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Abstract

The problem addressed in this paper is the integrated vehicle-crew-rostering problem (VCRP) aiming to define the schedules for the buses and the rosters for the drivers of a public transit company. The VCRP is described by a bi-objective mixed binary linear programming model with one objective function aggregating vehicle and crew scheduling costs and the other the rostering features. The VCRP is solved by a heuristic approach based on Benders decomposition where the master problem is partitioned into daily integrated vehicle-crew scheduling problems and the sub-problem is a rostering problem.

Computational experience with data from a bus company in Lisbon shows the ability of the decomposition approach for producing a variety of potentially efficient solutions for the VCRP within low computing times.

Keywords: integrated vehicle-crew-rostering problem, Benders decomposition, multi-objective optimization.

1. Introduction

This paper focuses on the operational planning phase of a public transit company that operates buses in an urban area. The problem addressed aims to assign drivers of a company to vehicles and vehicles to a set of pre-defined timetabled trips that cover passenger transport demand on a specific area, during a planning horizon. The objective is to minimize total costs and maximize drivers' preferences while satisfying passengers demand and driver constraints specified by general legislation, labor contracts and specific company rules. Due to the complexity of the corresponding combinatorial optimization problem, it is usually tackled on a sequential basis beginning with vehicle scheduling, followed by crew scheduling and, lastly, driver rostering. Given a set of timetabled trips, vehicle scheduling produces the set of daily schedules for the vehicles that perform all trips. The crew scheduling defines the daily crew duties that cover the respective vehicle schedules. Finally, for the planning horizon, crew duties are assigned to the company's drivers leading to a roster that must comply with rostering constraints. There is a high dependency among these three problems, hence following this sequential approach one cannot guarantee that the final result is the best solution to the overall problem.

Despite its computational burden, the integration of all or some of these problems is expected to outperform the corresponding sequential approach. Efficient algorithms have been developed to solve the integrated vehicle-crew scheduling problem (Borndörfer et al. (2006), Huisman et al. (2005), Hollis et al. (2006), Mesquita and Paia (2008)). Crew-rostering integration has been devised by Caprara et al. (2001), Ernst et al. (2001), Freling et al. (2004) and Lee and Chen (2003) albeit within other transport contexts (railway and air crews) and by Chu (2007) for airport staff. In Mesquita et al. (2008) advantages of integrating the three problems for public transit companies were pointed out. For an overview of problems arising in the transport domain, see Barnhart and Laporte (2007).

The problem addressed in this paper is the integrated vehicle-crew-rostering problem (VCRP). The VCRP solution consists of a set of daily vehicle schedules, covering all timetabled trips, and a roster defining the set of crew duties for each driver, covering all vehicle schedules of the planning horizon. Naturally, the VCRP objectives possess a conflicting nature. In fact, whereas the minimization of vehicle and driver costs

represents the interests of the management, other objectives, like evenly distributing overtime among drivers and fulfilling as much as possible the preferences of drivers for specific crew duties, arise from the drivers priorities. These non-reconcilable interests at the operational planning phase suggest a multi-objective mathematical model for VCRP.

In this paper, the authors developed an integrated approach to solve the VCRP based on Benders decomposition. The method iterates between the solution of an integrated vehicle-crew scheduling problem and the solution of a rostering problem.

Benders decomposition methods have already been proposed although within airline planning by Cordeau et al. (2001) and Mercier et al. (2005) for the integrated aircraft routing and crew scheduling problem and by Mercier and Soumis (2006) for integrated aircraft routing, crew scheduling and the flight retiming problem. The solution approaches proposed by these authors are based on three phases. In phase 1, the linear programming relaxation of the problem is solved by Benders decomposition. Phase 2 considers all cuts generated during phase1 and applies Benders decomposition with the integer master problem. Phase 3 reintroduces integrality constraints in the sub-problem. Also in the same application context, Papadakos (2009) presented a Benders decomposition approach to deal with the integrated fleet assignment, aircraft routing and crew scheduling problem where both the sub-problem and the master problem are solved by column generation.

This paper is organized as follows. The VCRP is presented in the section 2, along with its mathematical formulation. Section 3 is devoted to Benders methodology and section 4 to the description of the new solution approach. Finally, section 5 shows computational results and section 6 presents some conclusions.

2. Mathematical formulation

During a planning horizon H , partitioned into days, a set M of drivers must be assigned to a fleet of vehicles from a set D of depots in order to perform n timetabled trips (trips for short). For each trip the starting and ending times and locations are known. Trips s and t are compatible if the same vehicle can perform both, s and t , in sequence. The movement of a vehicle without passengers will be denoted by deadhead trip. There are three types of deadhead trips: those between the end location of a trip and the start

location of another trip, those from a depot to the start location of a trip (pull-out trips) and those from an end location of a trip to a depot (pull-in trips). The set of timetabled trips and deadhead trips performed by a vehicle on day $h \in H$ is a vehicle block. Each vehicle block starts and ends at the same depot.

A task is the smallest amount of work to be assigned to the same vehicle and crew and it corresponds to a deadhead trip followed by a trip. A changeover is the walking movement of a driver between two timetabled trips in order to change the vehicle. Each end location of a trip is a potential relief point where a changeover may occur. A crew duty is a daily combination of tasks that respects labor law, union contracts and internal rules of the company. These rules depend on the particular situation under study and usually constrain the maximum and minimum spread (time elapsed between the beginning and end of a crew duty), the maximum working time without a break, the break duration, etc. The crew duties can start (end) at a depot or at an end location of a trip.

A line of work is the sequence of crew duties and days-off, one per day, assigned to a particular driver during the planning or rostering horizon. A line of work for a particular driver must satisfy a certain number of constraints that result from the above mentioned regulations, namely: the driver must rest a given minimum number of hours between consecutive duties; he must work at most a given working time per week, a given working time during the planning horizon and a given number of consecutive days; he must get at least a given number of days-off per week and a given number of Sundays off in the planning horizon, as well as specific weekdays off and weekends off. A roster is the set of lines of work for the drivers of the company that covers all the crew duties during the planning horizon.

The integrated vehicle-crew-rostering problem aims to simultaneously determine a minimum cost set of vehicle blocks that daily covers all timetabled trips, a set of crew duties that daily covers all vehicle blocks and a minimum cost roster for the horizon. However, for a roster to be accepted in the company it should also comply with other kind of requirements arising from the interests of the drivers that express their preferred crew duties and call for balanced lines of work in what respects maximum overtime.

