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Fat handles and phase portraits of Non Singular Morse-Smale flows on S^3 with unknotted saddle orbits.

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Abstract

In this paper we build Non-singular Morse-Smale flows on S^3 with unknotted and unlinked saddle orbits by identifying fat round handles along their boundaries. This way of building the flows enables to get their phase portraits.

We also show that the presence of heteroclinic trajectories imposes an order in the round handle decomposition of these flows; this order is total for NMS flows composed of one repulsive, one attractive and n unknotted saddle orbits, for $n \geq 1$.

1 Introduction

The phase portrait of a dynamical system provides a complete description of the flow. It is not easy to get this picture for 3-dimensional manifolds and the drawing

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becomes more difficult when the number of periodic orbits increases. Our goal is to obtain an easy way for visualizing the flows.

Our study on S^3 is based on the round handle decomposition of the manifold for a given flow by D. Asimov [1] and by J. Morgan [4] and the topological characterization of the links of periodic orbits by M. Wada [5].

Every non-singular Morse Smale flow on S^3 admits a round handle decomposition whose core circles are the periodic orbits of the flow, and conversely, a round handle decomposition gives rise to a Non-singular Morse-Smale flow on a flow manifold.

The characterization of the links of periodic orbits of a non singular Morse-Smale flow on S^3 by M. Wada is obtained from Hopf links by applying six operations; each operation corresponds to a different attachment of a round 1-handle in the decomposition of the manifold. The flows characterized by the first three operations have all their saddle orbits unknotted and unlinked.

In this paper, we consider flows coming only from the first three operations of Wada. A description of this type of flows can be found in [2] and [3].

To claim our goal we define in section 3 the basic fat handles (fat handles with only one saddle orbit) and we prove that these flows can also be obtained from the identification of basic fat handles along their boundaries (Propositions 3.3 and 3.1). In section 4, we show that the appearance of heteroclinic trajectories connecting saddle orbits implies noncommutativity of the operations involved, establishing an order in the round handle decomposition (Proposition 4.1). We find a total or linear order when the flow contains only one attractive and one repulsive periodic orbit because of the existence of transversal intersections of invariant manifolds of the saddle orbits (Theorem 4.1).

2   Previous Results

A non singular Morse-Smale flow (or NMS for short) is a flow without fixed points, consisting of a finite number of hyperbolic periodic orbits where the intersections of stable and unstable manifolds of the saddle orbits are transversal.

D. Asimov [1] and J.W. Morgan [4] established a correspondence between NMS flows and round handle decompositions of the corresponding manifold. These flows are defined on flow manifolds.

Let M be a compact manifold whose boundary has been partitioned into two unions of components: $\partial M = \partial_- M \cup \partial_+ M$, $\partial_- M \cap \partial_+ M = \emptyset$. A flow manifold is a pair $(M, \partial_- M)$ satisfying:

- $\chi(\partial_- M) = \chi(M)$.

- $\chi(\partial_+ M) = \chi(M)$.

- There exists a nonsingular vector field on M pointing inwards on $\partial_- M$ and outwards on $\partial_+ M$.

For the case of dimension 3, a pair $(M, \partial_- M)$ of a manifold M and a compact submanifold $\partial_- M$ of ∂M, or by abuse of notation, a manifold M is called:

A round 0-handle if $(M, \partial_- M) \cong \left(D^2 \times S^1, \emptyset \right)$.

A round 1-handle if $(M, \partial_- M) \cong \left(D^1 \times D^1 \times S^1, D^1 \times \partial D^1 \times S^1 \right)$.

A round 2-handle if $(M, \partial_- M) \cong \left(D^2 \times S^1, \partial D^2 \times S^1 \right)$.

In this case, the round handles are diffeomorphic to tori and correspond to 0-handles when there is a repulsive periodic orbit in the core, to 2-handles if there is an attractive periodic orbit in the core and to 1-handles if the orbit in the core is a saddle; 0, 1 and 2 are the indices of the periodic orbits. A set of indexed periodic orbits is called an indexed link.
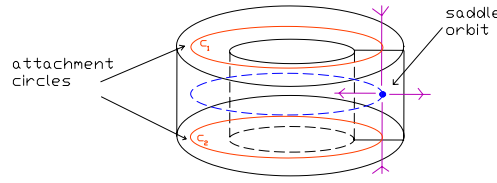

Figure 1: Round 1-handle.

