2019-03

Modelling optimal control of harvested prey predator system incorporating a prey refuge

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NM-AIST

https://doi.org/10.58694/20.500.12479/2469

MODELLING OPTIMAL CONTROL OF HARVESTED PREY PREDATOR SYSTEM INCORPORATING A PREY REFUGE

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A Dissertation Submitted in Partial Fulfilment of the Requirements for the Degree of Master's in Mathematical and Computer Sciences and Engineering of the Nelson Mandela African Institution of Science and Technology

Arusha, Tanzania

March, 2019

ABSTRACT

prey-predator interactions have been an important role in the dynamics of species populations. This work presents mathematical model for Modelling Optimal control of Harvested prey-predator system incorporating a prey refuge using deterministic differential equations. This study, develops two harvested prey-predator species, in which both species are affected by over-harvesting, furthermore the predator is affected by prey refuge. The intention is to investigate the impacts of over-harvesting to prey-predator species and suggest control strategies to alleviate the problem of loss of prey-predator species. The analysis of stability of equilibrium points were done by Jacobian matrix, Global stability analysis is done using Lyapunov function while the analysis of optimal control was done using Pontrygians maximum principle (PMP) and Hamiltonian principle. The control strategy suggested is the creation of reserve areas with restrictions of harvesting. The results obtained from theoretical and numerical analysis of the prey-predator with harvesting without control strategies showed that, harvesting affect the prey-predator species negatively. However, the results obtained from numerical analysis of the prey-predator model with control strategies showed that, the use of control strategy encourage the survival of both species

DECLARATION

I, Mfano Charles do hereby declare to the Senate of Nelson Mandela African Institution of Science and Technology that this dissertation is my own original work and that it has neither been submitted nor being concurrently submitted for degree award in any other institution.

______________________________ ______________

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The above declaration is confirmed

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(Supervisor 1)

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Dr. Thadei Damas Sagamiko **Date**
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CERTIFICATION

The undersigned certify that they have read and found the dissertation acceptable by the Nelson Mandela African Institution of Science and Technology.

Prof. Dmitry Kuznetsov **Date**

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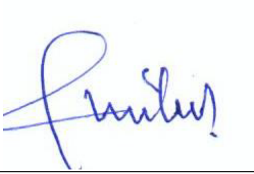

Dr. Thadei Damas Sagamiko **Date**

(Supervisor 2)

DEDICATION

Mr. and Mrs. Charles Petro Magudule

To you my beloved parents. I dedicate this dissertation for showing me the Light of the World.

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LIST OF ABBREVIATIONS AND SYMBOLS

PMP	Prontrygian's maximum principle.
ODE	Odinary differential equation.
FBSM	Forward Backward Sweep Method.
Rk4	Runge kutta fourth order.
YRS	Years.
AfDB	African Development Bank.
CoCSE	Communication and Computational Science and Engineering.
NM-AIST	Nelson Mandela African Institution of science and Technology.
UDSM	University of Dar-es-salaam
H	Halmiltonian
J	Objective functional

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CHAPTER ONE

INTRODUCTION

1.1 General Introduction

In ecological World there is a complex interaction between and amongst the species that constitute the ecological environment (Kar, 2006). However these ecological competitions come in various forms, the most common being competition for food and prey-predator relations (Begon *et al.*, 2006). Prey-predator is described as an interaction between the prey and predator in an Ecosystem (Kar, 2005*a*; Mahapatra and Santra, 2016*a*). The process of prey and predator to interact with another is called predation. Thus prey-predator system is one of the ways of species interactions (Beketov and Liess, 2006). Predation is most commonly considered to be an interaction where an organism (predator) consumes all or part of another living organism (its prey) thereby benefiting itself, but reducing the growth of the prey. For examples, lynx prey upon hares, cheetahs and wild dogs kill gazelles. Competition is a negative interaction that occurs when organisms of different species use the same resource(s) at the same time and the growth rate of each species is decreased (Kar, 2005*b*).

Generic models are models which are used by many researchers in the field of ecosystem such as those of Mahapatra and Santra (2016*a*) and Wang and Wu (2008). Simple models such as the Lotka Volterra model are not able to tell us what is going on in the majority of cases because of the complexity of interactions in different ecosystems. However models are usually constructed to explain the nature and interactions of different species (Chant, 1961; Ashine, 2017). Therefore, in this study we developed a generic mathematical model with Holling type II functional response of harvested prey-predator incorporating a prey refuge which can be applied in either of the ecosystems.

In dynamical system a definite activity by individual area causes severe destruction to the ecosystem of that area. If such activity is unavoidable then the prevailing authority of the area should plan a regular policy which would keep the destruction of the ecosystems minimal (Kar, 2006). One of such activity is harvesting, which has a strong impact on the dynamic evolution of a population subjected to it. It has been observed that over exploitation and over harvesting of population species are commonly practiced in fishing, forestry, and wildlife management which is done for the purpose of economic progress (Matsuda and Katsukawa, 2002). It is also agreed that biological species of preypredator system is harvested unscientifically and exported with the aim of positive economic profit which regularly decreases the resources and eventually the ecosystems collapse (Kar and Ghosh, 2010). Generally, Kar (2006) argued that using optimal harvesting efforts as controls can help discontinuities cyclic behaviour of the system of the prey-predator which may results to a required state of the ecosystems.

A refuge in biology and ecology is defined as a concept which revolves around the escape of an organism from predation (Sih, 1987; Kar, 2005*b*; Mahapatra and Santra, 2016*b*). Prey refuge in Game reserve and National parks is mostly practiced by Wildebeest and Cape buffaloes that help them to protect from predator attack, hence reduces their predation rate. Therefore, the addition of a small prey refuge stabilizes prey-predator interactions, the addition of a large refuge leads to almost changeability (i.e. random like prey population outbreak) (Das *et al.*, 2013*a*).

In population dynamics, a functional response of a prey predator to the prey density refers to the relationship between an individual's rate of consumption and food density (McNair, 1987). Thus, Holling (1957) suggested three types of functional responses namely as Holling type I, II and III. In Holling type I response, the feeding rate increases linearly with the prey availability then abruptly levels off. In this case, the capture rate is directly proportional (linear) to prey density, that is the predators feeding increases with increase in prey density. Holling type II functional response the rate of the prey consumption by a predator rises as prey density increases but eventually levels off plateau (or asymptote) at which the rate of consumptions remain constant regardless of the increase in prey density (When predator are saturated) and Holling type III functional response is when the predators feeding rate is at low prey population density (Dawes and Souza, 2013).

Over-harvesting of wildlife resources has been a challenging problem in most area of african therefore a better understanding of the nature would improve the way in which it is managed (Wilfred and MacColl, 2014). Therefore this study employed Holling Type II functional response in which the rate of consumption of predator depends on the availability of prey density as the only source of food.

1.2 Research Problem

This study is motivated by the fact that harvesting of prey-predator is needed in the ecosystem. This activity should not exceed the intrinsic growth rate of prey and predator species for the sustainability of ecosystem but it should be at the rate which the prey and predator will survive in the system (Das *et al.*, 2013*b*). It is argued that harvesting prey predator species incorporating prey refuge should be done by good policy planning (Kar and Ghosh, 2012). Several studies have been conducted on optimal control strategies and management policies for sustainability of the ecological species such as reported by Kar (2006), Xiao and Ruan (2001), Cai *et al.* (2008), Mayengo *et al.* (2014), Sagamiko *et al.* (2015). It is agreed that over-harvesting of prey -predator species has contributed to the loss of prey-predator populations. Therefore this study applied optimal control theory for harvesting prey-predator system incorporating prey refuge.

1.3 Research Objectives

1.3.1 General objective

To develop and analyze a model of harvested prey-predator system incorporating a prey refuge with optimal control strategies.

1.3.2 Specific objectives

Specific objectives were:

(i) To develop a harvested prey-predator model incorporating a prey refuge.

(ii) To analyze the effect of harvesting prey-predator system.

(iii) To modify model in (i) to include control variables.

(iv) To determine the impact of optimal control strategies on harvested prey-predator system with refuge.

1.4 Research Questions

The study was guided by the following questions:

(i) How can a harvested prey-predator Model be formulated?

(ii) How harvesting imposes negative effects on the prey-predator ecosystem?

(iii) How harvested prey-predator model incorporating a prey refuge with control variables can be developed?

(iv) Which dependent control variable maximizes the prey-predator species and minimizes the costs of control?

1.5 Significance of the Study

Significant of the study will:

(i) Increased knowledge of optimal control strategies on the prey predator system for maintaining the ecological balance.

(ii) Help policy planners on how to control over exploitation of prey-predator ecosystem for its sustainability.

(iii) The study will help people to have a better insight and understanding on controlling of over-harvesting of natural resources

(iv) Study provides a platform for detailed Model of Optimal control of harvested prey-predator system incorporating a prey refuge

(v) Act as basis for other researches

CHAPTER TWO

LITERATURE REVIEW

2.1 Introduction

This chapter analyzes different papers with a limited area in the field of prey-predator systems, extending over mathematical models that include over-harvesting of prey -predator species including the beahviour prey refuge as well as control strategies of over-haversting of prey-predator species. The review finds that there are number of limitations to the current research in prey-predator system . The narrow scope of prey refuge and control strategies. The effective inclusion of prey refuge objectives in models with optimal control stratergies. Furthermore, there are significant gaps in sensitivity analysis of models limiting the general applicability of the models. The chapter concludes with promising new avenues of research that demand effective inclusion of prey refuge and optimal control strategies.

2.2 Overview of mathematical modelling on harvested prey-predator incorporating a prey refuge

Different studies have been done on control strategies and management policy of over exploitation to maintain the prey predator species in the ecosystem (Pal and Mahapatra, 2014). This study intends to provide optimal control strategies which can be used in harvested prey predator incorporating a prey refuge.

Kar and Ghosh (2012) worked on suitability and optimal control of an exploited prey-predator system through provision of alternative food to a predator. They developed a two species of prey and predator model in which the predator was partially coupled with alternative prey aiming at studying the consequences of providing additional food to the predator as well as the effects of harvesting applied to both species, they observed that the provisional of alternative food to the predator was not always beneficial to the system however they used the effort pair as control parameters.

Sagamiko *et al.* (2015) proposed an optimal control of a threatened wildebeest- Lion prey-predator in the Serengeti ecosystem. In their studies two threatened species prey-predator model were used in which the prey was Wildebeest and the predator was Lion. They considered that the system is threatened by poaching, drought and retaliatory killing. However on their optimal control theory to the three threats to investigate optimal strategies for controlling the threats in the system was applied, in their study, they observed that the best results are archived when all controls are used at the same time. Whoever their studies did not consider the prey refuge as one of the control effort on the harvested prey predator system.

Kar (2006) addressed the modelling and analysis of harvested prey predator system incorporating a prey refuge, on their studies two species prey and predator were used and they concluded that using harvesting efforts as control is possible to break the cyclic behaviour of the system and drive the problem to a required state.

Kellner *et al.* (2010) explained the optimal harvesting rates for predator and prey species by using a multi-species bio economic model for a Caribbean reef community. They asked how more comprehensive optimization differs from traditional simple species approaches. They also identified trade-offs when the objective of the manager includes non-fishing values. Also found out that optimal solution when accounting for non-fishing values can include temporary or permanent fishing moratoriums in contrast to continuous fishing at low levels when only fishing products are considered. It was shown that the greatest gain from the ecosystem based fishery management not from improved estimation of the trophic coupling, but from reforming the social and economic management of individual fish stocks and by explicitly incorporating a broader set of values into management decision.

Chakraborty *et al.* (2011*a*) explored optimal control of harvesting and bifurcation of a prey-predator model with stage structure. They described a prey-predator model with stage structure for prey. The adult prey and predator populations are harvested in the system and observed that the singularity included bifurcation phenomenon appeared when the variation of the economic interest of harvesting was taken into account. They also incorporated the state feedback controller stabilizer the model system in case whenever there is economic interest. They used harvesting the optimal utilization of the resources, sustainability properties of the stock and resources rent earned from the resources.

Despite the fact that these literatures have insight and brilliant ideas on prey-predator system incorporating prey refuge, they have one thing in common, most of them have considered the combined effects of pollution, harvest and interaction (prey-predator). This study investigated the optimal control of harvested prey predator ecosystem incorporating the prey refuge which will be a generic model to be applied in either any other related ecosystem system which act significantly on the prey-predator ecosystem.

CHAPTER THREE

MATERIALS AND METHODS

3.1 Introduction

The prey-predator system is well described by prey-predator models. However it is known that predator population depends on the prey species for their survival thus the lower the prey species the lower the survival of the predator population.

Therefore, the predator population is affected by the changes in prey population in a complex prey-predator relationship (Chakraborty *et al.*, 2011*a*; Kim *et al.*, 2011).

Due to predation, prey species have involved with number of survival strategies on reducing their predation risk using the different way such as gregariousness, fight, camouflage and fighting back among others strategy is the use of prey refuge where prey species are protected from predation (Chakraborty *et al.*, 2011*a*).

Worthwhile the problem of prey predator interaction under constant rate of harvesting of both species has become common in the community due to economic progress and ecological balances. The study of the consequences of over-harvesting of both species prey and predator including hiding behaviour of prey on the dynamics of prey-predator interaction can be documented as a major issue in mathematical Ecology and theoretical Ecology however the population of the prey and predators species population growth is described in two famous of prey-predator models namely as exponential growth model and logistic growth model.

Meanwhile this study is making use of logistic growth model. This is due the reason that, the logistic growth is realistic to the study as compared to exponential growth as it is clear that, the environment imposes some limitations to the growth rate of the species. The limitations are such as diseases, availability of resources and so forth. Therefore, the current study investigates the impacts of over-harvesting on the prey-predator system incorporating a prey refuge.

3.2 Design and Methods

The study involved formulation of harvested prey-predator system incorporating a prey refuge as in equation 3.1, secondly model 3.1 was modified by including control variable u_1 as in the equation 3.31.

3.3 Methods on formulation of the basic model

We adopted the standard mathematical ecological process using deterministic model for harvested prey-predator models. The complete dynamical system of these differential equations is reffered to equations (3.1) and (3.31).

3.4 Methods on Model Analysis and Simulation

The system was analysed using Lemma theorem, Jacobian Matrix, Lypunov function, Prontrygian's maximum principle (PMP) and Halmitonian principle with the help of Maple SOFTWARE. Numerical simulation was dane using Rungekutha fourth order MATLAB and Foward-Backward Sweep Method ($FBSM$).

3.5 System control and Optimal Control Theory

Over -harvesting of prey-predator species in the ecosystem has been a challenging problem in management of natural resources where control stratergie to alleviate the problem arises .

Optimal control theory is well developed as branch of mathematics and engineering that identifies optimal control policies for dynamics systems. The theory has developed rapidly since the first paper by Pontryagin and collaborators in the late 1950 (Aniţa *et al.*, 2011). The word control has several meaning depending on the circumstances it is used. First, controlling a system can be defined as "determine the behaviour or supervise the running of a ceratin system. According to Sagamiko *et al.* (2015), Cara.F and Iriondo.F (2003) to control means to put things in order to guarantee that the system behaves as desired. In our context, the meaning holds, that if we have a prey-predator system which is affected by over-harvesting may lead the system to extinction and we need to act upon the situation to ensure that the system behaves as desired. It is in such situations when concepts like the optimal control theory" comes into play. The description of optimal control theory is as follows;

Consider dynamical system given by $G(x(t),u(t),t)$, where $x(t)$ is the state variable at a time t and $u(t)$ is another variable which represents the amount of effort used in applying a specific control to reduce the intensity of over-harvesting. The $u(t)$ is controlled to maximize the prey-predator species at a minimum costs (Sagamiko *et al.*, 2015; Kinene *et al.*, 2015). The model developed in this dessertation are continous dynamical systems where the state is governed by a set of ordinary differential equations (ODES).

Formulation of optimal control in this dessertation follows the form;

$$J(u_i) = Max[G(x(t),u(t),t) + \int_{t_0}^{t_1} F(x_i(t),u(t),t)dt], \tag{3.1}$$

subject to

$$\frac{dx_i}{dt} = f_i(x_i(t),u_i(t),t), \tag{3.2}$$

$$x_i \geq 0,$$

$$x(t_1) = free,$$

$$t\varepsilon(t_0,t_1),$$

$$i = 1,2.$$

The equation (1.1) is called the obejctive(or cost) functional, $x(t)$ is the state, $u(t)$ is the time dependent control. The control that maximizes the objective functional (1.1) is defined by $u^*(t)$ and is referred as optimal control. The result of substuting $u^*(t)$ into the state ODE in equation (1.2) is the optimal set $x^*(t)$, thus $(u^*(t),x^*(t))$ is the optimal pair.

3.6 Pontragin's Maximum Principle

In 1956, lev Pontragin's and his collaborators developed necessary conditions for optimal control theory and proved what is now called pontragin's maximum principle (PMP) (Boltyanskiy *et al.*, 1962). The key idea was introducing the adjoint function to attach the ODE to the objctive functional. The (PMP) provides necessary conditions that an optimal control and the corresponding state must satsify (Sagamiko *et al.*, 2015).

Theorem 1.1 Pontrygin's maximum principle

If u^* and x^* are optimal for the problem defined equations $(1.1)-(1.2)$ then there exist and adjoint varible $\lambda(t)$ such that $H(t,x^*(t),\lambda(t)) \leq H(t,x^*(t),u^*(t),\lambda(t))$ at each time for all u with values U_i where the Hamiltonian H is defined by $H(t,x(t),u(t),\lambda(t)) = f(t,x(t),u(t)) + \lambda(t)g(t,x(t),u(t))$, and the adjoint function defined by the ODE;

$$\frac{d\lambda}{dt} = -\frac{\partial H}{\partial x}, \ \lambda(T) \ = \frac{dG}{dx}(x(t_f)), \tag{3.3}$$

Note that

The final time condition on the adjoint variables is called the transversality condition. Potragin's maximum principle converts the problem maximize the objective functional subject to the state ODE and initial problem of optimizing the Hamiltonian pointwise.

From the Theory above, we introduce one corresponding adjoint variable.If we have $i = 1,2$. Then;

$$\begin{aligned} x_1(t) &= g_1(t,x_1(t),....x_n(t),u(t)), \\ x_2(t) &= g_2(t,x_1(t),....x_n(t),u(t)). \end{aligned} \tag{3.4}$$

Then, we introduce adjoints $\lambda_1(t)$ .....$\lambda_2(t)$

suppose the obejctive functional becomes ;

$$MaxJ = G(t,x_1(t),....x_n(t_f)) - \int_0^{t_f} f(t,x_1(t),u_t), \tag{3.5}$$

Then the Hamiltanian becomes;

$$\begin{aligned} H(t,x_t......x_n(t),u(t),\lambda(t) = f(t,x_1(t).....x_n(t),u(t)) + \lambda_1(t)g_1(t,x_1(t),....x_n(t),u(t)) \\ +\lambda_2(t)g_2(t,x_1(t),....x_n(t),u(t)). \end{aligned} \tag{3.6}$$

and the adjoint variable are defined by the ODE and transversality conditions

$$\begin{aligned}\frac{d\lambda_1}{dt} =& \frac{\partial H}{\partial x_1}, \lambda_1(T) = \frac{dG}{dx_1}(x_1(T)....x_(n)(T)) \\ \frac{d\lambda_n}{dt} =& \frac{\partial H}{\partial x_n}, \lambda_n(T) = \frac{dG}{dx_n}(x_1(T)....x_(n)(T)).\end{aligned} \tag{3.7}$$

3.7 The runge-kutta Method

Let an initial value problem described by

$$\frac{dy}{dt} = f(t,y), y(t_o) = y_0, \tag{3.8}$$

picking a step size $h > 0$ we define

$$y_{n+1} = y_n + \frac{h}{6}(K_1 + 2K_2 + 2K_3 + K_4) \tag{3.9}$$

for n = 0 ,1, 2, 3.... Where;

$$\begin{aligned} k_1 =& f(t_n, y_n), \\ K_2 =& f(t_n + \frac{h}{2}, y_n + \frac{h}{2}k_1), \\ K_3 =& f(t_n + \frac{h}{2}, y_n + \frac{h}{2}k_2), \\ k_4 =& f(t_n + h, y_n + k_3). \end{aligned} \tag{3.10}$$

3.8 Formulation of the harvested prey-predator model incorporating a prey refuge

In this section, we consider two different populations, the prey and predator interaction incorporating a prey refuge. The ecological setup considers the following assumptions.