To formulate the VCRP a mathematical model similar to the one proposed in Mesquita et al. (2008) is considered and the required notation follows:

H = planning horizon, partitioned into days

α = number of weeks in H

N_h = set of timetabled trips for day h

$N = \bigcup_{h \in H} N_h$, set of all timetabled trips for H

$n = |N|$

I^h = set of deadhead trips corresponding to all pairs of compatible trips on day h

I_c^h = subset of I^h where a changeover may occur

$T = \bigcup_{h \in H} I^h$, set of deadhead trips corresponding to all pairs of compatible trips for H

D = set of depots

ν_d = number of vehicles in depot d (vehicles are identical in each depot)

L_{st}^h = set of crew duties covering the deadhead trip from the end location of trip s to the start location of trip t and covering trip t , on day h

DL_t^h = set of crew duties covering the deadhead trip from any depot to the start location of trip t and covering trip t , on day h

LD_s^h = set of crew duties covering the deadhead trip from the end location of trip s to any depot, on day h

L^{1h} = early crew duties on day h

L^{2h} = late crew duties on day h

$L^h = \bigcup_{(s,t) \in I^h} L_{st}^h \cup \bigcup_{t \in N^h} DL_t^h \cup \bigcup_{s \in N^h} LD_s^h$, set of crew duties for day h , partitioned into L^{1h} and L^{2h}

o^h = day-off on day h , the $(|L^h| + 1)^{\text{th}}$ "crew duty"

M = set of drivers

u_ℓ = spread of crew duty ℓ

\bar{u} = normal working time of a crew duty

$u'_\ell = \max \{0, u_\ell - \bar{u}\}$, overtime of crew duty ℓ

b_{1w} = maximum total work per week per driver

b_{rw} = maximum total work during H per driver

g = maximum number of consecutive days without a day-off for a driver

Ω_w = minimum number of days-off per week per driver

Ω_S = minimum number of Sundays-off during H per driver

$e^{mh} = 1$ if driver m was assigned to a crew duty on day h from the previous planning horizon, or 0 otherwise

F^m = set of obligatory days-off (planned absences, for instance) for driver m during H

c_{st}^{1dh} = cost of the deadhead trip from trip s to trip t plus trip t cost, performed by a vehicle from depot d on day h

$c_{s,n+d}^{1h}, c_{n+d,s}^{1h}$ = pull-in and pull-out costs from trip s to depot d and from depot d to trip s , respectively, on day h

c_ℓ^2 = cost of crew duty ℓ

c^{3m} = cost of assigning work to driver m during H

c^4 = penalty cost for the maximum overtime per driver during H

c_ℓ^{5mh} = penalty cost based on driver m preference for crew duty ℓ on day h .

All the costs are assumed to be nonnegative. Now the decision variables are presented:

$z_{st}^{dh} = 1$ if a vehicle from depot d performs trips s and t in sequence on day h , or 0 otherwise

$z_{s,n+d}^h = 1$ if a vehicle returns to depot d after trip s on day h , or 0 otherwise

$z_{n+d,t}^h = 1$ if depot d directly supplies a vehicle for trip t on day h , or 0 otherwise

$w_\ell^h = 1$ if crew duty ℓ is selected on day h , or 0 otherwise

$y_\ell^{mh} = 1$ if driver m performs crew duty ℓ on day h , or 0 otherwise

$\omega^m = 1$ if driver m works during H , or 0 otherwise

δ = maximum overtime per driver during H

Then, the VCRP becomes the following bi-objective mixed integer linear programming problem:

$$\min \sum_{h \in H} \left(\sum_{d \in D} \sum_{(s,t) \in T} c_{st}^{1dh} z_{st}^{dh} + \sum_{d \in D} \sum_{s \in N} \left(c_{s,n+d}^{1h} z_{s,n+d}^h + c_{n+d,s}^{1h} z_{n+d,s}^h \right) + \sum_{\ell \in L^h} c_{\ell}^2 w_{\ell}^h \right) \quad (2.1)$$

$$\min \left(\sum_{m \in M} c^{3m} \omega^m + c^4 \delta + \sum_{m \in M} \sum_{h \in H} \sum_{\ell \in L^h} c_{\ell}^{5mh} y_{\ell}^{mh} \right) \quad (2.2)$$

subject to

$$\sum_{d \in D} \sum_{s:(s,t) \in I^h} z_{st}^{dh} + \sum_{d \in D} z_{n+d,t}^h = 1 \quad \forall t \in N^h, \forall h \in H \quad (2.3)$$

$$\sum_{s:(s,t) \in I^h} z_{st}^{dh} + z_{n+d,t}^h - \sum_{s:(t,s) \in I^h} z_{ts}^{dh} - z_{t,n+d}^h = 0 \quad \forall t \in N^h, \forall d \in D, \forall h \in H \quad (2.4)$$

$$\sum_{s \in N^h} z_{n+d,s}^h \leq \nu_d \quad \forall d \in D, \forall h \in H \quad (2.5)$$

$$\sum_{\ell \in DL_t^h} w_{\ell}^h - \sum_{d \in D} z_{n+d,t}^h = 0 \quad \forall t \in N^h, \forall h \in H \quad (2.6)$$

$$\sum_{\ell \in L_{st}^h} w_{\ell}^h - \sum_{d \in D} z_{st}^{dh} = 0 \quad \forall (s,t) \in I^h \setminus I_c^h, \forall h \in H \quad (2.7)$$

$$\sum_{\ell \in L_{st}^h} w_{\ell}^h - \sum_{d \in D} z_{st}^{dh} \geq 0 \quad \forall (s,t) \in I_c^h, \forall h \in H \quad (2.7)'$$

$$\sum_{\ell \in LD_s^h} w_{\ell}^h - \sum_{d \in D} z_{s,n+d}^h = 0 \quad \forall s \in N^h, \forall h \in H \quad (2.8)$$

$$\sum_{m \in M} y_{\ell}^{mh} - w_{\ell}^h = 0 \quad \forall \ell \in L^h, \forall h \in H \quad (2.9)$$

$$\sum_{\ell \in L^h \cup \{o^h\}} y_{\ell}^{mh} = 1 \quad \forall m \in M, \forall h \in H \quad (2.10)$$

$$\sum_{\ell \in L^{ih}} y_{\ell}^{mh} + \sum_{\ell \in L^{jh}} y_{\ell}^{m(h-1)} \leq 1 \quad \forall m \in M, h = 2, \dots, |H|, i, j = 1, 2, i \neq j \quad (2.11)$$

$$\sum_{\ell \in L^h} \sum_{h=7(l-1)+1}^{7l} u_{\ell} y_{\ell}^{mh} \leq b_{1w} \quad \forall m \in M, l = 1, \dots, \alpha \quad (2.12)$$

$$\sum_{\ell \in L^h} \sum_{h \in H} u_{\ell} y_{\ell}^{mh} \leq b_{rw} \quad \forall m \in M \quad (2.13)$$

$$\sum_{\ell \in L^h} \sum_{r=0}^g y_{\ell}^{m,h+r} \leq g \quad \forall m \in M, h = 1, \dots, H - g \quad (2.14)$$

$$\sum_{r=h}^0 e^{mr} + \sum_{\ell \in L^h} \sum_{r=1}^{h+g} y_{\ell}^{mr} \leq g \quad \forall m \in M, h = 1 - g, \dots, -1, 0 \quad (2.14)'$$

$$\sum_{h=7(l-1)+1}^{7l} y_{o^h}^{mh} \geq \Omega_w \quad \forall m \in M, l = 1, \dots, \alpha \quad (2.15)$$

$$\sum_{l=1}^{\alpha} y_{o^{7l}}^{m,7l} \geq \Omega_s \quad \forall m \in M \quad (2.16)$$

$$\sum_{\ell \in L^h} \sum_{h \in H} y_{\ell}^{mh} - |H| \omega^m \leq 0 \quad \forall m \in M \quad (2.17)$$