Proposition 2.1 (Morgan) *Given a flow manifold $(X, \partial_- X)$ with a NMS flow, then $(X, \partial_- X)$ has a round handle decomposition whose core circles are the closed orbits of the flow.*

The round handle decomposition for a compact, orientable 3-manifold M was modified by Morgan:

$$\emptyset = M_0 \subset M_1 \subset \cdots \subset M_i \subset M_{i+1} \subset \cdots \subset M_N = M \qquad (2.1)$$

where each manifold M_i, called *fat round handle*, is obtained from M_{i-1} by attaching a round $1-$handle by means of one or two attaching circles (see Figure 1).

From the round handle decomposition of the 3-dimensional sphere, M. Wada characterizes the set of periodic orbits of NMS flows in terms of a generator, the Hopf link with indices 0 and 2 attached to the components, and six operations defined from the type of attachment of the round 1-handles (see [5]).

Theorem 2.1 (M. Wada) *"Every indexed link which consists of all the closed orbits of a Non-Singular Morse-Smale flow on S^3 is obtained from Hopf links by applying the following six operations. Conversely, every indexed link obtained from Hopf links by applying the operations is the set of all the closed orbits of some Non-Singular Morse-Smale flow on S^3".*

OPERATIONS: For given indexed links l_1 and l_2, the six operations are defined as follows. Let $l_1 \cdot l_2$ denote the split sum of the links l_1 and l_2 and $N(k, M)$ a regular neighborhood of k in M.

1) $I(l_1,l_2) = l_1 \cdot l_2 \cdot u$, where u is an unknot with index 1.

2) $II(l_1,l_2) = l_1 \cdot (l_2 - k_2) \cdot u$, where k_2 is a component of l_2 of index 0 or 2.

3) $III(l_1,l_2) = (l_1 - k_1) \cdot (l_2 - k_2) \cdot u$, where k_1 is a component of l_1 of index 0 and k_2 is a component of l_2 of index 2.

4) $IV(l_1,l_2) = (l_1 \# l_2) \cup m$. The connected sum $(l_1 \# l_2)$ is obtained by composing a component k_1 of l_1 and a component k_2 of l_2, each of which has index 0 or 2. The index of the composed component $k_1 \# k_2$ is equal to either $i(k_1)$ or $i(k_2)$. Finally, m is a meridian of $k_1 \# k_2$ with $i = 1$.

5) $V(l_1)$: Choose a component k_1 of l_1 of index 0 or 2, and replace $N(k_1, S^3)$ by $D^2 \times S^1$ with three indexed circles in it; $\{0\} \times S^1$, k_2 and k_3. Here, k_2 and k_3 are parallel (p,q)-cables on $\partial N(\{0\} \times S^1, D^2 \times S^1)$, where p is the number of longitudinal turns and q the number of the transverse ones. The indices of $\{0\} \times S^1$ and k_2 are either 0 or 2, and one of them is equal to $i(k_1)$. The index of k_3 is 1.

6) $VI(l_1)$: Choose a component k_1 of l_1 of index 0 or 2. Replace $N(k_1, S^3)$ by $D^2 \times S^1$ with two indexed circles in it; $\{0\} \times S^1$ and the $(2,q)$-cable k_2 of $\{0\} \times S^1$. The index of $\{0\} \times S^1$ is 1, and $i(k_2) = i(k_1)$.

Operations I, II and III defined by Wada involve at least one inessential circle of attachment; so, the flows characterized only by these three operations have all their saddle orbits unknotted and unlinked. We refer to them as type A operations and denote by $\mathcal{F}_A(S^3)$ the set of such flows and by $\mathcal{L}_A(S^3)$ the set of the corresponding links of periodic orbits.

Operations IV, V and VI imply only essential attachments; so, saddle orbits generated by these operations are linked to other periodic orbits and they can be knotted.

We call *basic flows* those obtained from one operation of Wada on Hopf links.

Let us observe that even though the round handle decomposition of a flow is unique except for commutativity of some of the attachments involved, non-equivalent flows can be characterized by the same link.

In this paper, we focus on the set of flows $\mathcal{F}_A(S^3)$. We reproduce these flows by identifying two fat round handles along their boundaries and we obtain that there exists an order in the saddle orbits when heteroclinic trajectories appear.

3   Fat handles for $\mathcal{F}_A$ flows on $\mathbf{S}^3$

The 3-sphere S^3 is composed of two solid tori identified along their boundaries. So, we can obtain NMS flows on S^3 by identifying properly one repulsive and one attractive tori along their boundaries. These tori with a flow pointing inwards or outwards correspond to the fat round handles.

Given a flow φ on S^3, a repulsive fat handle is obtained by removing one attractive orbit and an attractive fat handle is obtained by removing one repulsive orbit. In this section, we obtain and classify the fat handles for $\mathcal{F}_A(S^3)$ flows.