(i) Both prey and predator are continuously harvested

(ii) Predator depends on the prey as the source food. Thus, in the absence of prey, predator population goes to extinction

(iii) We also assume that there is a refuge habitant where prey species are secured from predation and non -refuge habitat in which the prey species are visible to predation

(iv) In the absence of harvesting on both species, prey is assumed to grow logistically to the carrying capacity

(v) The rate of increase of the predators depends on the amount of biomass predators converts as food

Then from the above assumptions, we assume $x(t)$ and $y(t)$ to represent the population density of prey and predator respectively at time t. With this assumption we use Holling type (II) functional response to formulate the prey-preadator Model as follows (Denny, 2014);

$$\begin{aligned} \frac{dx}{dt} &= r(1-\frac{x}{k})x - \frac{\alpha(1-p)xy}{1+a(1-p)x} - q_1h_1x, \\ \frac{dy}{dt} &= -\mu y + \frac{b\,\alpha(1-p)xy}{1+a(1-p)x} - q_2h_2y. \end{aligned} \tag{3.11}$$

Where $x(t) > 0$, $y(t) > 0$, α, k, μ, a, b are all positive constants and r is the intrinsic growth rate of the prey. k is the environment carrying capacity of the prey in the absence of the predator and harvesting. The term $\frac{\alpha x}{1+ax}$ denotes the functional response of the predator which is a Holling type II response functional of the predator, μ is the death rate of the predator, $\frac{\alpha}{a}$ is the maximum number that can be eaten by each predator in unit time, b is the predation convention factor (biomass) denoting the number of newly born predators for each captured prey and q_1 and q_2 are catchability coefficient of the prey and predator respectively. p is the proportion of prey population not exposed to predation, that it protects px of the prey and leaves $(1-p)x$ of the prey available to predation. Note that $p \in [0,1]$

3.9 Model analysis

3.9.1 Boundedness of the system

The solution of the prey-predator model developed in (3.1) represent the populations of living individuals and they have their ecological meaning that is to say they must be positive and bounded.

Lemma: All the solutions of the system (3.1) which start with $\mathbf{R}^{2+}$ are uniformly bounded.

Proof: To prove the theorem, we define a function

$$W(t) = x(t) + \frac{\alpha}{\alpha\, b}\, y(t). \tag{3.12}$$

which follows as

$$W(t) = x(t) + \frac{1}{b}\, y(t). \tag{3.13}$$

Where $W(t)$ represents the total population of the prey and predator species

$$\frac{dW}{dt} = \frac{dx}{dt} + \frac{1}{b}\,\frac{dy}{dt}. \tag{3.14}$$

Then substitute equations (3.1) into equation (3.4)

$$\frac{dW}{dt} = r(1-\frac{x}{k})x - \frac{\alpha(1-p)xy}{1+a(1-p)x} - q_1h_1x + \frac{1}{b}(-\mu y + \frac{\alpha b(1-p)xy}{1+a(1-p)x} - q_2h_2y). \tag{3.15}$$

Then equation (3.5) will be simplified as

$$\frac{dW}{dt} = r(1-\frac{1}{k})x - \frac{\alpha(1-p)xy}{1+a(1-p)x} - q_1h_1x + \frac{1}{b}(-\mu - q_2h_2)y + \frac{\alpha(1-p)xy}{1+a(1-p)x}.$$

Then all terms with interspecific competition are cancelled out

$$\frac{dW}{dt} = r(1-\frac{x}{k})x - q_1h_1x + \frac{1}{b}(-\mu - q_2h_2)y.$$

Also on simplification we have

$$\frac{dW}{dt} = rx - \frac{rx^2}{k} - q_1h_1x + \frac{1}{b}(-\mu - q_2h_2)y.$$

$$Let\ E_1 = q_1h_1\ and\ E_2 = q_2h_2.$$

Then we have the simplified equation as follows

$$\frac{dW}{dt} = (r-E_1)x - \frac{rx^2}{K} - \frac{1}{b}(\mu + E_2)y.$$

Let the arbitrary constant to be Ω then the equation above will be written as follows

$$\frac{dW}{dt} = (r - E_1)x - \frac{rx^2}{K} - \frac{1}{b}(\mu + E_2)y + \Omega W(t) - \Omega W(t).$$

Thus,

$$\frac{dW}{dt} + \Omega W(t) \leq (r - E_1)x - \frac{rx^2}{K} - \frac{1}{b}(\mu + E_2)y + \Omega(x(t) + \frac{1}{b}y(t)). \tag{3.16}$$

Using the concept of perfect square

$$\frac{dW}{dt} + \Omega W(t) \leq (r - E_1 + \Omega)x - \frac{rx^2}{K} - \frac{1}{b}(\mu + E_2 - \Omega)y.$$

Then it follows

$$\frac{dW}{dt} + \Omega W(t) \leq \frac{K}{4r}(r - E_1 - \Omega)^2 - \frac{r}{K}(x^2 - (r - E_1 + \Omega)\frac{K}{r}),$$
$$+K^2(\frac{r - E_1 + \Omega}{4r^2})^2)^2 - \frac{1}{b}(\mu + E_2 - \Omega)y.$$

Then using techniques of completing the square it follows

$$\frac{dW}{dt} + \Omega W(t) \leq \frac{K}{4r}(r - E_1 + \Omega)^2 - \frac{r}{K}(x - (r - E_1 + \Omega)\frac{K}{2r})^2 - \frac{1}{b}(\mu + E_2 + \Omega)y$$
$$\frac{K}{4r}(r - E_1 + \Omega)^2 = Max[\frac{r}{K}(x - (r - E_1 + \Omega)\frac{K}{2r})^2].$$

Also, by letting the $\frac{K}{4r}(r - E_1 + \Omega)^2 = m_1$.

Thus

$$\frac{dW}{dt} + \Omega W(t) \leq m_1. \tag{3.17}$$

solving equation (3.7) differential inequality using integrating factor $I = e^{\Omega t}$ yields

$$W(t)e^{\Omega t} \leq \frac{m_1}{\Omega} + Ce^{-\Omega t} \tag{3.18}$$

At $t = 0$ equation in (3.8) becomes

$$W(0) = \frac{m_1}{\Omega} + (W(0) - \frac{m_1}{\Omega})e^{-\Omega(0)} \tag{3.19}$$

As $t \longrightarrow \infty$ equation (3.8)

$$0 \leq W(t) \leq \frac{m_1}{\Omega}$$

Therefore $W(t)$ is bounded and from positivity of x and y it follows

$$0 \leq x(t) \leq \frac{m_1}{\Omega}.$$

and

$$0 \leq y(t) \leq \frac{m_1}{\Omega}.$$

3.9.2 Analysis of the stability of the equilibrium points

In this section, we establish conditions for existence of equilibrium points of the model equations (3.1) The system has at least four equilibrium points obtained by setting $\frac{dy}{dt} = 0$ and $\frac{dx}{dt} = 0$ by so doing we get the possible equilibrium points of the system as;

(i) $E_0(0,0)$ Is the extinction of both species , prey and predator

(ii) $E_1(x,0)$ Is the predator extinction

(iii) $E_2(0,y)$ prey extinction

(iv) $E_3(x,y)$ The coexistence or equilibrium point of the system

But $E_0 = (0,0)$ point is trivial. The existence of the rest of the fixed equilibrium points are described below

(i) The existance of $E_1(x^*,0)$ with $x^* > 0$

let $y = 0$ the system of equation reduces to

$$0 = r(1 - \frac{x^*}{K})x^* - q_1 h_1 x.$$

on simplifying we have

$$x^*(r - \frac{rx^*}{K} - q_1 h_1) = 0.$$

Thus

$$x^* = \frac{K(r - q_1 h_1)}{r}.$$

Therefore

$$E_1(x^*,0) = (\frac{K(r - q_1 h_1)}{r},0).$$

From the expression of x^* we observe that harvesting has negative impact on the prey growth hence affect the prey population density. However, for the predator free equilibrium E_1 $(x^*, 0)$ to exist $r - q_1 h_1 > 0$ which implies that $r > q_1 h_1$.

Therefore in absence of predators the intrinsic growth rate of prey population should be greater than harvesting rate. Hence increasing harvesting of prey species results into decreasing of predator which affects the survival of predator species, this is the fact to prove that predator depend on prey as their only source of food.

(ii) $E_2(0, y^*)$ for $y^* > 0$.

Let $x = 0$ the system of equation (3.1) reduces to $y * (-\mu - q_1 h_1) = 0$ from which we obtain $y^* = 0$ which implies

$$E_0(0,0) = E_2(0, y^*). \tag{3.20}$$

The results above imply that predator depend on prey as their only source of food. Thus, in absence of prey, predator population become extinct.

(iii) Co-existence of equilibrium point $E_3(x^*, y^*)$

We equate the system of equation (3.1) equals to zero that is to say $\frac{dy}{dt} = 0$ and $\frac{dx}{dt} = 0$ then the system reduces to the following system of equations;

$$r(1 - \frac{x}{K})x - \frac{\alpha(1-p)xy}{1 + a(1-p)x} - q_1 h_1 x = 0$$
$$-\mu y + \frac{b\alpha(1-p)xy}{1 + a(1-p)x} - q_2 h_2 y = 0.$$

Using MAPLE software, the co-existence point will be as;

$$x^* = \frac{\mu + H_2}{((\mu + H_2)\,a - \alpha\,b)\,(p-1)}.$$

$$y^* = -\frac{b}{((\mu + H_2)\,a - \alpha\,b)\,(p-1)}\left(-r\left(\frac{((\mu + H_2)\,a - \alpha\,b)\,(p-1)\,k_1 - \mu - H_2}{((\mu + H_2)\,a - \alpha\,b)\,(p-1)\,k_1}\right) + H_1\right).$$

$$For\ H_1 = q_1 h_1\ and\ H_2 = q_2 h_2.$$

Thus point of the co-existence point

$$[x^* = \frac{\mu + H_2}{((\mu + H_2)\,a - \alpha\,b)\,(p-1)}, y^* = \frac{br(((\mu + H_2)a - \alpha b)(p-1)K - (\mu + H_2))}{[((\mu + H_2)a - \alpha b)(p-1)]^2 K} - D]. \tag{3.21}$$

Where

$$D = \frac{bH_1}{((\mu + H_2) - \alpha b)(p-1)}.$$

From the expression of $E_3(x^*, y^*)$ we observe that predators death rate and harvesting affect the convention factor b (Predator biomass to the prey) of newly born predators for each captured prey negatively which in turn results into negative effects on predator population density.

However the co-existence equilibrium point (non-trivial equilibrium point ($E_3(x^*, y^*)$)) exist if $((\mu + H_2)a - \alpha b) > 0$ implying that $\frac{\alpha b}{a} < \mu + H_2$. Therefore, in the presence of both populations and birth rate of predators should be greater than the sum of death rate and harvesting of predator. Increasing Harvesting to predator population causes rapid decrease of predators which results in increasing of prey population density.

3.9.3 Stability analysis of the equilibrium points

The stability properties of the equilibrium points are analyzed by computing the Jacobian matrix and determining the eigenvalues of the Jacobian matrix of each fixed point $E_0(0,0)$, $E_1(x^*,0)$, $E_2(0,y^*)$ and $E_3(x^*,y^*)$.

Theorem;

The equilibrium points are asymptotically stable if the real parts of the eigenvalues of each Jacobian matrix are negative. From the system equation (3.1) the following Jacobian matrix be used to proof to the theorem:

$$J(E_i) = \begin{bmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y} \end{bmatrix}.$$

This will be described as follows;

$$J(E_i) = \begin{bmatrix} r\left(1 - \frac{x^*}{K}\right) - \frac{rx^*}{K} - \frac{\alpha(1-p)y^*}{1+a(1-p)x^*} + \frac{\alpha(1-p)^2 x^* y^* a}{(1+a(1-p)x^*)^2} & -\frac{\alpha(1-p)x^*}{1+a(1-p)x^*} \\ \frac{b\alpha(1-p)y^*}{1+a(1-p)x^*} - \frac{b\alpha(1-p)^2 x^* y^* a}{(1+a(1-p)x^*)^2} & -\mu + \frac{b\alpha(1-p)x^*}{1+a(1-p)x^*} \end{bmatrix}. \tag{3.22}$$

Hence from the Jacobian matrix$J(E_i)$ above the equilibrium point;

(i) $E_0(0,0)$ is given by

$$J(E_0) = \begin{bmatrix} r & 0 \\ 0 & \mu \end{bmatrix}.$$

Thus using Maple SOFTWARE the eigenvalues of the Jacobian matrix $J(E_0)$ are r and $-\mu$ However $E_0(0,0)$ is saddle point under the condition that $r > 0$ and all saddles are unstable.

(ii) For the predator free equilibrium point $E_1(x^*,0) = E_1(\frac{K(r-q_1h_1)}{r},0)$

Thecorresponding Jacobian matrix is written as ;

$$J(E_1) = \begin{bmatrix} 2q_1h_1 & -\frac{\alpha K(1-q_1K)(1-p)}{r+\alpha K(1-p)(r-q_1K)} \\ & \\ 0 & -\mu + \frac{b\alpha K(1-p)(r-q_1h_1)}{r+aK(1-p)(r-q_1h_1)} \\ & \end{bmatrix}. \tag{3.23}$$

Eigenvalues of $E_1(x^*,0)$ are $2q_1h_1$ and $-\mu + \frac{b\alpha K(1-p)(r-q_1h_1)}{r+aK(1-p)(r-q_1h_1)}$ Hence J is locally asymptotically stable if $-\mu + \frac{b\alpha K(1-p)(r-q_1h_1)}{r+aK(1-p)(r-q_1h_1)} < 0.$
That is

$$\frac{b\alpha K(1-p)(r-q_1h_1)}{r+aK(1-p)(r-q_1h_1)} < \mu. \tag{3.24}$$

(iii) The corresponding Jacobian matrix of the equilibrium point $E_2(0,y^*)$

$$J(E_2) = \begin{bmatrix} r & 0 \\ 0 & -\mu \end{bmatrix}. \tag{3.25}$$

Hence we find that $E_2(0,y^*)$ =$E_0(0,0)$ hence the eigenvalues for Jacobian matrix $J(E_2)$ are r and $-\mu$ where $r > 0$ therefore the point at equilibrium $E_2(0,y^*)$ is unstable saddle.

(iv) For co-existence equilibrium point $E_3(x^*,y^*)$ Jacobian matrix is as follows:

$$J(E_3) = \begin{bmatrix} E_{11} & E_{12} \\ E_{21} & E_{22} \end{bmatrix}. \tag{3.26}$$

$$E_{11} = r\left(1-\frac{r}{K}\right) - \frac{r(H_2+\mu)}{G_2K} - \frac{G_1\alpha\,(1-p(G_4-G_3))}{G_1+a(H_2+\mu)} + M, \tag{3.27}$$

For

$$M = \frac{G_2{}^2 \alpha (1-p)(H_2+\mu)(G_4-G_3)\,a}{(G_1 (G_2 + a(1-p)(H_2+\mu)))^2}.$$

on simplification on the equation (3.17)

$$r\left(1-\frac{r}{k_1}\right) - \frac{r(H_2+\mu)}{G_2 k_1} - \frac{G_1 Q}{G_1 + a(H_2+\mu)} + \frac{G_2{}^2 (H_2+\mu) Q a}{(G_1 (G_2 + a(1-p)(H_2+\mu)))^2},$$

Where

$$Q = (G_4 - G_3)\,\alpha\,(1-p),$$

$$G_1 = a(H_2+\mu) - \alpha b,$$

$$G_2 = (1-p)\,[a(H_2+\mu) - \alpha\, b],$$

$$G_3 = \frac{bH_1}{(H_2(-1+p) + \mu(-1+p))\,a(-1+p) - \alpha(-1+p)\,b(-1+p)},$$

$$G_4 = \frac{br(((H_2(-1+p) + \mu(-1+p))\,a(-1+p) - \alpha(-1+p)\,b(-1+p))\,K - H_2 - \mu)}{[(H_2(-1+p) + \mu(-1+p))\,a(-1+p) - \alpha(-1+p)\,b(-1+p)]^2 K},$$

Thus the simplified E_{11} will be given by

$$E_{11} = r\left(1-\frac{r}{K}\right) - \frac{r(H_2+\mu)}{G_2 K} - \frac{G_1 Q}{G_1 + a(H_2+\mu)} + \frac{G_2{}^2 (H_2+\mu) Q a}{(G_1 (G_2 + a(1-p)(H_2+\mu)))^2}.$$

Again for

$$E_{12} = -\frac{\alpha\,(H_2+\mu)}{a\mu + aH_2 + [a(H_2+\mu) - \alpha\, b]}. \tag{3.28}$$

Hence by simplifying equation (3.18)

$$E_{12} = -\frac{\alpha\,(H_2+\mu)}{2a(H_2+\mu) - \alpha\, b}.$$

$$E_{21} = \frac{b\alpha\,(1-p)M}{1+G_5} - \frac{b(\alpha(1-p))^2(\mu+H_2)Ma}{(1-p)[a(\mu+H_2) - \alpha\, b](aG_5+1)^2}, \tag{3.29}$$

Where

$$M = \frac{br(((\mu(p-1) + H_2(p-1))\,a(p-1) - \alpha(p-1)\,b(p-1))\,K - \mu - H_2)}{[(\mu(p-1) + H_2(p-1))\,a(p-1) - \alpha(p-1)\,b(p-1)]^2 K} - D,$$

$$G_5 = \frac{\mu + H_2}{[a(\mu+H_2) - \alpha\, b]},$$

and

$$D = \frac{bH_1}{((\mu + H_2) - \alpha b)(p-1)}.$$

$$E_{22} = -\mu + \frac{b\alpha\,(H_2 + \mu)}{2a\,(H_2 + \mu) - \alpha\,b}. \tag{3.30}$$

The stability of the $J(E_3)$ is stated using the the characteristic of polynomial equation techniques using trace and determinant techniques proposition as follows:

Preposition 3.1 Suppose the Jacobian matrix evaluated at the co-existance equlibrium has characteristic polynomial equation

$$\lambda^2 - (trace(J(E_3))\lambda + Determinant(J(E_3) = 0. \tag{3.31}$$

such that

trace $(J(E_3)$=$E_{11} + E_{22}$ and $Determinant(J(E_3) = E_{11}E_{22} - E_{12}E_{21}$

The co-existence equilibrium point is locally symptotically stable or stable spiral if trace $(J(E_3) < 0$ and Determinant $(J(E_3) > 0$.

