mathcal{I}(\rho - \sum_{i=1}^r \mu_i P_i)$ *arising from the equality constraint.*

Note that (16) is not yet a SOS problem (because of the nonlinear functions $\mu_i$ appearing in $G_1$); however, it is a fuzzy summation so well-known semidefinite relaxations based on Polya's theorem [28] may be applied.

For instance, if $G_1(\rho, x)$ were chosen as the simple expression

$$G_1(\rho, x) = \phi(x, \rho) \cdot (\rho - \sum_{i=1}^r \mu_i P_i)$$

being $\phi(x, \rho)$ a polynomial vector in $\mathcal{R}^n_{\{\rho,x\}}$, then (16) is a single-dimensional fuzzy summation whose positiveness for $\mu_i \in \Gamma$ is proved if the $r$ SOS conditions below hold:

$$V(x) - V(\rho) - \epsilon(x) + \phi(x, \rho)(\rho - P_i(x)) \in \Sigma_{\{\rho,x\}}\ i = 1, \ldots, r \quad (17)$$

In fact, the above proposed structure of $G_1$ will be the actual choice in later examples.

## V. Discrete-time DA estimation

Given a particular region $B_1$ which belongs to the DA of a system (2), a larger estimate of the DA can be calculated following the next result.

**Lemma 6.** *Let* $B_1 = \{x \in \mathbb{R}^n : V_1(x) < 1\} \subset \mathcal{D}$ *be a (previously proven) bounded subset of the domain of attraction of* (2) *and let* $N$ *be a horizon parameter (number of future samples) fixed* a priori. *Then, any region* $B_2$ *such that*

$$B_2 \subseteq \{x \in \mathbb{R}^n : V_2(x) < 1\}, \quad (18)$$

*where*[6] $V_2(x) = V_1(f^{[N]}(x))$*, belongs also to the domain of attraction of the system* (2)*.*

*Proof:* Following system's dynamics (2), the points which in $N$ future samples will be inside $B_1$ are those defined by:

$$V_1(x_{k+N}) < 1 \equiv V_1(f^N(x_k)) < 1$$

So the region $B_2$ is a subset of the DA as any starting point in $B_2$ will enter the open set $B_1$ in a finite number of time steps. Hence, it will later reach the origin as $B_1 \subset \mathcal{D}$. ■

**Corollary 1.** *If* $B_1$ *contains the origin, when* $N \to \infty$*,* $B_2$ *exactly coincides with the actual DA of the origin of the nonlinear discrete system.*

[4] There is no loss of generality in setting $R(0)$ to zero, as a straightforward argumentation with affine scalings shows (details omitted for brevity).

[5] A slight abuse of notation is involved in the definition of the ideal as it is generated by an expression which is not a polynomial. In this context, the ideal will be considered to be the product of arbitrary polynomials –to be obtained by SOS optimization– by any product of the generating functions.

[6] Notation: $f^{[N]}(x) = (\underbrace{f \circ f \circ \ldots \circ f}_{N})(x)$.

*Proof:* Indeed, no point reaching the origin can avoid entering $B_1$ in a finite number of time steps, as the origin is in its interior. ■

*Remark* 1. Note that $B_1$ does not need to be a Lyapunov level set like the ones considered in classical results (which implicitly consider $N = 1$). In fact, there is no need of it being even an "invariant" set as understood in literature [23].

### *A. Application to Fuzzy Polynomial Systems*

Despite of Lemma 6 gives an exact description of the $N$-step DA, unless $f$ is linear, the result is a very high-degree expression if fuzzy polynomial models for system (2) are used (both in the state variables and in the membership functions). So, the results in the above lemma may be of little use if a reasonably simple approximation of the DA of a nonlinear system were needed for subsequent analysis or representation.

In order to obtain a simpler reliable representation for the DA, the following lemmas propose the use of fuzzy polynomial models in order to describe the nonlinear dynamics. Hence, inspired on the "best fitting region" of Definition 3, they obtain a user-defined low degree polynomial in order to characterize the LRDA.

The basic idea motivating the results below is obtaining a low-degree approximation of the 1-step DA $V(f(x)) < 1$ in Lemma 6 and, later, iterating such approximation.

**Lemma 7.** *Consider a known seed set $B_1 \subset \mathcal{D}_\Omega$ defined by $B_1 = \{x \in \Omega : V_1(x) < 1\}$ and a user-defined modelling region $\Omega$ defined by $o_q$ restrictions $\Omega = \{x : Q_l(x) \leq 1\}$ $l : 1, ..., o_q$, such that it is compact. Then, the region $B_2 = \{x : Q_l(x) \leq 1, V_2(x) \leq 1\}$ belongs to $\mathcal{D}_\Omega$ and $B_2 \supset B_1$, if a function $V_2(x)$ can be found solving the following problem:*

*minimize $\tau$ s.t.*

$$V_2(0) = 0 \tag{19}$$

$$V_2(x) - 1 - F_1(x, \rho) + G_1(x, \rho) > 0 \ \forall \{x, \rho\} \tag{20}$$

$$V_2(x) - 1 - F_2(x, \rho) + G_2(x, \rho) > 0 \ \forall \{x, \rho\} \tag{21}$$

$$1 - V_2(x) + \tau - F_3(x, \rho) + G_3(x, \rho) + G_4(x, \rho) > 0 \ \forall \{x, \rho\} \tag{22}$$

$$1 - V_2(x) - F_4(x) > 0 \ \forall x \tag{23}$$

*Where*

- $\tau > 0$,
- $F_1(x, \rho) \in \wp(V_1(\rho) - 1, 1 - Q_1(x), ..., 1 - Q_{o_q}(x))$,
- $F_2(x, \rho) \in \wp(Q_1(\rho) - 1, ..., Q_{o_q}(\rho) - 1, 1 - Q_1(x), ..., 1 - Q_{o_q}(x))$,
- $F_3(x, \rho) \in \wp(1 - Q_1(x), ..., 1 - Q_{o_q}(x))$,
- $F_4(x) \in \wp(1 - V_1(x), 1 - Q_1(x), ..., 1 - Q_{o_q}(x))$,
- $G_3(x, \rho) \in \Im(V_1(\rho) - 1)$,
- $\{G_1(x, \rho),\ G_2(x, \rho),\ G_4(x, \rho)\} \in \Im(\rho - \sum_{i=1}^{r} \mu_i P_i(x))$

Note that the same abuse of notation issue discussed in footnote 5 has been assumed.

*Proof:* By condition (20), the region $V_2 < 1$ will be an inner approximation (actually it has to fulfill the requirement only inside the modelling region $\Omega$) to the region defined by $V_1(x_{k+1}) < 1$ (the points which in *one* sample will be inside $B_1$): the condition implies that $V_2(x)$ is greater than 1 when $V_1(x_{k+1}) \geq 1$ and $x \in \Omega$.

Condition (21) implies that $V_2(x)$ should be greater than one for those points $x \in \Omega$ such that $x_{k+1} \notin \Omega$. Jointly with (20) the condition discards the points $x \in \Omega$ for which $V_1(x_{k+1}) < 1$ but $x_{k+1} \notin \Omega$.

So conditions (20),(21) together mean that all points in $V_2(x) < 1$ will fulfill $V_1(x_{k+1}) \leq 1$ and $x_{k+1} \in \Omega$, i.e. $x_{k+1} \in B_1$. Hence the obtained level set can be used as $B_2$ in (18), for $N = 1$.

Figure 1, in which $\Omega$ is, for clarity, only defined by a circle $Q(x) < 1$, illustrates the different regions involved in the conditions: the pink region $V_2 < 1$ must not intersect green zones ($V_1(x_{k+1}) > 1$) and red ones ($Q(x_{k+1}) > 1$).