$$\sum_{\ell \in L^h} \sum_{h \in H} u'_{\ell} y_{\ell}^{mh} - \delta \leq 0 \quad \forall m \in M \quad (2.18)$$

$$z_{st}^{dh} \in \{0,1\} \quad \forall (s,t) \in I^h, \forall d \in D, \forall h \in H \quad (2.19)$$

$$z_{s,n+d}^h, z_{n+d,s}^h \in \{0,1\} \quad \forall s \in N^h, \forall d \in D, \forall h \in H \quad (2.20)$$

$$w_{\ell}^h \in \{0,1\} \quad \forall \ell \in L^h, \forall h \in H \quad (2.21)$$

$$y_{\ell}^{mh} \in \{0,1\} \quad \forall m \in M, \forall \ell \in L^h \cup \{o^h\}, \forall h \in H \quad (2.22)$$

$$\omega^m \in \{0,1\} \quad \forall m \in M \quad (2.23)$$

$$\delta \geq 0. \quad (2.24)$$

The objectives of the VCRP derive from minimization of vehicle and driver costs, as well as minimization of inconvenience of work for the drivers. In fact, on the one hand, management aims at minimizing costs related with the fleet of operational vehicles, c_{st}^{ldh} ,

$c_{s,n+d}^{1h}$ and $c_{n+d,t}^{1h}$. Besides, management often wants to know the minimum workforce required to operate the fleet of vehicles, so as to assign drivers to other departments of the company or to replace those absent. Such policy results in minimizing crew duty costs c_ℓ^2 associated to variables w_ℓ^h and rostering costs c^{3m} associated to ω^m , variables representing drivers effectively assigned to work. On the other hand, interests of drivers must be taken into account and this motivates the definition of penalty costs c^4 associated with the overtime, since overtime is undesirable it should be minimized and equitably distributed. Also penalty costs c_ℓ^{5mh} related to drivers' preferences for specific duties can be considered in the model. All these objectives represent various conflicting interests that cannot usually be simultaneously fulfilled thus leading to a multi-objective perspective for VCRP. However, more than two objectives are not easily tackled. Hence, we opted by a bi-objective optimization problem: the first objective, (2.1), aggregates vehicle and crew duty costs; the second objective, (2.2), aggregates driver costs plus driver penalties, the rostering objective.

In the above bi-objective model, constraints (2.3)-(2.5) describe the vehicle scheduling problem. Constraints (2.3) state that each timetabled trip is performed, exactly once, by a vehicle that comes directly from a depot or from the end location of another timetabled trip. Constraints (2.4) together with (2.3) ensure that, for each day, each timetabled trip is performed, exactly once, by a vehicle that returns to the source depot, being constraints (2.5) depot capacity constraints. Note that, for each day $h \in H$, $\{(2.3), (2.4), (2.5)\}$ defines an integer multi-commodity network flow problem.

The constraint set $\{(2.6), (2.7), (2.7)', (2.8)\}$ links vehicle and crew duty variables ensuring that each task in a vehicle block is covered by one crew. (2.6) and (2.8) impose the covering of tasks involving deadhead trips from/to depots and (2.7) and (2.7)' refer to covering the remaining tasks. Set I^h is partitioned into two subsets, I_c^h and $I^h \setminus I_c^h$. Deadhead trips where changeovers may occur are included in I_c^h . Constraints (2.7) assign a single crew to each deadhead trip in $I^h \setminus I_c^h$ whereas constraints (2.7)' correspond to two different situations. On the one hand, constraints (2.7)' allow drivers to walk over deadhead trips that are not included in a vehicle block. Whenever the end

location of the first trip is the same as the start location of the second one, this movement represents a waiting time. On the other hand, they allow deadhead trips to be covered by more than one crew. As over-covering only occurs if assigning several drivers to a task is cheaper than assigning a single one, the major role of (2.7)' is to explicitly handle changeovers in the constraint set.

Constraints (2.9)-(2.16) define an assignment problem with additional constraints. Equalities (2.9) link crew and rostering variables by imposing that each crew duty, in a solution, must be assigned to one and only one driver and equalities (2.10) deal with the assignment of each driver to one crew duty or to a day-off on each day. Constraints (2.11) forbid the sequence of late/early duty followed by early/late duty to ensure that drivers rest a given minimum number of hours between consecutive duties (a hard constraint coming from legislation) and also that changes of shift are only allowed after a day-off (a soft rostering constraint here imposed as if it was a hard one). Inequalities (2.12) and (2.13) force drivers to work at most a given time per week and a given time during H . Inequalities (2.14) and (2.14)' impose for each driver the maximum number of days – g – without a day-off. As to (2.14)' they are defined for the first g days of H , taking into account the parameters for the crew duties assigned in the last days of the previous planning horizon. Furthermore, (2.15) and (2.16) ensure for each driver at least a given number of days-off per week and at least a given number of Sundays off in H , respectively.

Constraints (2.17) define the variables ω^m from the rostering variables y_ℓ^{mh} and inequalities (2.18) calculate the maximum overtime per driver so as to minimize and equitably distribute it through the second optimization objective.

Finally, (2.19)-(2.24) define the domains of the variables: a nonnegative space for δ and binary sets for the remaining.

3. Decomposition Approach

The VCRP has been modeled as a huge dimension mixed binary linear optimization problem that includes three main combinatorial structures: an integer multi-commodity network flow problem (Mesquita and Paia 2008), a set partitioning/covering structure and an assignment problem with minimum and maximum capacity constraints.

Within those combinatorial structures two well known scheduling problems can be identified: one defining the vehicle-crew scheduling process and the other the rostering. These two sub-problems share a set of variables and (complicating) constraints, in spite of involving other separable sets of variables and constraints. By taking into account the variables involved in the complicating constraints, a Benders decomposition based method arises as a natural approach to solve the overall problem, the VCRP. Such a technique has been applied in different combinatorial optimization contexts as referred to in the survey by Boschetti and Maniezzo (2009).

3.1 Benders decomposition

In 1962, Benders proposed a decomposition algorithm, for solving large single objective mixed integer linear programming problems, that alternates between a primal sub-problem and a master problem. The Benders sub-problem is a restriction of the original problem where some decision variables' values are fixed. In each iteration of the algorithm the solution of the master problem is used to adjust primal variables' values that will be fixed in the sub-problem, whereas the dual sub-problem solution is used to construct cuts - Benders cuts - to be added to the master. This method guarantees the convergence to the optimum under specific hypotheses latter generalized (Geoffrion 1972).