Also the interior equlibrium point is centre(neutral stable) if $trace(J(E_3) = 0$ and $Determinant(J(E_3) > 0$

3.10 Global Stability of equilibrium point

Points E_1 and E_2 is shown by Linearizing the system of equation (3.1) and defining appropriate Lyapounov function to sepately described each equilibrium point. The Linearization process is done using Jacobian technique such that

$$\frac{dX_i}{dt} = J(E_i)X_i. \tag{3.32}$$

Where $J(E_i)$ is the Jacobian Matrix and X_i is the small perburbation on x_i. Therefore the system (3.1) reduces to the following Linear system;

$$\begin{aligned} \frac{dX}{dt} &= [r(1 - \frac{x^*}{K}) - \frac{rx^*}{K} - \frac{\alpha(1-p)y^*}{(1+a(1-p)x*)^2}]X - [\frac{\alpha(1-p)x^*}{(1+a(1-p)x}]Y, \\ \frac{dY}{dt} &= [\frac{b\alpha(1-p)y^*}{1+a(1-p)x^*} - \frac{b\alpha(1-p)^2x^*y^*a}{(1+a(1-p)x^*)^2}]X + [-\mu + \frac{b\alpha(1-p)x^*}{1+a(1-p)x^*}]Y. \end{aligned} \tag{3.33}$$

The Lyapunov function is chosen as

$$V(X,Y) = \frac{X^2}{2} + \frac{Y^2}{2}. \tag{3.34}$$

The function $V(X,Y)$ is postive definite function since $V(X,Y) \geq 0$ for any values of (X,Y) and it is minimum at the origin that is $V(0,0) = 0$ The time derivative of $V(X,Y)$ is given by:

$$\frac{dV(X,Y)}{dt} = \frac{\partial V}{\partial X}.\frac{dX}{dt} + \frac{\partial V}{\partial Y}.\frac{dY}{dt}. \tag{3.35}$$

By substution of equation (3.23) and the partial V into equation (3.25) we obtain the relation below

$$\begin{aligned}\frac{dV(X,Y)}{dt} &= X[(r(1-\frac{x^*}{K}) - \frac{rx^*}{K} - \frac{\alpha(1-p)y^*}{(1+a(1-p)x*)^2})X - (\frac{\alpha(1-p)x^*}{(1+a(1-p)x)})Y] \\ &+ Y[(\frac{b\alpha(1-p)y^*}{1+a(1-p)x^*} - \frac{b\alpha(1-p)^2x^*y^*a}{(1+a(1-p)x^*)^2})X) + (-\mu + \frac{b\alpha(1-p)x^*}{1+a(1-p)x^*})Y].\end{aligned} \tag{3.36}$$

(i) **For a fixed point** $E_1(x^*,0)$

we substitute the equation $E_1(x^*,0) = (\frac{K(r-q_1h_1)}{r},0)$ into equation (3.26) above as follows;

$$\begin{aligned}\frac{dV(X,Y)}{dt} &= X^2(q_1h_1 - r) - (\frac{\alpha(1-p)(r-q_1h_1)}{1+aK(1-p)(1-q_1h_1)})XY \\ &+ (-\mu + \frac{b\alpha(1-p)(r-q_1h_1)K}{1+aK(1-p)(1-q_1h_1)})Y^2.\end{aligned} \tag{3.37}$$

Therefore from the equation (3.27) the equlibrium point $E_1(x^*,0)$ is globally asymptotically stable if the following condition is satsified that is

$$q_1h_1 - r < 0 \tag{3.38}$$

Thus, using simple simple algebraic mathematical manipulation results into the inequality

$$r > q_1h_1.$$

Hence, in the absence of the equilibrium point the $E_1(x^*,0)$ is globally stable if the intrinsic growth rate of the prey population is greater than the haversting rate.

(ii) **For Steady state** $E_3(x^*, y^*)$

Here, we substitute equation (3.11) into equation (3.27) to obtain

$$\frac{dV(X,Y)}{dt} = E_{11}X^2 + (E_{12} + E_{21})XY. \tag{3.39}$$

with usual notations for E_{11}, E_{12} and E_{21}. Therefore the point is globally stable if the condition below holds

$$\frac{dV(X,Y)}{dt} = (E_{11}X^2 + (E_{12} + E_{21})XY) < 0. \tag{3.40}$$

3.11 Harvested prey-predator model with control

3.11.1 Introduction

Over-harvesting of wildlife resources is an important challenge facing protected area in Africa, a better understanding of the nature would improve the way in which it is managed (Wilfred and MacColl, 2014). However familization on the local over-harvesting needed to adress the problem is still needed (Wilfred and MacColl, 2014). Nevertheless number of studies have attempted to access the nature of poaching, over-harvesting and consequent on implications for conservation such that of those (Kirk, 2012; Wilfred and MacColl, 2014; Sagamiko *et al.*, 2015). In recent years, optimal control theory has found applications in Applied Mathematics (Mathematical ecology and epidemology). The theory has developed rapidly since the first paper by Pontryagin and collaborators in the late 1950 (Aniţa *et al.*, 2011). The word control has a double meaning. First, controlling a system can be understood simply as testing or checking that its behaviour is satisfactory (Sagamiko *et al.*, 2015). In a deeper sense, to control is also to act, to put things in order to guarantee that the system behaves as desired (Sagamiko *et al.*, 2015; Cara.F and Iriondo.F, 2003). In our context, the second meaning holds, that if we have a prey-predator system which is affected by over-harvesting may lead the system to extinction and we need to act upon the situation to ensure that the system behaves as desired. It is in such situations when concepts like the optimal control theory" comes into play. Therefore we introduce into model equation (3.1) time dependent control effort $(u_1(t))$ on harvesting to alleviate the loss of species in the prey-predator system established in equations (3.1) above.

Hence, from model (3.1) we let $u_1(t)$ to represent over-harvesting control strategy (Creation of reserve areas with restriction of harvesting). Thus, the system of equations (3.1) becomes:

$$\begin{aligned} \frac{dx}{dt} &= r(1-\frac{x}{k}) - \frac{\alpha(1-p)xy}{1+a(1-p)x} - (1-u_1(t))q_1h_1x, \\ \frac{dy}{dt} &= -\mu y + \frac{b\,\alpha(1-p)xy}{1+a(1-p)x} - (1-u_1(t))q_2h_2y. \end{aligned} \tag{3.41}$$

Where α, k, μ, a, b are all positive constants and r is the intrinsic growth rate of the prey. k is the environment carrying capacity of the prey in the absence of the predation and harvesting. The term $\frac{\alpha x}{1+ax}$ denotes the functional response of the predator which is a Holling type II response functional of the predator, μ is the death rate of the predator, $\frac{\alpha}{a}$ is the maximum number that can be eaten by each predator in unit time, b is the predation convention factor(biomass) denoting the number of newly born predators for each captured prey and q_1 and q_2 are catchability coefficient of the prey and predator respectively. p is the proportion of prey population not exposed to predation, that it protects px of the prey and leaves $(1-p)x$ of the prey available to predation. Note that $p \in [0,1]$

3.12 Analysis of the Optimal control

3.12.1 Formulation of the objective function

Here we construct an objective function that provides the optimal population size of the prey-predator species at minimum costs for over-harvesting strategies. Thus the objective functional J is defined over a feasible set of control u_i and applied over the pre-defined finite time interval given by $[T_0, T_1]$.

The Objective function of this function will be formed by the following form

$$J(u_i) = [B(x_i(T_1), T_1) - \int_{T_0}^{T_1} (F(u_i(t), t))dt], \tag{3.42}$$

subject to

$$\frac{dx_i}{dt} = f_i(t, u_i(t), x_i(t)).$$

where $x_i(T_0) = x_i$ and $0 \leq u_i \leq 1$ for $t \in (T_0, T_1)$. The term $B(x_i(T_1))$ and $F(x_i(t), u_i(t), t)$ represent the prey-predator populations to be optimized at the terminal time control and total cost of control respectively.

Therefore from (3.31) the objective functional becomes

$$J(U) = Max_u[B_1 x(T_1) + B_2 y(T_1) - \int_0^{T_1} (\frac{Au_1^2}{2})dt], \tag{3.43}$$

subject to;

$$\begin{aligned}
\frac{dx}{dt} &= r(1-\frac{x}{k})x - \frac{\alpha(1-p)xy}{1+a(1-p)x} - (1-u_1(t))q_1 h_1 x, \\
\frac{dy}{dt} &= -\mu y + \frac{b\,\alpha(1-p)xy}{1+a(1-p)x} - (1-u_1(t))q_2 h_2 y.
\end{aligned}$$

for $x(T_0) = x_0$, $y(T_0) = y_0$ and $0 \leq u \leq 1$ for $t \in [0, T_1]$; $u \varepsilon U$

Also the terms $B_1 x(T_1)$ and $B_2 y(T_1)$ represents the prey and predator populations to be optimized at the terminal control and $\frac{Au_1^2}{2}$ is the total control cost for over-harvesting. The cost weight is A and state weights B_1, B_2 are all positive constants . The aim is to maximize u such that

$$J(u^*) = Max(J(u)).$$

with

$$0 \leq u \leq 1 \ for \ t \in [T_0, T_1]$$

3.12.2 Existence of optimal control

The aim is to show that the optimal control problem for the formulated in (3.32) has at least one solution before trying to solve the optimal control values.

Theorem :

Given optimal in (3.32) with u as control variable, then there exist $u \in U$ (Optimal control set) such that $J(u_1^*) = max(J(u_1))$

Proof

The proof for existance of optimal control provided by Kirk (2012), Fleming *et al.,* (1975), Sagamiko *et al.* (2015) and Berkovitz (2013) is valid such that:

(i) The Model equations (3.31) with control are linear in control variable u and bounded by a linear system in the state and control effort on over-haversting $u_1(t)$

(ii) The control U is convex,closed and bounded set

(iii) The integrad $-\frac{Au_1^2}{2}$ of the objective function (3.31) is concave in U

3.12.3 Characterization of the Optimal Control

The optimal control must satsify the necessary condition that are formulated by pontrygian's maximum principle (Boltyanskiy *et al.*, 1962). This principle converts equations (3.22) into a problem of maximizing point-wise a Hamiltonian (H) with respect to u_1

$$H = -\frac{Au_1^2}{2} + \lambda_1[r(1-\frac{x}{k})x - \frac{\alpha(1-p)xy}{1+a(1-p)x} - (1-u_1(t))q_1h_1x]$$
$$+ \lambda_2[-\mu y + \frac{b\ \alpha(1-p)xy}{1+a(1-p)x} - (1-u_1(t))q_2h_2y]. \quad (3.44)$$

Where λ_1 and λ_2 are the adjoint varibles or co-state variable. Applying pontryingin's maximum principle and existance results for the optimal control from Mappes *et al.* (2001). The following preposition is obtained.

Theorem 4.1

For Optimal control u_1 that maximize $J(u^*)$ over U , then there exist adjoint varibles λ_1 and λ_2 satsifying

$\frac{d\lambda_i}{dt} = -\frac{\partial H}{\partial x}$; with $\lambda_i(T) = B_i; i = 1,2$

Proof: Using the adjoint Condition set of the Theorem 4.1 the equation becomes

$$\frac{d\lambda_1}{dt} = -\frac{\partial H}{\partial x} = -\lambda_1[-\frac{rx}{k_1} + r\left(1-\frac{x}{k_1}\right) - \frac{\alpha\,(1-p)\,y}{1+(1-p)\,ax} + \frac{\alpha\,(1-p)^2\,xya}{(1+(1-p)\,ax)^2} - (1-u_1\,(t))\,q_1h_1]$$
$$-\lambda_2[\frac{b\alpha\,(1-p)\,y}{1+(1-p)\,ax} - \frac{b\alpha\,(1-p)^2\,xya}{(1+(1-p)\,ax)^2}],$$
$$\frac{d\lambda_2}{dt} = -\frac{\partial H}{\partial y} = -\lambda_1[-\frac{\alpha\,(1-p)\,x}{1+(1-p)\,ax}] - \lambda_2[-\mu + \frac{b\alpha\,(1-p)\,x}{1+(1-p)\,ax} - (1-u_1\,(t))\,q_2h_2]. \quad (3.45)$$

With transversality Conditions;

$$\lambda_1(T_1) = \frac{d(B_1x(T_1))}{dx} = B_1,$$
$$\lambda_2(T_1) = \frac{d(B_1y(T_1))}{dy} = B_2. \quad (3.46)$$

Using Optimality Condition, we have $\frac{\partial H}{\partial u_1} = 0$ at u_1^*

That is

$$\frac{\partial H}{\partial u_1} = -Au_1 + \lambda_1h_1q_1x + \lambda_2h_2q_2y = 0. \quad (3.47)$$

Which gives

$$u_1^* = \frac{\lambda_1h_1q_1x + \lambda_2h_2q_2y}{A}. \quad (3.48)$$

The following characterization holds on the interior of the control set

$$u_1^* = min\{1, max\{0, \frac{\lambda_1h_1q_1x + \lambda_2h_2q_2y}{A}\}\}. \quad (3.49)$$

Where λ_1 and λ_2 are the solutions of the system of adjoint equation (3.33)

Note that

The state system (3.33) has initial time condition and the co-state system (3.32) has the final time condition.

CHAPTER FOUR

RESULTS AND DISCUSSION

4.1 Introduction

This chapter covers simulations of the described model (3.1) and Model (3.22) and variations of prey refuge together with application of strategies on control

In the first case, the simulations for prey-predator incorporating a prey refuge (Model (3.1)) is done by showing phase diagrams of the equilibrium point, in this case we illustrate numerically the dynamical behavior of the equilibrium points discussed in the theoretical part.

Second case explore the effect of harvesting of prey -predator system together with variation of prey refuge

The third case of the simulations shows the numerical analysis of haversted prey-predator model with control (Model (3.22)), the aim is to describe the impact of time dependent control as on the prey-predator system as we claimed in control efforts to optimize prey-predator population densities at the final time control.

4.2 Representation of phase

Phase diagrams for model (3.1) and model (3.22) are drawn using parameters described in Table (1):

Table 1: Parameter Values

parameter	**values**	**source**
K	600	Estimated
r	1	Sagamiko *et al.* (2015)
α	$6.74x10^{-5}$	Assumed
μ	0.01	Kar (2006)
P	0.6	chosen from p$\in$ [0 1]
q_1	0.06	Assumed
q_2	0.0375	Assumed
h_1	2	Assumed
h_2	4	Assumed
b	0.16	Assumed
B_1	100	Assumed
B_2	200	Assumed
A	1000	Sagamiko *et al.* (2015)
a	0.02	Assumed

Most of paramenters from Table 1 are selected after numerical simulation, hence the phase diagram are drawn as shown in sections 4.2.1 and 4.2.2

4.2.1 Phase diagram for equlibrium point $E_1(x^*, c)$

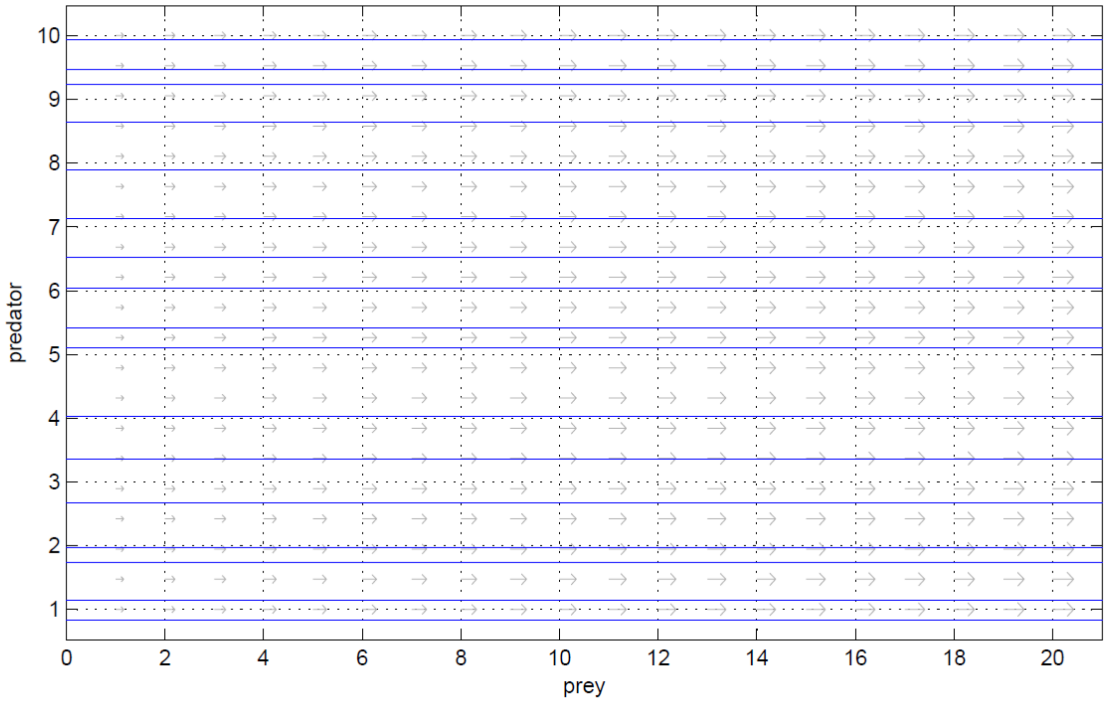

Figure 1: Phase diagram showing dynamic behaviour of $E_1(x^*, c)$

4.2.2 Phase diagram for equlibrium point $E_3(x^*, y^*)$

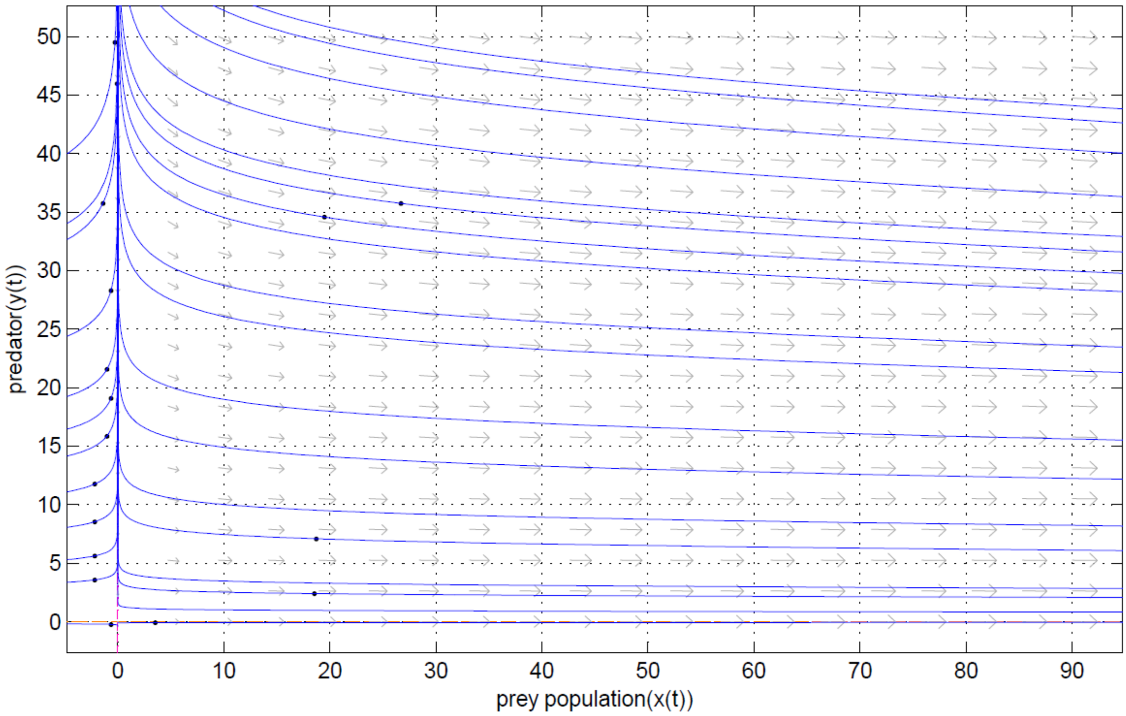

Figure 2: Phase diagram showing dynamical behaviour of $E_3(x^*, y^*)$

Figure 1 and 2 indicates that, in the absence of predator while presence of over-harvesting the dynamic equlibrium point of $E_1(x^*, c)$ is unstabe while the dyanmic behaviour of co-existance equilibrium point $E_3(x^*, y^*)$ in figure (2) is spiral unstable surounded by a stable convergence lines

4.2.3 The effect of varying harvesting on prey and predator species and variation of prey refuge

(i) **The effect of varying catchibility coefficient on harvesting of prey with effect of prey refuge on prey population denisties.**

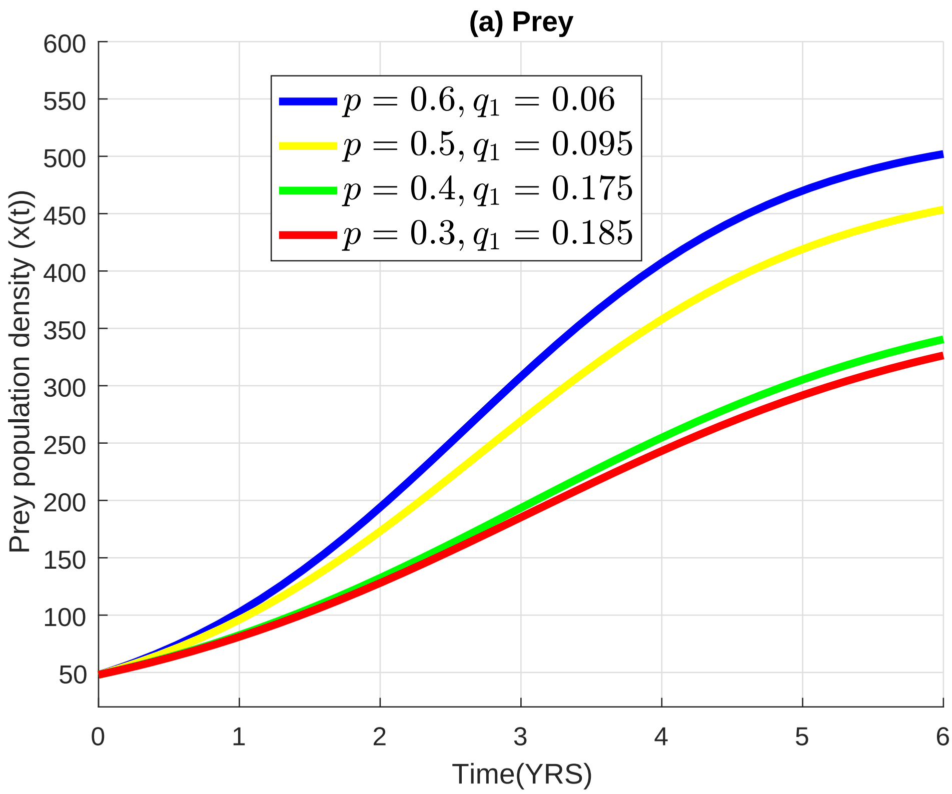

Figure 3: Simulation of the harvested prey with variation of catchibility coeffiecient and prey refuge

Figure 3 illustrate that at a minimum prey refuge p and high catchibiliy coefficient q_1 the population density of prey decreases, while at maximum prey refuge and low catchibility coefficient q_1 the population density of prey increases. Therefore from Fig. 3 we observed that the prey refuge and harvesting have a great impact on prey population .