Lastly, conditions (19), (22) and (23) are the adaptation of the best-fit conditions (15), (13) and (14), respectively, to the setting now in consideration, in order to obtain the "optimal" $V_2$ according to Definition 3. ■

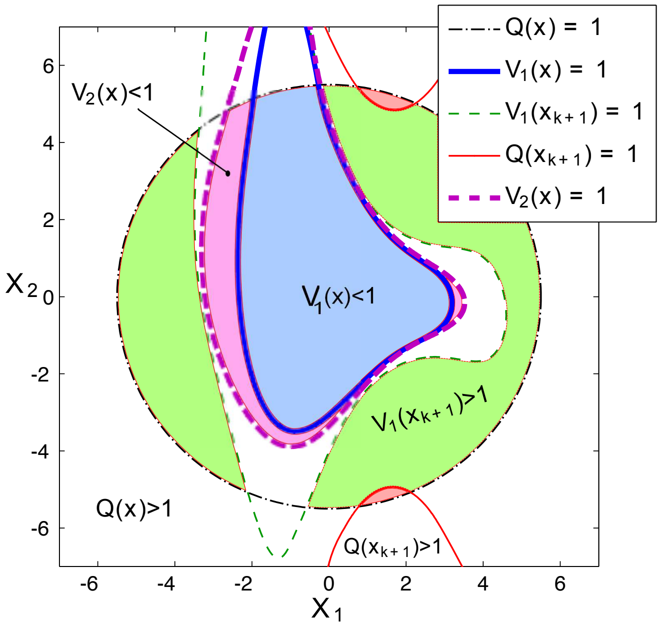



Fig. 1. Example of regions involved in Lemma 7 (Blue: $B_1$, Pink+Blue: $B_2$, plus other relevant boundaries in the legend).

The optimization problem in the above Lemma cannot be solved via SOS techniques. The reason is that conditions must involve only polynomial terms in order to be able to use semidefinite programming. However, it can be converted in an straightforward way to a SOS problem if all $V_1(x), V_2(x), Q_1(x), ..., Q_{o_q}(x)$ belong to the polynomials $\mathcal{R}_x$ and semidefinite relaxations are suitably applied: the case is identical to the one when transforming (16) to (17) in a previous section (details omitted for brevity).

*Remark* 2. Note that the slack variables $\rho$ in Lemma 7 are introduced instead of directly using $\sum_i \sigma_i^2 P_i$ ($\sigma^2 = \mu$), in order to reduce the degree of the conditions. Therefore, the computational complexity of the resulting semidefinite problem (Lemma 5) is lower.

*Remark* 3. Usually, $B_1$ will have been obtained with a shape-independent fuzzy technique in literature and this is why, in Lemma 7, the LRDA condition $B_1 \subset D_\Omega$ has been assumed. If $B_1$ had been obtained with a shape-dependent or other nonlinear stability analysis technique, then the resulting $B_2$ will be a larger set possibly including points of the DA outside the LRDA, so it would be a better solution: evidently, the larger the initial estimate $B_1$ is, the better the proposed methodology will work.

*Remark* 4. Note that, as in Lemma 6, $B_1$ does not need to be a Lyapunov level set fulfilling $V_1(x_{k+1}) - V_1(x_k) < 0$ in all its interior, even if the previous remark suggests it as a reasonable seed set. Note also that the resulting $B_2$ is also free from the above Lyapunov decrease condition. In fact, we don't even need to enforce neither $V_1 > 0$ nor $V_2 > 0$ inside the level set (with Positivstellensatz conditions). These are the reasons why the proposed methodology obtains better results than previous literature.

### B. Iterative Procedure

As a natural choice, using the $B_2$ obtained in Lemma 7 to define a new region $B_1$, a sequence of new functions and associated regions would be readily obtained by repeatedly applying Lemma 7.

*Remark* 5. There is also the possibility of remodelling while iterating, defining a new larger region $\Omega_2 \supset \Omega$ in order to obtain larger LRDA estimates, in particular when $\{x : V_2(x) \leq 1\} \not\subset \Omega$. Note that in that case conditions (21) must make reference to the *previous* modelling region $\Omega$ when setting up $F_2$, in order to fullfill Lemma 6. The other Positivstellensatz polynomials $F_1$, $F_3$ and $F_4$ must belong to the cone formed with the constraints associated to $\Omega_2$. For further details, see Example 3 in Section VII.

## VI. DA ESTIMATION IN CONTINUOUS-TIME SYSTEMS

The following theorem states conditions so that, given a particular region $B_1$ elsewhere proved to belong to the DA of (1), a larger one can be found via invariant set considerations.

**Theorem 1.** *Let $\Theta = \{x : Q_l(x) \leq 1,\ l : 1, ..., o_q\}$ be a compact user-defined region of interest, with $Q_l(x)$ differentiable. Let $B_1 = \{x \in \Theta : V_1(x) < 1\} \subset \mathcal{D}$ be a (previously proven) bounded subset of the domain of attraction of the origin of system* (1). *If we can find a differentiable function $V_2(x)$ such that, given $\epsilon > 0$, the following conditions hold:*

$$V_2(x) \geq 1\ \forall\ x \in C_m \qquad m : 1, .., o_q \tag{24}$$

$$\dot{V}_2(x) < -\epsilon(x) \text{ when } V_1(x) \geq 1, Q_l(x) \leq 1\ \forall l \tag{25}$$

*where*

$$C_m = \{x :\ Q_m = 1, \dot{Q}_m > 0,\ V_1 \geq 1,\ and\ Q_l \leq 1\ \forall l \neq m\}$$

*Then, the interior of the region $B_2 = \{x : Q_l(x) \leq 1,\ V_2(x) \leq 1\} \cup B_1$ belongs to $\mathcal{D}$.*

*Proof:* As $B_1 \subset \Theta$, we have $B_1 \subseteq B_2 \subseteq \Theta$.

Condition (25) means that $\dot{V}_2(x)$ is strictly negative in $\Theta \backslash B_1 = \{x :\ x \in \Theta,\ x \notin B_1\}$.

We will now prove that all trajectories starting in $x_0$ in the interior of $B_2 \backslash B_1$ reach in finite time $B_1$.

Indeed, as $V_2(x_0) < 1$, and for all $l$, $Q_l(x_0) < 1$ then $V_2(x(t)) < 1$ for all $t \geq t_0$ while in $B_2 \backslash B_1$ by (25). Hence, it will never exit $B_2 \backslash B_1$ neither via the frontier $V_2(x) = 1$, evidently, nor via $Q_l(x) = 1$ because $V_2(x) \geq 1$ or $\dot{Q}_l(x) < 0$ in such points due to (24). So, the only way of a trajectory to exit $B_2 \backslash B_1$ will be entering $B_1$.

As the fact that the trajectory remains forever in $B_2 \backslash B_1$ is not possible we can conclude that the trajectory from the above $x_0$ will enter $B_1$ in finite time (see below).

Indeed, we have $B_2 \backslash B_1 \subset \Theta \backslash B_1$. Also, $\Theta \backslash B_1$ is compact and, by (25), $\dot{V}_2(x(t)) < -\epsilon$ when $x(t) \in \Theta \backslash B_1$. $V_2(x)$ will achieve a minimum $\alpha$ in $\Theta \backslash B_1$. Consider a trajectory such that $V_2(x(0)) \leq 1$ and $\dot{V}_2(x(t)) < \epsilon$ for all $t \geq 0$. In that case, for all $t > (1 - \alpha)/\epsilon$ we would have $V_2(x(t)) < \alpha$, so such trajectory is not possible inside $\Theta \backslash B_1$: the state must have left $\Theta \backslash B_1$ (hence, $B_2 \backslash B_1$) in finite time. ■

Note that some sets $C_m$ may be empty so, in those cases, there is no need of checking condition (24).

**Corollary 2.** *If the condition:*

$$V_2(x) \leq 1 \text{ when } V_1(x) \leq 1, Q_l(x) \leq 1\ \forall l \tag{26}$$

*is also enforced, then $B_2 = \{x \in \Theta :\ V_2(x) \leq 1\}$, and $B_1 \subset B_2$. So $B_2$ and $V_2$ can be used again for finding new points in the domain of attraction, replacing $V_1$ and $B_1$ with them.*

The advantage of the above corollary is that there is no need of considering the union of regions discussed in Theorem 1 when defining $B_2$, simplifying further computations. An iterative algorithm naturally ensues (see Section VI-B).