In the following we show the fat handles obtained by removing one orbit in the basic flows of $\mathcal{F}_A(S^3)$. We refer to them as *basic fat round handles* and we denote them by describing the periodic orbits that contains. If the fat handle has
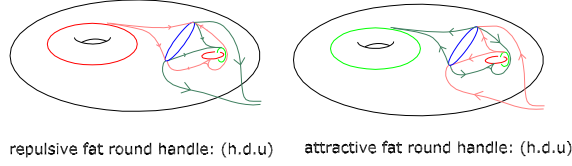
repulsive fat round handle: (h.d.u)   attractive fat round handle: (h.d.u)

Figure 2: Fat handles associated to operation I of Wada.

an attractive or repulsive orbit in its core we refer to it as thick torus; if it has no orbit in his core, we refer to it as solid torus.

Let h denote the Hopf link, let d denote a trivial separated knot corresponding to an attractive or repulsive orbit, let u denote a trivial separated knot corresponding to a saddle orbit and let $\cdot$ denote the separated sum of links.

Consider the flow $I(h,h)$. The link of periodic orbits of this flow consists in the separated sum of two Hopf links h and an unknot u, $h \cdot h \cdot u$. We obtain the fat handles associated to operation I by removing one attractive or one repulsive orbit. These fat handles are tori with an orbit in the core, i.e., thick tori, and we denote them hdu (see Figure 2).

From the flow $II(h,h) = h \cdot d \cdot u$, we obtain the fat handles associated to operation II by removing one attractive or repulsive orbit. Depending on the removed orbit, we obtain different types of fat handles. If we take off the separated orbit d, the fat handle is a torus without any orbit in the core, i.e. solid torus (see Figure 3) and we denote it hu. If one component of the Hopf link h is removed the fat handle is a torus with one orbit in its core (see Figure 4) and we denote it ddu. In this last case, the orbits of type d can be both repulsive, both attractive or one attractive and one repulsive.

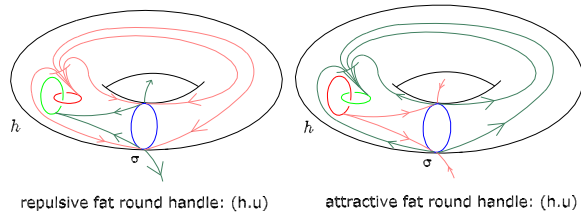repulsive fat round handle: (h.u)   attractive fat round handle: (h.u)

Figure 3: Fat handles associated to operation II of Wada removing an orbit d.

From the flow $III(h,h) = d \cdot d \cdot u$, we obtain the fat handles associated to operation III by removing one attractive or one repulsive orbit. They are tori without any orbit in the core and we denote them du (see Figure 5).

Let us remark that the identification along their boundaries of two fat handles without any orbit in their cores yields a transversal intersection of two invariant
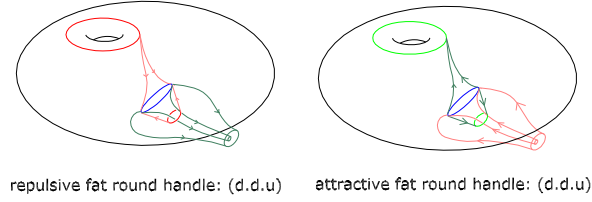
repulsive fat round handle: (d.d.u)   attractive fat round handle: (d.d.u)

Figure 4: Fat handles associated to operation II of Wada removing a component of h.

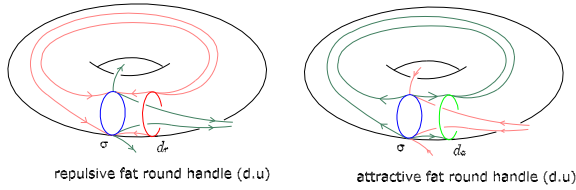repulsive fat round handle (d.u)   attractive fat round handle (d.u)

Figure 5: Fat handles associated to operation III of Wada.

manifolds of saddle orbits (see [3]).

We distinguish different classes of fat handles depending on the way the invariant manifolds of the saddle leave or enter the torus and the number of orbits in the canonical region where the identification of fat handles will be made:

- The fat round handles that are thick tori and the invariant manifolds of the saddle orbit going outwards the torus correspond to inessential circles. The attractive (repulsive) orbits filling the essential hole of the thick tori are in the canonical region where the identification with another fat handle will be done (see Figures 2 and 4).

- The fat round handles that are solid tori coming from operation II: one invariant manifold of the saddle orbit going outwards the torus is an essential circle and there is not any attractive or repulsive orbit in the canonical region of the identification (see Figure 3).