(ii) **The effect of varying catchibility coefficient on haversting of predator with effect of prey refuge on predator population densities**

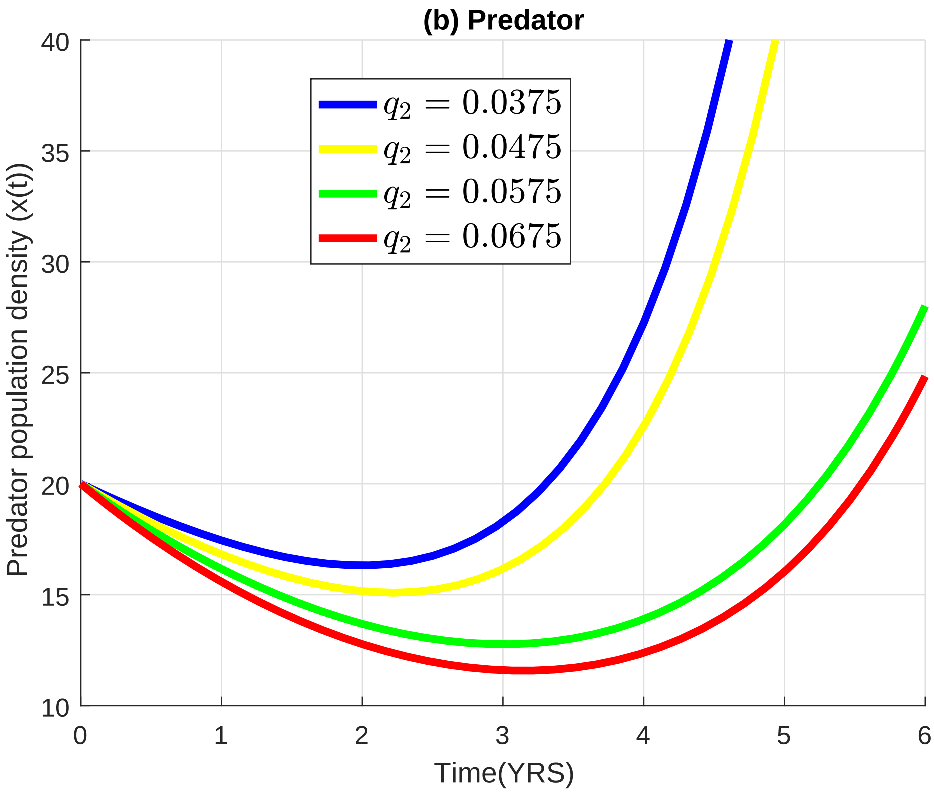

Figure 4: Simulation of the haversted predator with variation of catchibility coeffiecient

Figure 4 illustrate that at a high catchibiliy coefficient q_2 the population density of predator decreases, while at a low catchibility coeficient q_2 the population density of predator increases. Therefore from Fig. 4 we observed that the prey refuge and harvesting have a great impact on predator population density.

4.2.4 Numerical Results and simulations for prey and predator model with control (Model (3.22))

In this section optimal control stratergy is numerically solved by several numerical techniques using parameter values described in Table 1. A foward-backward sweep method(FBSM) is one of the numerical techiniques that can be used to solve an optimal control problem. The following scholars (Chakraborty *et al.*, 2011*b*; Sagamiko *et al.*, 2015; Kim *et al.*, 2011), suggested the method to be executed as follows:

(i) Using the new set of values, transversality condition $\lambda_{N+1} = \lambda(T)$ (T= final time)and guessed values for control vector, solve the adjoint vector backward in time using $RK4$

(ii) Make an initial guess for control vector u and use initial conditions (x_0 and y_0) for the state vector to solve for the variables foward in time using Rungekutta 4^{th} order numerical method ($RK4$)

(iii) The obtained value for state and adjoint variables are entered on the characterization of the optimal control (3.3) to update the control vector which becomes new value for the control.

(iv) Divide the total time interval into N equal subintervals and set the state at different times as $\vec{x}$ =$(x_1, x_2, ...x_{N+1})$ and co-state variables as $\vec{\lambda}$=$(\lambda_1, \lambda_2, ...\lambda_{N+1})$

(v) If the solutions of the variables (excluding the control are variable) are convergent that is to say stop the process when the values of the control varibale in the current and previous iterations are sufficiently close

The investigation of the impact of adding time dependent control variable $u_1(t)$ on the prey-predator system is studied numerically through the application of control strategy $u_1(t)$.

Using Table (1) of the parameter values (Sagamiko *et al.*, 2015) the constant and control variable are chosen depending on their relative importance and relative applications of the cost used for controlling the problem.

Thus the intial state variables are chosen as; B_1 = 100, B_2 = 200, A = 1000 and state variables are $y(0)$ = 20 and $x(0)$ = 50. Hence simulations for controlling strategy is carried out as follows:

4.2.5 Control strategy: Creation of reserve areas with restriction of harvesting for controlling of over-harvesting

(i) **The impact of u_1 on prey population**

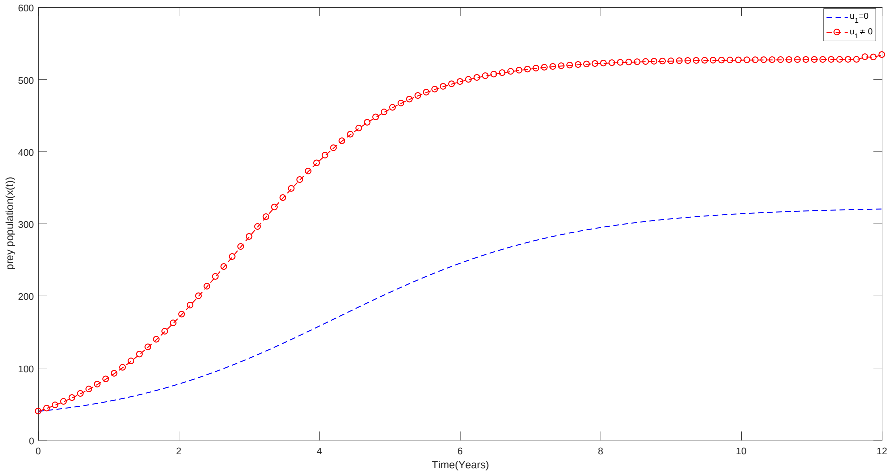

Figure 5: Simulation of a prey affected by over-harvesting showing the impact of creating reserve areas with restriction of harvesting

Figure 5 above shows the application of creating reserve area with restrictions of harvesting prey species, on this atratergy, control u_1 is used to optimize objective functional J. The results in Fig. 5 shows a significant difference in prey populations with optimal strartergy $(u_1 \neq 0)$ as compared to prey population without control $(u_1 = 0)$. This shows that preveting of over-harvesting incorporating a prey refuge in a system lead to repidly increase among prey species. However the increase of population due to control is due to the reason of prey refuge that protect most of prey species from predation hence reduce the source of food to the predator as their only source of food. The control profile is shown in Fig. 7, here we see that the optimal harvesting control u_1 increases gradually till time $t = 10$ Years Where it reseaches the bound of approximate 0.9 and continues to a final time. We observe that as the effort of control increases there is increase in number of individuals saved.

(ii) **The impact of u_1 on predator population**

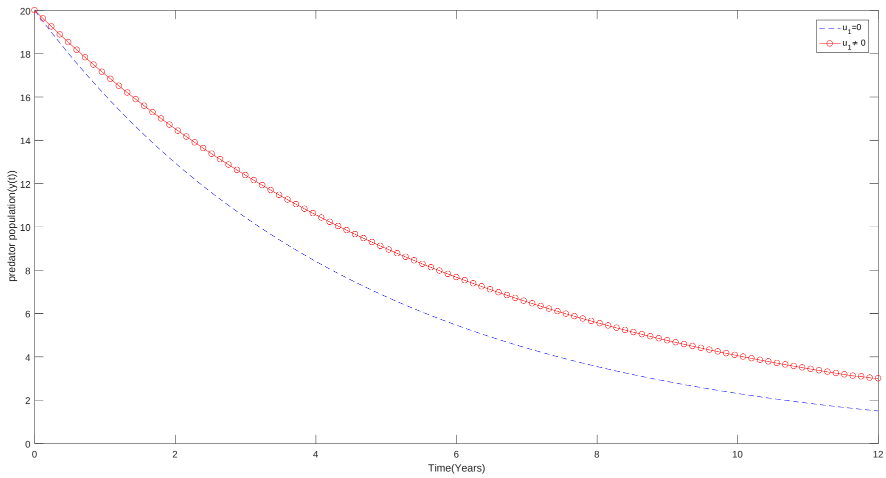

Figure 6: Simulation of a predator affected by over-harvesting showing the impact of creating reserve areas with restriction of harvesting

Figure 6 above shows the application of creating reserve area with restriction of harvesting predator species, on this atratergy, control u_1 is used to optimize objective functional J. The results in Fig. 6 shows a significant difference in predator populations with optimal stratergy as compared to predator population without control. This shows that preveting of over-harvesting incorporating a prey refuge in a system lead to increase among predator species. However the increase of population due to control is higher in prey species than in predator species as seen in Fig. 5 and 6 this is due to the reason of prey refuge that protect most of prey species from predation hence reduce the source of food to the predator as their only source of food. We observe that as the effort of control increases there is increase in number of individuals saved.

(iii) **Control profile for control strategy**

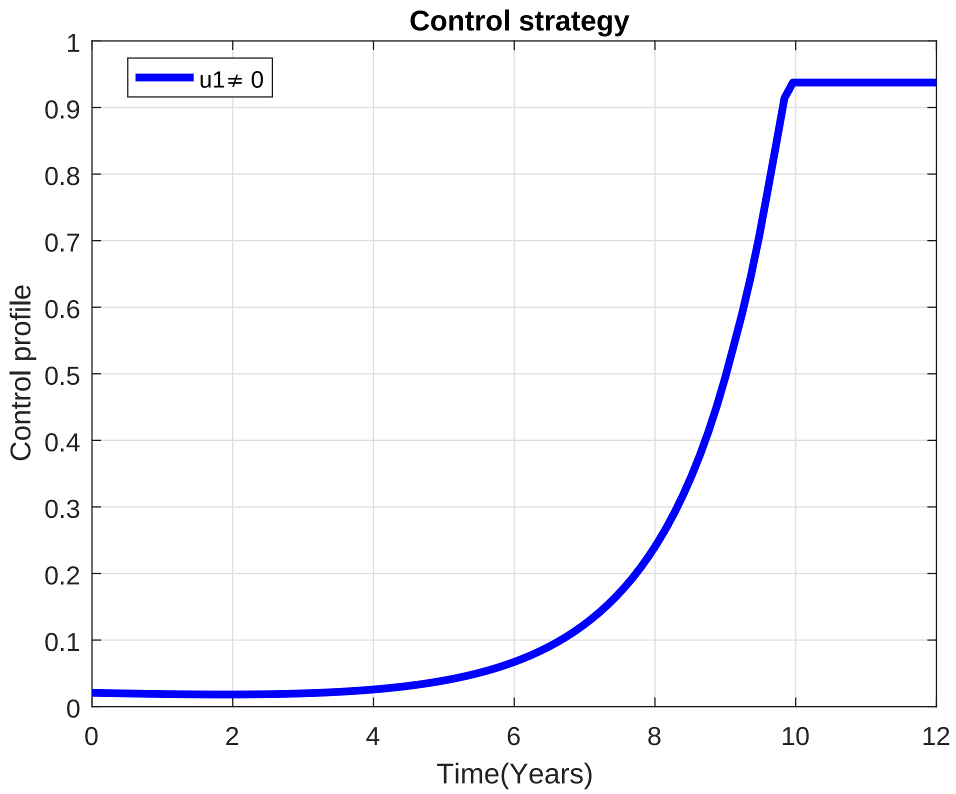

Figure 7: Control profile for u_1

Figure 5 and 6 above shows the application of creating reserve area with restriction of haversting prey and predator species, on this atratergy, control u_1 is used to optimize objective function J. The results in Fig. 5 and 6 shows a significant difference in prey and predator populations with optimal stratergy as compared to prey and predator population without control. This shows that preveting of over-harvesting incorporating a prey refuge in a system lead to repidly increase among prey and predator species. However the increase of population due to control is higher in prey species than in predator species this is due to the reason of prey refuge that protect most of prey species from predation hence reduce the source of food to the predator as their only source from of food. The control profile is shown in Fig. 7, here we see that the optimal harvesting control u_1 increases gradually till time $t = 10$ Years. Where it reseaches the bound of approximate 0.9 and continues to a final time. We observe that as the effort of control increases there is increase in number of individuals saved.

CHAPTER FIVE

CONCLUSION AND RECOMMENDATION

5.1 Conclusion

The aim of the study was to alleviate the loss of prey -predator species due to over-harvesting. We developed harvested prey-predator incorporating a prey refuge. Thereafter we carried out some analysis (theoretically and numerically) of how harvesting affect the prey -predator species we also modified the model to include the control efforts on over-harvesting before analysizing their(conrol efforts) impacts on the prey-predator system.

From the theoretical analysis of the model with harvesting without control we observed that harvesting has negative impacts on the intrinsic growth rate of prey- population density which qunsequently results into negative effects on the prey-predator population densities.

Meanwhile in the absence of predators, the intrinsic growth of the prey is greater than harvesting rate of the prey population.Also in the absence of prey population and presence in harvesting predators goes to extinction, the results which concur with the assumption that the predators depend on prey as the only source of food. Futhermore in the presence of both population densities the birth rate of predators must be greater than the sum of predator's mortality rate and harvesting rate.

The grobal stability observed that predator's population is obtained when harvesting rate exceed the intrinsic growth rate of prey population. Also increasing to prey population cause a rapid decrease of prey population which result in a rapid decrease of predator population density Figure 4 and 5. This situation can lead to extiction of both species. Hence, the increase of harvesting to predator population support the survival of prey species in the system.

The analysis of the optimal control of the optimal control stratergiies to show how impact in prey-predator system was conducted by carrying out simulations. here the stratergy suggested is the creation of reserve area with restriction of harvesting u_1 for controlling over-harvesting was used to optimize the prey-predator population densities. The results showed that there was an increase in a prey population density from 300 species to 550 species and little increase in predator population from 2 species to 4 species, the small increase in predator is due to the harvesting , prey refuge and mortality rate. Table 2 shows the final time states from numerical simulations at the terminal time control;

Table 2: Final time states from numerical simulations at the terminal time control

Control Stratergy	$x(12)$	$y(12)$
No Control	300	2
Control stratergy	550	4

5.2 Recommendations

The harvested prey-predator model incorporating a prey refuge and optimal control strategies to alleviate the problem of loss of prey and predator species has been formulated and analysed.

Numerical analysis of the model without control, showed that harvesting has negative impact on prey and predator population. The strategy suggested in the study is the creation of reserve area with restriction of harvesting for controlling the loss of prey and predator species. This is due to the reason that after numrical simulation, results showed that prey increased from 300 species without control from $t = 10$ Years to 550 species at time $t = 10$ Years after control and very small increase in predator population species from 2 species without control to 4 species at time $t = 13$ Years after control, the small increase of predator population density is due to the effect of prey refuge and over-harvesting. Generally, the study observed that using creation of reserve area with restriction of over-harvesting as control stratergy elleviate the loss of species when used. Table 2 shows the final time states from numerical simulations at the terminal time control. However we suggest the following to be done in future:

(i) Modelling harvested prey-predator system using stochastic approach instead of deterministic differential equation

(ii) Modelling prey-predator system with using deterministic differential equations incorporating while harvesting only one species

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Das, U., Kar, T. and Pahari, U. (2013*b*). Global dynamics of an exploited prey-predator model with constant prey refuge. *ISRN Biomathematics*. **2013**.