### A. Application to Fuzzy Polynomial Systems

In the following, fuzzy polynomial models (4) and restrictions will be used in the context of the above theorem to obtain LRDA estimates $\mathcal{D}_\Omega$ of the domain of attraction $\mathcal{D}$ of (1) in a modelling region $\Omega$. In this way, SOS programming can be used. In order for the polynomial model to be valid, the condition $\Theta \subset \Omega$ must be enforced by a suitable definition of $Q_l$, being $\Theta$ the region of interest discussed in Theorem 1.

*Remark* 6. The "region of interest" $\Theta$ is introduced, instead of the full modelling region $\Omega$, in order to reduce conservatism by eliminating the need of checking $\dot{V}_2 < 0$ in the whole $\Omega$, which may be infeasible. Indeed, note that if there are equilibrium points in $\Theta \backslash B_1$ then (25) will not hold. A suitable choice for $\Theta$ will be later discussed.

**Lemma 8.** *Consider a known set $B_1 \subset \mathcal{D}_\Omega$ defined by $B_1 = \{x \in \Omega : V_1(x) < 1\}$ and a user-defined region $\Theta$ defined by $o_q$ restrictions $\Theta = \{x : Q_l(x) \leq 1\}$ $l : 1, ..., o_q$, such that $\Theta \subset \Omega$ and it is compact. Then, the region $B_2 = \{x : Q_l(x) \leq 1, V_2(x) \leq 1\}$ belongs to $\mathcal{D}_\Omega$ and $B_2 \supset B_1$, if a continuous differentiable function $V_2(x)$ can* *be found solving the following SOS problem:*

*minimize $\tau$ s.t.*

$$V_2(0) = 0 \tag{27}$$

$$-(\frac{\partial V_2(x)}{\partial x}\rho + \epsilon) - F_1(x,\rho) + G_1(x,\rho) \in \Sigma_{x,\rho} \tag{28}$$

$$V_2(x) - 1 - F_{2m}(x,\rho) + G_{2m}(x,\rho) \in \Sigma_{x,\rho} \;\; m:1,...,o_q \tag{29}$$

$$1 - V_2(x) + \tau - F_{3m}(x) + G_{3m}(x) \in \Sigma_x \;\; m:1,...,o_q \tag{30}$$

$$1 - V_2(x) - F_4(x) \in \Sigma_x \tag{31}$$

*Where*

- $\tau > 0$, $\epsilon > 0$,
- $F_1(x,\rho) \in \wp(V_1(x) - 1, 1 - Q_1(x), ..., 1 - Q_{o_q}(x))$,
- $F_{2m}(x,\rho) \in \wp(V_1(x) - 1, \frac{\partial Q_m(x)}{\partial x}\rho)$,
- $F_{3m}(x) \in \wp(V_1(x) - 1)$,
- $F_4(x) \in \wp(1 - V_1(x), 1 - Q_1(x), ..., 1 - Q_{o_q}(x))$,
- $G_{2m}(x,\rho) \in \Im(Q_m(x) - 1, \rho - \sum_{i=1}^r \mu_i P_i(x))$,
- $G_{3m}(x) \in \Im(Q_m(x) - 1)$,
- $G_1(x,\rho) \in \Im(\rho - \sum_{i=1}^r \mu_i P_i(x))$.

*Proof:* Conditions (28) and (31) mean (25) and (26) respectively. As $\dot{Q}_m = \frac{\partial Q_m}{\partial x}\rho$, constraining $\rho = \sum_i \mu_i P_i(x)$ by "Positivstellensatz" multipliers, then condition (29) implies (24), also condition (30) implies (13), and condition (31) implies (14). ■

Note that, inspired in Definition 3, minimization of $\tau$ above allows obtaining a region $B_2$ which best fits $\Theta$ subject to the additional constraint of belonging to $\mathcal{D}_\Omega$.

*Remark* 7. As in the discrete case, the above optimization problem doesn't involve polynomial finite conditions. So, in order to be able to use semidefinite programming, a recasting is needed by taking $\{V_1(x), V_2(x), Q_1(x), ..., Q_{o_q}(x)\} \in \mathcal{R}_x$ and a finite number of terms from the cones and ideals. See, again, the transformation from (16) to (17) (details omitted for brevity).

The above lemma generalises particular cases in literature, as follows:

**Corollary 3.** *If $B_1 = \{0\}$ and all conditions of Lemma 8 are set with the particular choices $F_1(x,\rho) \in \wp(1-Q_1(x), ..., 1-Q_{o_q}(x))$, $F_{2m} = 0$, $F_{3m} = 0$, and* (31) *is omitted, $V_2$ is a Lyapunov function whose level set $\{x : V_2 \leq 1\}$ belongs to the DA of the origin, recovering classical local-stability results (Lemma 3).*

*Proof:* If $B_1 = \{0\}$ relaxing requirements of positiveness and decrescence inside $\{V_1 \leq 1\}$ should not be done because such $V_1$ does not exist. Hence, the term $V_1(x) - 1$ should be removed from the generator of the cones. Also, (31) which refers to conditions inside $B_1 \cap \Theta$ (i.e, the origin) is redundant with (27). The rest of conditions can then be interpreted as the usual ones on Lyapunov functions (locally in $\Theta$). ■

**Corollary 4.** *If conditions in Corollary 3 are solved getting $V_2$ and, later, only* (29) *is posed setting a new $V_2$ equal to an scaled version of the one just computed, then [15] is obtained.*

Indeed, [15] discusses only *a posteriori* scaling of Lyapunov functions.

## B. Iterative Procedure

Lemma 8 starts with a seed set $B_1 = \{x : V_1 < 1\}$ and a user-defined region $\Theta$ which, obviously, should intersect with the seed set (in most of practical cases, it will actually contain the seed set). The result is a new level set $\{x : V_2 < 1\}$ larger than $B_1$ such that its intersection with $\Theta$ belongs to the DA.

*a) Progressive enlargement of the DA estimate:* As a natural choice, if $\Theta$ were fixed, using the larger $V_2$ obtained with Lemma 8 to define a new seed region $B_1$, then the conditions of Lemma 8 are fulfilled and, thus, it can be applied again with the new seed. Hence, a sequence of new functions and associated regions would be readily obtained by repeatedly applying Lemma 8.

*b) Choice of region of interest $\Theta$:* There are various posibilities for choosing a region $\Theta$ but:

- a large $\Theta$ might eventually lead to (28) being infeasible, *e.g.*, if $\Theta$ included more than one equilibrium point.
- a small $\Theta$ would lead to little improvement in the domain of attraction estimates and, also, the restrictions (30) and (31) would be hard to fulfill if $V_2$ were a low-degree polynomial and $\Theta$ and $V_1$ defined complicated shapes.

Furthermore, as iterations progress and the DA estimates grow larger (encompassing most of $\Theta$), then constraining $\Theta$ to the initial "small" choice may not be a good option. This fact, jointly with the above issues arising in the choice of $\Theta$ motivate incorporating iterations in the size and shape of such region, as discussed below.

*c) Proposal for modification of $\Theta$:* Although there might be alternative options, for instance, the new region of interest can be defined by a user-defined "zoom" factor $\upsilon \geq 1$ as:

$$\Theta = \{x \in \Omega : V_1(x) \leq \upsilon, \;\; \upsilon \in \mathbb{R}\} \tag{32}$$

The smaller $\upsilon$ is, the smaller the region $\Theta - B_1$ is, so condition $\dot{V}_2 < 0$ there becomes less restrictive.