- The fat round handles that are solid tori coming from operation III: one invariant manifold of the saddle orbit leaves or enters the torus essentially and another one does it inessentially. There is one attractive or repulsive orbit filling a non essential hole in the torus, in the canonical region of the identification (see Figure 5).

Then, we define three classes of fat handles:

- A repulsive (attractive) fat handle belongs to class $[I]$ if it corresponds to a thick torus and the invariant manifolds of the saddles orbits go outwards (inwards) the torus by means of inessential circles.

- A repulsive (attractive) fat handle belongs to class $[II]$ if it corresponds to a solid torus, the invariant manifolds of the saddles orbits go outwards (inwards) the torus by means of essential circles and there is not any attractive or repulsive orbit in the canonical region of the identification.

- A repulsive (attractive) fat handle belongs to class $[III]$ if it corresponds to a solid torus, the invariant manifolds of the saddles orbits go outwards (inwards) the torus by means of essential and inessential circles and there is one attractive or repulsive orbit, filling a non essential hole in the torus, in the canonical region of the identification.

From the identification along their boundaries of one attractive and one repulsive basic fat handles we obtain the $\mathcal{F}_A(S^3)$ flows with two saddle orbits. Iterated fat round handles with two saddles are obtained by removing one repulsive (attractive) orbit in these flows.

Let us notice that when two fat handles belonging to class $[I]$ are identified along their boundaries, the orbits in their cores form a Hopf link in the canonical region of the identification. On the other hand, when fat handles of class $[II]$ and $[III]$ are identified, one heteroclinic orbit connecting the two saddles orbits appears (see Figures 7, 8 and 9).

In the following proposition we show that the iterated fat handles with two saddle orbits can be classified in one of these three classes of fat handles defined above.

Proposition 3.1 *For $\mathcal{F}_A\left(S^3\right)-$flows, a fat handle with two saddle orbits belongs to class $[I]$, $[II]$ or $[III]$.*

Proof. The fat handles with two saddles are obtained by identifying one repulsive and one attractive basic fat handles along their boundaries and then removing a repulsive or attractive periodic orbit.

- When an attractive fat handle hdu is identified with a repulsive fat handle hdu, the flow $I\left(I\left(h,h\right),h\right) = h \cdot h \cdot h \cdot u \cdot u$ is obtained. If an attractive (repulsive) orbit is removed after the identification, the resulting repulsive (attractive) fat handle, $hhduu$, corresponds to a thick torus with the manifolds of the saddle orbits going outwards (inwards) the torus inessentially. So, it belongs to class $[I]$.

- If an attractive fat handle hdu is identified with a repulsive fat handle ddu, the flow $I\left(II\left(h,h\right),h\right) = h \cdot h \cdot d \cdot u \cdot u$ is obtained. Now, we can remove the orbit d or a component of a Hopf h. If a component of h is removed, the repulsive (attractive) fat handle is $hdduu$, with one toroidal and one Hopf holes inside and invariant manifolds of the saddles crossing unessentially the torus; so,

it belongs to class $[I]$. If the orbit d is removed, the fat handle is $hhuu$, a solid torus with two Hopf holes inside, with no orbit in the canonical region of the identification and with the invariant manifold of one of the saddle orbits crossing the torus by means of an essential circle; so, it belongs to class $[II]$.

- The identification of an attractive (repulsive) fat handle hdu with a repulsive (attractive) fat handle hu is not admissible because this identification generates a bitorus (see Figure 6).
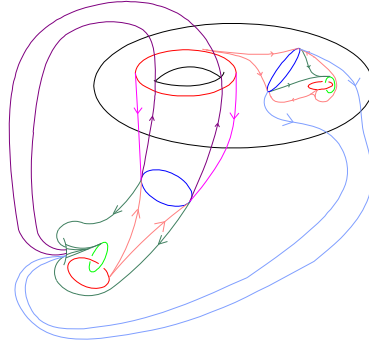

Figure 6: Identification of fat handles hdu and hu along their boundaries.

- When an attractive fat handle hdu is identified with a repulsive fat handle du, we obtain the flow $I\left(III\left(h,h\right),h\right)=h\cdot d\cdot d\cdot u\cdot u$. If an attractive (repulsive) orbit is now removed, the resulting repulsive (attractive) fat handle is either a thick torus with two toroidal holes inside, $ddduu$, belonging to class $[I]$ or a solid torus $hduu$ with one orbit in the canonical region of the identification, belonging to class $[III]$.