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APPENDICES

Appendix 1: Matlab scripts and functions used in simulations

Matlab code for figure (3) for showing the effect of varying catchibility coeffiecient on Harvesting of prey with effect of prey refuge on prey popuation densities

Script one(prey)

```
%Harvesting of prey with effect of prey refuge on prey population densities
function Dy=prey(t,y0,theta)

x=y0(1);
y=y0(2);
h1=theta(1);
h2=theta(2);
r=theta(3);
a=theta(4);
mu=theta(5);
K=theta(6);
q1=theta(7);
q2=theta(8);
p=theta(9);
alpha=theta(10);
b=theta(11);
Dy=[ r*x*(1-x/K)-alpha*(1-p)*x*y/(1+a*(1-p)*x)-q1*h1*x; -mu*y+alpha*b*(1-p)*x*y/(1+a*(1-p))*x-q2*h2*y;];
clc
clear all,close all
trange=0:6:6;
y1=[80 20];
hold on
%parameter estimation
theta=[2,4,1,0.02,0.01,600,0.06,0.02,0.6,0.0000674,0.16];
[x,y]=ode45(@prey,trange,y1,[],theta);
plot(x,y(:,1),'b','linewidth',3)
theta=[2,4,1,0.02,0.01,600,0.095,0.03,0.5,0.0000674,0.16];
[x,y]=ode45(@prey,trange,y1,[],theta);
plot(x,y(:,1),'y','linewidth',3)
theta=[2,4,1,0.02,0.01,600,0.175,0.004,0.4,0.0000674,0.16];
[x,y]=ode45(@prey,trange,y1,[],theta);
plot(x,y(:,1),'g','linewidth',3)
theta=[2,4,1,0.02,0.01,600,0.185,0.02,0.3,0.0000674,0.16];
[x,y]=ode45(@prey,trange,y1,[],theta);
plot(x,y(:,1),'r','linewidth',3)
```

```
legend({'$p=0.6,q_{1}=0.06$','$p=0.5,q_{1}=0.095$','$p=0.4, q_
    {1}=0.175$','$ p=0.3,q_{1}=0.185$'},'Interpreter','Latex','
    FontSize',14)
xlabel('Time(YRS)'),ylabel('Prey population density (x(t))')
xlim([0 6]),ylim([20 600])
title(' (a) Prey ')
grid on
hold off
%Matlab code for the implementation of harvesting predator by
    varying its
%catchability coefficient
% Script one(predator)
function Dy=predator(t,y0,theta)
x=y0(1);
y=y0(2);
h1=theta(1);
h2=theta(2);
r=theta(3);
a=theta(4);
mu=theta(5);
K=theta(6);
q1=theta(7);
q2=theta(8);
p=theta(9);
alpha=theta(10);
b=theta(11);
Dy=[r*x*(1-x/K)-alpha*(1-p)*x*y/(1+a*(1-p)*x)-q1*h1*x;-mu*y+
    alpha*b*(1-p)*x*y/(1+a*(1-p))*x-q2*h2*y];;
clc
clear all,close all
trange=0:6:6;
y1=[48 20];
hold on
%parameter estimation
theta=[2,4,1,0.02,0.01,600,0.06,0.0375,0.6,0.0000674,0.16];
[x,y]=ode45(@predator,trange,y1,[],theta);
plot(x,y(:,2),'b','linewidth',3)
theta=[2,4,1,0.02,0.01,600,0.095,0.0475,0.5,0.0000674,0.16];
[x,y]=ode45(@predator,trange,y1,[],theta);
plot(x,y(:,2),'y','linewidth',3)
theta=[2,4,1,0.02,0.01,600,0.175,0.0575,0.4,0.0000674,0.16];
[x,y]=ode45(@predator,trange,y1,[],theta);
plot(x,y(:,2),'g','linewidth',3)
theta=[2,4,1,0.02,0.01,600,0.185,0.0675,0.3,0.0000674,0.16];
[x,y]=ode45(@predator,trange,y1,[],theta);
plot(x,y(:,2),'r','linewidth',3)
```

```
legend({'$ q_{2}=0.0375 $','$ q_{2}=0.0475 $','$ q_{2}=0.0575 $'
    ,' $ q_{2}=0.0675 $'},'Interpreter','Latex','FontSize',14)
xlabel('Time(YRS)'),ylabel('Predator population density (x(t))')
xlim([0 6]),ylim([10 40])
title(' (b) Predator ')
grid on
hold off
```

MATLAB CODE for figure (5) showing the simulation of the effect of optimal application of creating a reserve area with restriction of harvesting prey -predator species incorporating a prey refuge

```
%Matlab code for implementation for optimal control strategies
function ydot = ppholling( t,y,U,Constant)
%this function solves a holling type two prey predator equation
x=y(1);
y=y(2);
alpha=Constant(1);
r=Constant(2);
mu=Constant(3);
b=Constant(4);
k=Constant(5);
p=Constant(6);
q1=Constant(7);
q2=Constant(8);
h1 = Constant(9);
h2 = Constant(10);
A = Constant(11); % weight
B1 = Constant(12); % weight
B2 = Constant(13); % weight
lf = [B1 B2];
u = U;
ydot1=r.*x.*(1-x./k)-alpha.*(1-p).*x.*y./(1+a.*(1-p).*x)-(1-u)
    .*q1*h1.*x;% first ODE
ydot2=-mu*y+alpha*b*(1-p)*x*y/(1+a*(1-p)*x)-(1-u)*q2*h2*y; %
    second ODE
ydot = [ydot1; ydot2];
% Test Rungekuta with ODE45 if they produce the same results
```

```
[Tx, X]=rk4foward(@ppholling,t0, tf,N, init,U,Constant);
%% IMPLIMENTATION OF THE ALGORITHM
%Test 1 stoping condition 1
delta = 0.01;
X=init;
i=0; %Initialize iteration counter
mm=size(X);
NumXX =10e10;
Xnew = rand(N+1,mm(2)).*(repmat(X,N+1,1));
DenXnew=norm(Xnew);
while NumXX/DenXnew>delta
Xold = Xnew;
oldu = U;
%FORWARD RUNGE KUTTA FOR STATES
[Tx, X]=rk4foward(@ppholling,t0, tf,N, init,U,Constant);
% BACKWARD RUNGEKUTA FOR COSTATES
[Tp, P]=rk4back(@ppholling_costate,t0, tf,N,init2,U,X,Constant)
    ;
%UPDATE THE CONTROLS
x = X(1,:);y = X(2,:);
lambda1 = P(1,:);lambda2 = P(2,:);
% Case0:No control,
u1 =zeros(1,N+1);
% Case1:u1=0,
u1 =min(1,max(0,((q1*h1*lambda1.*x+q2*h2*lambda2.*y)/A)));
Uu= u1';
U = 0.5*Uu + 0.5*oldu; % Convex combination of the controls
Xnew = X';
NumXX =abs(norm(Xnew-Xold));
DenXnew =norm(Xnew);
i=i+1 %Update iteration counter
end
%% PLOTING
X=X';
Tx =Tx';
XX=X(:,1); YY=X(:,2);
J =B1*XX(end)+B2*YY(end)-sum(A*Uu(:,1).*Uu(:,1)) %Change to the
    suitable objective function
S=[Tx,X];
cd('C:\Users\Mfano\Desktop\Charlesmfano_mat2_edit')
save('case4State', 'S');
save('case4Control', 'Uu');
save('Cost', 'J');
plot(Tx,U(:,1),'-b', 'linewidth',3);
legend('u1\neq 0');
title(' Control strategy')
```

```
xlabel('Time(Years)')
ylabel('Control profile')
function [t, w]=rk4foward(f,t0, tf, N, init,U,Constant)
%function rk4() approximates the solutions of systems of
%differential equations
%with input ode function f and t in the interval [t0; tf] and the initial
%conditions are in the m-dimensional vector alpha
%as with function rk4(), the inputs are the endpoints a and b, the
%number of subdivisions N in the interval [a; b], and the initial
h = (tf-t0)/N; %the step size
t(1) = t0;
w(:,1) = init; %initial conditions
for i = 1:N
uu = U(i,:);
k1 = h*f(t(i),w(:,i),uu,Constant);
k2 = h*f(t(i)+h/2, w(:,i)+0.5*k1,uu,Constant);
k3 = h*f(t(i)+h/2, w(:,i)+0.5*k2,uu,Constant);
k4 = h*f(t(i)+h, w(:,i)+k3,uu,Constant);
w(:,i+1) = w(:,i) + (k1 + 2*k2 + 2*k3 + k4)/6;
t(i+1) = t0 + i*h;
end
%MFANORK4BACKWARD
function [t, w]=rk4back(f,t0, tf, N, init2,U,YY,Constant)
%function rk4back() approximates the solutions of systems of
%differential equations
%with input ode function f and t in the interval [a; b] and the initial
%conditions are in the m-dimensional vector alpha
%as with function rk4back(), the inputs are the endpoints a and b, the
%number of subdivisions N in the interval [a; b], and the initial
h = -(tf-t0)/N; %the step size
t(N) = tf;
w(:,N) = init2; %initial conditions
for i = N:-1:1
uu = U(i,:);
yy = YY(:,i);
k1 = h*f(t(i),w(:,i),uu,yy,Constant);
k2 = h*f(t(i)+h/2, w(:,i)+0.5*k1,uu,yy,Constant);
k3 = h*f(t(i)+h/2, w(:,i)+0.5*k2,uu,yy,Constant );
k4 = h*f(t(i)+h, w(:,i)+k3,uu,yy,Constant);
if i>1
```

```
w(:,i-1) = w(:,i) + (k1 + 2*k2 + 2*k3 + k4)/6;
t(i-1) = tf + i*h;
else
ww = w(:,i) + (k1 + 2*k2 + 2*k3 + k4)/6;
tt = tf + i*h;
end
end
w = [ww w];
t = [tt t];
%the estimated initial condition for STATE SYSTEM
%Matlab code for implementation of control strategy on predator
%subplot(1,2,2)
load case0State
X=S;
load case1State
load case1Control
% U=Uu;
plot(X(:,1),X(:,3),'--b',S(:,1),S(:,3),'-ro');
title('Predator')
ylabel('predator population(y(t))')
xlabel('Time(Years)')
hleg1=legend('u_1=0','u_1\neq 0');
%plot(X(:,1),U(:,1),'-b','linewidth',3);
%title('Controls')
% simulation for the control strategies for harvested prey species
load case0State
X=S;
load case1State
load case1Control
% U=Uu;
plot(X(:,1),X(:,2),'--b',S(:,1),S(:,2),'--ro','linewidth' ,1);
title('Prey')
ylabel('prey population(x(t))')
xlabel('Time(Years)')
hleg1=legend('u_1=0','u_1\neq 0');
```

MATLAB CODE for figure (5) showing the simulation of the effect of optimal application of creating a reserve area with restriction of harvesting prey -predator species incorporating a prey refuge

```
Codes used for drawing graphs in paper
%Matlab code for implementation for optimal control stratergies
```

```
function ydot = ppholling( t,y,U,Constant)
%this function solves a holling type two prey predator equation
x=y(1);
y=y(2);
alpha=Constant(1);
r=Constant(2);
mu=Constant(3);
b=Constant(4);
k=Constant(5);
p=Constant(6);
q1=Constant(7);
q2=Constant(8);
h1 = Constant(9);
h2 = Constant(10);
A = Constant(11); % weight
B1 = Constant(12); % weight
B2 = Constant(13); % weight
lf = [B1 B2];
u = U;
ydot1=r.*x.*(1-x./k)-alpha.*(1-p).*x.*y./(1+a.*(1-p).*x)-(1-u).*
    q1*h1.*x;% first ODE
ydot2=-mu*y+alpha*b*(1-p)*x*y/(1+a*(1-p)*x)-(1-u)*q2*h2*y; %
    second ODE
ydot = [ydot1; ydot2];
clear all; close all; clc
t0 = 0; tf=12;N=100;
time =linspace(t0,tf,N);
y0 = [40 20]; %the estimated initial condition for STATE SYSTEM
%%
%--- SOURCES OF CONSTANTS --
% alpha mu r b a k q1 q2 h1 h2 p B1 B2 A p
Constant = [0.0000674 0.01 1 0.16 0.02 600 0.06 0.0375 2 4 0.6
    100 200 1000 0.6];
alpha=Constant(1);
mu=Constant(2);
r=Constant(3);
b=Constant(4);
a=Constant(5);
k=Constant(6);
q1=Constant(7);
q2=Constant(8);
h1=Constant(9);
h2=Constant(10);
p= Constant(11);
B1 = Constant(12); % weight
B2 = Constant(13); % weight
```

```
A = Constant(14); % weight
lf = [B1 B2];
%% TEST SECTION
% U =[0 ];
% [Tx,X] = ode45(@ppholling,time,y0,[],U,Constant);
init =y0;
init2 =lf;
h = (tf-t0)/N;
u = linspace(0,0,N+1);
u1=u';
%u1=u';
U = [u1];
%U=u1;
% Test Rungekuta with ODE45 if they produce the same results
[Tx, X]=rk4foward(@ppholling,t0, tf,N, init,U,Constant);
%% IMPLIMENTATION OF THE ALGORITHM
%Test 1 stoping condition 1
delta = 0.01;
X=init;
i=0; %Initialize iteration counter
mm=size(X);
NumXX =10e10;
Xnew = rand(N+1,mm(2)).*(repmat(X,N+1,1));
DenXnew=norm(Xnew);
while NumXX/DenXnew>delta
Xold = Xnew;
oldu = U;
%FORWARD RUNGE KUTTA FOR STATES
[Tx, X]=rk4foward(@ppholling,t0, tf,N, init,U,Constant);
% BACKWARD RUNGEKUTA FOR COSTATES
[Tp, P]=rk4back(@ppholling_costate,t0, tf,N,init2,U,X,Constant);
%UPDATE THE CONTROLS
x = X(1,:);y = X(2,:);
lambda1 = P(1,:);lambda2 = P(2,:);
% Case0:No control,
u1 =zeros(1,N+1);
% Case1:u1=0,
u1 =min(1,max(0,((q1*h1*lambda1.*x+q2*h2*lambda2.*y)/A)));
Uu= u1';
U = 0.5*Uu + 0.5*oldu; % Convex combination of the controls
Xnew = X';
NumXX =abs(norm(Xnew-Xold));
DenXnew =norm(Xnew);
i=i+1 %Update iteration counter
end
%% PLOTING
```

```
X=X';
Tx =Tx';
XX=X(:,1); YY=X(:,2);
J =B1*XX(end)+B2*YY(end)-sum(A*Uu(:,1).*Uu(:,1)) %Change to the suitable objective function
S=[Tx,X];
cd('C:\Users\Mfano\Desktop\Charlesmfano_mat2_edit')
save('case4State', 'S');
save('case4Control', 'Uu');
save('Cost', 'J');
plot(Tx,U(:,1),'-b', 'linewidth',3);
legend('u1\neq 0');
title(' Control strategy')
xlabel('Time(Years)')
ylabel('Control profile')
%Matlab code for implementation for optimal control strategies
clear
all;
close all;
clc
t0 = 0;
tf=12;
N=100;
time =linspace(t0,tf,N);
y0 = [40 20]; %the estimated initial condition for STATE SYSTEM
%%
%---- SOURCES OF CONSTANTS --
% alpha mu r b a k q1 q2 h1 h2 p B1 B2 A p
Constant = [0.0000674 0.01 1 0.16 0.02 600 0.06 0.0375 2 4 0.6 100 200 1000 0.6];
alpha=Constant(1);
mu=Constant(2);
r=Constant(3);
b=Constant(4);
a=Constant(5);
k=Constant(6);
q1=Constant(7);
q2=Constant(8);
h1=Constant(9);
h2=Constant(10);
p= Constant(11);
B1 = Constant(12); % weight
B2 = Constant(13); % weight
A = Constant(14); % weight
lf = [B1 B2];
%% TEST SECTION
```

```
% U =[0 ];
% [Tx,X] = ode45(@ppholling ,time ,y0 ,[] ,U,Constant);
init =y0;
init2 =lf;
h = (tf-t0)/N;
u = linspace(0,0,N+1);
u1=u';
%u1=u';
U = [u1];
%U=u1;
% Test Rungekuta with ODE45 if they produce the same results
[Tx, X]=rk4foward(@ppholling ,t0, tf ,N, init ,U,Constant);
%% IMPLIMENTATION OF THE ALGORITHM
%Test 1 stoping condition 1
delta = 0.01;
X=init;
i=0; %Initialize iteration counter
mm=size(X);
NumXX =10e10;
Xnew = rand(N+1,mm(2)).*(repmat(X,N+1,1));
DenXnew=norm(Xnew);
while NumXX/DenXnew>delta
Xold = Xnew;
oldu = U;
%FORWARD RUNGE KUTTA FOR STATES
[Tx, X]=rk4foward(@ppholling ,t0, tf ,N, init ,U,Constant);
% BACKWARD RUNGEKUTA FOR COSTATES
[Tp, P]=rk4back(@ppholling_costate ,t0, tf ,N,init2 ,U,X,Constant);
%UPDATE THE CONTROLS
x = X(1,:);y = X(2,:);
lambda1 = P(1,:);lambda2 = P(2,:);
% Case0:No control ,
u1 =zeros(1,N+1);
% Case1:u1=0,
u1 =min(1,max(0,((q1*h1*lambda1.*x+q2*h2*lambda2.*y)/A)));
Uu= u1';
U = 0.5*Uu + 0.5*oldu; % Convex combination of the controls
Xnew = X';
NumXX =abs(norm(Xnew-Xold));
DenXnew =norm(Xnew);
i=i+1 %Update iteration counter
end
%% PLOTING
X=X';
Tx =Tx';
XX=X(:,1); YY=X(:,2);
```

```
J =B1*XX(end)+B2*YY(end)-sum(A*Uu(:,1).*Uu(:,1)) %Change to the
   suitable objective function
S=[Tx,X];
cd('C:\Users\Mfano\Desktop\Charlesmfano_mat2_edit')
save('case4State', 'S');
save('case4Control', 'Uu');
save('Cost', 'J');
plot(Tx,U(:,1),'-b', 'linewidth',3);
legend('u1\neq 0');
title(' Control strategy')
xlabel('Time(Years)')
ylabel('Control profile')
```

Mathematical Theory and Modeling www.iiste.org
ISSN 2224-5804 (Paper) ISSN 2225-0522 (Online)
Vol.8, No.8, 2018

Modelling and Numerical Simulation of Harvested Prey – Predator System Incorporating A Prey Refuge

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Abstract

prey-predator is defined as an interaction between the prey and predator in ecosystem. However, over-harvesting of wildlife resources is an important challenge facing protected area in Africa, a better understanding of the nature would improve the way in which it is managed.

This paper describes Modelling harvested prey–predator model incorporating a prey refuge in which a prey and predator species are affected by over-harvesting. The intention is to investigate the impacts of over-harvesting and make a possible suggestion on how to alleviate the problem. The results obtained from theoretical and numerical analysis of the prey-predator with harvesting showed that, overharvesting affect the prey-predator species negatively. However, the results obtained from numerical analysis of the prey-predator model with control strategies showed that catchibility coefficient and prey refuge has a great impact on both prey and predator species on their population densities.

Keywords: prey-predator system, harvesting, incorporating a prey refuge

1.Introduction

The prey-predator models has become one of the great interest to researchers in mathematics and ecology because they deal with number of factors in environmental problem, such as community morbidity and how to control it and optimal harvest policy to sustain a community (Sagamiko, 2015; A. B. Ashine, 2017.). Therefore, the developed mathematical model of prey-predator interaction of Lotka-Volterra model has motivated extensive study in the area of ecological modelling.

In dynamical system a definite activity done by individual area causes severe destruction to the ecosystem of that area. If such activity is unavoidable then the prevailing authority of the area should plan a regular policy which would keep the destruction of the ecosystems minimal (Kar, 2006). One of such activity is harvesting, which has a strong impact on the dynamic evolution of a population subjected to it however, it has been observed that over exploitation and over-harvesting of population species are commonly practiced in fishing, forestry and wildlife management which is done for the purpose of economic progress (Katsukawa, 2002). It is also agreed that biological species of prey–predator system is harvested unscientifically and exported with the aim of positive economic profit which regularly decreases the resources and eventually the ecosystems collapse.

(Ghosh, 2010; Kar, 2006) argued that using optimal harvesting efforts as controls can help discontinuities cyclic behaviour of the system of the prey-predator which may results to a required state of the ecosystem.

The study of the consequences of hiding behaviour of prey on the dynamics of predator-prey interactions can be recognized as a major issue in applied mathematics and theoretical ecology. However, prey refuge in Game reserve and National parks is mostly practiced by Wildebeest, Cape buffaloes that help them to protect from predator attack, hence reduces their predation rate. Therefore, under such situation it is expected that the addition of a small prey refuge stabilizes prey-predator interactions, the addition of a large refuge leads to almost changeability (i.e. random like prey population outbreak) (Li, 2013). Hence this study employed Holling Type II functional response on its model in which the rate of consumption of predator was assumed to depends on the availability of prey density as the only source of food.