If $V_1(x)$ were $\mathcal{C}^1$ differentiable, and enhanced proposal for the choice of $\Theta$ may be based on the evident fact that for any fixed time $\delta > 0$, the set $\{x_0 \in \mathbb{R}^n : V_1(x(\delta)) < 1, x(0) = x_0\}$ is included in the domain of attraction $\mathcal{D}$. Intuitively, from the first order Taylor series expansion of $V_1(x(t))$,

$$V_1(x(\delta)) \approx V_1(x_0) + \delta\frac{\partial V_1}{\partial x}\dot{x}(0)$$

the new region of interest in Corollary 8 can be, choosing $\delta > 0$:

$$\Theta = \left\{ x \in \Omega : V_1(x) + \delta\frac{\partial V_1(x)}{\partial x}\sum_{i=1}^r \mu_i P_i(x) \leq \upsilon, \;\; x \in G \right\} \tag{33}$$

where $G \in \Omega$ is, in general case, a sphere limiting the search zone and ensuring compatness ($G \equiv \Omega$ if $\Omega$ is compact). The constant $\upsilon$ has the same meaning as in (32) and $\delta$ is a new user-defined constant.

Note that the accuracy of these steps is not very relevant because region $\Theta$ can actually be arbitrarily defined by the user in Theorem 1. Also, in order to be less conservative, the original nonlinear system may also be remodelled: indeed, for a given $\Theta \subset \Omega$, the closer the modelling region $\Omega$ is to $\Theta$ the less uncertain the fuzzy model will be.

To clarify the proposed methodology, the Examples 1 and 2 at Section VII use the following algorithm (particular case of Lemma 8):

**Algorithm 1.** *Starting from a known* $B_1 = \{V_1(x) < 1\}$, $B_1 \in \mathcal{D}$ *and* $V_1(x) \in \mathcal{R}_x$, *carry out the following steps:*

1) *Choose a starting combination of region increase parameters* $\delta \geq 0$ *(gradient) and* $\upsilon \geq 1$ *(zoom), defining a candidate region of interest*[7] (33).
2) *Find a new polynomial* $V_2(x)$ *solving the following SOS problem:*

*minimize* $\tau$ *such that*

$$V_2(0) = 0 \tag{34}$$

$$-(\frac{\partial V_2}{\partial x}\rho + \epsilon) - \psi_{1i}(V_1 - 1) - \psi_{2i}(R - x^T x)$$
$$-\psi_{3i}(\upsilon - V_1 - \delta\frac{\partial V_1}{\partial x}\rho) + \phi_1(\rho - P_i) \in \Sigma_{x,\rho}\ i : 1, ..., r \tag{35}$$

$$1 - V_2 - \psi_4(1 - V_1) - \psi_5(R - x^T x) \in \Sigma_x \tag{36}$$

$$V_2 - 1 + \phi_{2i}(V_1 + \delta\frac{\partial V_1}{\partial x}\rho - \upsilon) - \psi_{6i}(R - x^T x)$$
$$+ \phi_3(\rho - P_i) \in \Sigma_{x,\rho}\ i : 1, ..., r \tag{37}$$

$$V_2 - 1 - \psi_{7i}(\upsilon - V_1 + \delta\frac{\partial V_1}{\partial x}\rho) + \phi_{4i}(R - x^T x)$$
$$+ \phi_5(\rho - P_i) \in \Sigma_{x,\rho}\ i : 1, ..., r \tag{38}$$

$$1 - V_2 + \tau + \phi_{6i}(V_1 + \delta\frac{\partial V_1}{\partial x}\rho - \upsilon) - \psi_{8i}(R - x^T x)$$
$$+ \phi_7(\rho - P_i) \in \Sigma_{x,\rho}\ i : 1, ..., r \tag{39}$$

*where* $\epsilon > 0$, $\tau > 0$, $R$ *is a user-defined radius of a sphere belonging to* $\Omega$, $\psi_j \in \Sigma_x$, $\psi_{ji} \in \Sigma_{x,\rho}$, $\phi_k \in \mathcal{R}^n_{x,\rho}$ *and* $\phi_{ki} \in \mathcal{R}_{x,\rho}$.
3) *If the above problem is feasible, set* $V_1(x) = V_2(x)$ *and return to Step 1.*
4) *If problem in Step 2 is not feasible, then:*
   a) *If* $\upsilon > 1$, *set* $\upsilon = \max(1, \upsilon - \Delta_\upsilon)$ *(*$\Delta_\upsilon$ *user-defined step) and go back to Step 2.*
   b) *If* $\upsilon = 1$, *reduce* $\delta$ *by* $\Delta_\delta$ *(user-defined step) and go back to Step 2.*
   c) *If* $\upsilon = 1$ *and* $\delta \leq 0$ ***stop*** *the algorithm, due to lack of progress. The finally proved DA estimate* $B_2$ *is obtained in closed-form by the current* $V_1(x)$, *computed in the last feasible iteration (i.e., after setting* $V_1 = V_2$ *in Step 3):*

$$B_2 = \{x : V_1(x) < 1,\ x_1^2 + x_2^2 < R^2\}$$

[7] A slack variable $\rho = \sum_i \mu_i P_i$ will be introduced as in Lemma 5 in next steps. Note also that region (33) might be defined by a high-degree polynomial for $\delta \neq 0$, so it may have a strange shape and can take large values far from the origin. Hence, to avoid numerical problems, a low-degree best-fitting region (Definition 3) to (33) may also be obtained in this step for later use if needed. Details omitted for brevity.

**Note.** Conditions on the above algorithm are a particularization of those in Lemma 8 as follows:

- (34), (35), (36) and (39) correspond to (27), (28), (31), and (30), respectively
- (37) and (38) are conditions (29) but forcing $\{x : V_2(x) = 1\}$ to be contained inside $\Theta$ to avoid the result of each iteration being an "intersection" (i.e., forcing $B_1$ in next iteration to be defined by only *one* polynomial inequality), setting $F_{2m} = 0$.

*Remark* 8. With condition (36), i.e., $V_2 \leq 1$ when $V_1 \leq 1$, Corollary 2 applies and the proved domain of attraction increases in each iteration. Note that improvements come from the fact that there is no need for either $V_1 > 0$, $\dot{V}_1 < 0$, $V_2 > 0$ or $\dot{V}_2 < 0$ in *all* the interior of the level sets, contrary to usual Lyapunov approaches.

## VII. EXAMPLES

***Example 1:*** *Non-fuzzy polynomial system.*

First, a simple example from [23, Example 8.9] is provided in order to show the performance of the proposed methodology in this paper over standard level-set ones in the referred source.

Consider the polynomial system:

$$\begin{aligned} \dot{x}_1 &= -x_2 \\ \dot{x}_2 &= x_1 + (x_1^2 - 1)x_2 \end{aligned} \tag{40}$$

For the above system, linearization shows that the origin is stable: there is a neighbourhood of it belonging to its DA provable with a Lyapunov function $V(x) = 1.5x_1^2 - x_2x_1 + x_2^2$, see [23] for details. However, phase plane simulation shows that it has an unstable limit cycle so the DA of the origin is limited by it.

The Lyapunov-based methodology proposed in [23] obtains an initial estimate of the DA from a rough bounding of $\dot{V}$ given by $\{x : V(x) \leq 0.801\}$. Then, zooming out this region by performing a trial-and-error contour plotting, the above estimate is expanded to $\{x : V(x) \leq 2.25\}$.

