Journal de Théorie des Nombres de Bordeaux

anciennement Séminaire de Théorie des Nombres de Bordeaux

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Tome 33, nº 1 (2021), p. 17-38.

<http://jtnb.centre-mersenne.org/item?id=JTNB_2021__33_1_17_0>

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de Bordeaux **33** (2021), 17–38

The twisted forms of a semisimple group over an $\mathbb{F}_q$-curve

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Résumé. Soit C une courbe projective, lisse et connexe définie sur un corps fini $\mathbb{F}_q$. Étant donné un $C - S$-schéma en groupes semisimples où S est un ensemble fini de points fermés de C, nous décrivons l'ensemble de ($\mathcal{O}_S$-classes de) formes tordues de $\underline{G}$ en termes d'invariants géométriques de son groupe fondamental $F(\underline{G})$.

Abstract. Let C be a smooth, projective and geometrically connected curve defined over a finite field $\mathbb{F}_q$. Given a semisimple $C-S$-group scheme $\underline{G}$ where S is a finite set of closed points of C, we describe the set of ($\mathcal{O}_S$-classes of) twisted forms of $\underline{G}$ in terms of geometric invariants of its fundamental group $F(\underline{G})$.

1. Introduction

Let C be a projective, smooth and geometrically connected curve defined over a finite field $\mathbb{F}_q$. Let Ω be the set of all closed points on C. For any $\mathfrak{p} \in \Omega$ let $v_\mathfrak{p}$ be the induced discrete valuation on the (global) function field $K = \mathbb{F}_q(C)$, $\widehat{\mathcal{O}}_\mathfrak{p}$ the ring of integers in the completion $\widehat{K}_\mathfrak{p}$ of K with respect to $v_\mathfrak{p}$, and $k_\mathfrak{p}$ the residue field. Any finite subset $S \subset \Omega$ gives rise to a *Dedekind scheme*, namely, a Noetherian integral scheme of dimension 1 whose local rings are regular; If S is *nonempty* it will be the spectrum of the Dedekind domain

$$\mathcal{O}_S := \{x \in K : v_\mathfrak{p}(x) \geq 0 \ \ \forall\, \mathfrak{p} \notin S\}.$$

Otherwise, if $S = \emptyset$, the corresponding Dedekind scheme is the curve C itself, and we denote by $\mathcal{O}_S$ the structural sheaf of C.

Throughout this paper $\underline{G}$ is an $\mathcal{O}_S$-group scheme whose generic fiber $G := \underline{G} \otimes_{\mathcal{O}_S} K$ is almost-simple, and whose fiber $\underline{G}_\mathfrak{p} = \underline{G} \otimes_{\mathcal{O}_S} \widehat{\mathcal{O}}_\mathfrak{p}$ at any $\mathfrak{p} \in \Omega - S$ is *semisimple*, namely, (connected) reductive over $k_\mathfrak{p}$, and the rank of its root system equals that of its lattice of weights ([12, Exp. XIX Def. 2.7, Exp. XXI Def. 1.1.1]). Let $\underline{G}^{\mathrm{sc}}$ be the universal (central) cover (being simply-connected) of $\underline{G}$, and suppose that its *fundamental group* $F(\underline{G}) := \ker[\underline{G}^{\mathrm{sc}} \twoheadrightarrow \underline{G}]$ (cf. [10, p. 40]) is of order prime to $\mathrm{char}(K)$.

Manuscrit reçu le 17 juin 2019, révisé le 26 novembre 2020, accepté le 21 décembre 2020.
2010 *Mathematics Subject Classification.* 11G20, 11G45, 11R29.
Mots-clefs. Class number, Hasse principle, Tamagawa number, étale cohomology.

A *twisted form* of $\underline{G}$ is an $\mathcal{O}_S$-group that is isomorphic to $\underline{G}$ over some finite étale cover of $\mathcal{O}_S$. We aim to describe explicitly (in terms of some invariants of $F(\underline{G})$ and the group of outer automorphisms of $\underline{G}$) the finite set of all twisted forms of $\underline{G}$, modulo $\mathcal{O}_S$-isomorphisms. This is done first in Section 2 for forms arising from the torsors of the adjoint group $\underline{G}^{\mathrm{ad}}$, and then in Section 3, through the action of the outer automorphisms of $\underline{G}$ on its Dynkin diagram, for all twisted forms. More concrete computations are provided in Sections 4, 5 and 6. The case of type A deserves a special consideration, this is done in Section 7. The Zariski topology is treated in Section 8.

Before we start may we quote B. Conrad in the abstract of [11]: "The study of such $\mathbb{Z}$-groups provides concrete applications of many facets of the theory of reductive groups over rings (scheme of Borel subgroups, automorphism scheme, relative non-abelian cohomology, etc.), and it highlights the role of number theory (class field theory, mass formulas, strong approximation, point-counting over finite fields, etc.) in analyzing the possibilities".

2. Torsors

A *$\underline{G}$-torsor P in the étale topology* is a sheaf of sets on $\mathcal{O}_S$ equipped with a (right) $\underline{G}$-action, which is locally trivial in the étale topology, namely, locally for the étale topology on $\mathcal{O}_S$, this action is isomorphic to the action of G on itself by translation. The associated $\mathcal{O}_S$-group scheme ${}^P\underline{G} = \underline{G}'$, being an inner form of $\underline{G}$, is called the *twist* of $\underline{G}$ by P (e.g., [24, §2.2, Lem. 2.2.3, Exs. 1, 2]). We define $H^1_{\text{ét}}(\mathcal{O}_S, \underline{G})$ to be the set of isomorphism classes of $\underline{G}$-torsors relative to the étale topology (or the flat one; these two cohomology sets coincide when $\underline{G}$ is smooth; cf. [2, VIII Cor. 2.3]). This set is finite ([5, Prop. 3.9]). The sets $H^1(K, G)$ (denoting the Galois cohomology) and $H^1_{\text{ét}}(\widehat{\mathcal{O}}_\mathfrak{p}, \underline{G}_\mathfrak{p})$ are defined similarly.

There exists a canonical map of pointed-sets:

$$\lambda : H^1_{\text{ét}}(\mathcal{O}_S, \underline{G}) \to H^1(K, G) \times \prod_{\mathfrak{p} \notin S} H^1_{\text{ét}}(\widehat{\mathcal{O}}_\mathfrak{p}, \underline{G}_\mathfrak{p}).$$

defined by $[X] \mapsto [(X \otimes_{\mathcal{O}_S} \operatorname{Sp} K) \times \prod_{\mathfrak{p} \notin S} X \otimes_{\mathcal{O}_S} \operatorname{Sp} \widehat{\mathcal{O}}_\mathfrak{p}]$. Let $[\xi_0] := \lambda([\underline{G}])$. The *principal genus* of $\underline{G}$ is then $\ker(\lambda) = \lambda^{-1}([\xi_0])$, namely, the classes of $\underline{G}$-torsors that are generically and locally trivial at all points of $\mathcal{O}_S$. More generally, a *genus* of $\underline{G}$ is any fiber $\lambda^{-1}([\xi])$ where $[\xi] \in \operatorname{Im}(\lambda)$. The *set of genera* of $\underline{G}$ is then:

$$\operatorname{gen}(\underline{G}) := \{\lambda^{-1}([\xi]) \,:\, [\xi] \in \operatorname{Im}(\lambda)\},$$

hence $H^1_{\text{ét}}(\mathcal{O}_S, \underline{G})$ is a disjoint union of all genera.

The ring of S-integral adèles $\mathbb{A}_S := \prod_{\mathfrak{p} \in S} \widehat{K}_\mathfrak{p} \times \prod_{\mathfrak{p} \notin S} \widehat{\mathcal{O}}_\mathfrak{p}$ is a subring of the adèles $\mathbb{A}$. A $\underline{G}$-torsor $P = \operatorname{Iso}(\underline{G}, \underline{G}')$ belongs to the principal genus of

$\underline{G}$ if it is both $\mathbb{A}_S$- and K-trivial, hence the principal genus bijects as a pointed-set to the S-*class set* of $\underline{G}$ (see [21, Thm. I.3.5]):

$$\mathrm{Cl}_S(\underline{G}) := \underline{G}(\mathbb{A}_S)\backslash\underline{G}(\mathbb{A})/G(K).$$

Being finite ([5, Prop. 3.9]), its cardinality, called the S-*class number* of $\underline{G}$, is denoted $h_S(\underline{G})$. As $\underline{G}$ is assumed to have connected fibers, by Lang's Theorem (recall that all residue fields are finite) all $H^1_{\text{ét}}(\widehat{\mathcal{O}}_{\mathfrak{p}}, \underline{G}_{\mathfrak{p}})$ vanish, which indicates that any two $\underline{G}$-torsors share the same genus if and only if they are K-isomorphic.

The universal cover of $\underline{G}$ forms a short exact sequence of étale $\mathcal{O}_S$-groups (cf. [10, p. 40]):

$$1 \to F(\underline{G}) \to \underline{G}^{\mathrm{sc}} \to \underline{G} \to 1. \tag{2.1}$$

This gives rise by étale cohomology to the co-boundary map of pointed sets:

$$\delta_{\underline{G}} : H^1_{\text{ét}}(\mathcal{O}_S, \underline{G}) \twoheadrightarrow H^2_{\text{ét}}(\mathcal{O}_S, F(\underline{G})) \tag{2.2}$$

which is surjective by ([13, Cor. 1]) as $\mathcal{O}_S$ is of *Douai-type* (see [17, Def. 5.2 and Exam. 5.4 (iii),(v)]). It follows from the fact that $H^2_{\text{ét}}(\mathcal{O}_S, \underline{G}^{\mathrm{sc}})$ (resp., $H^2_{\text{ét}}(\mathcal{O}_S, \underline{G}^{\mathrm{sc}})$) has only trivial classes and in finite number ([13, Thm. 1.1]).