2. Model and its Properties

In this section, we consider two different populations, the prey (x(t)) and predator (y(t)) interaction incorporating a prey refuge in which the model is formulated using deterministic differential equation and its stability analysis is done using Jacobian Matrix while simulation is done using MATLAB software

2.1 Model Assumptions

The ecological setup considers the following assumptions as follows;

(i) Both prey and predator are continuously harvested
(ii) Predator depend on the prey as its favourite food. Thus, in absence of f prey the predator goes to extinction
(iii) We also assumed that there is a refuge habitat where prey species are secured from predation and non-refuge habitat in which the prey are visible to predation
(iv) In absence of harvesting on both species, prey is assumed to grow logistically to the carrying capacity
(v) The rate of increase of the predator depends on the amount of biomass predator converts as food

Then from the above assumptions, we assume $x(t)$ and $y(t)$ represent the population density of prey and predator respectively at time t. with assumption we use Holling type (II) function response to formulate the pre-predator model as follows

$$\frac{dx}{dt} = r\left(1-\frac{x}{K}\right)x - \frac{\alpha(1-p)xy}{1+a(1-p)x} - q_1h_1x \quad (1)$$

$$\frac{dy}{dt} = -\mu y + \frac{\alpha(1-p)xy}{1+a(1-p)x} - q_2h_2y$$

Where $x(t) > 0$ and $y(t) > 0$, also α, K, μ, a, b are all positive constants and r is the intrinsic growth rate of the prey. K is the carrying capacity of the prey in the absence of the predator and harvesting, the term $\frac{\alpha(1-p)xy}{1+\alpha(1-p)x}$ is the functional response of the predator which is a Holling type (II) response functional of the predator, μ is the death rate of the predator, $\frac{\alpha}{a}$ is the maximum number that can be eaten by each predator per unit time, b is the predators for each captured prey, q_1 and q_2 are catchibility coefficient of the prey and predator respectively. P is the proportion of prey population not exposed to predation, that it protects px and leaves $(1-p)x$ of the prey available to predation. Note that $p \in [0, 1]$

3. Model analysis

3.1 Boundedness of the system

The solution of the prey-predator model developed in (1) represents the populations of living individuals and they have their ecological meaning that is to say they must be positive and bounded.

Lemma: All solutions of the system (1) which starts with $\mathcal{R}^{2+}$ are uniformly bounded.

Proof: To prove the theorem, we define a function

$$W(t) = x(t) + \frac{\alpha}{\alpha b} y(t) \quad (2)$$

which simplifies to

$$W(t) = x(t) + \frac{1}{b} y(t) \quad (3)$$

Where $W(t)$ represents total population of the prey and predator species, we differentiate equation (3) with respect to t above as;

$$\frac{dW}{dt} = \frac{dx}{dt} + \frac{1}{b}\frac{dy}{dt} \quad (4)$$

Then substitute equation (1) into equation (4)

$$\frac{dW}{dt} = r\left(1 - \frac{x}{K}\right)x - \frac{\alpha(1-p)xy}{1+a(1-p)x} - q_1h_1x + \frac{1}{b}\left(-\mu y + \frac{\alpha(1-p)xy}{1+a(1-p)x} - q_2h_2y\right) \quad (5)$$

Then equation (5) will be simplified as follows;

$$\frac{dW}{dt} = r\left(1 - \frac{x}{K}\right)x - \frac{\alpha(1-p)xy}{1+a(1-p)x} - q_1h_1x + \frac{1}{b}(-\mu - q_2h_2)y + \frac{\alpha(1-p)xy}{1+a(1-p)x}$$

Then all terms of interspecific competition are cancelled out

$$\frac{dW}{dt} = r\left(1 - \frac{x}{K}\right)x - q_1h_1x + \frac{1}{b}(-\mu - q_2h_2)y$$

Also, on simplification we have

$$\frac{dW}{dt} = rx - \frac{rx^2}{K} - q_1h_1x + \frac{1}{b}(-\mu - q_2h_2)y$$

$$\text{We let } E_1 = q_1h_1 \text{ and } E_2 = q_2h_2$$

Then we have the simplified equation as follows $\frac{dW}{dt} = (r - E_1)\,x - \frac{r\,x^2}{K} - \frac{1}{b}(\mu + E_2)\,y$

Let the arbitrary constant to be Ω then the equation above will be written as follows

$$\frac{dW}{dt} = (r - E_1)\,x - \frac{r\,x^2}{K} - \frac{1}{b}(\mu + E_2)\,y + \Omega W(t) - \Omega W(t)$$

Thus;

$$\frac{dW}{dt} + \Omega W(t) \leq (r - E_1)\, x - \frac{r\, x^2}{K} - \frac{1}{b}(\mu + E_2)\, y + \Omega\left(x(t) + \frac{1}{b} y(t)\right) \tag{6}$$

Using the concept of perfect square

$$\frac{dW}{dt} + \Omega W(t) \leq (r - E_1 + \Omega)\, x - \frac{r\, x^2}{K} - \frac{1}{b}(\mu + E_2 - \Omega)\, y$$

Then it follows

$$\frac{dW}{dt} + \Omega W(t) \leq \frac{K}{4\, r}(r - E_1 + \Omega)^2 \; - \frac{r}{K}\left(x^2 - (r - E_2 + \Omega)\frac{K}{r}\right)^2 - \frac{1}{b}(\mu + E_2 + \Omega)\, y$$

$$\text{But } \frac{K}{4\, r}(r - E_1 + \Omega)^2 = \max\left[\frac{r}{K}\left(x^2 - (r - E_2 + \Omega)\frac{K}{r}\right)^2\right]$$

$$\text{Also letting the } \frac{K}{4\, r}(r - E_1 + \Omega)^2 = m_1$$

Thus

$$\frac{dW}{dt} + \Omega W(t) \leq m_1 \tag{7}$$

Solving equation (7) differential inequality using integrating factor $I = e^{\Omega t}$ yields

$$W(t)e^{\Omega t} \leq \frac{m_1}{\Omega} + Ce^{-\Omega t} \tag{8}$$

At $t = 0$ equation in (8) becomes

$$W(0) = \frac{m_1}{\Omega} + \left(W(0) - \frac{m_1}{\Omega}\right) e^{-\Omega(0)} \tag{9}$$

As $t \to \infty$ (8)

$$0 \leq W(t) \leq \frac{m_1}{\Omega}$$

Therefore $W(t)$ is bounded and from positivity of x and y it follows

$$0 \leq x(t) \leq \frac{m_1}{\Omega}$$

and

$$0 \leq y(t) \leq \frac{m_1}{\Omega}$$

3.2 Analysis of the stability of the equilibrium points

In this section, we establish condition for the existence of equilibrium pons of the model equation (1) the system has at least four equilibrium points obtained by setting $\frac{dy}{dt} = 0$ and $\frac{dx}{dt} = 0$ by so doing we get the possible equilibrium points of the system as;

(i) $E_0(0,0)$ is the extinction of both species, prey and predator
(ii) $E_1(x,0)$ is the predator extinction
(iii) $E_2(0,y)$ is the prey extinction
(iv) $E_3(x,y)$ the coexistence or equalibrium point of the system

But $E_0(0,0)$ point is trial. The existence of the rest of the fixed equilibrium points are described below

(i) The existence of $E_1(x^*, 0)$ with $x^* > 0$

Let $y = 0$ the system of equation reduces to

$$0 = r\left(1 - \frac{x^*}{K}\right)x^* - q_1h_1x^*$$

On simplifying we have

$$x^*\left(r - \frac{rx^*}{K} - q_1h_1\right) = 0$$

Thus $x^* = \frac{K(r-q_1h_1)}{r}$

Therefore $E_1(x^*, 0) = \left(\frac{K(r-q_1h_1)}{r}, 0\right)$

From the expression of x^* we observe that harvesting has negative impact on the prey growth hence affect the prey population density. However, for the predator free equilibrium point $E_1(x^*, 0)$ to exist $r - q_1h_1 > 0$ which implies $r > q_1h_1$. Therefore, in absence of predator the intrinsic growth rate of prey population should be greater than harvesting rate. Hence increasing harvesting of prey species results into decreasing of predator which affects survival of predator species. This is the fact prove that predator depends on the prey as their only source of food.

(ii) The existence of $E_2(0, y^*)$ with $y^* > 0$

Let $x = 0$ the system of equation (1) reduces to $y^*(-\mu - q_1h_1) = 0$ from which we obtain $y^* = 0$ which implies

$$E_2(0, y^*) = E_0(0,0) \tag{10}$$

The results above imply that the predator depend on prey as their only source of food. Thus, in absence of prey, predator populations become exist.

(iii) Co-existence of equilibrium point $E_3(x^*, y^*)$

We equate the equation (1) equals to zero that is to say $\frac{dy}{dt} = 0$ and $\frac{dx}{dt} = 0$ then the system reduces the following equations;

$$r\left(1 - \frac{x}{K}\right)x - \frac{\alpha(1-p)xy}{1 + a(1-p)x} - q_1h_1x = 0$$

$$-\mu y + \frac{\alpha(1-p)xy}{1 + a(1-p)x} - q_2h_2y = 0$$

Using MAPLE software, the co-existence point will be as;

$$x^* = \frac{\mu + H_2}{\big((\mu + H_2)a - \alpha b\big)(p-1)}$$

$$y^* = -\frac{b}{\big((\mu + H_2)a - \alpha b\big)(p-1)}\left(-r\left(\frac{\big((\mu + H_2)a - \alpha b\big)(p-1)K - \mu - H_2}{\big((\mu + H_2)a - \alpha b\big)(p-1)K}\right) + H_2\right)$$

For $H_1 = q_1 h_1$ and $H_2 = q_2 h_2$

Thus, the existence of the point

$$E_3(x^*, y^*) = \left(\frac{\mu+H_2}{\big((\mu+H_2)a-\alpha b\big)(p-1)}, -\frac{b}{\big((\mu+H_2)a-\alpha b\big)(p-1)}\left(-r\left(\frac{\big((\mu+H_2)a-\alpha b\big)(p-1)K-\mu-H_2}{\big((\mu+H_2)a-\alpha b\big)(p-1)K}\right) + H_2\right)\right) \quad (11)$$

From the expression of $E_3(x^*, y^*)$ we observe that predators death rate and harvesting affect the convention factor b (predator biomass to the prey) of newly born predator negatively which in turn results into negative effects on predator population density. However, the co-existence equilibrium point (non -trivial) exist if $\big((\mu + H_2)a - \alpha b\big) > 0$ implying that $\frac{\alpha b}{a} < \mu + H_2$. Therefore, in the absence of both populations birth rate of predator should be greater than the sum of death rate and harvesting of predator. Increasing harvesting to predator population causes rapid decrease of predator which results in increasing of prey population density.

3.3 Stability analysis of the equilibrium points

The stability of the equilibrium points is analyzed by computing the Jacobian matrix and determining the eigenvalues of the Jacobian matrix of each fixed point $E_0(0,0)$, $E_1(x^*, 0)$, $E_2(0, y^*)$ and $E_3(x^*, y^*)$.The equilibrium points are asymptotically stable if the real parts of the eigenvalues of each jacobian matrix are negative. From the system equation (1) the general Jacobian matrix of the equations is given by;

$$J(E_i) = \begin{pmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y} \end{pmatrix}$$

This will be described as follows;

$$J(E_i) = \begin{pmatrix} r\left(1 - \frac{x^*}{K}\right) - \frac{r\,x^*}{K} - \frac{\alpha(1-p)^2 x^* y^* a}{\left(1+a(1-p)\,x^*\right)^2} & -\frac{\alpha(1-p)\,x^*}{1+a(1-p)\,x^*} \\ \frac{b\alpha(1-p)y^*}{1+a(1-p)\,x^*} - \frac{\alpha(1-p)^2 x^* y^* a}{\left(1+a(1-p)\,x^*\right)^2} & -\mu + \frac{b\alpha(1-p)\,x^*}{1+a(1-p)\,x^*} \end{pmatrix} \quad (12)$$

Hence from the Jacobian matrix $J(E_i)$ above the equilibrium point;

(i) **$E_0(0, 0)$ is given by**

$$J(E_0) = \begin{pmatrix} r & 0 \\ 0 & -\mu \end{pmatrix}$$

Thus, using Maple software, the eigenvalues of the Jacobian matrix $J(E_0)$ are r and $-\mu$ However, $E_0(0,\ 0)$ is saddle point under condition that $r > 0$ and all saddles are unstable.

(ii) **For predator free equilibrium point $E_1(x^*, 0) = \left(\frac{K(1-q_1h_1)}{r}, 0\right)$**

The corresponding matrix is written as

$$J(E_1) = \begin{pmatrix} 2q_1h_1 & -\frac{\alpha K(1-q_1K)(1-p)}{r + \alpha K(1-q_1K)(1-p)} \\ 0 & -\mu + \frac{\alpha b(r-q_1h_1)(1-p)}{r + \alpha K(r-q_1h_1)(1-p)} \end{pmatrix} \quad (13)$$

Eigenvalues of $E_1(x^*,0)$ are $2q_1h_1$ and
$-\mu+\frac{\alpha b(r-q_1h_1)(1-p)}{r+\alpha K(r-q_1h_1)(1-p)}$ hence J is locally asymptotically stable if

$$\frac{\alpha b(r-q_1h_1)(1-p)}{r+\alpha K(r-q_1h_1)(1-p)}<\mu \tag{14}$$

(iii) The corresponding Jacobian matrix of the equilibrium point $\mathbf{E_2(0,y^*)}$

$$J(E_2) = \begin{pmatrix} r & 0 \\ 0 & -\mu \end{pmatrix} \tag{15}$$

Hence, we find that $E_0(0,0) = E_2(0,y^*)$ hence the eigen values for Jacobian matrix $J(E_2)$ are r and $-\mu$ where $r>0$ therefore the point at equilibrium $E_2(0,y^*)$ is unstable saddle.

(iv) For co-existence equilibrium point $\mathbf{E_3(x^*,y^*)}$

The jacobian matrix $J(\mathbf{E_3})$ is given by

$$\begin{pmatrix} E_{11} & E_{12} \\ E_{21} & E_{22} \end{pmatrix} \tag{16}$$

Where

$$E_{11}=r\left(1-\frac{r}{K}\right)-\frac{r(H_2+\mu)}{G_2K}-\frac{G_1\alpha\big(1-p(G_4-G_3)\big)}{G_1+a(H_2+\mu)}+M \tag{17}$$

$$M=\frac{{G_2}^2\alpha(1-p)(H_2+\mu)(G_4-G_3)a}{\Big(G_1\big(G_2+a(1-p)(H_2+\mu)\big)\Big)^2}$$

Therefore, on simplification of equation (17)

$$r\left(1-\frac{r}{K}\right)-\frac{r(H_2+\mu)}{G_2K}-\frac{G_1Q}{G_1+a(H_2+\mu)}+\frac{{G_2}^2(H_2+\mu)Qa}{\Big(G_1\big(G_2+a(1-p)(H_2+\mu)\big)\Big)^2}$$

Where

$$Q=(G_4-G_3)\alpha(1-p)$$

$$G_1=a(H_2+\mu)-\alpha b$$

$$G_2=(1-p)[a(H_2+\mu)-\alpha b]$$

$$G_3=\frac{bH_1}{\big(H_2(-1+p)+\mu(-1+p)\big)a(-1+p)-\alpha(-1+p)b(-1+p)}$$

$$G_4=\frac{br\Big(\Big(\big(H_2(-1+p)+\mu(-1+p)\big)a(-1+p)-\alpha(-1+p)b(-1+p)\Big)K-H_2-\mu\Big)}{\big[\big(H_2(-1+p)+\mu(-1+p)\big)a(-1+p)-\alpha(-1+p)b(-1+p)\big]^2K}$$

Again for

$$E_{12}=-\frac{\alpha(H_2+\mu)}{a\mu+aH_2+[a(H_2+\mu)-\alpha b]} \tag{18}$$

$$E_{21}=\frac{b\alpha(1-p)M}{1+G_5}-\frac{b\big(\alpha(1-p)\big)^2(\mu+H_2)Ma}{(1-p)[a(\mu+H_2)-\alpha b](aG_5+1)^2} \tag{19}$$

where

$$M = \frac{br\left(\left(\left(H_2(-1+p)+\mu(-1+p)\right)a(-1+p)-\alpha(-1+p)b(-1+p)\right)K - H_2 - \mu\right)}{\left[\left(H_2(-1+p)+\mu(-1+p)\right)a(-1+p)-\alpha(-1+p)b(-1+p)\right]^2 K} - D$$

And

$$G_5 = \frac{\mu + H_2}{[a(\mu + H_2) - \alpha b]}$$

$$D = \frac{bH_1}{\left((\mu + H_2) - \alpha b\right)}$$

$$E_{22} = -\mu + \frac{b\alpha(H_2 + \mu)}{2a(H_2 + \mu) - \alpha b} \quad (20)$$

The stability of the $J(E_3)$ is stated using the characteristic of polynomial equation techniques using trace and determinant techniques proposition as follows

Preposition 3.1: suppose the jacobian matrix is evaluated at the co-existence equilibrium has characteristic polynomial equation

(21) $$\lambda^2 - \left(\text{trace}\left(J(E_3)\right)\right)\lambda + \text{determinant}\left(J(E_3)\right) = 0$$

Such that trace $(J(\mathbf{E_3})) = E_{11} + E_{22}$ and determinant $\left(J(E_3)\right) = E_{11}E_{22} - E_{12}E_{21}$
The co-existence equilibrium point is locally stable or stable spiral if
$\text{trace}\left(J(E_3)\right) < 0$ and $\text{determinant}\left(J(E_3)\right) > 0$. Also, the interior equilibrium point is Centre (neutral stable) if $\text{trace}\left(J(E_3)\right) = 0$ and $\text{determinant}\left(J(E_3)\right) > 0$

4. Global stability of equilibrium point

Points E_1 and E_2 is shown by linearizing the system of equation (1) and defining appropriate Lyapunov function to separately described each equilibrium point. The linearizing process is done using jacobian technique such that;

$$\frac{dX_i}{dt} = J(E_i)X_i \quad (22)$$

Where $J(E_i)$ is the Jacobian Matrix and X_i is the small perturbation on x_i. Therefore, the system (1) reduces to the following linear system;

$$\frac{dX}{dt} = \left[r\left(1 - \frac{x^*}{K}\right) - \frac{rx^*}{K} - \frac{\alpha(1-p)y^*}{(1| + a(1-p)x^*)^2}\right]X - \left[\frac{\alpha(1-p)x^*}{(1 + a(1-p)x)}\right]Y \quad (23)$$

$$\frac{dY}{dt} = \left[\frac{b\alpha(1-p)y^*}{1 + a(1-p)x^*} - \frac{\alpha b(1-p)^2 y^* x^* a}{(1| + a(1-p)x^*)^2}\right]X + \left[-\mu + \frac{\alpha(1-p)x^*}{(1 + a(1-p)x)}\right]Y$$

The Lyapunov function is chosen as

$$V(X, Y) = \frac{X^2}{2} + \frac{Y^2}{2} \quad (24)$$

The function $V(X, Y)$ is positive definite function since $V(X, Y) \geq 0$ for any values of (X, Y) and it is minimum at the origin that is $V(0, 0) = 0$ the time derivative of $V(X, Y)$ is given by

$$\frac{dV(X,Y)}{dt} = \frac{\partial V}{\partial X}.\frac{dX}{dt} + \frac{\partial V}{\partial Y}.\frac{dY}{dt} \qquad (25)$$

By substituting equation (23) and the partial V into (25) we obtain the relation below;

$$\frac{dV(X,Y)}{dt} = X\left[\left(r\left(1 - \frac{x^*}{K}\right) - \frac{rx^*}{K} - \frac{\alpha(1-p)y^*}{(1| + a(1-p)x^*)^2}\right)X - \left(\frac{\alpha(1-p)x^*}{(1 + a(1-p)x)}\right)Y\right] +$$

$$Y\left[\left(\frac{b\alpha(1-p)y^*}{1+a(1-p)x^*} - \frac{\alpha b(1-p)^2 y^* x^* a}{(1|+a(1-p)x^*)^2}\right)X + \left(-\mu + \frac{\alpha(1-p)x^*}{(1+a(1-p)x)}\right)Y\right] \qquad (26)$$

(i) For fixed $E_1(x^*, 0)$

We substitute the equation $E_1(x^*, 0) = \left(\frac{K(r-q_1h_1)}{r}, 0\right)$ into equation (26) above as follows

$$\frac{dV(X,Y)}{dt} = X^2(q_1h_1 - r) - \left(\frac{\alpha(1-p)(r-q_1h_1)}{1 + \alpha K(1-p)(1-q_1h_1)}\right) \qquad (27)$$

Therefore, from the equation (27) the equilibrium point $E_1(x^*, 0)$ is asymptotically stable if it satisfies the condition that

$$q_1h_1 - r < 0 \qquad (28)$$

Thus, using simple algebraic mathematical manipulation results into $r > q_1h_1$
Hence in absence of the equilibrium point $E_1(x^*, 0)$ is globally stable if the intrinsic growth rate of the prey population is greater than the harvesting rate.