- If an attractive fat handle ddu is identified with a repulsive fat handle ddu, the resulting flow is $II\left(II\left(h,h\right),h\right)=h\cdot d\cdot d\cdot u\cdot u$. Now, we can remove a component of the Hopf h or an orbit d. Then, the resulting repulsive (attractive) fat handle is either a thick torus with two toroidal holes inside, $ddduu$, belonging to class $[I]$ or a solid torus with one toroidal and one Hopf holes inside, $hduu$, with no orbit in the canonical region of the identification, belonging to class $[II]$.

- The identification of an attractive (repulsive) fat handle ddu with a repulsive (attractive) fat handle hu is not admissible because a bitorus is obtained.

- When an attractive fat handle ddu is identified with a repulsive fat handle du, the defined flow is $II\left(III\left(h,h\right),h\right)=d\cdot d\cdot d\cdot u\cdot u$. If an attractive (repulsive) orbit is removed after the identification the resulting repulsive (attractive) fat

handle *dduu* is either a solid torus with two toroidal holes inside and with no orbit in the canonical region of the identification belonging to class $[II]$ or a solid torus with two toroidal holes inside with one orbit d in the canonical region of the identification, belonging to class $[III]$.

- If an attractive fat handle *hu* is identified with a repulsive fat handle *hu*, the flow$II\left(II\left(h,h\right),h\right)=h\cdot h\cdot u\cdot u$ is obtained (see Figure 7). In this case, one heteroclinic trajectory appears. If a repulsive (attractive) orbit is removed after the identification the resulting attractive (repulsive) fat handle *dhuu* is a thick torus and it belongs to class $[I]$.
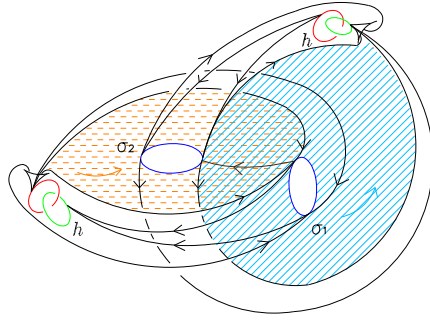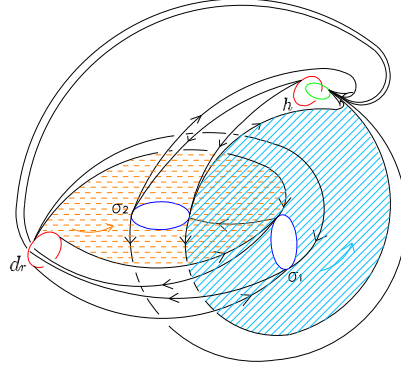

Figure 7: Flow $II\left(II\left(h,h\right),h\right)=h\cdot h\cdot u\cdot u$.

- If an attractive (repulsive) fat handle *hu* is identified with a repulsive (attractive) fat handle *du*, the resulting flow is $II\left(III\left(h,h\right),h\right)=h\cdot d\cdot u\cdot u$ and one heteroclinic trajectory appears (see Figure 8). If a repulsive (attractive) orbit is removed after the identification the resulting attractive (repulsive) fat handle is either a solid torus with one Hopf hole inside, *huu*, belonging to class $[II]$, or a thick torus with one toroidal hole inside, *dduu*, belonging to class $[I]$.

- Finally, if an attractive fat handle *du* is identified with a repulsive fat handle *du*, we obtain the flow $III\left(III\left(h,h\right),h\right)=d\cdot d\cdot u\cdot u$ (see Figure 9). If an attractive (repulsive) orbit is removed after the identification the resulting repulsive (attractive) fat handle *duu* is a solid torus with one orbit d in the canonical region of the identification; so, it belongs to class $[III]$.

□

Let us remark that fat handles of class $[I]$ are of the form ldu^2,where d is in the core of the fat handle and l represent the other orbits. Similarly, fat handles of class $[II]$ are of the form lhu^2,where h is in the canonical region of the identification and fat handles of class $[III]$ are of the form lu^2,where l contains only one d in the region of identification.

Figure 8: Flow $II\left(III\left(h,h\right),h\right)=h\cdot d\cdot u\cdot u$
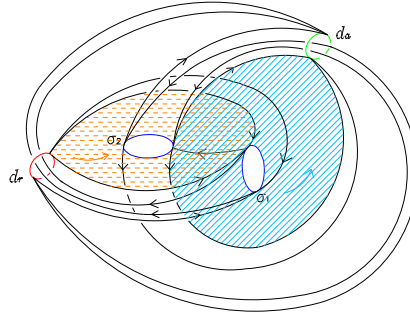
Figure 9: Identification of two fat handles du.