A representation $\rho : \underline{G}^{\mathrm{sc}} \to \underline{\mathbf{GL}}_1(A)$ where A is an Azumaya $\mathcal{O}_S$ algebra, is said to be *center-preserving* if $\rho(Z(\underline{G})^{\mathrm{sc}}) \subseteq Z(\underline{\mathbf{GL}}_1(A))$. The restriction of ρ to $F(\underline{G}) \subseteq Z(\underline{G}^{\mathrm{sc}})$, composed with the natural isomorphism $Z(\underline{\mathbf{GL}}_1(A)) \cong \underline{\mathbb{G}}_m$, is a map $\Lambda_\rho : F(\underline{G}) \to \underline{\mathbb{G}}_m$, thus inducing a map: $(\Lambda_\rho)_* : H^2_{\text{ét}}(\mathcal{O}_S, F(\underline{G})) \to H^2_{\text{ét}}(\mathcal{O}_S, \underline{\mathbb{G}}_m) \cong \mathrm{Br}(\mathcal{O}_S)$. Together with the preceding map $\delta_{\underline{G}}$ we get the map of pointed-sets:

$$(\Lambda_\rho)_* \circ \delta_{\underline{G}} : H^1_{\text{ét}}(\mathcal{O}_S, \underline{G}) \to \mathrm{Br}(\mathcal{O}_S), \tag{2.3}$$

which associates any class of $\underline{G}$-torsors with a class of Azuamaya $\mathcal{O}_S$-algebras in $\mathrm{Br}(\mathcal{O}_S)$.

When $F(\underline{G}) = \underline{\mu}_m$, the following composition is surjective:

$$w_{\underline{G}} : H^1_{\text{ét}}(\mathcal{O}_S, \underline{G}) \xrightarrow{\delta_{\underline{G}}} H^2_{\text{ét}}(\mathcal{O}_S, F(\underline{G})) \xrightarrow{i_*^{(2)}} {}_m\mathrm{Br}(\mathcal{O}_S), \tag{2.4}$$

and coincides with $(\Lambda_\rho)_* \circ \delta_{\underline{G}}$.

The original *Tits algebras* introduced in [26], are central simple algebras defined over a field, associated to algebraic groups defined over that field. This construction was generalized to group-schemes over rings as shown in [22, Thm. 1]. We briefly recall it here over $\mathcal{O}_S$: Being semisimple, $\underline{G}$ admits an inner form $\underline{G}_0$ which is *quasi-split* (in the sense of [12, XXIV, 3.9], namely, not only requiring a Borel subgroup to be defined over $C - S$ but some additional data involving the scheme of Dynkin diagrams, see [10, Def. 5.2.10.]).

Definition 1. Any center-preserving representation $\rho_0 : \underline{G}_0 \to \underline{\mathbf{GL}}(V)$ gives rise to a "twisted" center-preserving representation: $\rho : \underline{G} \to \underline{\mathbf{GL}}_1(A_\rho)$, where A_ρ is an Azumaya $\mathcal{O}_S$-algebra, called the *Tits algebra corresponding to the representation* ρ, and its class in $\mathrm{Br}(\mathcal{O}_S)$, is its *Tits class.*

Lemma 2.1. *If $\underline{G}$ is adjoint, then for any center-preserving representation ρ of $\underline{G}_0^{\mathrm{sc}}$, and a twisted $\underline{G}$-form ${}^P\underline{G}$ by a $\underline{G}$-torsor P, one has: $((\Lambda_\rho)_* \circ \delta_{\underline{G}})([{}^P\underline{G}])] = [{}^PA_\rho] - [A_\rho] \in \mathrm{Br}(\mathcal{O}_S)$ where $[{}^PA_\rho]$ and $[A_\rho]$ are the Tits classes of $({}^P\underline{G})^{\mathrm{sc}}$ and $\underline{G}^{\mathrm{sc}}$ corresponding to ρ, respectively.*

Proof. By descent $F(\underline{G}_0) \cong F(\underline{G})$, so we may write the short exact sequences of $\mathcal{O}_S$-groups:

$$\begin{aligned} &1 \to F(\underline{G}) \to \underline{G}^{\mathrm{sc}} \to \underline{G} \to 1 \\ &1 \to F(\underline{G}) \to \underline{G}_0^{\mathrm{sc}} \to \underline{G}_0 \to 1 \end{aligned} \tag{2.5}$$

which yield the following commutative diagram of pointed sets (cf. [16, IV, Prop. 4.3.4]):

$$\begin{array}{ccc} H^1_{\text{ét}}(\mathcal{O}_S, \underline{G}_0) & \xrightarrow{=} & H^1_{\text{ét}}(\mathcal{O}_S, \underline{G}) \\ \downarrow{\scriptstyle \delta_0} & & \downarrow{\scriptstyle \delta_{\underline{G}}} \\ H^2_{\text{ét}}(\mathcal{O}_S, F(\underline{G})) & \xrightarrow[=]{r_{\underline{G}}} & H^2_{\text{ét}}(\mathcal{O}_S, F(\underline{G})) \end{array} \tag{2.6}$$

in which $r_{\underline{G}}(x) := x - \delta_0([\underline{G}])$, so that $\delta_{\underline{G}} = r_{\underline{G}} \circ \delta_0$ maps $[\underline{G}]$ to $[0]$. The image of any twisted form ${}^P\underline{G}$ where $[P] \in H^1_{\text{ét}}(\mathcal{O}_S, \underline{G})$ (see in Section 1), under the coboundary map

$$\delta : H^1_{\text{ét}}(\mathcal{O}_S, \underline{G}_0) \to H^2_{\text{ét}}(\mathcal{O}_S, Z(\underline{G}_0^{\mathrm{sc}}))$$

induced by the universal covering of $\underline{G}_0$ corresponding to ρ, is $[{}^PA_\rho]$, where ${}^PA_\rho$ is the Tits-algebra of $({}^P\underline{G})^{\mathrm{sc}}$ (see [22, Thm. 1]). But $\underline{G}_0$ is adjoint, so $Z(\underline{G}_0^{\mathrm{sc}}) = F(\underline{G}_0) \cong F(\underline{G})$, thus the images of δ and δ_0 coincide in $\mathrm{Br}(\mathcal{O}_S)$, whence:

$$((\Lambda_\rho)_*(\delta_{\underline{G}}([\underline{G}'])) = ((\Lambda_\rho)_*(\delta_0([{}^P\underline{G}]) - \delta_0([\underline{G}])) = [{}^PA_\rho] - [A_\rho]. \qquad \square$$

The fundamental group $F(\underline{G})$ is a finite, of multiplicative type (cf. [12, XXII, Cor. 4.1.7]), commutative and smooth $\mathcal{O}_S$-group (as its order is assumed prime to $\mathrm{char}(K)$).

Lemma 2.2. *If $\underline{G}$ is not of type* A*, or $S = \emptyset$, then $H^1_{\text{ét}}(\mathcal{O}_S, \underline{G})$ is isomorphic to $H^2_{\text{ét}}(\mathcal{O}_S, F(\underline{G}))$.*

Proof. Applying étale cohomology to sequence (2.1) yields the exact sequence:

$$H^1_{\text{ét}}(\mathcal{O}_S, \underline{G}^{\mathrm{sc}}) \to H^1_{\text{ét}}(\mathcal{O}_S, \underline{G}) \xrightarrow{\delta_{\underline{G}}} H^2_{\text{ét}}(\mathcal{O}_S, F(\underline{G}))$$

in which $\delta_{\underline{G}}$ is surjective (see (2.2)). If $\underline{G}$ is not of absolute type A, it is locally isotropic everywhere ([6, 4.3 and 4.4]), in particular at S. This is of course redundant when $S = \emptyset$. Thus $H^1_{\text{ét}}(\mathcal{O}_S, \underline{G}^{\text{sc}})$ vanishes ([4, Lem. 2.3]). Changing the base-point in $H^1_{\text{ét}}(\mathcal{O}_S, \underline{G})$ to any $\underline{G}$-torsor P, it is bijective to $H^1_{\text{ét}}(\mathcal{O}_S, {}^P\underline{G})$ where ${}^P\underline{G}$ is an inner form of $\underline{G}$ (see Section 1), thus an $\mathcal{O}_S$-group of the same type. Similarly all fibers of $\delta_{\underline{G}}$ vanish. This amounts to $\delta_{\underline{G}}$ being injective thus an isomorphism. □

The following two invariants of $F(\underline{G})$ were defined in [4, Def. 1]:

Definition 2. Let R be a finite étale extension of $\mathcal{O}_S$. We define:

$$i(F(\underline{G})) := \begin{cases} {}_m\mathrm{Br}(R) & F(\underline{G}) = \mathrm{Res}_{R/\mathcal{O}_S}(\underline{\mu}_m) \\ \ker({}_m\mathrm{Br}(R) \xrightarrow{N^{(2)}} {}_m\mathrm{Br}(\mathcal{O}_S)) & F(\underline{G}) = \mathrm{Res}^{(1)}_{R/\mathcal{O}_S}(\underline{\mu}_m) \end{cases}$$

where for a group $*$, ${}_m*$ stands for its m-torsion part, and $N^{(2)}$ is induced by the norm map $N_{R/\mathcal{O}_S}$.

For $F(\underline{G}) = \prod_{k=1}^r F(\underline{G})_k$ where each $F(\underline{G})_k$ is one of the above, $i(F(\underline{G})) := \prod_{k=1}^r i(F(\underline{G})_k)$.

We also define for such R:

$$j(F(\underline{G})) := \begin{cases} \mathrm{Pic}(R)/m & F(\underline{G}) = \mathrm{Res}_{R/\mathcal{O}_S}(\underline{\mu}_m) \\ \ker(\mathrm{Pic}(R)/m \xrightarrow{N^{(1)}/m} \mathrm{Pic}(\mathcal{O}_S)/m) & F(\underline{G}) = \mathrm{Res}^{(1)}_{R/\mathcal{O}_S}(\underline{\mu}_m) \end{cases}$$

where $N^{(1)}$ is induced by $N_{R/\mathcal{O}_S}$, and again

$$j\left(\prod_{k=1}^r F(\underline{G})_k\right) := \prod_{k=1}^r j(F(\underline{G})_k).$$

Definition 3. We call $F(\underline{G})$ *admissible* if it is a finite direct product of factors of the form:

(1) $\mathrm{Res}_{R/\mathcal{O}_S}(\underline{\mu}_m)$,
(2) $\mathrm{Res}^{(1)}_{R/\mathcal{O}_S}(\underline{\mu}_m)$, $[R : \mathcal{O}_S]$ is prime to m,

where R is any finite étale extension of $\mathcal{O}_S$.

Lemma 2.3. *If $F(\underline{G})$ is admissible then there exists a short exact sequence of abelian groups:*

$$1 \to j(F(\underline{G})) \to H^2_{\text{ét}}(\mathcal{O}_S, F(\underline{G})) \xrightarrow{\bar{i}_*} i(F(\underline{G})) \to 1. \tag{2.7}$$

This sequence splits thus reads: $H^2_{\text{ét}}(\mathcal{O}_S, F(\underline{G})) \cong j(F(\underline{G})) \times i(F(\underline{G}))$.