(ii) For steady state $E_3(x^*, y^*)$

Here, we substitute equation (11) into equation (26) to obtain

$$\frac{dV(X,Y)}{dt} = E_{11}X^2 + (E_{12} + E_{21})XY \qquad (29)$$

With usual notation for E_{11}, E_{12} and E_{21} . Therefor the point is globally stable if the condition below holds

$$\frac{dV(X,Y)}{dt} = (E_{11}X^2 + (E_{12} + E_{21})XY) < 0 \qquad (30)$$

5. Numerical Results and Simulation

Numerical simulation in this paper is done in two cases using MATLAB software. The two cases are phase diagram and variation of catchibility coefficient of prey and predator on harvesting rate. The corresponding parameter used in the developed model in equation (1) is described in table (1) below;

Table 1: The table of the corresponding parameters for developed model in equation (1) with their sources;

Parameter	Parameter Names	Parameter values
K	Carrying capacity of the prey	600 (Assumed)
R	Intrinsic growth rate of the prey	1
α		0.00000674
μ	Predator's death rate	0.01
p	Prey refuge	0.6 Chosen from $p \in [0,1]$
q_1	Catchibility coefficient of prey	0.06
q_2	Catchibility Coefficient of predator	0.0375
h_1	Harvesting rate	2
h_2	Harvesting rate	4
B_1	Cost weight	100
B_2		200
A		1000
a		0.02

Case 1 phase diagram of the model in equation (1) after numerical simulation was

(i) Phase diagram for equilibrium point $E_1(x^*, c)$

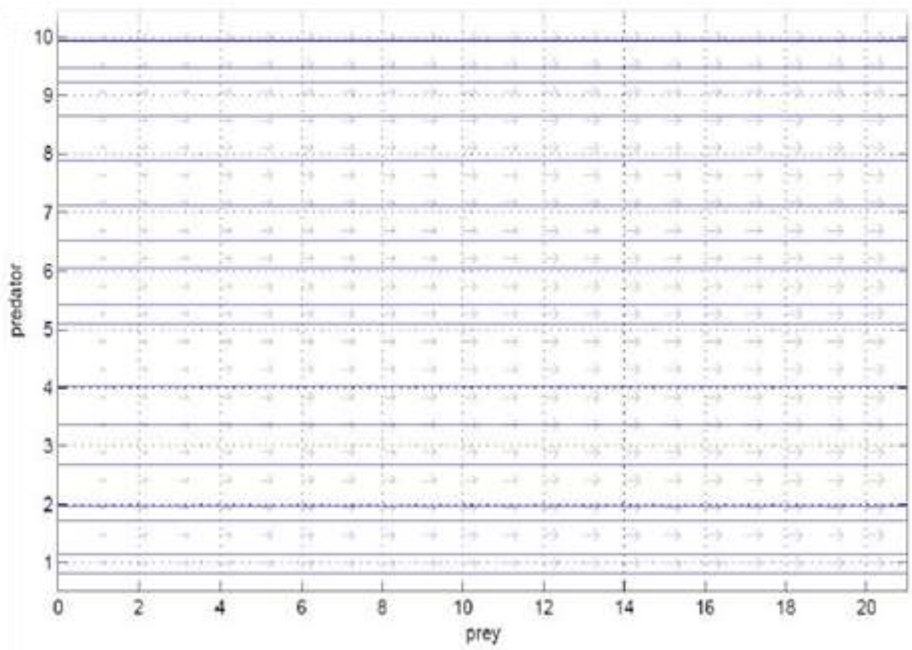


Figure 1: Phase diagram showing dynamic beaviour of $E_1(x^*, c)$

(ii) Phase diagram for equilibrium point $E_3(x^*, y^*)$

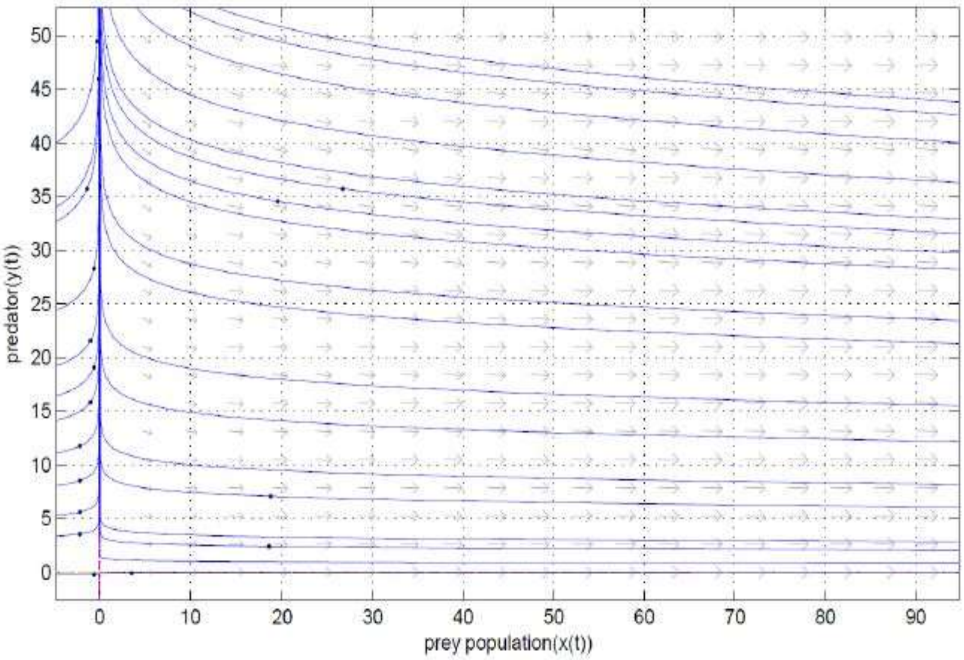


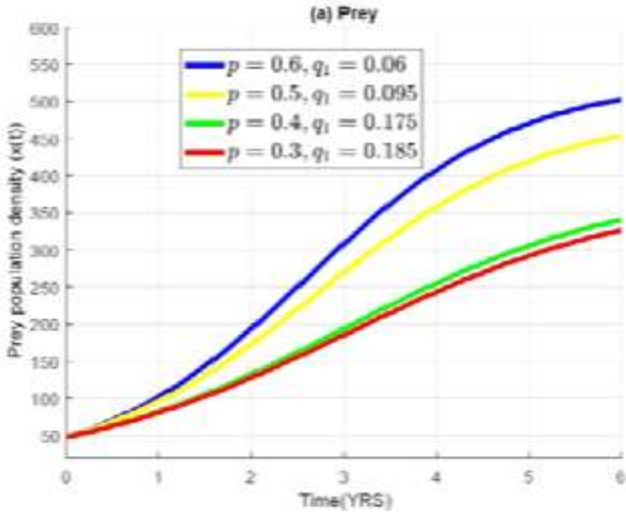
Figure 2: Phase diagram showing dynamical behaviour of $E_3(x^*, y^*)$

Figure (1) above indicate that in the absence of predator while presence of over-harvesting the dynamic equilibrium point of $E_1(x^*, c)$ is unstabe while the dynamic behaviour of co- existence equilibrium point $E_3(x^*, y^*)$ is spiral unstable surrounded by a stable convergence lines at point as shown in figure (2).

Case II: Effects of harvesting without any control strategy

In this section we present figures of harvesting prey and predator species without control using the parameter described in Table 1.

(i) The effect of varying catchibility coefficient on harvesting of prey with effect of prey refuge on prey population density;



igure 3: Showing Simulation of the harvested prey with variation of catchibility co-fiecient and prey refuge

Figure 3 illustrate that at a minimum prey refuge p and high catchibility coefficient q_1 the population density of prey decreases as we see in the figure 3 above. However the Red line shows the catchibility q_1 =0.185 and prey refuge p = 0.3 with only approximately 320 number of prey species , the Green line has catchibility coefficient q_1 = 0.175 and prey refuge 0.4 with approximately 350 number of prey species , the yellow line shows the catchibility coefficient q_1 =0.095 and p = 0.5 with approximately 450 number of prey species and blue line shows catchibility coefficient q_1 = 0.06 and prey refuge p= 0.6 with approximately 500 number prey species, while at maximum prey refuge and low catchibility coefficient q_1 the population density of prey increases. Therefore, from figure 3 we observed that the high the prey refuge and the lower the catchibility coefficient the greater the number of the prey species are saved as shown in the figure above thus we conclude that prey refuge and harvesting have a great impact on prey population density.

(ii) The effect of varying catchibility coefficient on harvesting of predator with effect on predator population density

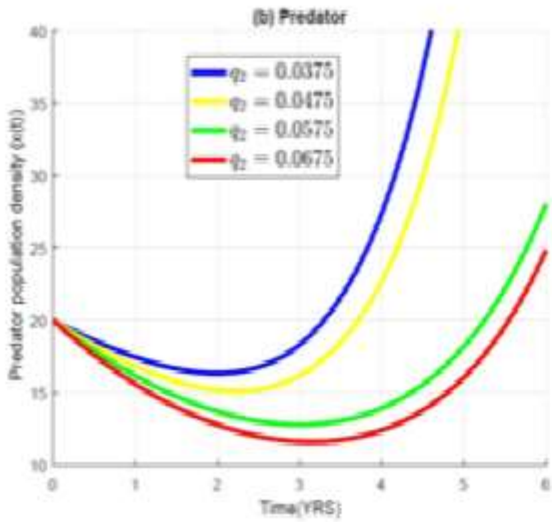


Figure 4: Showing Simulation of the harvested predator with variation of catchibility coefficient

Figure 4 illustrate that at a high catchibility coefficient q_2 the population density of predator decreases, while at low catchibility coefficient q_2 the population density of predator increases. Therefore, from figure 4 we observed that harvesting have a great impact on predator population density as we discussed in theoretically.

6. Discussion, Conclusion and Recommendation

In this paper, we presented Modelling and Numerical simulation of harvested prey predator model incorporating a prey refuge using a deterministic differential equation. The aim was to analyze the effect of harvested prey-predator species we observed that overharvesting, prey refuge and variation of catchibility coefficient of both prey and predator species has great impact on both species on their population growth.

Acknowledgement: The author would like to acknowledge the Nelson Mandela African Institution of science and Technology (NM-AIST), the peer reviewers from the journal of mathematical theory and Modelling for the African Development Bank (AfDB) and supervisors for their support and contribution of their ideas on writing this paper.

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OPTIMALCONTROL OF HARVESTED PREY –PREDATOR SYSTEM INCORPORATING A PREY REFUGE

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November 24, 2018

Abstract

This paper focuses on the formulation, investigation and analysis of Optimal control of a Harvested prey-predator system incorporating a prey refuge using Deterministic Differential Equations. In this study we developed two harvested prey and predator species, in which both prey and predator are affected by over-harvesting, further the predator is affected by prey refuge. The intention is to investigate the impacts of over-harvesting to prey-predator species and suggest control strategies to alleviate the problem of loss of prey and predator species due to over-harvesting. The analysis of optimal control was done using Prontrygian's maximum principle (PMP) and Halmitonian principle. The control strategy suggested is the creation of reserve areas with restriction of harvesting for controlling over-harvesting. The results obtained from theoretical and numerical analysis of the prey-predator with harvesting without control strategies showed that, harvesting affect the prey-predator species negatively. However, the results obtained from numerical

analysis of the prey-predator model with control strategies showed that, the use of control strategy encourage the survival of the prey-predator species.
Key words: prey-predator system,harvesting, incoroprating a prey refuge Optimal control

1 Introduction

In this study we analyse optimal harvested prey-predator model incorporating a prey refuge, in which only one prey and one predator population is considered. The deterministic differential system based on Lotka-Voltera developed model form as ;

$$\begin{aligned} \frac{dN_1}{dt} &= N_1(a_1 - b_{12}N_2) \\ \frac{dN_2}{dt} &= N_2(-a_2 - b_{21}N_1) \end{aligned} \tag{1.1}$$

Where all parameters are taken as positive constants, however the analysis of the developed mode is done using Potryagian's maximum principle (PMP) which provide the Halmitonian H and necessary conditions with which optimal control an the co-state variables must satsify, numerical simulation is done using MATLAB software.

Over-harvesting of wildlife resources is an important challenge facing protected area in Africa, a better understanding of the nature would improve the way in which it is managed is still needed [1]. However familization on the local over-harvesting needed to adress the problem is still needed [1].

In recent years, optimal control theory has found applications in Applied Mathematics (Mathematical ecology and epidemology). The theory has developed rapidly since the first paper by Pontryagin and collaborators in the late 1950' [2]. The word control has a double meaning. First, controlling a system can be understood simply as testing or checking that its behaviour is satisfactory [3]. In a deeper sense, to control is also to act, to put things in order to guarantee that the system behaves as desired [3, 4]. In our context, the second meaning holds, that if we have a prey-predator system which is affected by over-harvesting may lead the system to extinction and we need to act upon the situation to ensure that the system behaves as desired. It is in such situations when concepts like the ''optimal control theory" comes into play.

It is also agreed that biological species of prey–predator system is harvested unscientifically and exported with the aim of positive economic profit which regularly decreases the resources and eventually the ecosystems collapse [5]. Generally [6] argued that using optimal harvesting efforts as controls can help discontinuities cyclic behaviour of the system of the prey-predator which may results to a required state of the ecosystems.

2 Mathematical model

In this section , we propose a deterministic model of one prey and one predator population. that is to say prey($x(t)$) and predator ($y(t)$) with harvesting on both species whereby prey species is considered to under prey refuge meaning that they are secured from predation

2.1 Basic Model

In this section, we consider two different population the prey($x(t)$) and predator ($y(t)$) interaction incorporating a prey refuge. The ecological setup considers the following assumptions.

(i) Both prey and predator are continuously harvested

(ii) Predator depends on the prey as its favourite food.Thus,in the absence of prey predator population goes to extinction

(iii) We also assume that there is a refuge habitant where prey species are protected from predation and non -refuge habitant in which the prey species are exposed to predation

(iv) In absence of harvesting and predator interaction, prey species is assumed to grow logistically to the carrying capacity

(v) The rate of increase of the predators depends on the amount of biomass predators converts as food

Then from the above assumptions, We assume $x(t)$ and $y(t)$ to represents the population density of prey and predator respectively at time t. With assumptions we use the Holling type(II) functional response to formulate the prey predator model as follows

$$\begin{aligned} \frac{dx}{dt} &= r(1-\frac{x}{k})x - \frac{\alpha(1-p)xy}{1+a(1-p)x} - q_1h_1x \\ \frac{dy}{dt} &= -\mu y + \frac{b\ \alpha(1-p)xy}{1+a(1-p)x} - q_2h_2y \end{aligned} \tag{2.1}$$

Where $x(t) > 0$,$y(t) > 0$, α, k,μ ,a,b are all positive constants and r is the intrinsic growth rate of the prey. k is the environment carrying capacity of the prey in the absence of the predation and harvesting. The term $\frac{\alpha x}{1+ax}$ denotes the functional response of the predator which is a Holling type II response functional of the predator,μ is the death rate of the predator, $\frac{\alpha}{a}$ is the maximum number that can be eaten by each predator in unit time, b is the predation convention factor(biomass) denoting the number of newly born predators for each captured prey and q_1 and q_2 are catchability coefficient of the prey and predator respectively. p is the proportion of prey population not exposed to predation, that it protects px of the prey and leaves $(1-p)x$ of the prey available to predation. Note that $p \in [0, 1]$

2.2 proposed model

Therefore we introduce into model equation (2.1) time dependent control effort $(u_1(t))$ on harvesting to alleviate the loss of species in the prey-predator system established in equations (2.1) above.

Hence from the developed model (2.1) we let $u_1(t)$ to represent over-harvesting control strategy (Creation of reserve areas with restriction of harvesting). Thus, the system of equations (2.1) becomes:

$$\begin{aligned} \frac{dx}{dt} &= r(1-\frac{x}{k}) - \frac{\alpha(1-p)xy}{1+a(1-p)x} - (1-u_1(t))q_1h_1x \\ \frac{dy}{dt} &= -\mu y + \frac{b\ \alpha(1-p)xy}{1+a(1-p)x} - (1-u_1(t))q_2h_2y \end{aligned} \tag{2.2}$$

Where α, k, μ, a, b are all positive constants and r is the intrinsic growth rate of the prey. k is the environment carrying capacity of the prey in the absence of the predation and harvesting. The term $\frac{\alpha x}{1+ax}$ denotes the functional response of the predator which is a Holling type II response functional of the predator, μ is the death rate of the predator, $\frac{\alpha}{a}$ is the maximum number that can be eaten by each predator in unit time, b is the predation convention factor(biomass) denoting the number of newly born predators for each captured prey and q_1 and q_2 are catchability coefficient of the prey and predator respectively. p is the proportion of prey population not exposed to predation, that it protects px of the prey and leaves $(1-p)x$ of the prey available to predation. Note that

$p \in [0,1]$

3 Analysis of the Optimal control

3.1 Formulation of the objective function

Here we construct an objective function that provides the optimal population size of the prey-predator species at minimum costs for over-harvesting strategies. Thus the objective functional J is defined over a feasible set of control u_i and applied over the pre-defined finite time interval given by $[T_0, T_1]$. The Objective function of this function will be formed by the following form

$$J(u_i) = [B(x_i(T_1), T_1) - \int_{T_0}^{T_1} (F(u_i(t), t))dt] \tag{3.1}$$

subject to

$$\frac{dx_i}{dt} = f_i(t, u_i(t), x_i(t))$$

where $x_i(T_0) = x_i$ and $0 \leq u_i \leq 1$ for $t \in (T_0, T_1)$. The term $B(x_i(T_1))$ and $F(x_i(t), u_i(t), t)$ represent the prey-predator populations to be optimized at the terminal time control and total cost of control respectively ;

Therefore from (3.1) the objective functional becomes

$$J(U) = Max_u[B_1 x(T_1) + B_2 y(T_1) - \int_0^{T_1} (\frac{Au_1^2}{2})dt] \tag{3.2}$$

subject to;

$$\begin{aligned}
\frac{dx}{dt} &= r(1 - \frac{x}{k})x - \frac{\alpha(1-p)xy}{1 + a(1-p)x} - (1 - u_1(t))q_1 h_1 x \\
\frac{dy}{dt} &= -\mu y + \frac{b\ \alpha(1-p)xy}{1 + a(1-p)x} - (1 - u_1(t))q_2 h_2 y
\end{aligned}$$

for $x(T_0) = x_0$, $y(T_0) = y_0$ and $0 \leq u \leq 1$ for $t \in [0, T_1]$; $u \varepsilon U$

Also the terms $B_1 x(T_1)$ and $B_2 y(T_1)$ represents the prey and predator populations to be optimized at the terminal control and $\frac{Au_1^2}{2}$ is the total control cost for over-harvesting. The cost weight is A and state weights B_1,B_2 are all positive constants . The aim is to maximize u such that

$$J(u^*) = Max(J(u))$$

with

$$0 \leq u \leq 1 \ for \ t \in [T_0, T_1]$$

3.2 Existence of optimal control

The aim is to show that the optimal control problem for the formulated in (3.2) has at least one solution before trying to solve the optimal control values.