The VCRP is indeed a very complex and huge dimension multi-objective combinatorial problem that will be optimized from a Pareto perspective (Ehrgott and Gandibleux 2000). As it is not reasonable to search for the entire Pareto frontier due to such a demanding process on computing resources, the VCRP will be tackled within a single objective issue by weighting the two original objective functions, that is, by substituting (2.1) and (2.2) by:

$$\begin{aligned} \min \lambda_1 \sum_{h \in H} \left(\sum_{d \in D} \sum_{(s,t) \in T} c_{st}^{1dh} z_{st}^{dh} + \sum_{d \in D} \sum_{s \in N} \left(c_{s,n+d}^{1h} z_{s,n+d}^h + c_{n+d,s}^{1h} z_{n+d,s}^h \right) + \sum_{\ell \in L^h} c_{\ell}^2 w_{\ell}^h \right) + \\ + \lambda_2 \left(\sum_{m \in M} \left(c^{3m} \omega^m + \sum_{h \in H} \sum_{\ell \in L^h} c_{\ell}^{5mh} y_{\ell}^{mh} \right) + c^4 \delta \right) \end{aligned}$$

which is equivalent to

$$\begin{aligned}
\min \sum_{h \in H} & \left(\sum_{d \in D} \sum_{(s,t) \in T} \lambda_1 c_{st}^{1dh} z_{st}^{dh} + \sum_{d \in D} \sum_{s \in N} \left(\lambda_1 c_{s,n+d}^{1h} z_{s,n+d}^h + \lambda_1 c_{n+d,s}^{1h} z_{n+d,s}^h \right) + \sum_{\ell \in L^h} \lambda_1 c_{\ell}^2 w_{\ell}^h \right) + \\
& + \sum_{m \in M} \left(\lambda_2 c^{3m} \omega^m + \sum_{h \in H} \sum_{\ell \in L^h} \lambda_2 c_{\ell}^{5mh} y_{\ell}^{mh} \right) + \lambda_2 c^4 \delta
\end{aligned} \tag{3.1}$$

where λ_1 and λ_2 are nonnegative real parameters.

The non-supported efficient solutions of the bi-objective problem (2.1) to (2.24) cannot be obtained from minimization of (3.1) subject to constraints (2.3) to (2.24) even by taking into account all the possible choices for the weights λ_1, λ_2 (Steuer 1986). However, a partial Pareto optimization strategy, not requiring the entire set of efficient solutions, copes with the typical decision makers' demand for a set containing a few solutions, achieving different levels of quality for the objectives.

The mathematical model presented includes different types of decision variables. Variables z define the vehicle schedules, variables w are associated to crew duties and variables y , ω and δ are connected with rostering. These variables may be partitioned into two sets: the zw -set and the $y\omega\delta$ -set. The decomposition approach proposed, based on Benders method, alternates between the solution of a master problem involving the zw -set, a vehicle-crew scheduling problem for all the days of H , and the solution of the corresponding sub-problem involving the $y\omega\delta$ -set, a rostering problem.

In order to present the sub-problem and the master problem, the VCRP defined by (2.3) to (2.24) and (3.1) is supposed to possess a non-empty feasible region. Now, it is rewritten through the following matrix form:

$$\min \lambda_1 c^1 Z + \lambda_1 c^2 W + \lambda_2 c^3 \omega + \lambda_2 c^4 \delta + \lambda_2 c^5 Y \tag{3.2}$$

subject to

$$A^1 Z \geq A^2 \tag{3.3}$$

$$B^1 Z + B^2 W \geq B^3 \tag{3.4}$$

$$QY - W = 0 \tag{3.5}$$

$$E^1 Y \geq E^2 \tag{3.6}$$

$$G^1 Y + G^2 \omega \geq G^3 \tag{3.7}$$

$$PY + \delta \geq 0 \tag{3.8}$$

$$Z \geq 0 \text{ and binary} \quad (3.9)$$

$$W \geq 0 \text{ and binary} \quad (3.10)$$

$$Y \geq 0 \text{ and binary} \quad (3.11)$$

$$\omega \geq 0 \text{ and binary} \quad (3.12)$$

$$\delta \geq 0 \quad (3.13)$$

where $c^1, c^2, c^3, c^5, A^1, A^2, B^1, B^2, B^3, Q, E^1, E^2, G^1, G^2, G^3, P, 1, 0, Z, W, Y$ and ω are appropriate dimension matrices.

Here (3.2) stands for (3.1), (3.3) corresponds to (2.3)-(2.5), (3.4) represents (2.6)-(2.8), (3.5) corresponds to (2.9), (3.6) stands for (2.10)-(2.16), (3.7) for (2.17), (3.8) represents inequalities (2.18), and, finally (3.9)-(3.13) correspond to (2.19)-(2.24).

3.2. Sub-problem

Fixing the values of the z and w variables in VCRP at values given by vectors \bar{Z} and \bar{W} , respectively, the following sub-problem is obtained:

$$\begin{aligned} & (\text{Sub}_{\bar{z}\bar{w}}) \\ \min & \lambda_2 c^3 \omega + \lambda_2 c^4 \delta + \lambda_2 c^5 Y + (\lambda_1 c^1 \bar{Z} + \lambda_1 c^2 \bar{W}) \end{aligned} \quad (3.2)'$$

subject to

$$QY = \bar{W} \quad (3.5)'$$

$$E^1 Y \geq E^2 \quad (3.6)$$

$$G^1 Y + G^2 \omega \geq G^3 \quad (3.7)$$

$$PY + \delta 1 \geq 0 \quad (3.8)$$

$$Y \geq 0 \text{ and binary} \quad (3.11)$$

$$\omega \geq 0 \text{ and binary} \quad (3.12)$$

$$\delta \geq 0. \quad (3.13)$$

As one considers the set of crew duties and vehicle blocks induced by (\bar{Z}, \bar{W}) , this sub-problem is a rostering problem. Moreover, if (Y, ω, δ) is a feasible solution for $\text{Sub}_{\bar{z}\bar{w}}$ and (\bar{Z}, \bar{W}) satisfies (3.3), (3.4), (3.9) and (3.10) then $(\bar{Z}, \bar{W}, Y, \omega, \delta)$ is feasible for VCRP.

Let us denote by $\text{LSub}_{\bar{z}\bar{w}}$ the linear programming relaxation of $\text{Sub}_{\bar{z}\bar{w}}$, where (3.11), (3.12) are replaced by $Y \geq 0$ and $\omega \geq 0$, respectively. A new set of constraints, (3.14), is added to set unitary bounds on the ω variables:

$$I\omega \leq 1 \quad (3.14)$$

where 1 is an appropriate dimension unitary vector and I an identity matrix. Note that, unitary bounds for the y variables are not necessary since constraints (2.10) and (2.11), included in matrix representation (3.6), force these variables to be less than or equal to 1.