Proof. This sequence was shown in [4, Cor. 2.9] for the case S is nonempty. The proof based on applying étale cohomology to the related Kummer exact sequence is similar for $S = \emptyset$. The splitting when $F(\underline{G})$ is quasi-split was proved in [15, Thm. 1.1]. When $F(\underline{G}) = \mathrm{Res}^{(1)}_{R/\mathcal{O}_S}(\underline{\mu}_m)$, $[R : \mathcal{O}_S]$ prime to

m, consider the exact diagram obtained by étale cohomology applied to the Kummer exact sequences related to $\underline{\mu}_m$ over $\mathcal{O}_S$ and R:

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \mathrm{Pic}(R)/m & \longrightarrow & H^2_{\text{ét}}(R,\underline{\mu}_m) & \xrightarrow{i_*} & \mathrm{Br}(R)[m] & \longrightarrow & 1 \\ & & \downarrow{\scriptstyle N^{(1)}/m} & & \downarrow{\scriptstyle N^{(2)}} & & \downarrow{\scriptstyle N^{(2)}[m]} & & \\ 1 & \longrightarrow & \mathrm{Pic}(\mathcal{O}_S)/m & \longrightarrow & H^2_{\text{ét}}(\mathcal{O}_S,\underline{\mu}_m) & \longrightarrow & \mathrm{Br}(\mathcal{O}_S)[m] & \longrightarrow & 1. \end{array} \tag{2.8}$$

The splitting of the two rows then implies the one in the assertion. □

As a result we have two bijections as pointed-sets: the first is $\mathrm{gen}(\underline{G}) \cong i(F(\underline{G}))$; the affine case shown in [4, Cor. 3.2] holds as aforementioned for $S = \emptyset$ as well, in which case $\mathrm{Br}(C)$ is trivial ([8, Thm. 4.5.1.(v)]) thus $\underline{G}$ admits a single genus. The second bijection is $\mathrm{Cl}_S(\underline{G}) \cong i(F(\underline{G}))$ unless $\underline{G}$ is anisotropic at S, for which it does not have to be injective [4, Prop. 4.1]; hence when S is empty this bijection is guaranteed. Combining Lemma 2.2 with Lemma 2.3 these form (unless $\underline{G}$ is anisotropic at S) an isomorphism of finite abelian groups:

$$H^1_{\text{ét}}(\mathcal{O}_S,\underline{G}) \cong j(F(\underline{G})) \times i(F(\underline{G})). \tag{2.9}$$

3. Twisted-forms

Before continuing with the classification of $\underline{G}$-forms, we would like to recall the following general construction due to Giraud and prove one related Lemma. Let R be a unital commutative ring. A central exact sequence of étale R-group schemes:

$$1 \to A \xrightarrow{i} B \xrightarrow{\pi} C \to 1 \tag{3.1}$$

induces by étale cohomology a long exact sequence of pointed-sets ([16, III, Lem. 3.3.1]):

$$1 \to A(R) \to B(R) \to C(R) \to H^1_{\text{ét}}(R,A) \xrightarrow{i_*} H^1_{\text{ét}}(R,B) \to H^1_{\text{ét}}(R,C) \tag{3.2}$$

in which $C(R)$ acts "diagonally" on the elements of $H^1_{\text{ét}}(R,A)$ in the following way: For $c \in C(R)$, a preimage X of c under $B \to C$ is a A-bitorsor, i.e., $X = bA = Ab$ for some $b \in B(R')$, where R' is a finite étale extension of R ([16, III, 3.3.3.2]). Then given $[P] \in H^1_{\text{ét}}(R,A)$:

$$c * P = P \overset{A}{\wedge} X = (P \times X)/(pa, a^{-1}x). \tag{3.3}$$

The exactness of (3.2) implies that $B(R) \xrightarrow{\pi} C(R)$ is surjective if and only if $\ker(i_*) = 1$. This holds true starting with any twisted form ${}^PB$ of B, $[P] \in H^1_{\text{ét}}(R,A)$.

Lemma 3.1. *The following are equivalent:*

(1) *the push-forward map* $H^1_{\text{ét}}(R,A) \xrightarrow{i_*} H^1_{\text{ét}}(R,B)$ *is injective,*
(2) *the quotient map* ${}^P B(R) \xrightarrow{\pi} C(R)$ *is surjective for any* $[P] \in H^1_{\text{ét}}(R,A)$*,*
(3) *the* $C(R)$*-action on* $H^1_{\text{ét}}(R,A)$ *is trivial.*

Proof. Consider the exact and commutative diagram (cf. [16, III, Lem. 3.3.4])

$$\begin{array}{ccccccc} B(R) & \xrightarrow{\pi} & C(R) & \longrightarrow & H^1_{\text{ét}}(R,A) & \xrightarrow{i_*} & H^1_{\text{ét}}(R,B) \\ & & & & \cong \big\downarrow \theta_P & & \cong \big\downarrow r \\ {}^P B(R) & \xrightarrow{\pi} & C(R) & \longrightarrow & H^1_{\text{ét}}(R,{}^P A) & \xrightarrow{i'_*} & H^1_{\text{ét}}(R,{}^P B), \end{array}$$

where the map i'_* is obtained by applying étale cohomology to the sequence (3.1) while replacing B by the twisted group scheme ${}^P B$, and θ_P is the induced twisting bijection.

(1) $\Leftrightarrow$ (2). The map i_* is injective if and only if $\ker(i'_*)$ is trivial for any A-torsor P. By exactness of the rows, this is condition (2).

(1) $\Leftrightarrow$ (3). By [16, Prop. III.3.3.3 (iv)], i_* induces an injection of $H^1_{\text{ét}}(R,A)/C(R)$ into $H^1_{\text{ét}}(R,B)$. Thus $i_* : H^1_{\text{ét}}(R,A) \to H^1_{\text{ét}}(R,B)$ is injective if and only if $C(R)$ acts on $H^1_{\text{ét}}(R,A)$ trivially. ☐

Following B. Conrad in [10], we denote the group of outer automorphisms of $\underline{G}$ by Θ.

Proposition 3.2 ([10, Prop. 1.5.1])**.** *Assume* Φ *spans* $X_{\mathbb{Q}}$ *and that* $(X_{\mathbb{Q}},\Phi)$ *is reduced. The inclusion* $\Theta \subseteq \mathbf{Aut}(\mathrm{Dyn}(\underline{G}))$ *is an equality, if the root datum is adjoint or simply-connected, or if* $(X_{\mathbb{Q}},\Phi)$ *is irreducible and* $(\mathbb{Z}\Phi^\vee)^*/\mathbb{Z}\Phi$ *is cyclic.*

Remark 3.3. The only case of irreducible Φ in which the non-cyclicity in Proposition 3.2 occurs, is of type $\mathrm{D}_{2n}(n \geq 2)$, in which $(\mathbb{Z}\Phi^\vee)^*/\mathbb{Z}\Phi \cong (\mathbb{Z}/2)^2$ (cf. [10, Ex. 1.5.2]).

Remark 3.4. Since $\underline{G}$ is reductive, $\mathbf{Aut}(\underline{G})$ is representable as an $\mathcal{O}_S$-group and admits the short exact sequence of smooth $\mathcal{O}_S$-groups (see [12, XXIV, 3.10],[11, §3]):

$$1 \to \underline{G}^{\text{ad}} \to \mathbf{Aut}(\underline{G}) \to \Theta \to 1. \tag{3.4}$$

Applying étale cohomology we get the exact sequence of pointed-sets:

$$\begin{aligned} \mathbf{Aut}(\underline{G})(\mathcal{O}_S) \to \Theta(\mathcal{O}_S) \to H^1_{\text{ét}}(\mathcal{O}_S,\underline{G}^{\text{ad}}) & \\ \xrightarrow{i_*} H^1_{\text{ét}}(\mathcal{O}_S,\mathbf{Aut}(\underline{G})) \to H^1_{\text{ét}}(\mathcal{O}_S,\Theta) & \end{aligned} \tag{3.5}$$

in which by Lemma 3.1 the $\Theta(\mathcal{O}_S)$-action is trivial on $H^1_{\text{ét}}(\mathcal{O}_S, \underline{G}^{\text{ad}})$ if and only if i_* is injective, being equivalent to the surjectivity of

$$({}^P\mathbf{Aut}(\underline{G}))(\mathcal{O}_S) = \mathbf{Aut}({}^P\underline{G})(\mathcal{O}_S) \to \Theta(\mathcal{O}_S)$$

for all $[P] \in H^1_{\text{ét}}(\mathcal{O}_S, \Theta)$ (this action is trivial inside each genus).

It is a classical fact that $H^1_{\text{ét}}(\mathcal{O}_S, \text{Aut}(\underline{G}))$ is in bijection with twisted forms of $\underline{G}$ up to isomorphism (for a general statement of this correspondence, see [7, §2.2.4]). Therefore this pointed-set shall be denoted from now and on by $\mathbf{Twist}(\underline{G})$. This bijection is done by associating any twisted form $\underline{H}$ of $\underline{G}$ with the $\mathbf{Aut}(\underline{G})$-torsor $\mathbf{Iso}(\underline{G}, \underline{H})$. If $\underline{H}$ is an inner-form of $\underline{G}$, then $[H]$ belongs to $\text{Im}(i_*)$ in (3.5).