Now, using the proposal in this paper, the region $B_1 = \{x : V(x) \leq 2.25\}$ is used as the algorithm *seed* region. The initial step-size parameters are set to $\nu = 1.1$, $\delta = 0.2$ and a $4^{th}$ degree polynomial boundary $V_2$ is chosen. With $\Delta_\nu = 0.1$, $\Delta_\delta = 0.1$, Algorithm 1 runs for 9 iterations until it stops due to lack of progress. The largest region obtained with a $4^{th}$ degree polynomial boundary is $\{x : V_2(x) < 1\}$, where

$$V_2(x) = 0.18157x_1^2 - 0.58255x_1x_2 + 0.0058x_2^2 + 0.0327x_1^4$$
$$+0.15975x_1^3x_2 + 0.14346x_1^2x_2^2 - 0.0709x_1x_2^3 + 0.053x_2^4$$

Then, four more iterations are executed by reducing starting algorithm parameters to $\nu = 1.02$, $\delta = 0.05$ and also setting a $6^{th}$ degree for the new polynomial boundaries $V_2$. Finally, the new DA estimate is, explicitly:

$$B_2 = \{x : -0.02023x_1^5x_2 - 0.401x_1x_2 + 0.595x_1^2 - 0.1633x_1^4$$
$$+0.3378x_2^2 - 0.0514x_2^4 + 0.0206x_1^6 + 0.055x_2^6 - 0.15867x_1^2x_2^2$$
$$+0.09x_1x_2^3 - 0.0208x_1x_2^5 + 0.182x_1^3x_2 - 0.0578x_1^3x_2^3$$
$$+0.049x_1^2x_2^4 + 0.0388x_1^4x_2^2 < 1\}$$

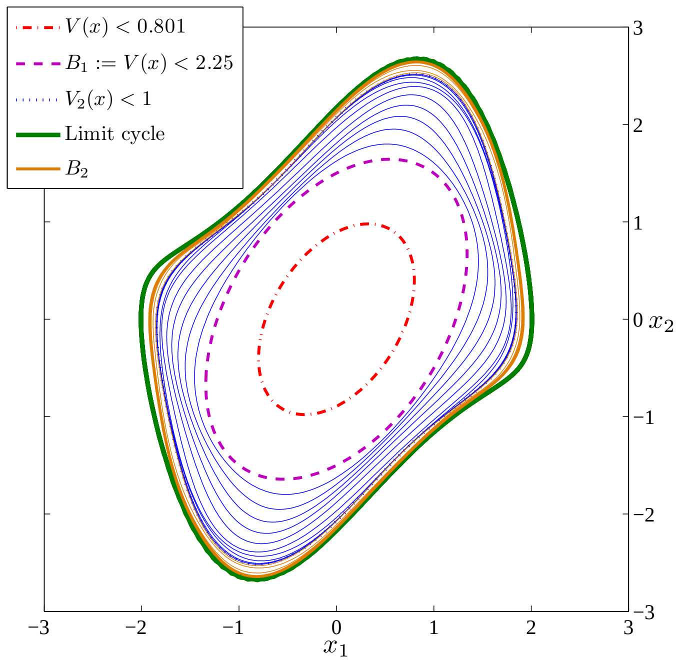



Fig. 2. [Example 1] Domain of attraction evolution using $4^{th}$ order polynomial curves (blue) and $6^{th}$ order ones (brown). Seed set $B_1$ taken from [23].

The improvement over estimates in [23] can be checked on Figure 2. In fact, the obtained boundary of $B_2$ is pretty close to the actual limit cycle (see Figure 8.2 in the cited source, and green contour below for numerical simulation-based approximations to it) which is the *exact* shape of the DA for which a closed-form solution is, however, unavailable.

***Example 2:*** *Continuous-time non-polynomial system.*

Consider the nonlinear system:

$$\begin{aligned}\dot{x}_1 &= -3x_1 + 0.5x_2 \\ \dot{x}_2 &= x_2(-2 + 3\sin(x_1))\end{aligned} \quad (41)$$

which has equilibrium points at $x = 0$, at $(x_1 = 0.7297 + 2k\pi, x_2 = 6x_1)$ and at $(x_1 = -0.7297 + (2k+1)\pi, x_2 = 6x_1)$ for $k \in \mathbb{Z}$.

The objective is to estimate the domain of attraction of the origin in a state-space modelling region $\Omega$ defined as a sphere of radius $R_e$ centered in $x = 0$:

$$\Omega = \{(x_1, x_2) \,|\, x_1^2 + x_2^2 < R_e^2\}$$

For instance, for $R_e = 10$, we have the two equilibrium points inside $\Omega$: $e_0 = (0, 0)$ and $e_1 = (0.7297, 4.378)$. Linearisation shows that $e_0$ is a stable node (two negative real Jacobian eigenvalues), and $e_1$ is a saddle point (one stable and one unstable eigenvalues).

Taking into account the range $-10 \leq x_1 \leq 10$ and using the $5^{th}$ degree Taylor expansion of $\sin(x_1)$, there exists an exact fuzzy-polynomial representation in $\Omega$ such that $\sin(x_1) = \mu_1(x)P_1(x) + \mu_2(x)P_2(x)$, where:

$$P_1(x) = x_1 - \frac{1}{6}x_1^3 + 9.16 \cdot 10^{-3}x_1^5$$

$$P_2(x) = x_1 - \frac{1}{6}x_1^3 + 1.56 \cdot 10^{-3}x_1^5$$

which gives a two-vertices fuzzy polynomial model (4) with membership functions ($z \equiv x$):

$$\mu_1(x) = \frac{\sin(x_1) - P_2(x)}{7.6 \cdot 10^{-3}x_1^5}, \quad \mu_2(x) = 1 - \mu_1(x)$$

For other sizes of the modelling region $\Omega$, resulting in different ranges of $x_1$, suitable vertex models may be obtained by the same Taylor-series methodology.

A starting region $B_1$ is obtained with well-known methodologies [19], [29]: a search was made for a polynomial Lyapunov function $V_1(x)$ giving the maximum radius $R_1$ of a circle included in its level set $\{V_1(x) \leq 1\}$, and such that $\dot{V}_1$ decreases in a spherical modelling region around the origin of radius $R_e$.

As there is a saddle point $e_1$, whatever the choice for $V_1$ is, we will have $\dot{V}_1(e_1) = 0$. Forcefully, *any* Lyapunov function search from literature (for instance, Lemma 1, Lemma 4) will *not* be feasible for $R_e \geq \|e_1\| = 4.44$. So, to obtain a first seed set, $R_e$ was set to 4.42 in the numerical implementations, corresponding to curve $C_1$ in Figure 3. In fact, because of the inherent conservatism from fuzzy modelling, the single saddle point in the original nonlinear system becomes a "strip" of possible equilibrium points (for different values of $\mu$) in the fuzzy model.

The Lyapunov function is found by using Lemma 4, i.e., solving the SOS problem of maximising $R_i$ subject to

$$\begin{aligned} V - \epsilon x^T x + \psi_1(x^T x - R_e^2)) &\in \Sigma_x \\ V - 1 - \psi_2(x^T x - R_e^2) &\in \Sigma_x \\ 1 - V + \psi_3(x^T x - R_i^2) &\in \Sigma_x \end{aligned}$$

$$-(\frac{\partial V}{\partial x}P_i(x) + \epsilon x^T x) - \phi_i(R_e^2 - x^T x) \in \Sigma_x \quad i = 1, 2$$

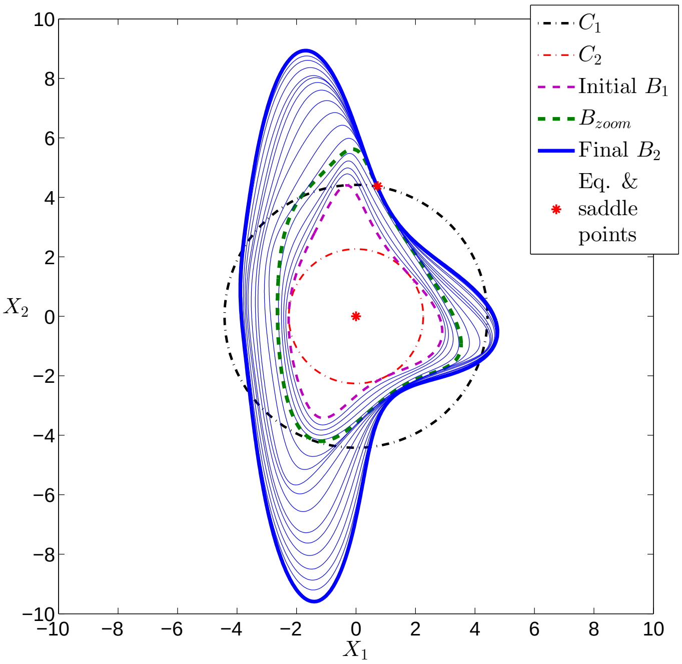
for $\epsilon = 0.001$, and multipliers $\{\phi_i, \ \psi_j\} \in \Sigma_x$. The Lyapunov function's degree has been set to 4. Obviously, higher degrees would yield better results, but the objective of the paper is showing that improvements in DA estimation can be made *without increasing the polynomial degree*.