Let ζ, β, ϕ, χ and ε be the dual vectors corresponding to (3.5)', (3.6), (3.7), (3.8) and (3.14), respectively. Then, the dual of the linear programming problem $\text{LSub}_{\bar{z}\bar{w}}$, denoted by $\text{DLSub}_{\bar{z}\bar{w}}$, can be written as:

$$\begin{aligned} & (\text{DLSub}_{\bar{z}\bar{w}}) \\ \max \quad & \zeta \bar{W} + \beta E^2 + \phi G^3 + \varepsilon 1 + (\lambda_1 c^1 \bar{Z} + \lambda_1 c^2 \bar{W}) \end{aligned} \quad (3.15)$$

subject to

$$\phi G^2 + \varepsilon I \leq \lambda_2 c^3 \quad (3.16)$$

$$\chi I \leq \lambda_2 c^4 \quad (3.17)$$

$$\zeta Q + \beta E^1 + \phi G^1 + \chi P \leq \lambda_2 c^5 \quad (3.18)$$

$$\beta, \phi, \chi \geq 0, \varepsilon \leq 0. \quad (3.19)$$

If the drivers, defined by set M , with the respective availabilities from e^{mh} and F^m , are enough to cover all crew duties for all days of H , given by \bar{W} and at the same time all the rostering constraints are satisfied, assuming all variables may be non-integer, then $\text{LSub}_{\bar{z}\bar{w}}$ has feasible solution and optimal solution also (note that the respective feasible regions are bounded along the optimization direction). In this case, a Benders cut is obtained from the optimal solution of $\text{DLSub}_{\bar{z}\bar{w}}$ which corresponds to an extreme point of the respective feasible region. If the available drivers are not enough to cover all the crew duties in \bar{W} respecting all rostering constraints even accepting fractional variables, then $\text{LSub}_{\bar{z}\bar{w}}$ is unfeasible and its dual, $\text{DLSub}_{\bar{z}\bar{w}}$, is unbounded (there is always at least a feasible dual solution, the nil vector). In this case the corresponding Benders cut is obtained from an extreme ray of the feasible region of $\text{DLSub}_{\bar{z}\bar{w}}$.

3.3. Master problem

Let PD and RD be, respectively, the set of the extreme points and the set of extreme rays of the dual feasible region defined by (3.16) to (3.19), according to the Benders decomposition theory the master problem follows:

(Master)

$$\min \varphi_0 \quad (3.20)$$

subject to

$$\varphi_0 \geq \lambda_1 c^1 Z + (\bar{\varsigma} + \lambda_1 c^2)W + \bar{\beta} E^2 + \bar{\phi} G^3 + \bar{\varepsilon} 1 \quad (\bar{\varsigma}, \bar{\beta}, \bar{\phi}, \bar{\chi}, \bar{\varepsilon}) \in PD \quad (3.21)$$

$$0 \geq \bar{\mu} W + \bar{\eta} E^2 + \bar{h} G^3 + \bar{\lambda} 1 \quad (\bar{\mu}, \bar{\eta}, \bar{h}, \bar{\lambda}) \in RD \quad (3.22)$$

$$A^1 Z \geq A^2 \quad (3.3)$$

$$B^1 Z + B^2 W \geq B^3 \quad (3.4)$$

$$Z \geq 0 \text{ and binary} \quad (3.9)$$

$$W \geq 0 \text{ and binary.} \quad (3.10)$$

The constraint set $\{(3.3), (3.4), (3.9), (3.10)\}$ is also included in the mathematical model of the VCRP where it describes the integrated vehicle-crew scheduling problem for all the days of the planning horizon. This fact suggested the decomposition procedure for the VCRP that will be detailed in the next section.

4. Solution approach

Let $VCRP_{y, \omega \geq 0}$ be the problem obtained from relaxing the integrality constraints for the y and ω variables in VCRP defined through (3.2) to (3.13) with a specific choice for the parameters λ_1 and λ_2 . Fixing the values of the z and w variables in $VCRP_{y, \omega \geq 0}$ at values given by vectors \bar{Z} and \bar{W} , respectively, we obtain the linear sub-problem $LSub_{\bar{z}\bar{w}}$ and Benders decomposition theory guarantees that an optimal solution for $VCRP_{y, \omega \geq 0}$ is achieved, in case it exists (see Benders 1962). However, such optimal solution might not be feasible for VCRP due to the possibility of obtaining non-integer values for the y and ω variables. In fact, the sub-problem $Sub_{\bar{z}\bar{w}}$ is a mixed binary linear programming problem that does not satisfy the integrality property (the y and ω variables must be

forced to be binary) and for this case, to the authors' knowledge, no Benders convergence results have been generalized.

For each specific choice of values for parameters λ_1 and λ_2 , this paper proposes a non-exact approach for VCRP that iterates between a vehicle-crew scheduling problem for the planning horizon and a rostering problem thus obtaining, at the end, a feasible solution for the VCRP that naturally might not be an optimal one. In addition, this decomposition method is also much useful insofar as, along the several iterations, it produces a pool of feasible solutions for the VCRP. Such solutions can be analyzed from the two original objectives' perspective and one can determine the potentially efficient solutions corresponding to the points of the objectives' space that are not dominated by other points in the pool, the so-called potentially non-dominated points.

The Decomposition algorithm is summarized in figure 1.

Decomposition algorithm

//input//

Data: $\lambda_1, \lambda_2, c^1, c^2, c^3, c^4, c^5, A^1, A^2, B^1, B^2, B^3, Q, E^1, E^2, G^1, G^2, G^3, P$

//initialization//

step 1) $PD_0 = RD_0 = Pool = \Phi$.

step 2) $k=1$.

//iteration k //

step 3) Define $Master_k$ with the cuts from PD_{k-1} and RD_{k-1} .

step 4) Define $RMaster_k(\bar{U}, \bar{V})$, a lagrangean relaxation of the cuts associated to multipliers \bar{U} and

\bar{V} satisfying $\sum_{i=1}^{k_1} u_i = 1$.

step 5) Solve $RMaster_k(\bar{U}, \bar{V})$.

5.1) Apply the integrated vehicle-crew scheduling algorithm for each day of H .

5.2) Concatenate the $|H|$ solutions thus building a feasible solution of the vehicle-crew scheduling problem for $H, (\bar{Z}, \bar{W})$.

step 6) Call procedure Sub-problem($k; (\bar{Z}, \bar{W}); PD_{k-1}; RD_{k-1}; Pool$).

//stopping criterion//

step 7) If $k \leq \text{maxiterations} - 1$

7.1) then $k=k+1$ and go to step 3;

7.2) otherwise, calculate the potentially efficient solutions from the *Pool*.
 Stop.

Figure 1. Decomposition algorithm for the VCRP.

Steps 3, 4 and 5 of each iteration of the Decomposition algorithm are devoted to the master problem whereas step 6 calls the procedure to tackle the sub-problem. These features will be detailed in the next sub-sections.

4.1. Solving the master problem

Suppose that, in iteration k , PD_k has k_1 extreme points and RD_k has k_2 extreme rays.

Then the master problem becomes:

(Master _{k})

$$\min \varphi_0 \quad (3.20)$$

subject to

$$\varphi_0 - \lambda_1 c^1 Z - (\bar{\zeta}_i + \lambda_1 c^2)W \geq \bar{\beta}_i E^2 + \bar{\phi}_i G^3 + \bar{\varepsilon}_i 1 \quad i = 1, \dots, k_1 \quad (3.21)'$$

$$-\bar{\mu}_i W \geq \bar{\eta}_i E^2 + \bar{h}_i G^3 + \bar{\lambda}_i 1 \quad i = 1, \dots, k_2 \quad (3.22)'$$

$$A^1 Z \geq A^2 \quad (3.3)$$

$$B^1 Z + B^2 W \geq B^3 \quad (3.4)$$

$$Z \geq 0 \text{ and binary} \quad (3.9)$$

$$W \geq 0 \text{ and binary} \quad (3.10)$$

where $(\bar{\zeta}_i, \bar{\beta}_i, \bar{\phi}_i, \bar{\lambda}_i, \bar{\varepsilon}_i) \in PD_k$ and $(\bar{\mu}_i, \bar{\eta}_i, \bar{h}_i, \bar{\lambda}_i) \in RD_k$.