Sequence (3.4) splits, provided that $\underline{G}$ is quasi-split (as in Section 2). Recall that $\underline{G}$ admits an inner form $\underline{G}_0$ which is quasi-split. Then $\mathbf{Aut}(\underline{G}_0) \cong \underline{G}_0^{\text{ad}} \rtimes \Theta$ (the outer automorphisms group of the two groups are canonically isomorphic). This implies by [14, Lem. 2.6.3] the decomposition

$$\begin{aligned} \mathbf{Twist}(\underline{G}_0) &= H^1_{\text{ét}}(\mathcal{O}_S, \mathbf{Aut}(\underline{G}_0)) \\ &= \coprod_{[P] \in H^1_{\text{ét}}(\mathcal{O}_S, \Theta)} H^1_{\text{ét}}(\mathcal{O}_S, {}^P(\underline{G}_0^{\text{ad}}))/\Theta(\mathcal{O}_S) \end{aligned} \tag{3.6}$$

where the quotients are taken modulo the action (3.3) of $\Theta(\mathcal{O}_S)$ on the ${}^P(\underline{G}^{\text{ad}})$-torsors. But $\mathbf{Twist}(\underline{G}_0) = \mathbf{Twist}(\underline{G})$ and as $\underline{G}_0$ is inner:

$$H^1_{\text{ét}}(\mathcal{O}_S, {}^P(\underline{G}_0^{\text{ad}})) = H^1_{\text{ét}}(\mathcal{O}_S, {}^P(\underline{G}^{\text{ad}})),$$

hence (3.6) can be rewritten as:

$$\mathbf{Twist}(\underline{G}) = \coprod_{[P] \in H^1_{\text{ét}}(\mathcal{O}_S, \Theta)} H^1_{\text{ét}}(\mathcal{O}_S, {}^P(\underline{G}^{\text{ad}}))/\Theta(\mathcal{O}_S). \tag{3.7}$$

The pointed-set $H^1_{\text{ét}}(\mathcal{O}_S, \Theta)$ classifies étale extensions of $\mathcal{O}_S$ whose automorphism group embed into Θ. As all $H^1_{\text{ét}}(\mathcal{O}_S, {}^P(\underline{G}^{\text{ad}}))$ are finite, $\mathbf{Twist}(\underline{G})$ is finite. Together with Lemma 2.2 we get:

Proposition 3.5. *If $\underline{G}$ is not of type* A *then:*

$$\mathbf{Twist}(\underline{G}) \cong \coprod_{[P] \in H^1_{\text{ét}}(\mathcal{O}_S, \Theta)} H^2_{\text{ét}}(\mathcal{O}_S, F({}^P(\underline{G}^{\text{ad}})))/\Theta(\mathcal{O}_S),$$

the $\Theta(\mathcal{O}_S)$-action on each component is carried by Lemma 2.2 from the one on $H^1_{\text{ét}}(\mathcal{O}_S, {}^P(\underline{G}^{\text{ad}}))$, cf. (3.3).

Corollary 3.6. *When $S = \emptyset$, i.e., over C, any outer form of $\underline{G}$ has a unique genus on which $\Theta(\mathcal{O}_S)$ acts trivially, hence one has (including for type* A*):*

$$\mathbf{Twist}(\underline{G}) \cong \coprod_{[P] \in H^1_{\text{ét}}(\mathcal{O}_S, \Theta)} j(F({}^P(\underline{G}^{\text{ad}}))).$$

Since C is smooth, $\mathcal{O}_S$ is a Dedekind ring and any finite étale covering of it is the normalization of $\mathcal{O}_S$ (or of C when $S = \emptyset$) in some finite separable extension of K, which is unramified outside S. So we may look on the fundamental groups over the according extension of fields; The following is the list of all types of absolutely almost-simple K-groups (e.g., [23, p. 333]):

Type of G	$F(G^{\mathrm{ad}})$	$\mathbf{Aut}(\mathrm{Dyn}(\underline{G}))$
${}^1\mathrm{A}_{n-1>0}$	μ_n	$\underline{\mathbb{Z}/2}$
${}^2\mathrm{A}_{n-1>0}$	$R^{(1)}_{L/K}(\mu_n)$	$\underline{\mathbb{Z}/2}$
$\mathrm{B}_n, \mathrm{C}_n, \mathrm{E}_7$	μ_2	0
${}^1\mathrm{D}_n$	$\mu_4,\ n = 2k+1$ $\mu_2 \times \mu_2,\ n = 2k$	$\underline{\mathbb{Z}/2}$
${}^2\mathrm{D}_n$	$R^{(1)}_{L/K}(\mu_4),\ n = 2k+1$ $R_{L/K}(\mu_2),\ n = 2k$	$\underline{\mathbb{Z}/2}$
${}^{3,6}\mathrm{D}_4$	$R^{(1)}_{L/K}(\mu_2)$	$\underline{S_3}$
${}^1\mathrm{E}_6$	μ_3	$\underline{\mathbb{Z}/2}$
${}^2\mathrm{E}_6$	$R^{(1)}_{L/K}(\mu_3)$	$\underline{\mathbb{Z}/2}$
$\mathrm{E}_8, \mathrm{F}_4, \mathrm{G}_2$	1	0

4. Split fundamental group

In the following we show Proposition 3.5. We start with the simple case in which $\Theta = 0$:

Corollary 4.1. *If* $\underline{G}$ *is of the type* $\mathrm{B}_{n>1}, \mathrm{C}_{n>1}, \mathrm{E}_7, \mathrm{E}_8, \mathrm{F}_4, \mathrm{G}_2$*, for which* $F(\underline{G}^{\mathrm{ad}}) \cong \underline{\mu}_m$ *there exists an isomorphism of finite abelian groups* $\mathbf{Twist}(\underline{G}) \cong \mathrm{Pic}(\mathcal{O}_S)/m \times {}_m\mathrm{Br}(\mathcal{O}_S)$.

Proof. As $\underline{G}$ is not of type A this derives from Proposition 3.5, together with the fact that $\Theta(\mathcal{O}_S) = 0$ whence there is a single component $H^1_{\text{ét}}(\mathcal{O}_S, \underline{G}^{\mathrm{ad}})$ on which the action of $\Theta(\mathcal{O}_S)$ is trivial, and the description of the isomorphic group $H^2_{\text{ét}}(\mathcal{O}_S, F(\underline{G}^{\mathrm{ad}}))$ is as in the split case in Lemma 2.3. $\square$

Example 4.2. Given a regular quadratic $\mathcal{O}_S$-form Q of rank $2n+1$, its *special orthogonal group* $\underline{G} = \underline{\mathrm{SO}}_Q$ is smooth and connected of type B_n ([9, Thm. 1.7]). Since $F(\underline{G}) = \underline{\mu}_2$ we assume $\mathrm{char}(K)$ is odd. According to Corollary 4.1 we then get

$$\mathbf{Twist}(\underline{G}) \cong \mathrm{Pic}(\mathcal{O}_S)/2 \times {}_2\mathrm{Br}(\mathcal{O}_S).$$

In case $|S| = 1$ and Q is split by an hyperbolic plane, an algorithm producing explicitly the inner forms of Q is provided in [3, Algorithm 1].

5. Quasi-split fundamental group

Unless $\underline{G}$ is of absolute type D_4, Θ is either trivial or equals $\{\mathrm{id}, \tau : A \mapsto (A^{-1})^t)\}$. In the latter case, τ acts on the $\underline{G}^{\mathrm{ad}}$-torsors via $X = \underline{G}^{\mathrm{ad}} b$, where b is an outer automorphism of $\underline{G}$, defined over some finite étale extension of $\mathcal{O}_S$ (see (3.3)). In particular:

$$\tau * \underline{G}^{\mathrm{ad}} = (\underline{G}^{\mathrm{ad}} \times X)/(ga, a^{-1}x),$$

which is the opposite group $(\underline{G}^{\mathrm{ad}})^{\mathrm{op}}$, as the action is via $a^{-1}x = x(a^t)$, (a is viewed as an element of $\underline{G}^{\mathrm{ad}}$, not as an inner automorphism). Now if τ is defined over $\mathcal{O}_S$, then $\mathbf{Aut}(\underline{G})(\mathcal{O}_S) \to \Theta(\mathcal{O}_S)$ is surjective and $(\underline{G}^{\mathrm{ad}})^{\mathrm{op}}$ is $\mathcal{O}_S$-isomorphic to $\underline{G}^{\mathrm{ad}}$, hence as τ is the only non-trivial element in $\Theta(\mathcal{O}_S)$, the map $H^1_{\text{ét}}(\mathcal{O}_S, {}^P\mathrm{Aut}(\underline{G})) \to H^1_{\text{ét}}(\mathcal{O}_S, \Theta)$ is surjective for all $[P] \in H^1_{\text{ét}}(\mathcal{O}_S, \underline{\Theta})$. This implies by Remark 3.4 that $\Theta(\mathcal{O}_S)$ acts trivially on $H^1_{\text{ét}}(\mathcal{O}_S, \underline{G}^{\mathrm{ad}})$. Otherwise, $\underline{G}^{\mathrm{ad}}$ and $(\underline{G}^{\mathrm{ad}})^{\mathrm{op}}$ represent two distinct classes in $H^1_{\text{ét}}(\mathcal{O}_S, \underline{G}^{\mathrm{ad}})$, being identified by $\Theta(\mathcal{O}_S)$.

For any extension R of $\mathcal{O}_S$ and L of K, we denote $\underline{G}_R := \underline{G} \otimes_{\mathcal{O}_S} R$ and $G_L := G \otimes_K L$, respectively. Let $[A_{\underline{G}}]$ be the Tits class of the universal covering $\underline{G}^{\mathrm{sc}}$ of $\underline{G}$ (see Definition 1). This class does not depend on the choice of the representation ρ of $\underline{G}^{\mathrm{sc}}$, thus its notation is omitted. Recall that when $F(\underline{G})$ splits $w_{\underline{G}^{\mathrm{ad}}}$ defined in (2.4) coincides with $\Lambda_* \circ \delta_{\underline{G}}$. Similarly, when $F(\underline{G}) = \mathrm{Res}_{R/\mathcal{O}_S}(\underline{\mu}_m)$ (*quasi-split*) where $R/\mathcal{O}_S$ is finite étale, $\Lambda_* \circ \delta_{\underline{G}_R}$ and $w_{\underline{G}^{\mathrm{ad}}_R}$ defined over R, coincide.