The largest circle proved to belong to the DA with this standard methodology is $C_2$, and the Lyapunov level set is limited by the dashed-purple curve "Initial $B_1$" in Figure 3.

The proved domain of attraction is then enlarged following Algorithm 1, looking for $4^{th}$ degree new polynomials $V_2(x)$. Figure 3 shows how the estimated domain of attraction increases from the Lyapunov-only solution, i.e.,"Initial $B_1$", as iterations progress. First, with a zoom factor $\upsilon = 1.2$ and $\delta = 0$, and $\Delta_\upsilon = 0.1$, Algorithm 1 works for five iterations reaching region labelled as $B_{zoom}$ in the figure.

Using $B_{zoom}$ as seed, restarting the algorithm with $\delta = 0.03$, $\Delta_\delta = 0.01$ and $\upsilon = 1$, the algorithm runs for 12 more iterations, and gives the best feasible DA proved (curve "Final $B_2$" in the Figure).

Although simulations show that the domain of attraction is quite larger, iterations find hard to obtain a better estimate using a *closed* 4th degree boundary. Indeed, each new candidate region has to be valid for the family of "all" systems




$C_1$: Starting modelling region $\Omega$ with a single equilibrium (Eq) point inside (classical Lyapunov techniques used locally in $C_1$);
Initial $B_1$: Level set of the Lyapunov function proving $C_2$;
$B_{zoom}$: Last iteration with $\nu \neq 1$, $\delta = 0$;
Final $B_2$: Last iteration with $\nu = 1$, $\delta \neq 0$.

Fig. 3. [Example 2] Domain of attraction evolution using $4^{th}$ order polynomial curves.

between $P_1$ and $P_2$: however, the difference between the vertex polynomials grows larger as we depart further from the origin. Anyway, the obtained result "Final $B_2$" is substantially larger than the initial Lyapunov level set "Initial $B_1$" from usual methodologies in literature.

In summary, the largest set proved to belong to the LRDA $\mathcal{D}_\Omega$ is the set:

$$\begin{aligned}B_2 = \{x : &-0.2828x_1 - 0.1238x_2 - 0.1315x_1^2 - 0.0918x_2x_1 \\ &- 0.0468x_2^2 + 0.0056x_1^3 + 0.0252x_1^2x_2 + 0.1111x_2^2x_1 \\ &+ 0.0039x_2^3 + 0.0099x_1^4 + 0.0123x_1^3x_2 \\ &+ 0.0358x_1^2x_2^2 + 0.002x_1x_2^3 + 0.0017x_2^4 < 1\}\end{aligned}$$

**Note.** In general the proved DA with Lemma 8 is an intersection between a level set and the region of interest, i.e., $B_2 = \{x : V_2(x) < 1 \cap \Theta\}$. However, in this particular case, the intersection notation is not needed (in fact the possibility is intentionally not allowed enforcing $\{x : V_2(x) < 1\} \subset \Theta \subset \Omega$). The next example considers the more general case.

Note also that the techniques by the authors in [15] obtain a DA estimate larger than "Initial $B_1$" but smaller than $B_{zoom} \cap C_1$ (not shown for brevity), much smaller than the one "Final $B_2$" obtained in this work.

***Example 3**: Discrete-time system.*

Consider the following nonlinear system obtained by the Euler discretization of (41) at sample time $T = 0.1$ seconds:

$$\begin{aligned}x_{1k+1} &= 0.7x_{1k} + 0.05x_{2k} \\ x_{2k+1} &= x_{2k}(0.8 + 0.3\sin(x_{1k}))\end{aligned} \tag{42}$$

which has the same equilibrium as (41). However, due to the large sampling period in the Euler approximation, the domain of attraction may change, as discussed below. Also, for illustration, the degree of the fuzzy-polynomial approximation of $\sin(x_1)$ has been chosen differently.

The objective again is to estimate the domain of attraction of the origin in a state-space circular modelling region of radius $R_e$ centered in $x = 0$. The discrete system has the same equilibrium points as the continuous-time one.

For instance, using the $3^{th}$ degree Taylor expansion of $\sin(x_{1k})$ computed in the range $|x_1| < 10$, there exists an exact fuzzy-polynomial representation in $\Omega$ such that $\sin(x_{1k}) = \mu_1(x_k)P_1(x_k) + \mu_2(x_k)P_2(x_k)$, where:

$$P_1(x_k) = x_{1k} - \frac{1}{6}x_{1k}^3$$
$$P_2(x_k) = x_{1k} - 0.01054x_{1k}^3$$

which gives a two-vertices fuzzy polynomial model (5) with membership functions ($z_k \equiv x_k$):

$$\mu_1(x_k) = \frac{\sin(x_{1k}) - P_2(x_k)}{-0.15612x_{1k}^3}, \quad \mu_2(x_k) = 1 - \mu_1(x_k)$$

A starting region $B_1$, is again obtained with well-known Lyapunov methodologies [30]. The way is to search for a polynomial $V_1(x)$ which gives the maximum radius $R_1$ of a circle included in the region $\{x : V_1(x) < 1\}$ such that $V_1$ decreases in a circular region around the origin of radius $R_e$. Let us detail how initial $V_1$ was crafted in this example:

As in Example 2, whatever the choice for $V_1$ is, any Lyapunov function search from literature will not be feasible for $R_e \geq \|e_1\| = 4.44$, so $R_e$ was set to 4.15 in the numerical implementations[8], hence $\Omega$ in the previous sections corresponds to curve $C_1$ in Figures 4 and 5.

The starting Lyapunov function may be found by two approaches:

1) Solving the SOS problem of maximising $R_i$ subject to

$$\begin{aligned}V(x) - \epsilon x^T x + \psi_1(x^T x - R_e^2)) &\in \Sigma_x \\ V(x) - 1 - \psi_2(x^T x - R_e^2) &\in \Sigma_x \\ 1 - V(x) + \psi_3(x^T x - R_i^2) &\in \Sigma_x\end{aligned}$$

$$\begin{aligned}Z(\sigma)V(x) - V(\sum_i \sigma_i^2 P_i(x)) - Z(\sigma)\epsilon x^T x - \\ \phi_1 Z(\sigma)(R_e^2 - x^T x) \in \Sigma_{x,\sigma}\end{aligned}$$

where $\epsilon = 0.001$, $Z(\sigma)$is used to make conditions homogeneous[9] in $\sigma^2$ , and $\{\phi_1\ \psi_j\} \in \Sigma_x$ are Positivstellensatz multipliers.
The drawback with this approach is that the degree of the polynomial conditions above grows quickly with the Lyapunov function's degree (because computations involve products of $\sigma^2$ and powers of $x$).

[8] $R_e$ cannot be increased without leading to an infeasible problem due to the intrinsic conservativeness issues of the fuzzy-polynomial approach [2].

[9] The change $\mu \equiv \sigma^2$ is enforced. Also, suitable manipulations (multiplication by powers of $1 = \sum_i \sigma_i^2$) in the term $V(\sum_i \sigma_i^2 P_i(x))$ are implicitly assumed for homogeneization.