The master problem is a difficult binary linear programming problem and must be solved repeatedly, i.e., in each iteration of the algorithm. Moreover, the convergence results of the Benders algorithm to an optimal solution do not apply here, as mentioned above, due to the combinatorial nature of the sub-problem, $\text{Sub}_{\bar{z}W}$. Consequently, a non-exact approach to tackle the master problem is advisable. The option favoured a method based on the relaxation of (3.21)' and (3.22)'. At iteration k , a lagrangean relaxation of

Master_k is considered, where (3.21)' and (3.22)' are embedded in the objective function associated with the non-negative lagrangean multipliers U and V , respectively:

$$(\text{RMaster}_k(U, V))$$

$$\min \varphi_0 + \sum_{i=1}^{k_1} u_i (-\varphi_0 + \lambda_1 c^1 Z + (\bar{\varsigma}_i + \lambda_1 c^2)W + r_i) + \sum_{i=1}^{k_2} v_i (\bar{\mu}_i W + s_i) \quad (4.1)$$

subject to

$$A^1 Z \geq A^2 \quad (3.3)$$

$$B^1 Z + B^2 W \geq B^3 \quad (3.4)$$

$$Z \geq 0 \text{ and binary} \quad (3.9)$$

$$W \geq 0 \text{ and binary} \quad (3.10)$$

where $r_i = \bar{\beta}_i E^2 + \bar{\phi}_i G^3 + \bar{\epsilon}_i 1$ for all $i=1, \dots, k_1$ and $s_i = \bar{\eta}_i E^2 + \bar{h}_i G^3 + \bar{\lambda}_i 1$ for all $i=1, \dots, k_2$.

Note that, the integrality property is not valid for $\text{RMaster}_k(U, V)$. As a result, in each iteration k , one has $v(\text{LMaster}_k) \leq \max_{U, V \geq 0} \text{RMaster}_k(U, V) \leq v(\text{Master}_k)$, where $v(\text{LMaster}_k)$

is the optimal value of the linear programming relaxation of Master_k . Hence, the lagrangean relaxation might do better than the linear relaxation in what respects lower bounds for $v(\text{Master}_k)$. Usually, the values of the lagrangean multipliers associated to the relaxed constraints are set equal to the values of the corresponding dual variables and an optimizing iterative procedure updates the multipliers so that the lower bound improves. However, in this approach no multiplier improvement is performed.

A specific choice for the lagrangean multipliers values, \bar{U} and \bar{V} , such that $\sum_{i=1}^{k_1} \bar{u}_i = 1$ (satisfied by the corresponding dual variables), converts the objective function of

$\text{RMaster}_k(\bar{U}, \bar{V})$, in (4.1), into:

$$\begin{aligned} & \varphi_0 + \sum_{i=1}^{k_1} \bar{u}_i (-\varphi_0 + \lambda_1 c^1 Z + (\bar{\varsigma}_i + \lambda_1 c^2)W + r_i) + \sum_{i=1}^{k_2} \bar{v}_i (\bar{\mu}_i W + s_i) = \\ & = \lambda_1 c^1 Z + \left(\sum_{i=1}^{k_1} \bar{u}_i \bar{\varsigma}_i + \lambda_1 c^2 \right) W + \sum_{i=1}^{k_1} \bar{u}_i r_i + \sum_{i=1}^{k_2} \bar{v}_i \bar{\mu}_i W + \sum_{i=1}^{k_2} \bar{v}_i s_i = \\ & = \lambda_1 c^1 Z + \left(\sum_{i=1}^{k_1} \bar{u}_i \bar{\varsigma}_i + \lambda_1 c^2 + \sum_{i=1}^{k_2} \bar{v}_i \bar{\mu}_i \right) W + \sum_{i=1}^{k_1} \bar{u}_i r_i + \sum_{i=1}^{k_2} \bar{v}_i s_i. \end{aligned}$$

Now, this objective function can be rewritten by identifying the components of the vectors C^1, Z, C^2 and W :

$$\sum_{h \in H} \left(\sum_{d \in D} \sum_{(s,t) \in T} \lambda_1 c_{st}^{1dh} z_{st}^{dh} + \sum_{d \in D} \sum_{s \in N} \left(\lambda_1 c_{s,n+d}^{1h} z_{s,n+d}^h + \lambda_1 c_{n+d,s}^{1h} z_{n+d,s}^h \right) + \sum_{\ell \in L^h} \bar{c}_\ell^2 w_\ell^h \right) + \sum_{i=1}^{k_1} \bar{u}_i r_i + \sum_{i=1}^{k_2} \bar{v}_i s_i$$

$$\text{where } \bar{c}_\ell^2 = \sum_{i=1}^{k_1} \bar{u}_i \bar{s}_i + \lambda_1 c_\ell^2 + \sum_{i=1}^{k_2} \bar{v}_i \bar{\mu}_i.$$

For any choice for the parameter λ_1 , since $\sum_{i=1}^{k_1} \bar{u}_i r_i + \sum_{i=1}^{k_2} \bar{v}_i s_i$ is constant, this objective function, along with the set of constraints of $\text{RMaster}_k(\bar{U}, \bar{V})$, can be partitioned into $|H|$ independent subsets. Therefore, solving $\text{RMaster}_k(\bar{U}, \bar{V})$ for a specific choice of the multipliers \bar{U} and \bar{V} is equivalent to solving $|H|$ independent integrated vehicle-crew scheduling problems, one for each day of the planning horizon. The (daily) integrated vehicle-crew scheduling problems can be solved by the algorithm proposed in Mesquita and Paia (2008) which combines a heuristic column generation procedure with a branch-and-bound scheme.

Note that these vehicle-crew scheduling solutions may give slightly different daily schedules for the vehicles and also for the crews. However, in real cases public transit companies, usually, have the same vehicle-crew schedules in each day type of the planning horizon - there is a pattern for the weekdays and one pattern for the weekend days. Hence, it is desirable that solutions resulting from the master problem will follow this scheme. Consequently, in step 5 of the Decomposition algorithm the master problem is solved for each day type.

4.2. Solving the sub-problem

In each iteration of the Decomposition algorithm, step 6 (figure 1) refers to the sub-problem. Figure 2 details the procedure.

```
Procedure Sub-problem( $k; (\bar{Z}, \bar{W}); PD_{k-1}; RD_{k-1}; Pool$ )
```

```
//sub-problem of iteration  $k$ //
```

```

step 1) Define  $\text{Sub}_{\bar{z}\bar{w}}$ , the rostering sub-problem for iteration  $k$ 
step 2) Solve  $\text{LSub}_{\bar{z}\bar{w}}$ , the corresponding linear relaxation:
    2.1) in case it has a finite optimal value, save the corresponding dual solution and go to step 3;
    2.2) in case it is unfeasible, save an extreme ray of the dual linear feasible region and go to step 5.
step 3) Solve  $\text{Sub}_{\bar{z}\bar{w}}$  to get a feasible roster if the threshold  $\pi$  is not attained.
//solutions for the VCRP//
step 4) Update the Pool of feasible solutions of the VCRP.
step 5) Update the sets of extreme points and of extreme rays,  $PD_{k-1}$  and  $RD_{k-1}$ .
Stop.