Proposition 5.1. *Suppose $\Theta \cong \overline{\mathbb{Z}/2}$ and that $F(\underline{G}^{\mathrm{ad}}) = \mathrm{Res}_{R/\mathcal{O}_S}(\underline{\mu}_m)$, R is finite étale over $\mathcal{O}_S$. Then TFAE:*

1. *$\underline{G}_R$ admits an outer automorphism,*
2. *$[A_{\underline{G}_R}]$ is 2-torsion in ${}_m\mathrm{Br}(R)$,*
3. *$\Theta(R)$ acts trivially on $H^1_{\text{ét}}(R, \underline{G}^{\mathrm{ad}}_R)$.*

If, furthermore, $\underline{G}$ is not of type A*, or $S = \emptyset$, then these facts are also equivalent to:*

4. *$\underline{G}$ admits an outer automorphism,*
5. *$[A_{\underline{G}}]$ is 2-torsion in ${}_m\mathrm{Br}(R)$,*
6. *$\Theta(\mathcal{O}_S)$ acts trivially on $H^1_{\text{ét}}(\mathcal{O}_S, \underline{G}^{\mathrm{ad}})$.*

Proof. By Lemma 2.1 the map $\Lambda_* \circ \delta_{\underline{G}^{\mathrm{ad}}_R} : H^1_{\text{ét}}(R, \underline{G}^{\mathrm{ad}}_R) \to \mathrm{Br}(R)$ maps $[\underline{H}^{\mathrm{ad}}]$ to $[A_{\underline{H}}] - [A_{\underline{G}_R}]$ where $[A_{\underline{H}}]$ is the Tits class of $\underline{H}^{\mathrm{sc}}$ for a $\underline{G}^{\mathrm{ad}}_R$-torsor $\underline{H}^{\mathrm{ad}}$. Consider this combined with the long exact sequence obtained by applying étale cohomology to the sequence (3.4) tensored with R:

$$\begin{array}{ccccccc} & & & & \mathrm{Cl}_R(\underline{G}_R^{\mathrm{ad}}) & & \\ & & & & \big\downarrow & & \\ \mathrm{Aut}(\underline{G}_R)(R) & \longrightarrow & \Theta(R) & \longrightarrow & H^1_{\text{ét}}(R, \underline{G}_R^{\mathrm{ad}}) & \xrightarrow{i_*} & \mathbf{Twist}(\underline{G}_R) \\ & & & & \Big\downarrow{\scriptstyle w_{\underline{G}_R^{\mathrm{ad}}} = \Lambda_* \circ \delta_{\underline{G}_R^{\mathrm{ad}}}} & & \\ & & & & {}_m\mathrm{Br}(R) & & \end{array} \tag{5.1}$$

where $\mathrm{Cl}_R(\underline{G}_R^{\mathrm{ad}})$ is the principal genus of $\underline{G}_R^{\mathrm{ad}}$ (see [4, Prop. 3.1]) noting that $F(\underline{G}_R^{\mathrm{ad}}) = \underline{\mu}_m$). Being an inner form of $\underline{G}_R^{\mathrm{ad}}$, $(\underline{G}_R^{\mathrm{ad}})^{\mathrm{op}}$ is obtained by a representative in $H^1_{\text{ét}}(R, \underline{G}_R^{\mathrm{ad}})$. Its $w_{\underline{G}_R^{\mathrm{ad}}}$-image: $[A^{\mathrm{op}}_{\underline{G}_R}] - [A_{\underline{G}_R}]$ is trivial if and only if $A_{\underline{G}_R}$ is of order ≤ 2 in ${}_m\mathrm{Br}(R)$, which is equivalent to $\mathrm{Aut}(\underline{G}_R)(R)$ surjecting on $\Theta(R)$, and $\Theta(R)$ acting trivially on $H^1_{\text{ét}}(R, \underline{G}_R^{\mathrm{ad}})$ (see at the beginning of Section 5).

If, furthermore, $\underline{G}$ is not of type A or $S = \emptyset$, then by Lemma 2.2, together with the Shapiro Lemma we get the isomorphisms of abelian groups:

$$H^1_{\text{ét}}(\mathcal{O}_S, \underline{G}^{\mathrm{ad}}) \cong H^2_{\text{ét}}(\mathcal{O}_S, F(\underline{G}^{\mathrm{ad}})) \cong H^2_{\text{ét}}(R, \underline{\mu}_m) \cong H^1_{\text{ét}}(R, \underline{G}_R^{\mathrm{ad}}). \tag{5.2}$$

So if $\Theta(R)$ acts trivially on $H^1_{\text{ét}}(R, \underline{G}_R^{\mathrm{ad}})$, then so does $\Theta(\mathcal{O}_S)$ on $H^1_{\text{ét}}(\mathcal{O}_S, \underline{G}^{\mathrm{ad}})$. On the other hand if it does not, this implies that $\mathrm{Aut}(\underline{G}_R)(R) \to \Theta(R) \cong \underline{\mathbb{Z}/2}$ is not surjective, thus neither is $\mathrm{Aut}(\underline{G})(\mathcal{O}_S) \to \Theta(\mathcal{O}_S)$, which is equivalent to $\Theta(\mathcal{O}_S)$ acting non-trivially on $H^1_{\text{ét}}(\mathcal{O}_S, \underline{G}^{\mathrm{ad}})$ by Remark 3.4. Moreover, since $i(F(\underline{G}_R^{\mathrm{ad}})) = i(F(\underline{G}^{\mathrm{ad}})) = {}_m\mathrm{Br}(R)$ (Definition 2), the identification (5.2) shows that $\mathrm{Cl}_R(\underline{G}_R^{\mathrm{ad}})$ bijects to $\mathrm{Cl}_S(\underline{G}^{\mathrm{ad}})$, whence $[A_{\underline{G}_R}]$ is 2-torsion in ${}_m\mathrm{Br}(R)$ if and only if $[A_{\underline{G}}]$ is. □

If we wish to interpret a $\underline{G}$-torsor as a twisted form of some basic form, we shall need to describe $\underline{G}$ first as the automorphism group of such an $\mathcal{O}_S$-form.

Example 5.2. Let A be a division $\mathcal{O}_S$-algebra of degree $n > 2$. Then $\underline{G} = \underline{\mathbf{SL}}(A)$ of type $\mathrm{A}_{n-1>1}$ is smooth and connected ([10, Lem. 3.3.1]). It admits a non-trivial outer automorphism τ. If the transpose anti-automorphism $A \cong A^{\mathrm{op}}$ is defined over $\mathcal{O}_S$ (extending τ by inverting again), then $\tau \in \mathbf{Aut}(\underline{G})(\mathcal{O}_S)$. Otherwise, as $(\underline{G}^{\mathrm{ad}})^{\mathrm{op}}$ is not $\mathcal{O}_S$-isomorphic by some conjugation to $\underline{G}^{\mathrm{ad}} = \underline{\mathbf{PGL}}(A)$, it represents a non-trivial class in $H^1_{\text{ét}}(\mathcal{O}_S, \underline{G}^{\mathrm{ad}} = \mathbf{Inn}(\underline{G}))$, whilst its image in $\mathbf{Twist}(\underline{G})$ is trivial by the inverse isomorphism $x \mapsto x^{-1}$ defined over $\mathcal{O}_S$ (say, by the Cramer rule). So finally $\Theta(\mathcal{O}_S)$ acts trivially on $H^1_{\text{ét}}(\mathcal{O}_S, \underline{\mathbf{PGL}}(A))$ if and only if $\mathrm{ord}(A) \leq 2$ in $\mathrm{Br}(\mathcal{O}_S)$, as Proposition 5.1 predicts.

5.1. Type $\mathbf{D_{2k}}$. Let A be an Azumaya $\mathcal{O}_S$-algebra ($\mathrm{char}(K) \neq 2$) of degree $2n$ and let (f,σ) be a *quadratic pair* on A, namely, σ is an involution on A and $f : \mathrm{Sym}(A,\sigma) = \{x \in A : \sigma(x) = x\} \to \mathcal{O}_S$ is a linear map. The scalar $\mu(a) := \sigma(a)\cdot a$ is called the *multiplier* of a. For $a \in A^\times$ we denote by $\mathbf{Int}(a)$ the induced inner automorphism. If σ is orthogonal, the associated *similitude group* is:

$$\underline{\mathbf{GO}}(A,f,\sigma) := \{a \in A^\times : \mu(a) \in \mathcal{O}_S^\times,\ f \circ \mathbf{Int}(a) = f\},$$

and the map $a \mapsto \mathbf{Int}(a)$ is an isomorphism of the *projective similitude group* $\underline{\mathbf{PGO}}(A,f,\sigma) := \mathbf{GO}(A,f,\sigma)/\mathcal{O}_S^\times$ with the group of rational points $\mathbf{Aut}(A,f,\sigma)$. Such a similitude is said to be *proper* if the induced automorphism of the Clifford algebra $C(A,f,\sigma)$ is the identity on the center; otherwise it is said to be *improper*. The subgroup $\underline{G} = \mathbf{PGO}^+(A,f,\sigma)$ of these proper similitudes is connected and adjoint, called the *projective special similitude group*. If the discriminant of σ is a square in $\mathcal{O}_S^\times$, then $\underline{G}$ is of type ${}^1\mathrm{D}_n$. Otherwise of type ${}^2\mathrm{D}_n$.

When $n=2k$, in order that Θ captures the full structure of $\mathbf{Aut}(\mathrm{Dyn}(\underline{G}))$, we would have to restrict ourselves to the two edges of simply-connected and adjoint groups (see Remark 3.3).

Corollary 5.3. *Let $\underline{G}$ be of type ${}^2\mathrm{D}_{2k}, k \neq 2$, simply-connected or adjoint. For any $[P] \in H^1_{\text{ét}}(\mathcal{O}_S, \underline{\mathbb{Z}/2})$ let R_P be the corresponding quadratic étale extension of $\mathcal{O}_S$. Then:*

$$\mathbf{Twist}(\underline{G}) \cong \coprod_{[P]\in H^1_{\text{ét}}(\mathcal{O}_S,\underline{\mathbb{Z}/2})} \mathrm{Pic}(R_P)/2 \times {}_2\mathrm{Br}(R_P).$$

Proof. Any form ${}^P(\underline{G}^{\mathrm{ad}})$ has Tits class $[A_{{}^P\underline{G}}]$ of order ≤ 2 in ${}_2\mathrm{Br}(R_P)$. Hence as $\Theta \cong \underline{\mathbb{Z}/2}$ and $\underline{G}$ is not of type A, by Proposition 5.1 $\Theta(\mathcal{O}_S)$ acts trivially on $H^1_{\text{ét}}(\mathcal{O}_S, {}^P(\underline{G}^{\mathrm{ad}}))$ for all P in $\Theta(\mathcal{O}_S)$. All fundamental groups are admissible, so the Corollary statement is Proposition 3.5 together with the description of each $H^2_{\text{ét}}(\mathcal{O}_S, F({}^P(\underline{G}^{\mathrm{ad}})))$ as in Lemma 2.3. □

6. Non quasi-split fundamental group

When $F(\underline{G}^{\mathrm{ad}})$ is not quasi-split, we cannot apply the Shapiro Lemma as in (5.2) to gain control on the action of $\Theta(\mathcal{O}_S)$ on $H^1_{\text{ét}}(\mathcal{O}_S, F(\underline{G}^{\mathrm{ad}}))$. Still under some conditions this action is provided to be trivial.