Along the proof of the previous proposition we have obtained all the flows φ with two unlinked saddles. Therefore,

Proposition 3.2 *A flow* $\varphi\in\mathcal{F}_A\left(S^3\right)$ *with two saddle orbits can be obtained by identifying one repulsive and one attractive basic fat handles along their boundaries.*

Proof. From Proposition 3.1 we have that:

The flow $I\left(I\left(h,h\right),h\right)=h\cdot h\cdot h\cdot u\cdot u$ is obtained by identifying the fat handles hdu and hdu along their boundaries.

The flow $I\left(II\left(h,h\right),h\right)=h\cdot h\cdot d\cdot u\cdot u$ is obtained by identifying the fat handles hdu and ddu along their boundaries.

The flow $I\left(III\left(h,h\right),h\right)=h\cdot d\cdot d\cdot u\cdot u$ is obtained by identifying the fat handles hdu and du along their boundaries.

The flow $II\left(II\left(h,h\right),h\right)=h\cdot d\cdot d\cdot u\cdot u$ is obtained by identifying the fat handles ddu and ddu along their boundaries.

The flow $II\left(II\left(h,h\right),h\right)=h\cdot h\cdot u\cdot u$ is obtained by identifying the fat handles hu and hu along their boundaries.

The flow $II\left(III\left(h,h\right),h\right)=d\cdot d\cdot d\cdot u\cdot u$ is obtained by identifying the fat handles ddu and du along their boundaries.

The flow $II\left(III\left(h,h\right),h\right)=h\cdot d\cdot u\cdot u$ is obtained by identifying the fat handles hu and du along their boundaries.

The flow $III\left(III\left(h,h\right),h\right)=d\cdot d\cdot u\cdot u$ is obtained by identifying the fat handles du and du along their boundaries. $\square$

Let us recall that there are non-equivalent flows characterized by the same link of periodic orbits.

As we see in the following propositions, the fat handles with any number of unlinked saddle orbits are in the same classes previously defined and the flows $\mathcal{F}_A\left(S^3\right)$ can be obtained by replacing successively one attractive (repulsive) orbit by the corresponding basic fat handle, taking into account the restrictions in the proof of Proposition 3.1: fat handles belonging to class $[I]$ can not be identified with fat handles belonging to class $[II]$, i.e., one component of a Hopf link can not be replaced by a fat handle belonging to class $[II]$.

Proposition 3.3 *For $\mathcal{F}_A\left(S^3\right)-$flows, a fat handle with n saddle orbits belongs to class $[I],[II]$ or $[III]$.*

Proof. Let us show this result by means of an induction process, by identifying iterated fat handles with the basic ones.

We know that the first identification leads to fat handles with $n=2$ saddles and they belong to class $[I]$, $[II]$ or $[III]$. Let us suppose that it holds for fat handles with $n-1$ saddles and let us prove it for n.

Let $l\cdot u^{n-1}$ denote the link corresponding to a flow φ with $n-1$ saddle orbits. By assumption, after removing one attractive or repulsive orbit k, the corresponding fat handle is of class $[I],[II]$ or $[III]$, depending on the removed orbit k. According to the previous notation, we denote them as $\left(l-h\right)du^{n-1}$, $\left(l-h-k\right)hu^{n-1}$ or $\left(l-k\right)u^{n-1}$, respectively.

Now, we make the identifications with the different basic fat handles and then, we remove an attractive or repulsive periodic orbit in order to obtain the fat handles with n saddles. As we identify repulsive with attractive fat handles, the removed orbit is located in the canonical region where the identification is made; the other canonical regions do not change.

- Let us suppose that the repulsive fat handle with $(n-1)$ saddle orbits belongs to class $[I]$. It means that it is a thick torus with a repulsive orbit filling the essential hole and all the manifolds of the saddles go outwards the torus in an inessential way. Let us denote it $\left(l-h\right)du^{n-1}$, where d fills the core of the thick torus. If this fat handle is identified with an attractive fat handle hdu, the attractive orbit d of this fat handle forms a Hopf link with the orbit in the core of the repulsive fat handle and the flow $l_2=l\cdot h\cdot u^n=l_1\cdot h\cdot u=I(l_1,h)$ is obtained.

If a repulsive (attractive) orbit belonging to a Hopf link is removed after the identification the resulting attractive (repulsive) fat handle is also a thick torus and it belongs to class $[I]$.