```

Figure 2. Procedure for tackling the sub-problem.

Exact standard algorithms are used to solve the sub-problem (step 3) and the respective linear relaxation (step 2), $\text{Sub}_{\bar{z}\bar{w}}$ and $\text{LSub}_{\bar{z}\bar{w}}$. The dual linear variables or dual extreme rays obtained in step 2 will give rise to the Benders cuts that will be added to the master problem, in the next iteration.

Let $v(\text{LSub}_{\bar{z}\bar{w}}, k-1)$ denote the linear programming relaxation value in iteration $k-1$ of the Decomposition algorithm. To obtain a feasible roster, a branch-and-bound is executed with $\text{Sub}_{\bar{z}\bar{w}}$ whenever the following criterion involving the two optimization objectives and a threshold π (step 3) is satisfied:

$$v(\text{LSub}_{\bar{z}\bar{w}}, k) \leq \min_{i=1, \dots, k-1} v(\text{LSub}_{\bar{z}\bar{w}})_i + \pi \text{ or}$$

$$v(\text{RMaster}_k(U, V)) \leq \min_{i=1, \dots, k-1} v(\text{RMaster}_i(U, V)) + \pi.$$

In this case, the solution of the master, a set of vehicle-crew schedules covering the planning horizon, along with the solution of the sub-problem correspond to a feasible solution for the VRCP which will be included in the *Pool* - step 4 of the procedure in figure 2.

5. Computational experiment

A computational experiment was performed using real-world data from a public transit company operating in Lisbon.

All linear programming relaxations and branch-and-bound schemes, in the Decomposition algorithm, were tackled with CPLEX solvers (CPLEX Manual version 11.0, 2007). As for the integer resolution of the rostering sub-problems a time limit of 7200 seconds was imposed. The (daily) integrated vehicle-crew scheduling problems were solved by the algorithm proposed in Mesquita and Paias (2008) by setting the parameters $\epsilon=7$, $\gamma=3000$ and $p=4/15$, where ϵ is the parameter related with the definition of the tasks, γ is the maximum number of columns generated per iteration and p is a parameter related with the heuristic pricing of the columns. See Mesquita et al. 2009 for a detailed description of these parameters.

All algorithms were coded in C, using VStudio 6.0/C++ and all the programs ran on a PC Pentium IV 3.2 GHz.

5.1 Test instances

The test instances used for the experiments were derived from an urban bus service inside the city of Lisbon and involve scheduling problems with 122, 168, 224, 226 and 238 trips and 4 depots. The input of each VCRP instance includes the start and end times, the start and end locations for each trip and the deadhead times between locations and depots. Two different demand patterns (timetabled trips) are considered, one for weekdays and the other for weekend days. Consequently, in each iteration of the Decomposition algorithm the integrated vehicle-crew scheduling problem is solved twice: for a weekday type and for a weekend day type.

Concerning daily crew duties and the rostering, some parameters have to be defined in order to respect the rules imposed by Portuguese Law, union contracts and specific rules of the bus company. A detailed description of them may be seen in Mesquita et al. (2008).

In what respects the vehicle-crew scheduling process one has:

- for each crew duty the minimum spread is set at 1 hour
- the maximum spread is 5 hours for duties without a break; otherwise, it is 10 hours and 45 minutes
- break times range from 1 hour to 2 hours and 20 minutes
- the maximum duration allowed for a crew duty before a break occurs is 5 hours

- a penalty of 5000 m.u. is added to the cost of each pull-in and each pull-out trip in order to minimize the number of vehicle blocks in the schedule

- $\lambda_1 = 1$.

Respecting the rostering process one has:

- $|H| = 28$

- $\alpha = 4$

- $|M| = 80$

- $u_\ell \in [300, 645]$ minutes

- $\bar{u} = 480$ minutes (8 hours)

- $a = 11$ hours - the minimum rest period of 11 hours allows the separation of the set of crew duties into early duties (L^{1h}), starting at a point between 6:00 a.m. and 3:30 p.m. and late duties (L^{2h}), starting in the interval from 3:30 p.m. to midnight

- $b_{1w} = 2880$ minutes (48 hours)

- $b_{rw} = 10560$ minutes (176 hours)

- $\Omega_w = 2$ days

- $\Omega_S = 1$ day

- $g = 6$ days

- $F^m = \emptyset$

- $c^{3m} = 0.96$

- $c^4 = 0.04$

- $c_\ell^{5mh} = 0$

- $\lambda_2 = 1$

- $\pi = 0$.

5.2 Computational results for the VCRP

Tables 1 and 2 show computational results obtained from 10 iterations of the proposed Decomposition algorithm. In both tables, “nvehic”, “ncrew” and “ndriver” refer to the number of vehicles, number of crew duties and number of drivers, respectively. In

table 1, column (10) contains the average maximum overtime per driver measured in units of 15 minutes. The last columns in table 1 are devoted to CPU time: the values reported in columns (11) and (12) are total CPU values obtained from the 10 iterations, respectively, for the VCP master problem and for the linear programming relaxation rostering sub-problem. Column (13) shows total CPU values for determining mixed-integer solutions of the rostering sub-problems (feasible rosters) and, in brackets, the number of MILP sub-problems solved according to the threshold π .

Table 1. Results from 10 iterations of the Decomposition algorithm.

	master (daily) vehicle-crew schedules						sub-problem monthly roster			total CPU (seconds)		
	weekday			weekend								
	nvehic average	ncrew min	ncrew max	nvehic average	ncrew min	ncrew max	ndriver min	ndriver max	max overtime average (quarters)	VCP	rostering LP	MILP
122	9	17	17	6	9	9	21	21	64.7	61	7	3893 (3)
168	17	38	38	10	19	19	46	46	81.0	109	83	42684 (6)
224	18	39	39	10	16	16	50	52	113.6	4296	83	36000 (5)
226	15.6	33	35	7	15	15	44	47	117.4	13208	73	36000 (5)
238	22	54	54	11	27	27	69	75	109.3	1038	164	28800 (4)

As one can see, in table 1, the Decomposition algorithm has produced solutions that, despite being different, have the same number of vehicles and crews. Variations have occurred only for instance 226. This diversity of vehicle-crew solutions led to rostering solutions that, for the same instance, may have a great variation in the number of drivers. For instance 224 the rostering solutions differ at most in 2 drivers. For the last two instances, 226 and 238, the number of drivers varies from 44 to 47 and 69 to 75, respectively.

On average, considering all instances, the rostering mixed integer linear program (MILP) was solved 5 times out of 10. One can notice that, for each instance, a small

increase in the number of mixed integer linear rostering problems solved, during 10 iterations, led to a great increase in total CPU time.

In table 2, for each instance, the first row of columns (2) to (7) shows the results obtained in the first iteration of the Decomposition algorithm which corresponds to that of a sequential approach applied to the same instance. The subsequent rows (or row), in columns (2) to (7), correspond to the potentially non-dominated points (or point) obtained. Columns (9) to (11) report on the difference between the solution corresponding to a potentially non-dominated point - potentially efficient solution - and the solution obtained on iteration 1 (sequential approach), concerning the number of vehicles, the number of crew duties, the number of drivers and the overtime. The last column refers to overtime and is given in percentage.