Remark 6.1. As opposed to ${}_m\mathrm{Br}(K)$ which is infinite for any integer $m > 1$, ${}_m\mathrm{Br}(\mathcal{O}_S)$ is finite. To be more precise, if $S \neq \emptyset$, $\mathrm{Sp}(\mathcal{O}_S)$ is obtained by removing $|S|$ points from the projective curve C, hence $|{}_m\mathrm{Br}(\mathcal{O}_S)| = m^{|S|-1}$ (see the proof of [4, Cor. 3.2]). When $S = \emptyset$ we have $\mathrm{Br}(C) = 1$. In particular, if $\underline{G}$ is not of absolute type A and $F(\underline{G}^{\mathrm{ad}})$ splits over an extension R such that the number of places in $\mathrm{Frac}(R)$ which lie above places in S is 1, or when $S = \emptyset$, then $\underline{G}^{\mathrm{ad}}$ can posses only one genus and consequently the $\Theta(\mathcal{O}_S)$-action on $H^1_{\text{ét}}(\mathcal{O}_S, \underline{G}^{\mathrm{ad}})$ is trivial.

E. Artin in [1] calls a Galois extension L of K *imaginary* if no prime of K is decomposed into distinct primes in L. We shall similarly call a finite étale extension of $\mathcal{O}_S$ *imaginary* if no prime of $\mathcal{O}_S$ is decomposed into distinct primes in it.

Lemma 6.2. *If R is imaginary over $\mathcal{O}_S$ and m is prime to $[R : \mathcal{O}_S]$, then ${}_m\mathrm{Br}(R) = {}_m\mathrm{Br}(\mathcal{O}_S)$.*

Proof. If $S = \emptyset$ and R/C is imaginary then $\mathrm{Br}(R) = \mathrm{Br}(\mathcal{O}_S) = 1$. Otherwise, the composition of the induced norm $N_{R/\mathcal{O}_S}$ with the diagonal morphism coming from the Weil restriction

$$\underline{\mathbb{G}}_{m,\mathcal{O}_S} \to \mathrm{Res}_{R/\mathcal{O}_S}(\underline{\mathbb{G}}_{m,R}) \xrightarrow{N_{R/\mathcal{O}_S}} \underline{\mathbb{G}}_{m,\mathcal{O}_S}$$

is the multiplication by $n := [R : \mathcal{O}_S]$. It induces together with the Shapiro Lemma the maps:

$$H^2_{\text{ét}}(\mathcal{O}_S, \underline{\mathbb{G}}_{m,\mathcal{O}_S}) \to H^2_{\text{ét}}(R, \underline{\mathbb{G}}_{m,R}) \xrightarrow{N^{(2)}} H^2_{\text{ét}}(\mathcal{O}_S, \underline{\mathbb{G}}_{m,\mathcal{O}_S})$$

whose composition is the multiplication by n on $H^2_{\text{ét}}(\mathcal{O}_S, \underline{\mathbb{G}}_{m,\mathcal{O}_S})$. Identifying $H^2_{\text{ét}}(*, \underline{\mathbb{G}}_m)$ with $\mathrm{Br}(*)$ and restricting to the m-torsion subgroups gives the composition

$${}_m\mathrm{Br}(\mathcal{O}_S) \to {}_m\mathrm{Br}(R) \xrightarrow{N^{(2)}} {}_m\mathrm{Br}(\mathcal{O}_S)$$

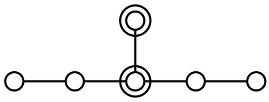
being still multiplication by n, thus an automorphism when n is prime to m. This means that ${}_m\mathrm{Br}(\mathcal{O}_S)$ is a subgroup of ${}_m\mathrm{Br}(R)$. As R is imaginary over $\mathcal{O}_S$, it is obtained by removing $|S|$ points from the projective curve defining its fraction field, so $|{}_m\mathrm{Br}(R)| = |{}_m\mathrm{Br}(\mathcal{O}_S)| = m^{|S|-1}$ by Remark 6.1, and the assertion follows. ☐

Corollary 6.3. *If $F(\underline{G}) = \mathrm{Res}^{(1)}_{R/\mathcal{O}_S}(\underline{\mu}_m)$ is admissible and $R/\mathcal{O}_S$ is imaginary, then $i(F(\underline{G})) = \ker({}_m\mathrm{Br}(R) \to {}_m\mathrm{Br}(\mathcal{O}_S))$ (see Definition 2) is trivial, hence $\underline{G}$ admits a single genus (cf.* [4, Cor. 3.2]*).*

6.1. Type $\mathbf{E_6}$. A *hermitian* Jordan triple over $\mathcal{O}_S$ is a triple $(A, \mathfrak{X}, U)$ consisting of a quadratic étale $\mathcal{O}_S$-algebra A with conjugation σ, a free of finite rank $\mathcal{O}_S$-module $\mathfrak{X}$, and a quadratic map $U : \mathfrak{X} \to \mathrm{Hom}_A(\mathfrak{X}^\sigma, \mathfrak{X}) : x \mapsto U_x$, where $\mathfrak{X}^\sigma$ is $\mathfrak{X}$ with scalar multiplication twisted by σ, such that $(\mathfrak{X}, U)$ is an (ordinary) Jordan triple as in [20]. In particular if $\mathfrak{X}$ is an *Albert $\mathcal{O}_S$-algebra*, then it is called an *hermitian Albert triple*. In that case the associated trace form $T : A \times A \to \mathcal{O}_S$ is symmetric non-degenerate and it follows that the structure group of $\mathfrak{X}$ agrees with its group of norm similarities. Viewed as an $\mathcal{O}_S$-group, it is reductive with center of rank 1 and its semisimple part, which we shortly denote $G(A, \mathfrak{X})$, is simply connected of type E_6. It is of relative type ${}^1\mathrm{E}_6$ if $A \cong \mathcal{O}_S \times \mathcal{O}_S$ and of type ${}^2\mathrm{E}_6$ otherwise.

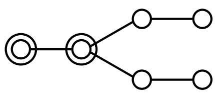
Groups of type ${}^1\mathrm{E}_6$ are classified by four relative types, among them only ${}^1\mathrm{E}_{6,2}^{16}$ has a non-commutative Tits algebra, thus being the only type in which $\Theta(\mathcal{O}_S) \cong \mathbb{Z}/2$ may act non-trivially on $H^1_{\text{ét}}(\mathcal{O}_S, \underline{G}^{\text{ad}})$. More precisely, the Tits-algebra in that case is a division algebra D of degree 3 (cf. [25, p. 58]) and the $\Theta(\mathcal{O}_S)$-action is trivial if and only if $\text{ord}([D]) \leq 2$ in $\text{Br}(\mathcal{O}_S)$. But $\text{ord}([D])$ is odd, thus this action is trivial if and only if D is a matrix $\mathcal{O}_S$-algebra.

${}^1\mathrm{E}_{6,2}^{16}$



In the case of type ${}^2\mathrm{E}_6$, one has six relative types (cf. [25, p. 59]), among which only ${}^2\mathrm{E}_{6,2}^{16''}$ has a non-commutative Tits algebra (cf. [26, p. 211]). Its Tits algebra is a division algebra of degree 3 over R, and its Brauer class has trivial corestriction in $\text{Br}(\mathcal{O}_S)$. By Albert and Riehm, this is equivalent to D possessing an $R/\mathcal{O}_S$-involution.

${}^2\mathrm{E}_{6,2}^{16''}$



From now on $\sim$ denotes the equivalence relation on the Brauer group which identifies the class of an Azumaya algebra with the class of its opposite.

Corollary 6.4. *Let $\underline{G}$ be of (absolute) type E_6. For any $[P] \in H^1_{\text{ét}}(\mathcal{O}_S, \underline{\mathbb{Z}/2})$ let R_P be the corresponding quadratic étale extension of $\mathcal{O}_S$. Then*

$$\mathbf{Twist}(\underline{G}) \cong \text{Pic}(\mathcal{O}_S)/3 \times {}_3\text{Br}(\mathcal{O}_S)/\sim$$
$$\coprod_{1 \neq [P]} \ker(\text{Pic}(R_P)/3 \to \text{Pic}(\mathcal{O}_S)/3) \times (\ker({}_3\text{Br}(R_P) \to {}_3\text{Br}(\mathcal{O}_S)))/\sim,$$

where $[P]$ runs over $H^1_{\text{ét}}(\mathcal{O}_S, \underline{\mathbb{Z}/2})$. The relation $\sim$ is trivial in the first component unless $\underline{G}^{\text{ad}}$ is of type ${}^1\mathrm{E}_{6,2}^{16}$ and is trivial in the other components unless ${}^P(\underline{G}^{\text{ad}})$ is of type ${}^2\mathrm{E}_{6,2}^{16''}$.

Proof. The group $\Theta(\mathcal{O}_S)$ acts trivially on members of the same genus, so it is sufficient to check its action on the set of genera for each type. Since $F({}^P(\underline{G}^{\text{ad}}))$ is admissible for any $[P] \in H^1_{\text{ét}}(\mathcal{O}_S, \Theta)$, by [4, Cor. 3.2] the set of genera of each ${}^P(\underline{G}^{\text{ad}})$ bijects as a pointed-set to $i(F({}^P(\underline{G}^{\text{ad}})))$, so the assertion is Proposition 3.5 together with Lemma 2.3. The last claims are retrieved from the above discussion on the trivial action of $\Theta(\mathcal{O}_S)$ when ${}^P(\underline{G}^{\text{ad}})$ is not of type ${}^1\mathrm{E}_{6,2}^{16}$ or ${}^2\mathrm{E}_{6,2}^{16''}$. □

Example 6.5. Let C be the elliptic curve $Y^2Z = X^3 + XZ^2 + Z^3$ defined over $\mathbb{F}_3$. Then:

$$C(\mathbb{F}_3) = \{(1:0:1),(0:1:2),(0:1:1),(0:1:0)\}.$$

Removing the $\mathbb{F}_3$-point $\infty = (0:1:0)$ the obtained smooth affine curve C^{af} is $y^2 = x^3+x+1$. Letting $\mathcal{O}_{\{\infty\}} = \mathbb{F}_3[C^{\mathrm{af}}]$ we have $\mathrm{Pic}(\mathcal{O}_{\{\infty\}}) \cong C(\mathbb{F}_3)$ (e.g., [3, Ex. 4.8]). Among the affine supports of points in $C(\mathbb{F}_3) - \{\infty\}$:

$$\{(1,0),(0,1/2) = (0,2),(0,1)\},$$

only $(1,0)$ has a trivial y-coordinate thus being of order 2 (according to the group law there), to which corresponds the fractional ideal $P = (x-1,y)$ of order 2 in $\mathrm{Pic}(\mathcal{O}_{\{\infty\}})$. As $\mathrm{Pic}(\mathcal{O}_{\{\infty\}})/3 = 1$ and $\mathrm{Br}(\mathcal{O}_{\{\infty\}}) = 1$, a form of type ${}^1\mathrm{E}_6$ has no non-isomorphic *inner* form, while a form of type ${}^2\mathrm{E}_6$ may have more; for example $R = \mathcal{O}_{\{\infty\}} \oplus P$ being geometric and étale cannot be imaginary over $\mathcal{O}_{\{\infty\}}$, which means it is obtained by removing two points from a projective curve, thus

$$\begin{aligned}\ker({}_3\mathrm{Br}(R) \to {}_3\mathrm{Br}(\mathcal{O}_{\{\infty\}}))/{\sim} &= {}_3\mathrm{Br}(R)/{\sim} \\ &= \{[R],[A],[A^{\mathrm{op}}]\}/{\sim} = \{[R],[A]\},\end{aligned}$$

hence an $\mathcal{O}_{\{\infty\}}$-group of type E_6 splitting over R admits a non-isomorphic inner form.