- Let us suppose that the attractive fat handle $(l - h) \, du^{n-1}$ of class $[I]$ is identified with a repulsive fat handle ddu of class $[I]$. The two orbits that are in the core of the thick tori become a Hopf link after identifying them along their boundaries and the resulting flow is $l_2 = l \cdot d \cdot u^n = II \, (l_1, h)$, where d is an attractive or repulsive orbit filling a non-essential toroidal hole. If it is identified with the basic fat handle du (class $[III]$), the resulting flow is $l_2 = (l - h) \cdot d \cdot d \cdot u^n = (l - k) \cdot d \cdot u^n$, where each d is an attractive or repulsive orbit filling a non-essential toroidal hole. As in the previous proposition, depending on the position of the removed orbit, the resulting fat handle is in class $[I]$, $[II]$ or $[III]$.

- An attractive fat handle of class $[I]$ can not be identified with a repulsive fat handle hu because a bitorus appears.

- An attractive fat handle of type $[II]$, $(l - h - k) \, hu^{n-1}$ can be identified with a repulsive fat handle hu. The flow corresponds to $(l - k) \cdot h \cdot u^n$ and one heteroclinic trajectory appears in the canonical region where the identification has been made; this canonical region becomes a 3-ball. After removing one orbit belonging to a Hopf link, the resulting fat handle is in class $[I]$.

- If an attractive fat handle of type $[II]$, $(l - h - k) \, hu^{n-1}$ is identified with a repulsive fat handle du, the resulting flow corresponds to $(l - k) \cdot d \cdot u^n$ and one heteroclinic trajectory appears. If the repulsive (attractive) orbit removed is in the canonical region where the identification has been made, the fat handle is a solid torus belonging to class $[II]$.

- Finally, if an attractive fat handle of type $[III]$, $(l - k) \, u^{n-1}$, is identified with a repulsive fat handle du, the resulting flow corresponds to $(l - k) \cdot d \cdot u^n$ and one heteroclinic trajectory appears. If the repulsive (attractive) orbit removed is in the canonical region where the identification has been made, the fat handle is a solid torus belonging to class $[III]$. $\square$

Let us emphasize that, along the proof, we have obtained all the flows φ with n unlinked saddles. Therefore,

Theorem 3.1 *A flow $\varphi \in \mathcal{F}_A \left(S^3\right)$ with n saddle orbits can be obtained by identifying fat handles along their boundaries.*

Proof. Following the results of Proposition 3.3, if $l \cdot u^{n-1}$ is a flow with $n - 1$ saddle orbits, the corresponding fat handle obtained by removing one attractive (or repulsive) orbit is one of the defined in the previous proposition: $(l - h) \, du^{n-1}$, $(l - h - k) \, hu^{n-1}$ and $(l - k) \, u^{n-1}$ if it is of class $[I]$, $[II]$ or $[III]$, respectively.

Iterating one more Wada operation, we have: $I\left(lu^{n-1},h\right)=l{\cdot}h{\cdot}u^n$, $II\left(lu^{n-1},h\right)=l\cdot d\cdot u^n$ or $II\left(lu^{n-1},h\right)=(l-k)\cdot h\cdot u^n$ and $III\left(lu^{n-1},h\right)=(l-k)\cdot d\cdot u^n$, that are the flows obtained in the Proposition 3.3. $\square$

Let us notice that one heteroclinic trajectory appears when fat handles that are solid tori are identified along their boundaries. As we see in the following section, these type of trajectories impose an order in the flow.

4   Order in the NMS flows

The round handle decomposition of the manifold S^3 for a given NMS flow is unique except for commutativity of some of the attachments involved. There exist attachments that can lead to different flows depending on the order they are made. The appearance of heteroclinic trajectories connecting saddle orbits implies non commutativity of the operations involved establishing an order in the round handle decomposition. As we have seen in the previous section, heteroclinic trajectories connecting saddles appear with the identification of fat handles of classes $[II]$ and $[III]$.

Proposition 4.1 *For $\varphi\in\mathcal{F}_A(S^3)$, the heteroclinic trajectories induce an order in the flow.*

Proof. Heteroclinic trajectories appear when fat round handles belonging to class $[II]$ or $[III]$ are identified along their boundaries. The unstable manifold of a saddle orbit u_i in the repulsive fat handle intersects transversely the stable manifold of the saddle orbit u_{i+1} in the attractive fat handle (see Figures 10 and 11).

Then, the saddles are ordered, $u_i<u_{i+1}$, and this order of the orbits implies a partial order in the filtration of the flow.

$\square$

For the particular case of a flow where only operation III is implied the order is total.

Theorem 4.1 *Let $\mathcal{F}_3(S^3)$ be the set of NMS flows on S^3 coming only from operation III. Then, the set of orbits of $\varphi\in\mathcal{F}_3(S^3)$ is totally ordered.*

Proof. A flow $\varphi\in\mathcal{F}_3(S^3)$ is of the form $d_a\cdot d_r\cdot u\cdot...\cdot u$, where u denotes an unknot corresponding to a saddle orbit. The flow goes from the repulsive orbit d_r to the attractive orbit d_a.