Table 2. The Decomposition algorithm versus the Sequential algorithm.

	potentially non-dominated points						improvement from sequential approach (iteration 1)			
	weekday		weekend		planning horizon					
	nvehic	ncrew	nvehic	ncrew	ndriver	max overtime	Δ nvehic	Δ ncrew	Δ ndriver	Δ overtime
122	9	17	6	9	21	74				
	9	17	6	9	21	56	0	0	0	-24%
168	17	38	10	19	46	88				
	17	38	10	19	46	75	0	0	0	-15%
224	18	39	10	16	51	110				
	18	39	10	16	51	108	0	0	0	-2%
	18	39	10	16	50	107	0	0	-1	-3%
226	15	35	7	15	47	131				
	15	35	7	15	44	115	0	0	-3	-12%
	16	33	7	15	44	111	+1	0	-3	-15%
	15	35	7	15	46	112	0	0	-1	-15%
238	22	54	11	27	69	111				
	22	54	11	27	70	107	0	0	+1	-4%
	22	54	11	27	71	117	0	0	+2	+5%

With the exception of instance 238, the potentially non-dominated points obtained by the Decomposition algorithm dominate the points obtained by the sequential approach (see the first row - first iteration - per instance). In instance 238, the first solution obtained by the Decomposition algorithm, is itself a potentially efficient solution. The last row of the table displays a point for this instance that corresponds to a reduction in the cost of the weekday VCP thus being a potentially non-dominated point.

In general, one can see from the above results that the improvement over the first solution is obtained by minimizing the maximum overtime per driver. In fact, the solution of the master problem could be adjusted using the feedback obtained by introducing Bender cuts. This feedback guided the building of the vehicle and crew schedules thus conducting to rosters with less overtime per driver and with fewer drivers. Note that, although a sequential approach greatly reduces CPU time, the resulting integrated problem might not be solvable if no feasible roster can be built from the vehicle-crew scheduling solution.

6. Conclusions

This paper proposes a new methodology to deal with the integrated vehicle-crew-rostering problem within public transit companies. The VCRP is modelled as a bi-objective mixed binary linear problem and the solution approach is based on Benders decomposition. It alternates between the solution of an integrated vehicle-crew scheduling master problem and the solution of the corresponding linear programming relaxation rostering sub-problem, used to produce Benders cuts. In spite of the fact that the feasible region of the Benders sub-problem is not convex, hence it does not satisfy the hypotheses for the convergence of the Benders algorithm, here Benders decomposition is used within a non-exact method for the VCRP that produces a pool of feasible solutions. In fact, in each iteration of the proposed decomposition algorithm, a pre-defined criterion is analysed and whenever satisfied branch-and-bound techniques are applied to obtain a feasible roster that together with the master problem vehicle-crew scheduling solution give a feasible solution to the VCRP.

The effects of integration of the three difficult combinatorial optimization problems were analyzed for real instances of the VCRP. The generation of Benders cuts proved to

be effective within the proposed non-exact method for producing a pool of feasible and potentially efficient solutions for the VCRP at reasonable computing times.

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References

- Barnhart, C. and G. Laporte (eds.) (2007). *Transportation. Handbooks in Operations Research and Management Science*, 14. Elsevier, North-Holland, Amsterdam, Holland.
- Benders, J.F. (1962). Partitioning procedures for solving mixed-variables programming problems. *Numerische Mathematik* 4: 238-252.
- Borndörfer, R., M. Grötschel and M. Pfetsch (2006). Can O.R. methods help public transportation systems break even? German Research Group Moves Industry Close to Elusive Goal. *OR/MS Today* 33: 30-40.
- Boschetti, M. and V. Maniezzo (2009). Benders decomposition, Lagrangean relaxation and metaheuristic design. *Journal of Heuristics* 15: 283-312.
- Caprara, A., M. Monacci and P. Toth (2001). A global method for crew planning in railway applications. In Voss S., Daduna J. (eds.), *Computer-Aided Scheduling of Public Transport. Lecture Notes in Economics and Mathematical Systems*, 505. Springer, Berlin, Germany, 17-36.
- Chu, S.C.K. (2007). Generating, scheduling and rostering of shift crew-duties: applications at the Hong Kong International Airport. *European Journal of Operational Research* 177: 1764-1778.
- Cordeau, J-F., G. Stojković, F. Soumis and J. Desrosiers (2001). Benders decomposition for simultaneous aircraft routing and crew scheduling. *Transportation Science* 35(4): 375-88.
- CPLEX Manual (version 11.0) (2007). Using the CPLEX^R Callable Library and CPLEX Mixed Integer Library. ILOG INC., Incline Village, Nevada, USA.
- Ehrgott, M. and X. Gandibleux (2000). A survey and annotated bibliography of multiobjective combinatorial optimization. *OK Spectrum* 22: 425-460.
- Ernst, A., H. Jiang, M. Krishnamoorthy, H. Nott and D. Sier (2001). An integrated optimization model for train crew management. *Annals of Operations Research* 108: 211-224.
- Freling, R., R.M. Lentink and A.P.M. Wagelmans (2004). A decision support system for crew planning in passenger transportation using a flexible branch-and-price algorithm. *Annals of Operations Research* 127: 203-222.
- Geoffrion, A.M. (1972). Generalized Benders decomposition. *Journal of Optimization Theory and Applications* 10: 237-260.
- Hollis, B.L., M.A. Forbes and B.E. Douglas (2006). Vehicle routing and crew scheduling for metropolitan mail distribution at Australia Post. *European Journal of Operational Research* 173: 133-150.

- Huisman, D., R. Freling and A.P.M. Wagelmans (2005). Multiple-depot integrated vehicle and crew scheduling. *Transportation Science* 39 (4): 491-502.
- Lee, C.-K and C.-H. Chen (2003). Scheduling of train drivers for Taiwan Railway Administration. *Journal of Eastern Asia Society of Transportation Studies* 5: 292-306.
- Mercier, A., J-F Cordeau and F. Soumis (2005). A computational study of Benders decomposition for the integrated aircraft routing and crew scheduling problem. *Computers & Operations Research* 32(6): 1451-76.
- Mercier, A. and F. Soumis (2006). An integrated aircraft routing, crew scheduling and flight retiming model. *Computers & Operations Research* 34(8): 2251-65.
- Mesquita, M., M. Moz, A. Paias, J. Paixão, M. Pato and A. Respício (2008). Solving public transit scheduling problems. CIO-Working Paper 1/2008, available online <http://cio.fc.ul.pt/home.do>.
- Mesquita, M. and A. Paias (2008). Set partitioning/covering-based approaches for the integrated vehicle and crew scheduling problem. *Computers and Operations Research* 35: 1562-1575.
- Mesquita, M., A. Paias and A. Respício (2009) Branching approaches for integrated vehicle and crew scheduling. *Public Transport* 1: 21-37.
- Papadakos, N. (2009). Integrated airline scheduling. *Computers & Operations Research* 36: 176-195.
- Steuer, R.E. (1986). *Multiple Criteria Optimization: Theory, Computation, and Application*. John Wiley & Sons, New York, USA.