6.2. Type $\mathbf{D_{2k+1}}$. Recall from Section 5.1 that an adjoint $\mathcal{O}_S$-group $\underline{G}$ of absolute type D_n can be realized as $\underline{\mathbf{PGO}}^+(A,\sigma)$ where A is Azumaya of degree $2n$ and σ is an orthogonal involution on A. Suppose n is odd. If $\underline{G}$ is of relative type ${}^1\mathrm{D}_n$ then $F(\underline{G}) = \underline{\mu}_4$ is admissible, thus not being of absolute type A, $\mathrm{Cl}_S(\underline{G})$ bijects to $j(\underline{\mu}_4) = \mathrm{Pic}(\mathcal{O}_S)/4$ and $\mathrm{gen}(\underline{G})$ bijects to $i(\underline{\mu}_4) = {}_4\mathrm{Br}(\mathcal{O}_S)$. Otherwise, when $\underline{G}$ is of type ${}^2\mathrm{D}_n$, then $F(\underline{G}) = \mathrm{Res}^{(1)}_{R/\mathcal{O}_S}(\underline{\mu}_4)$ where $R/\mathcal{O}_S$ is quadratic. Again not being of absolute type A, $\mathrm{Cl}_S(\underline{G}) \cong j(F(\underline{G})) = \ker(\mathrm{Pic}(R)/4 \to \mathrm{Pic}(\mathcal{O}_S)/4)$, but here, as $F(\underline{G})$ is not admissible, by [4, Cor. 3.2] $\mathrm{gen}(\underline{G})$ only injects in $i(F(\underline{G})) = \ker({}_4\mathrm{Br}(R) \to {}_4\mathrm{Br}(\mathcal{O}_S))$. If $R/\mathcal{O}_S$ is imaginary, then by Lemma 6.2 $i(F(\underline{G})) = 1$. Altogether by Proposition 3.5 we get:

Corollary 6.6. *Let $\underline{G}$ be of (absolute) type* D_{2k+1}*. For any $[P] \in H^1_{\text{ét}}(\mathcal{O}_S,\underline{\mathbb{Z}/2})$ let R_P be the corresponding quadratic étale extension of $\mathcal{O}_S$. Then there exists an exact sequence of pointed-sets*

$$\begin{aligned}\mathbf{Twist}(\underline{G}) \hookrightarrow \mathrm{Pic}(\mathcal{O}_S)/4 \times \coprod_{1\neq[P]} \ker(\mathrm{Pic}(R_P)/4 \to \mathrm{Pic}(\mathcal{O}_S)/4) \\ \times \left({}_4\mathrm{Br}(\mathcal{O}_S)/{\sim} \times \coprod_{1\neq[P]} (\ker({}_4\mathrm{Br}(R_P) \to {}_4\mathrm{Br}(\mathcal{O}_S)))/{\sim}\right),\end{aligned}$$

where $[P]$ *runs over* $H^1_{\text{ét}}(\mathcal{O}_S, \underline{\mathbb{Z}/2})$ *and* $[A] \sim [A^{\text{op}}]$*. This map surjects onto the first component. Whenever* $R_P/\mathcal{O}_S$ *is imaginary* $\ker({}_4\text{Br}(R_P) \to {}_4\text{Br}(\mathcal{O}_S)) = 1$ *and this map is a bijection.*

Example 6.7. Let $\mathcal{O}_{\{\infty\}} = \mathbb{F}_q[x]$ (q is odd) obtained by removing $\infty = (1/x)$ from the projective line over $\mathbb{F}_q$. Suppose $q \in 4\mathbb{N}-1$ so $-1 \notin \mathbb{F}_q^2$, and let $\underline{G} = \underline{\mathbf{SO}}_{10}$ be defined over $\mathcal{O}_{\{\infty\}}$. The discriminant of an orthogonal form Q_B induced by an $n \times n$ matrix B is $\text{disc}(Q_B) = (-1)^{\frac{n(n-1)}{2}} \det(B)$. As $\text{disc}(Q_{1_{10}}) = -1$ is not a square in $\mathcal{O}_{\{\infty\}}$, $\underline{G}$ is considered of type ${}^2\text{D}_5$. It admits a maximal torus $\underline{T}$ containing five 2×2 rotations blocks $\left(\begin{smallmatrix} a & b \\ -b & a \end{smallmatrix}\right)$: $a^2+b^2 = 1$ on the diagonal. Over $R = \mathcal{O}_{\{\infty\}}[i]$ such block is diagonalizable; it becomes $\text{diag}(t, t^{-1})$. The obtained diagonal torus $\underline{T}'_s = P\underline{T}_s P^{-1}$ where $\underline{T}_s = \underline{T} \otimes R$ and P is some invertible 10×10 matrix over R, is split and 5-dimensional, so may be identified with the 5×5 diagonal torus, whose positive roots are:

$$\alpha_1 = \varepsilon_1 - \varepsilon_2, \quad \alpha_2 = \varepsilon_2 - \varepsilon_3, \quad \alpha_3 = \varepsilon_3 - \varepsilon_4, \quad \alpha_4 = \varepsilon_4 - \varepsilon_5, \quad \alpha_5 = \varepsilon_4 + \varepsilon_5.$$

Let g be the matrix differing from the 10×10 unit only at the last 2×2 block, being $\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$. Then $\det(g) = -1$ thus $\text{disc}(Q_g) = 1$ where Q_g is the induced quadratic form. This means that $\underline{G}' = \underline{\mathbf{SO}}(Q_g)$ of type ${}^1\text{D}_5$ is the unique outer form of $\underline{G}$ (up to $\mathcal{O}_S$-isomorphism). Then $\Theta = \mathbf{Aut}(\text{Dyn}(G))$ acts on $\text{Lie}(g\underline{T}'_s g^{-1})$ by mapping the last block $\begin{pmatrix} 0 & \ln(t) \\ -\ln(t) & 0 \end{pmatrix}$ to $\begin{pmatrix} 0 & -\ln(t) \\ \ln(t) & 0 \end{pmatrix}$ and so swapping the above two roots α_4 and α_5. Since $\mathcal{O}_{\{\infty\}}$ and R are PIDs, their Picard groups are trivial. As only one point was removed in both domains also $\text{Br}(\mathcal{O}_{\{\infty\}}) = \text{Br}(R) = 1$. We remain with only the two above forms, i.e., $\mathbf{Twist}(\underline{G}) = \{[\underline{G}], [\underline{G}']\}$.

The same holds for $\mathcal{O}_S = \mathbb{F}_q[x, x^{-1}]$: again it is a UFD thus $\underline{G} = \underline{\mathbf{SO}}_{10}$ defined over it still posseses only one non-isomorphic outer form. As $\mathcal{O}_S$ is obtained by removing two points from the projective $\mathbb{F}_q$-line, this time ${}_4\text{Br}(\mathcal{O}_S)$ is not trivial, but still equals ${}_4\text{Br}(\mathcal{O}_S)$, so: $\ker({}_4\text{Br}(R) \to {}_4\text{Br}(\mathcal{O}_S)) = 1$.

6.3. Type D$_4$. This case deserves a special regard as Θ is the symmetric group $\underline{S_3}$ when $\underline{G}$ is adjoint or simply-connected (cf. Propoposition 3.2). Suppose C is an Octonion $\mathcal{O}_S$-algebra with norm N. For any similitude t of N (see Section 5.1) there exist similitudes t_2 and t_3 such that

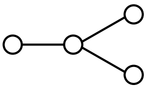
$$t_1(xy) = t_2(x) \cdot t_3(y) \quad \forall\, x, y \in C.$$

Then the mappings:

$$\alpha : [t_1] \mapsto [t_2], \;\; \beta : [t_1] \mapsto [\widehat{t_3}] \tag{6.1}$$

where $\widehat{t}(x) := \mu(t)^{-1} \cdot t(x)$, satisfy $\alpha^2 = \beta^3 = \mathrm{id}$ and generate $\Theta = \mathbf{Out}(\mathbf{PGO}^+(N)) \cong \underline{S_3}$.