2) If the idea of introducing slack variables $\rho$ is applied (Lemma 5), the above problem can be expressed as maximising $R_i$ subject to:

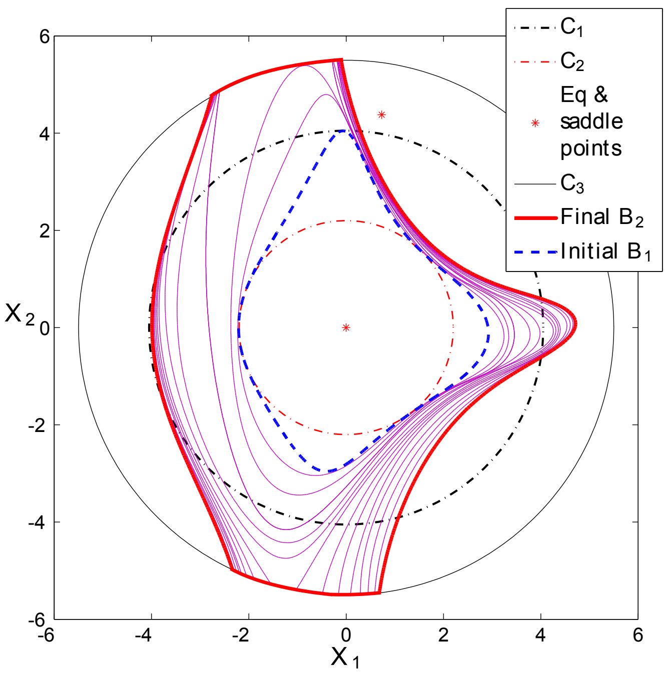
$$\begin{aligned} V - \epsilon x^T x + \psi_1(x^T x - R_e^2)) &\in \Sigma_x \\ V - 1 - \psi_2(x^T x - R_e^2) &\in \Sigma_x \\ 1 - V + \psi_3(x^T x - R_i^2) &\in \Sigma_x \end{aligned}$$

$$\begin{aligned} V - V(\rho) - \epsilon x^T x - \phi_{1i}(R_e^2 - x^T x) \\ + \phi_2(\rho - P_i(x)) \in \Sigma_{x,\rho} \quad i : 1, 2 \end{aligned}$$

where $\epsilon = 0.001$ and multipliers $\phi_2 \in \Sigma^n_{x,\rho}$, $\phi_{1i} \in \Sigma_{x,\rho}, \psi_j \in \Sigma_x$.

In the example, the second approach has been used, and the Lyapunov function's degree has been set to 4. The largest circle proved to belong to the DA with this standard methodology is $C_2$, and the Lyapunov level set is limited by the dashed-blue curve Initial $B_1$ in figures 4 and 5.

The proven domain of attraction is then enlarged following Lemma 7, as proposed in section V-B, iteratively searching for new polynomials $V_2(x)$ of $4^{th}$ degree. Two trials of the iterations with different modelling regions have been considered.




$C_1$: Starting modelling region close to largest circle with a single equilibrium (Eq) point inside;
$C_2$: largest circle in DA proved with classical Lyapunov techniques over $C_1$;
$C_3$: New modelling region, including the saddle point (always infeasible with previous literature results);
Initial $B_1$: Level set of the classical Lyapunov function proving $C_2$;
Final $B_2$: Last iteration of iterative algorithm in Section V-B.

Fig. 4. [Example 3.a)] Domain of attraction evolution using $4^{th}$ order polynomial curves and fixed $R_e$.

*a) Circle of radius $R_e = 5.5$:* Consider the user-defined spherical region ($C_3$ in Figure 4):

$$C_3 = \{x : x_1^2 + x_2^2 < 5.5^2\}$$

so $\Omega \equiv C_3$ in this case. Note, importantly, that it includes the saddle point so no Lyapunov function can be ever found to decrease in all $C_3$.

Figure 4 shows how the estimated domain of attraction increases from the Lyapunov-only solution "Initial $B_1$" as iterations progress.

The final estimation of the LRDA is given by:

$$B_2 = \{x : V_2(x) < 1,\ x_1^2 + x_2^2 < 5.5^2\}$$

with

$$\begin{aligned} V_2(x) = 0.003054x_1^4 - 0.00132x_1^3x_2 - 0.02021x_1^3 + \\ 0.01636x_1^2x_2^2 - 0.001495x_1^2x_2 + 0.004x_1^2 + 0.00075x_1x_2^3 + \\ 0.03096x_1x_2^2 + 0.02511x_1x_2 + 0.32495x_1 - 0.00034x_2^4 + \\ 0.0025x_2^3 + 0.02942x_2^2 + 0.030556x_2 \end{aligned}$$

*b) Circle of radius $R_e = 10$:* Note that, as iterations progress in the above case ($a$), the obtained sets approach the boundary of the modelling region $C_3$ (actually, they *cross* it). Hence, that suggest that larger regions might be obtained if the modelling region is expanded. This second case considers expanding a little the modelling region in each iteration until a final target $R_e = 10$ is reached (or the algorithm stops improving).

Figure 5 shows the final DA estimation.

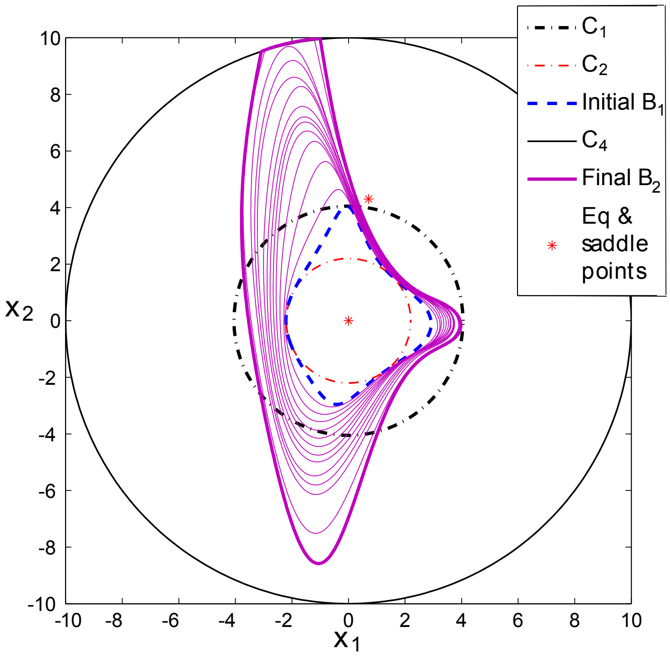
The new LRDA found ("Final $B_2$" on the picture) is $B_2 = \{x : V_2(x) < 1,\ x_1^2 + x_2^2 < 10^2\}$, being

$$\begin{aligned} V_2(x) = 0.00864x_1^4 - 0.00214x_1^3x_2 - 0.0454x_1^3 + \\ 0.01313x_1^2x_2^2 + 0.00427x_1^2x_2 - 0.0042x_1^2 + 0.003525x_1x_2^3 \\ + 0.0388x_1x_2^2 + 0.0394x_1x_2 + 0.4642x_1 + 0.0006x_2^4 \\ + 0.00454x_2^3 + 0.01627x_2^2 - 0.05847x_2 \end{aligned} \quad (43)$$

## VIII. Conclusions

In this paper, a sum-of-squares iterative methodology has been presented, with the objective of improving an initial estimate of the domain of attraction of a nonlinear system. The result is a DA estimate defined in closed-form by polynomial boundaries. A Taylor-series based fuzzy polynomial model is needed in first place. Then, the newly obtained level sets avoid the need of constraints (positiveness, decrease) inside the already-proven regions. In this way, the requirements of a true Lyapunov function are relaxed. The procedures are different for the discrete and the continuous cases.

With the proposals in this work, conservatism with respect to solutions from previous literature is reduced: Lyapunov-based solutions can be used as a "seed" for the algorithms here developed.




$C_1$: Starting modelling region close to largest circle with a single equilibrium (Eq) point inside (same as Fig. 4);
$C_2$: largest circle in DA proved with classical Lyapunov techniques over $C_1$ (same as Fig. 4);
$C_4$: Circular modeling region for $R_e = 10$;
Initial $B_1$: Lyapunov level set proving $C_2$ (same as Fig. 4);
Final $B_2$: DA estimate in last iteration.