Theorem :

Given optimal in (3.2) with u as control variable, then there exist $u \in U$ (Optimal control set) such that $J(u_1^*) = max(J(u_1))$

Proof

The proof for existance of optimal control provided by [7], Fleming .w.H and Rishel.R.W (1975), [3] and [8] is valid such that:

(i) The Model equations (3.1) with control are linear in control variable u and bounded by a linear system in the state and control effort on over-haversting $u_1(t)$

(ii) The control U is convex,closed and bounded set

(iii) The integrad $-\frac{Au_1^2}{2}$ of the objective function (3.1) is concave in U

3.3 Characterization of the Optimal Control

The optimal control must satsify the necessary condition that are formulated by pontrygian's maximum principle [9]. This principle converts equations (3.2) into a problem of maximizing point-wise a Hamiltonian (H) with respect to u_1

$$\begin{aligned} H = & -\frac{Au_1^2}{2} + \lambda_1[r(1-\frac{x}{k})x - \frac{\alpha(1-p)xy}{1+a(1-p)x} - (1-u_1(t))q_1h_1x] \\ & + \lambda_2[-\mu y + \frac{b\ \alpha(1-p)xy}{1+a(1-p)x} - (1-u_1(t))q_2h_2y] \end{aligned} \tag{3.3}$$

Where λ_1 and λ_2 are the adjoint varibles or co-state variable. Applying pontryingin's maximum principle and existance results for the optimal control from (Mappes et al.(2001)). The following preposition is obtained

Preposition 4.1

For Optimal control u_1 that maximize $J(u^*)$ over U , then there exist adjoint varibles λ_1 and λ_2 satsifying

$\frac{d\lambda_i}{dt} = -\frac{\partial H}{\partial x}$; with $\lambda_i(T) = B_i; i = 1, 2$

Proof: Using the adjoint Condition set of the Preposition 4.1 the equation becomes

$$\frac{d\lambda_1}{dt} = -\frac{\partial H}{\partial x} = -\lambda_1[-\frac{rx}{k_1} + r\left(1 - \frac{x}{k_1}\right) - \frac{\alpha\,(1-p)\,y}{1+(1-p)\,ax} + \frac{\alpha\,(1-p)^2\,xya}{(1+(1-p)\,ax)^2} - (1-u_1\,(t))\,q_1h_1]$$
$$- \lambda_2[\frac{b\alpha\,(1-p)\,y}{1+(1-p)\,ax} - \frac{b\alpha\,(1-p)^2\,xya}{(1+(1-p)\,ax)^2}]$$
$$\frac{d\lambda_2}{dt} = -\frac{\partial H}{\partial y} = -\lambda_1[-\frac{\alpha\,(1-p)\,x}{1+(1-p)\,ax}] - \lambda_2[-\mu + \frac{b\alpha\,(1-p)\,x}{1+(1-p)\,ax} - (1-u_1\,(t))\,q_2h_2] \tag{3.4}$$

With transversality Conditions;

$$\lambda_1(T_1) = \frac{d(B_1x(T_1))}{dx} = B_1$$
$$\lambda_2(T_1) = \frac{d(B_1y(T_1))}{dy} = B_2 \tag{3.5}$$

Using Optimality Condition, we have $\frac{\partial H}{\partial u_1} = 0$ at u_1^*

That is

$$\frac{\partial H}{\partial u_1} = -Au_1 + \lambda_1h_1q_1x + \lambda_2h_2q_2y = 0 \tag{3.6}$$

Which gives

$$u_1^* = \frac{\lambda_1h_1q_1x + \lambda_2h_2q_2y}{A} \tag{3.7}$$

The following characterization holds on the interior of the control set

$$u_1^* = min\{1, max\{0, \frac{\lambda_1h_1q_1x + \lambda_2h_2q_2y}{A}\}\} \tag{3.8}$$

Where λ_1 and λ_2 are the solutions of the system of adjoint equation (3.3)

Note that

The state system (3.3) has initial time condition and the co-state system (3.2) has the final time condition

4 Numerical Results and Simulation

In this section optimal control stratergy is numerically solved by several numerical techniques using parameter values. A foward-backward sweep method(FBSM) is one of the numerical techiniques that can be used to solve an optimal control problem .The following scholars [10, 3, 11] ,suggested the method to be esxcuted as follows;

(1) Using the new set of values ,transversality condition $\lambda_{N+1} = \lambda(T)$ ($T=$ final time)and guessed values for control vector ,solve the adjoint vector backward in time using

$RK4$

(2) Make an initial guess for control vector u and use initial conditions (x_0 and y_0) for the state vector to solve for the variables foward in time using Rungekutta 4^{th} order numerical method ($RK4$)

(3) The obtained value for state and adjoint variables are entered on the characterization of the optimal control (3.3) to update the control vector which becomes new value for the control.

(4) Divide the total time interval into N equal subintervals and set the state at different times as $\vec{x}$ =$(x_1, x_2, ...x_{N+1})$ and co-state variables as $\vec{\lambda}$=$(\lambda_1, \lambda_2, ...\lambda_{N+1})$

(5) If the solutions of the variables (excluding the control are variable) are convergent that is to say stop the process when the values of the control varibale in the current and previous iterations are sufficiently close

The investigation of the impact of adding time dependent control variable $u_1(t)$ on the prey-predator system is studied numerically through the application of control strategy $u_1(t)$.

4.1 Control strategy: Creation of reserve areas with restriction of harvesting for controlling of over-harvesting

4.1.1 The impact of u_1 on prey population

In this case the constant and control variable are chosen depending on their relative importance and relative applications of the cost used for controlling the problem. Thus the intial state and parameter variables are chosen as follows; $B_1 = 100$, $B_2 = 200$, $A = 1000$, K=600, r=1, μ=0.01, α=0.00000674, p= 0.6 Chosed from p $\in [0 \;\; 1]$, $q_1 = 0.06$, $q_2 = 0.0375$, $h_1 = 2$, $h_2 = 4$, $b = 0.16$, $a = 0.02$ and state variables are $y(0) = 20$ and $x(0) = 50$.

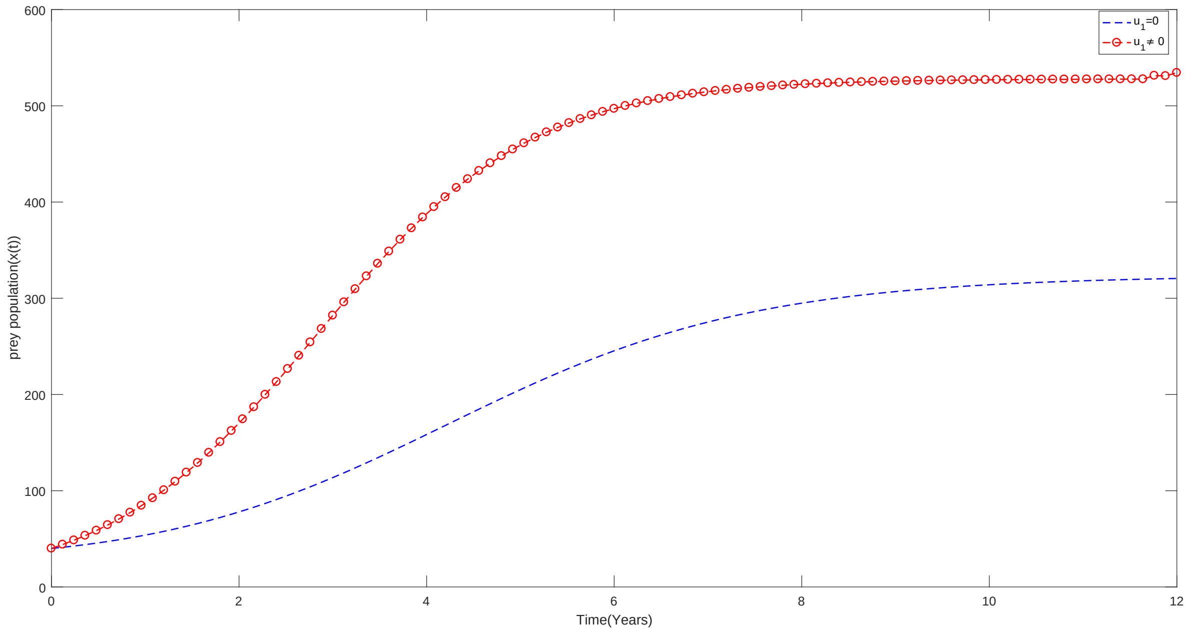

Figure 1: Simulation of a prey affected by over-harvesting showing the impact of creating reserve areas with restriction of harvesting

Figure 1 above shows the application of creating reserve area with restrictions of harvesting prey species, on this atratergy, control u_1 is used to optimize objective functional J. The results in Figure 1 shows a significant difference in prey populations with optimal stratergy ($u_1 \neq 0$) as compared to prey population without control ($u_1 = 0$). This shows that preveting of over-harvesting incorporating a prey refuge in a system lead to repidly increase among prey species. However the increase of population due to control is due to the reason of prey refuge that protect most of prey species from predation hence reduce the source of food to the predator as their only source of food. We observe that as the effort of control increases there is increase in number of prey individuals are saved.

4.1.2 The impact of u_1 on predator population

In this case the constant and control variable are chosen depending on their relative importance and relative applications of the cost used for controlling the problem. Thus the intial state and parameter variables are chosen as follows; $B_1 = 100$, $B_2 = 200$, $A =$ 1000, K=600, r=1, μ=0.01, α=0.00000674, p= 0.6 Chosed from p $\in [0 \;\; 1]$, $q_1 = 0.06$, q_2 $= 0.0375$, $h_1 = 2$, $h_2 = 4$, $b = 0.16$, $a = 0.02$ and state variables are $y(0) = 20$ and $x(0)$ $= 50$.

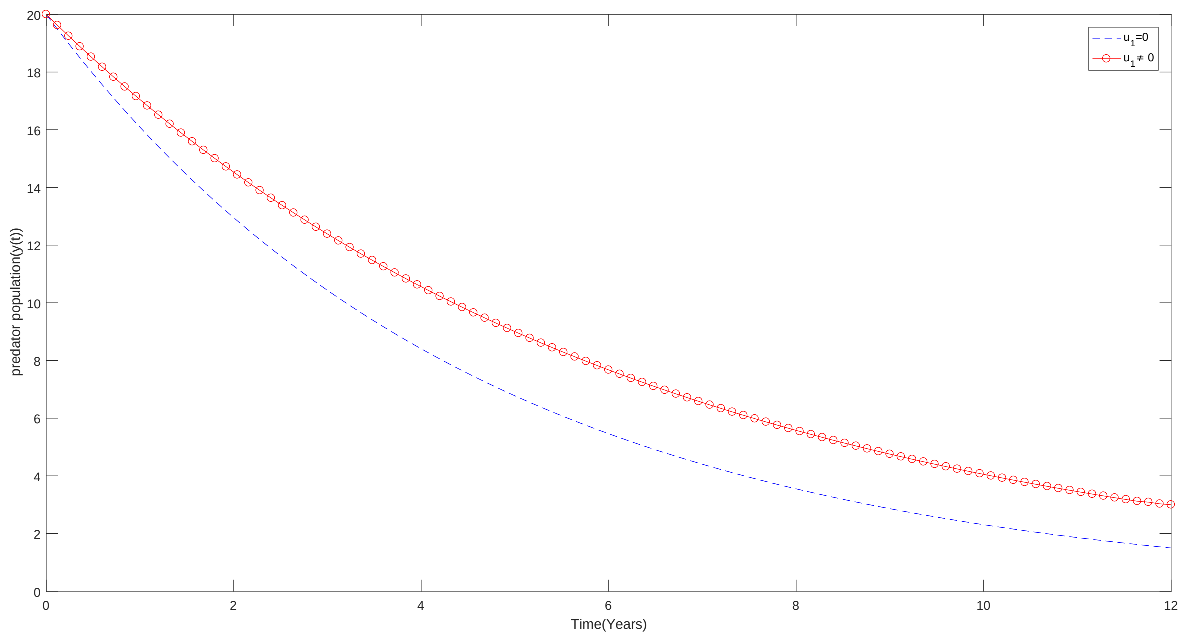

Figure 2: Simulation of a predator affected by over-harvesting showing the impact of creating reserve areas with restriction of harvesting

Figure 2 above shows the application of creating reserve area with restriction of harvesting predator species, on this atratergy, control u_1 is used to optimize objective functional J. The results in Figure 2 shows a significant difference in predator populations with optimal stratergy as compared to predator population without control.This shows that preveting of over-harvesting incorporating a prey refuge in a system lead to increase among predator species. However the increase of population due to control is higher in prey species than in predator species as seen in figure (1) and (2) this is due to the reason of prey refuge that protect most of prey species from predation hence reduce the source of Food to the predator as their only source of food. We observe that as the effort of control increases there is increase in number of predator individuals are saved.

4.1.3 Control profile for control strategy

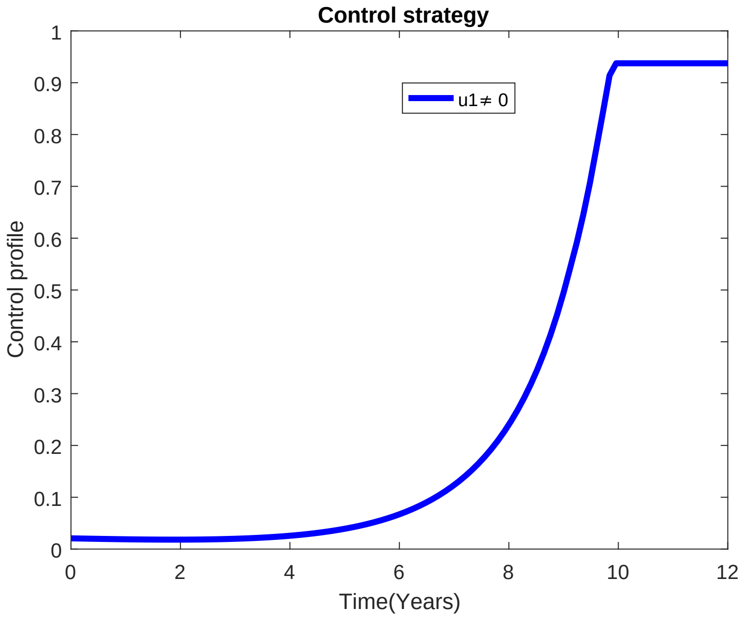

Figure 3: Control profile for u_1

Figure 1 and 2 above shows the application of creating reserve area with restriction of harvesting prey and predator species, on this atratergy, control u_1 is used to optimize objective function J. The results in Figure 1 and 2 shows a significant difference in prey and predator populations with optimal stratergy as compared to prey and predator population without control.This shows that preventing of over-harvesting incorporating a prey refuge in a system lead to repidly increase among prey and predator species. However the increase of population due to control is higher in prey species than in predator species this is due to the reason of prey refuge that protect most of prey species from predation hence reduce the source of Food to the predator as their only source from of food.The control profile is shown in figure 3 , here we see that the optimal harvesting control u_1 increases gradually till time $t = 10$ Years Where it reaches the bound of approximate 0.9 and continues to a final time. We observe that as the effort of control increases there is increase in number of individuals saved.

5 Discussion, Conclusion and Recommendation

In this paper, we presented optimal control of haversted prey predator model incorporating a prey refuge using a deterministic differential equations. The aim isto suggest optimal control strategies to alleviate the loss of prey-predator species by maximizing the number of prey and predator species at the minimal cost. we obeserved that using congtrol stragegy one at a time manages the increase of the number of species.

Acknowledgement: The author would like to acknowledge the Nelson Mandela African Institution of science and Technology (NM-AIST), African Development Bank (AfDB) and supervisors for their support on writting this paper.

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MODELLING OPTIMAL CONTROL OF HARVESTED PREY- PREDATOR SYSTEM INCORPORATING A PREY REFUGE

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Introduction

Prey-predator is described as an interaction between the prey and predator in Ecosystem (Kumar, 2005; Mahapatra & Santra, 2016) . The process of prey and predator to interact one another is called predation. Thus prey-predator system is one of the ways of species interactions (Begon M, 2006). Predation is most commonly considered to be an interaction where an organism (predator) consumes all or part of another living organism (its prey) thereby benefiting itself, but reducing the growth of the prey. For examples, lynx prey upon hares, cheetahs and wild dogs kill gazelles. Competition is a negative interaction that occurs when organisms of dierent species use the same resource(s) at the same time and the growth rate of each species is decreased (Kumar, 2005).

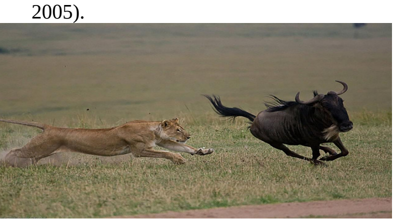


Figure1: Shows the interaction between wildebeest (Prey) and predator (Lion)

Statement of the problem

This study is motivated due to the fact that harvesting of prey-predator is needed in the ecosystem . Previous studies on optimal control strategies concentrated on management policies for sustainability of the ecological species . But optimal control of the harvested prey predator was less considered.

Therefore this study is aimed at applying optimal control theory for harvesting prey-predator system incorporating prey refuge.

Objectives

i. To develop a harvested prey-predator model incorporating a prey refuge.
ii. To analyze the effect of harvesting prey-predator system.
iii. To Modify model (i) to include control variables.
iv. To determine the impact of optimal control strategies on harvested prey-predator system with refuge.

Ecological Assumptions of the prey-predator Model

(i) Both prey and predator are continuously harvested.

(ii) Predator depends on the prey as its source of food. Thus, in the absence of prey, predator population goes to extinction.

(iii) We also assume that there is a refuge habitant where prey species are secured from predation and non-refuge habitant in which the prey species are visible to predation.

(iv) In the absence of harvesting both species , prey is assumed to grow logistically to the carrying capacity.

Model has been developed under basic assumptions.

$$\frac{dx}{dt} = r\left(1-\frac{x}{k}\right)x - \frac{\alpha(1-p)\,x\,y}{1+\alpha(1-p)\,x} - (1-u_1(t))q_1h_1x\,,$$

$$\frac{dy}{dt} = -\mu y + \frac{b\,\alpha(1-p)\,x\,y}{1+\alpha(1-p)\,x} - (1-u_1(t))q_2h_2y$$

Model analysis

❖ Stability analysis is determined by Jacobian matrix

$$J(E_i) = \begin{pmatrix} r\left(1-\frac{x^*}{K}\right) - \frac{r x^*}{K} - \frac{\alpha(1-p)^2x^*y^*a}{(1+a(1-p)x^*)^2} & -\frac{\alpha(1-p)x^*}{1+a(1-p)x^*} \\ \frac{b\alpha(1-p)y^*}{1+a(1-p)x^*} - \frac{\alpha(1-p)^2x^*y^*a}{(1+a(1-p)x^*)^2} & -\mu + \frac{b\alpha(1-p)x^*}{1+a(1-p)x^*} \end{pmatrix}$$

❖ **Global stability** is also asymptotically stable provided that

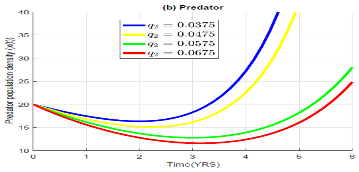
$$\frac{dV(X,Y)}{dt} = (E_{11}X^2 + (E_{12}+E_{21})XY) < 0$$

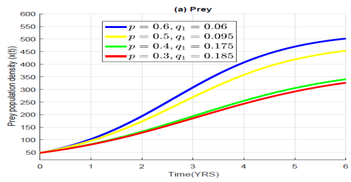
Lemma : All solution s of the system model which start with $\mathcal{R}^+$ are uniformly bounded

Table: Parameter values for prey-predator Model

parameter	values	source
K	600	Estimated
r	1	Sagamiko *et al.* (2015)
α	0.0000674	Assumed
μ	0.01	Kar (2006)
P	0.6	chosen from p∈ [0 1]
q_1	0.06	Assumed
q_2	0.0375	Assumed
h_1	2	Assumed
h_2	4	Assumed
b	0.16	Assumed
B_1	100	Assumed
B_2	200	Assumed
A	1000	(Sagamiko *et al.*, 2015)
a	0.02	Assumed

Numerical analysis: For variaraion of harvesting





Conclusion

Despite of the competition between ecological balance and economic progress but Over-harvesting of wildlife resources is an important challenge facing protected areas in Africa, a better understanding of the nature would improve the ways in which nature is managed. However, familization on the local over-harvesting is managed by creation of reserve area with restriction of harvesting .

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