Consider the flow $III\left(III\left(h,h\right),h\right)=d_a\cdot d_r\cdot u\cdot u$ (see Figure 9). One unstable manifold of the first saddle orbit u_1 cuts transversely the stable manifold of the second saddle orbit u_2 and one heteroclinic trajectory appears from u_1 to u_2 (see [3]). We write it as $u_1<u_2$.

Each time operation III is applied, two fat handles that are solid tori are identified along their boundaries; so, the unstable manifold of the saddle u_i cuts transversely the stable manifold of the new saddle u_{i+1} and one heteroclinic trajectory appears connecting u_i and u_{i+1}; so, $u_i<u_{i+1}$ (see Figure 12).
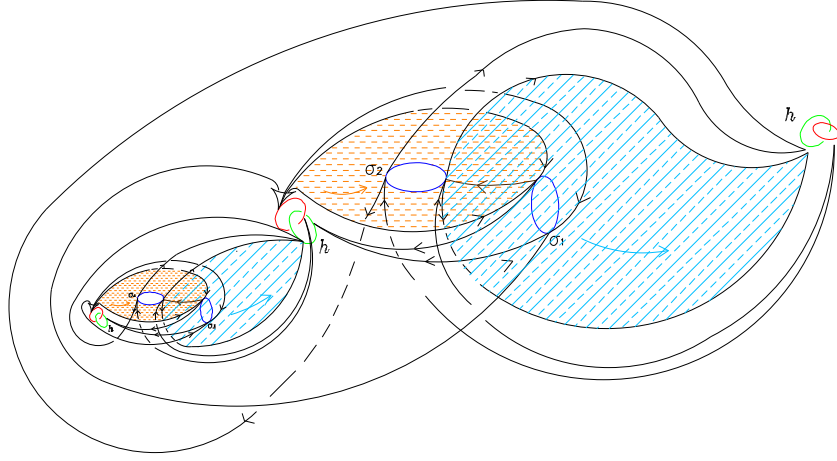
Figure 10: Flow $II(II(II(II(h,h),h),h),h) = h \cdot h \cdot h \cdot u \cdot u \cdot u \cdot u$. There is a partial order $\sigma_1 < \sigma_2$, $\sigma_3 < \sigma_4$.
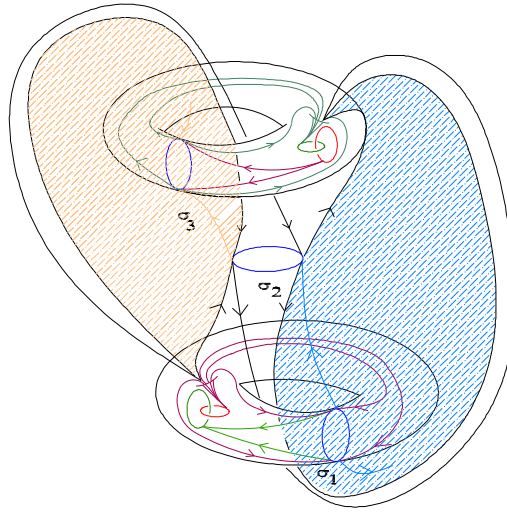

Figure 11: Flow $II(II(III(h,h),h),h) = h \cdot h \cdot u \cdot u \cdot u$. There is an order $\sigma_1 < \sigma_2 < \sigma_3$.

Therefore, for a flow $d_a \cdot d_r \cdot u \cdot ... \cdot u$ we can write

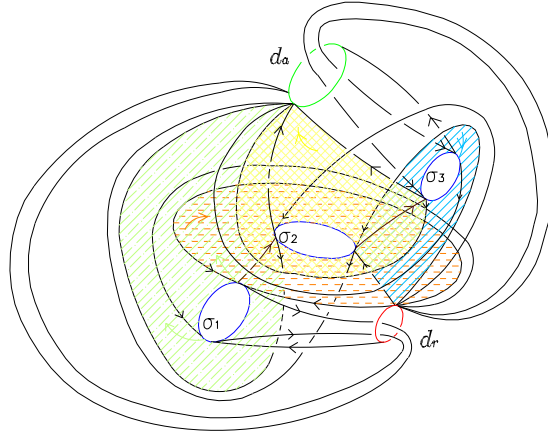$$d_r < u_1 < ... < u_i < u_{i+1} < ... < u_n < d_a. \tag{4.2}$$

Figure 12: Flow $III(III(III(h,h),h),h)$. The order is $d_r < u_1 < u_2 < u_3 < d_a$.

□

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