Fig. 5. [Example 3.b)] Domain of attraction evolution using $4^{th}$ order polynomial curves for increasingly larger modelling region radius.

## REFERENCES


[1] K. Tanaka and H. O. Wang, *Fuzzy control systems design and analysis: a linear matrix inequality approach*, 2nd ed., ser. Wiley-Interscience publication. John Wiley and Sons, 2001.

[2] A. Sala and C. Ariño, "Polynomial fuzzy models for nonlinear control: A Taylor series approach," *IEEE Trans. Fuzzy Syst.*, vol. 17, pp. 1284–1295, August 2009.

[3] G. Chesi, "Estimating the domain of attraction for non-polynomial systems via LMI optimizations," *Automatica*, vol. 45, no. 6, pp. 1536 – 1541, 2009.

[4] H. K. Lam and F. H. F. Leung, "LMI-Based stability and performance conditions for Continuous-Time nonlinear systems in Takagi-Sugeno's form," *IEEE Trans. Syst., Man, Cybern. B (Cybern.)*, vol. 37, no. 5, pp. 1396–1406, Oct. 2007.

[5] T. Guerra and L. Vermeiren, "LMI-based relaxed nonquadratic stabilization conditions for nonlinear systems in the Takagi-Sugeno's form," *Automatica*, vol. 40, pp. 823–829, May 2004.

[6] F. Amato, F. Calabrese, C. Cosentino, and A. Merola, "Stability analysis of nonlinear quadratic systems via polyhedral lyapunov functions," *Automatica*, vol. 47, pp. 614–617, 2011.

[7] H. K. Lam and L. D. Seneviratne, "BMI-Based stability and performance design for Fuzzy-Model-Based control systems subject to parameter uncertainties," *IEEE Trans. Syst., Man, Cybern. B (Cybern.)*, vol. 37, no. 3, pp. 502–514, Jun. 2007.

[8] H. K. Lam, "Stabilization of nonlinear systems using sampled-data output-feedback fuzzy controller based on polynomial-fuzzy-model-based control approach," *IEEE Trans. Syst., Man, Cybern. B (Cybern.)*, vol. 42, no. 1, pp. 258 –267, Feb. 2012.

[9] J. Dong, Y. Wang, and G.-H. Yang, "Output feedback fuzzy controller design with local nonlinear feedback laws for discrete-time nonlinear systems," *IEEE Trans. Syst., Man, Cybern. B (Cybern.)*, vol. 40, no. 6, pp. 1447 –1459, Dec. 2010.

[10] S. Prajna, A. Papachristodoulou, P. Seiler, and P. A. Parrilo, "Sostools and its control applications," in *Positive Polynomials in Control*, ser. Lecture Notes in Control and Information Sciences, D. Henrion and A. Garulli, Eds. Springer Berlin / Heidelberg, 2005, vol. 312, pp. 272–292.

[11] F. Wu and S. Prajna, "SOS-based solution approach to polynomial LPV system analysis and synthesis problems," *Inter. Journ. Control*, vol. 78, pp. 600–611, May 2005.

[12] K. Tanaka, H. Yoshida, H. Ohtake, and H. O. Wang, "A sum of squares approach to stability analysis of polynomial fuzzy systems," *Proc. Amer. Control Conf*, pp. 4071–4076, 2007.

[13] ——, "Stabilization of polynomial fuzzy systems via a sum of squares approach," *IEEE 22nd inter. sym. on intel. control*, pp. 160–165, 2007.

[14] J. L. Pitarch, A. Sala, and C. V. Ariño, "Spherical domain of attraction estimation for nonlinear systems," in *1st IFAC Conf. on Embedded Systems, Comp. Int. and Tel. in Control (CESCIT)*, Wüzburg, Germany, Apr 2012, pp. 108–114.

[15] J. L. Pitarch, A. Sala, C. V. Ariño, and F. Bedate, "Estimación del dominio de atracción de sistemas no lineales mediante modelos borrosos polinomiales," *Rev. Iber. Aut. Inf. Indust.(RIAI)*, vol. 9, no. 2, pp. 152 – 161, 2012.

[16] R. Genesio, M. Tartaglia, and A. Vicino, "On the estimation of asymptotic stability regions: State of the art and new proposals," *IEEE Trans. on Aut. Control*, vol. 30, no. 8, pp. 747 – 755, Aug. 1985.

[17] B. Tibken, "Estimation of the domain of attraction for polynomial systems via lmis," in *Proc. 39th IEEE Conf. on Dec. Control*, vol. 4. IEEE, 2000, pp. 3860–3864.

[18] A. Chakraborty, P. Seiler, and G. J. Balas, "Nonlinear region of attraction analysis for flight control verification and validation," *Control Eng. Pract.*, vol. 19, pp. 335–345, 2011.

[19] G. Chesi, *Domain of Attraction Analysis and Control Via Sos Programming.* Springer Verlag, 2011.

[20] S. Prajna, A. Papachristodoulou, P. Seiler, and P. A. Parrilo, *SOSTOOLS: Sum of squares optimization toolbox for MATLAB*, Avaliable on `http://www.cds.caltech.edu/sostools` and `http://www.mit.edu/~parrilo/sostools`, 2004.

[21] J. F. Sturm, *Using SeDuMi 1.02, a MATLAB toolbox for optimization over symmetric cones.*, Avaliable on http://http://sedumi.ie.lehigh.edu, August 1998.

[22] Z. Jarvis-Wloszek, R. Feeley, W. Tan, K. Sun, and A. Packard, "Control applications of sum of squares programming," in *Positive Polynomials in Control*, ser. Lecture Notes in Control and Information Sciences, D. Henrion and A. Garulli, Eds. Springer Berlin / Heidelberg, 2005, vol. 312, pp. 3–22.

[23] H. K. Khalil, *Nonlinear Systems*, 3rd ed. Prentice Hall, 2002.

[24] K. Tanaka, H. Yoshida, H. Ohtake, and H. Wang, "A sum-of-squares approach to modeling and control of nonlinear dynamical systems with polynomial fuzzy systems," *IEEE Trans. Fuzzy Syst.*, vol. 17, no. 4, pp. 911 –922, Aug. 2009.

[25] A. Papachristodoulou and S. Prajna, "Analysis of Non-polynomial Systems Using the Sum of Squares Decomposition," in *Positive Polynomials in Control*, ser. Lecture Notes in Control and Information Sciences, D. Henrion and A. Garulli, Eds. Springer Berlin / Heidelberg, 2005, vol. 312, pp. 23–43.

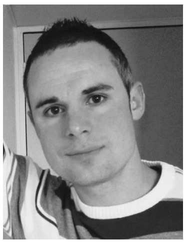
[26] A. Sala, "On the conservativeness of fuzzy and fuzzy-polynomial control of nonlinear systems," *Annual Rev. Control*, vol. 33, no. 1, pp. 48 – 58, 2009.

[27] G. Feng, "Stability analysis of discrete-time fuzzy dynamic systems based on piecewise lyapunov functions," *IEEE Trans. on Fuzzy Syst.*, vol. 12, no. 1, pp. 22 – 28, Feb. 2004.

[28] A. Sala and C. V. Ariño, "Asymptotically necessary and sufficient conditions for stability and performance in fuzzy control: Applications of Polyaś theorem," *Fuzzy Sets Syst.*, vol. 158, pp. 2671–2686, June 2007.

[29] A. Papachristodoulou and S. Prajna, "On the construction of lyapunov functions using the sum of squares decomposition," in *Proc. 41st IEEE Conf. on Dec. Control, 2002*, vol. 3, Dec. 2002, pp. 3482 – 3487.

[30] K. Tanaka, H. Ohtake, and H. O. Wang, "An SOS-based stable control of polynomial discrete fuzzy systems," in *Proc. 2008 American Control Conf. (ACC), Seattle, Washington, USA.* IEEE, Jun. 2008, pp. 4875–4880.



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