$^1\mathrm{D}_4$



Having three conjugacy classes, there are three classes of outer forms of $\underline{G}$ (cf. [10, p. 253]), which we denote as usual by $^1\mathrm{D}_4$, $^2\mathrm{D}_4$ and $^{3,6}\mathrm{D}_4$. The groups in the following table are the generic fibers of these outer forms, L/K is the splitting extension of $F(G^{\mathrm{ad}})$ (note that in the case $^6\mathrm{D}_4$ L/K is not Galois):

Type of G	$F(G^{\mathrm{ad}})$	$[L:K]$
$^1\mathrm{D}_4$	$\mu_2 \times \mu_2$	1
$^2\mathrm{D}_4$	$R_{L/K}(\mu_2)$	2
$^{3,6}\mathrm{D}_4$	$R^{(1)}_{L/K}(\mu_2)$	3

Starting with $\underline{G}$ of type $^1\mathrm{D}_4$, one sees that $F({}^P(\underline{G}^{\mathrm{ad}}))$ (splitting over some corresponding extension $R/\mathcal{O}_S$) is admissible for any $[P] \in H^1_{\text{ét}}(\mathcal{O}_S, \Theta)$, thus according to Lemma 2.3

$$\forall\, [P] \in H^1_{\text{ét}}(\mathcal{O}_S, \Theta): \;\; H^2_{\text{ét}}(\mathcal{O}_S, F({}^P(\underline{G}^{\mathrm{ad}}))) \cong j(F({}^P(\underline{G}^{\mathrm{ad}}))) \times i(F({}^P(\underline{G}^{\mathrm{ad}}))).$$

The action of $\Theta(\mathcal{O}_S)$ is trivial on the first factor, classifying torsors of the same genus, so we concentrate on its action on $i(F({}^P(\underline{G}^{\mathrm{ad}})))$. Since $\Theta \not\cong \underline{\mathbb{Z}/2}$ we cannot use Proposition 5.1, but we may still imitate its arguments:

The group $\Theta(\mathcal{O}_S)$ acts non-trivially on $H^1_{\text{ét}}(\mathcal{O}_S, {}^P(\underline{G}^{\mathrm{ad}}))$ for some $[P] \in H^1_{\text{ét}}(\mathcal{O}_S, \Theta)$ if it identifies two non isomorphic torsors of ${}^P(\underline{G}^{\mathrm{ad}})$. The Tits algebras of their universal coverings lie in $({}_2\mathrm{Br}(\mathcal{O}_S))^2$ if ${}^P(\underline{G}^{\mathrm{ad}})$ is of type $^1\mathrm{D}_4$, i.e., if P belongs to the trivial class in $H^1_{\text{ét}}(\mathcal{O}_S, \Theta)$, in ${}_2\mathrm{Br}(R)$ for R quadratic étale over $\mathcal{O}_S$ if ${}^P(\underline{G}^{\mathrm{ad}})$ is of type $^2\mathrm{D}_4$, i.e., if $[P] \in {}_2H^1_{\text{ét}}(\mathcal{O}_S, \Theta)$, and in $\ker({}_2\mathrm{Br}(R) \to {}_2\mathrm{Br}(\mathcal{O}_S))$ for a cubic étale extension R of $\mathcal{O}_S$ if ${}^P(\underline{G}^{\mathrm{ad}})$ is one of the types $^{3,6}\mathrm{D}_4$, i.e., if $[P] \in {}_3H^1_{\text{ét}}(\mathcal{O}_S, \Theta)$. Therefore these Tits algebras must be 2-torsion, which means that the two torsors are $\mathcal{O}_S$-isomorphic in the first case and R-isomorphic in the latter three. If $F({}^P(\underline{G}^{\mathrm{ad}}))$ is quasi-split this means (by the Shapiro Lemma) that $\Theta(\mathcal{O}_S)$ acts trivially on $H^1_{\text{ét}}(\mathcal{O}_S, {}^P(\underline{G}^{\mathrm{ad}}))$. If $F({}^P(\underline{G}^{\mathrm{ad}}))$ is not quasi-split, according to Corollary 6.3 if R is imaginary over $\mathcal{O}_S$ then $i(F({}^P(\underline{G}^{\mathrm{ad}}))) = 1$.

If a quadratic form Q has a trivial discriminant on a vector space V, the Tits algebras of the group are $\mathrm{End}(V)$ and the two components $\mathbf{C}_+(Q)$, $\mathbf{C}_-(Q)$ of the even Clifford algebra of Q, and the triality automorphism cyclically permutes those three. More generally, if the group is represented as $\mathbf{PGO}^+(A, \sigma)$ for some orthogonal involution of trivial discriminant on a central simple algebra A of degree 8, triality automorphisms permute A

and the two components of the Clifford algebra $\mathbf{C}(A,\sigma)$; Altogether we finally get:

Corollary 6.8. *Let $\underline{G}$ be of (absolute) type D_4 being simply-connected or adjoint. For any $[P] \in H^1_{\text{ét}}(\mathcal{O}_S,\Theta)$ let R_P be the corresponding étale extension of $\mathcal{O}_S$. Then:*

$$\begin{aligned}\mathbf{Twist}(\underline{G}) \cong (\mathrm{Pic}(\mathcal{O}_S)/2 \times {}_2\mathrm{Br}(\mathcal{O}_S))^2 \\ \coprod_{1\neq[P]\in_2 H^1_{\text{ét}}(\mathcal{O}_S,\Theta)} \mathrm{Pic}(R_P)/2 \times {}_2\mathrm{Br}(R_P) \\ \coprod_{1\neq[P]\in_3 H^1_{\text{ét}}(\mathcal{O}_S,\Theta)} \ker(\mathrm{Pic}(R_P)/2 \to \mathrm{Pic}(\mathcal{O}_S)/2) \\ \times (\ker({}_2\mathrm{Br}(R_P) \to {}_2\mathrm{Br}(\mathcal{O}_S)))/\Theta(\mathcal{O}_S).\end{aligned}$$

If R_P is imaginary over $\mathcal{O}_S$, then $\ker({}_2\mathrm{Br}(R_P) \to {}_2\mathrm{Br}(\mathcal{O}_S)) = 1$.

7. The anisotropic case

Now suppose that $\underline{G}$ does admit a twisted form such that the generic fiber of its universal covering is anisotropic at S. As previously mentioned, such group must be of absolute type A and $S \neq \emptyset$. Over a local field k, an outer form of a group of type ${}^1\mathrm{A}$ which is anisotropic, must be the special unitary group arising by some hermitian form h in r variables over a quadratic extension of k or over a quaternion k-algebra ([27, §4.4]).

A *unitary* $\mathcal{O}_S$-group is $\underline{\mathbf{U}}(B,\sigma) := \mathbf{Iso}(B,\sigma)$ where B is a non-split quaternion Azumaya defined over an étale quadratic extension R of $\mathcal{O}_S$ and σ is a unitary involution on B, i.e., whose restriction to the center R is not the identity. The *special unitary group* is the kernel of the reduced norm:

$$\underline{\mathbf{SU}}(B,\sigma) := \ker(\mathrm{Nrd} : \underline{\mathbf{U}}(B,\tau) \twoheadrightarrow \underline{\mathbf{GL}}_1(R)).$$

These are of relative type ${}^2\mathrm{C}_{2m}$ ($m \geq 2$) ([27, §4.4]) and isomorphic over R to type ${}^1\mathrm{A}_{2m-1}$.

So in order to determine exactly when $H^1_{\text{ét}}(\mathcal{O}_S,\underline{G}^{\text{sc}})$ *does not* vanish, we may restrict ourselves to $\mathcal{O}_S$-groups whose universal covering is either $\underline{\mathbf{SL}}_1(A)$ or $\underline{\mathbf{SU}}(B,\sigma)$. In the first case, the reduced norm applied to the units of A forms the short exact sequence of smooth $\mathcal{O}_S$-groups:

$$1 \to \underline{\mathbf{SL}}_1(A) \to \underline{\mathbf{GL}}_1(A) \xrightarrow{\mathrm{Nrd}} \underline{\mathbb{G}}_m \to 1. \tag{7.1}$$

Then étale cohomology gives rise to the long exact sequence:

$$\begin{aligned}1 \to \mathcal{O}_S^\times / \mathrm{Nrd}(A^\times) \to H^1_{\text{ét}}(\mathcal{O}_S,\underline{\mathbf{SL}}_1(A)) \xrightarrow{i_*} H^1_{\text{ét}}(\mathcal{O}_S,\underline{\mathbf{GL}}_1(A)) \\ \xrightarrow{\mathrm{Nrd}_*} H^1_{\text{ét}}(\mathcal{O}_S,\underline{\mathbb{G}}_m) \cong \mathrm{Pic}(\mathcal{O}_S)\end{aligned} \tag{7.2}$$

in which Nrd_* is surjective since $\underline{\mathbf{SL}}_1(A)$ is simply-connected and $\mathcal{O}_S$ is of Douai-type (see above).

Definition 4. We say that the *local-global Hasse principle* holds for $\underline{G}$ if $h_S(\underline{G}) = |\mathrm{Cl}_S(\underline{G})| = 1$.

Thus the Hasse principle says that an $\mathcal{O}_S$-group is $\mathcal{O}_S$-isomorphic to $\underline{G}$ if and only if it is K-isomorphic to it. This is automatic for simply-connected groups which are not of type A or when $S = \emptyset$ for which by Lemma 2.2 $H^1_{\text{ét}}(\mathcal{O}_S, \underline{G}) \cong H^2_{\text{ét}}(\mathcal{O}_S, F(\underline{G}))$ is trivial.

Corollary 7.1. *Let $\underline{G} = \underline{\mathbf{SL}}_1(A)$ where A is a quaternion $\mathcal{O}_S$-algebra.*

1. *If $\mathrm{Nrd} : A^\times \to \mathcal{O}_S^\times$ is not surjective, then the Hasse principle does not hold for $\underline{G}$.*
2. *If the generic fiber G is isotropic at S, then $\mathbf{Twist}(\underline{G})$ is in bijection as a pointed-set with the abelian group $\mathrm{Pic}(\mathcal{O}_S)/2 \times {}_2\mathrm{Br}(\mathcal{O}_S)$.*

Proof. (1). The generic fiber $\mathbf{SL}_1(A)$ is simply-connected thus due to Harder $H^1(K, \mathbf{SL}_1(A)) = 1$, which indicates that $\underline{\mathbf{SL}}_1(A)$ admits a single genus (cf. Section 2), i.e., $H^1_{\text{ét}}(\mathcal{O}_S, \underline{\mathbf{SL}}_1(A))$ is equal to $\mathrm{Cl}_S(\underline{\mathbf{SL}}_1(A))$. By the exactness of sequence (7.2), $H^1_{\text{ét}}(\mathcal{O}_S, \underline{\mathbf{SL}}_1(A))$ cannot vanish if $\mathrm{Nrd}(A^\times) \neq \mathcal{O}_S^\times$.

(2). Being of type A_1, $\underline{G} = \underline{\mathbf{SL}}_1(A)$ does not admit a non-trivial outer form, which implies that $\mathbf{Twist}(\underline{G}) = H^1_{\text{ét}}(\mathcal{O}_S, \underline{G}^{\text{ad}})$. The short exact sequence of the universal covering of $\underline{G}^{\text{ad}} = \underline{\mathbf{PGL}}_1(A)$ with fundamental group $\underline{\mu}_2$, induces the long exact sequence (cf. (2.2)):

$$H^1_{\text{ét}}(\mathcal{O}_S, \underline{\mathbf{SL}}_1(A)) \to H^1_{\text{ét}}(\mathcal{O}_S, \underline{\mathbf{PGL}}_1(A)) \xrightarrow{\delta_{\underline{G}^{\text{ad}}}} H^2_{\text{ét}}(\mathcal{O}_S, \underline{\mu}_2)$$

in which since $H^1_{\text{ét}}(\mathcal{O}_S, \underline{\mathbf{SL}}_1(A))$ is trivial (due to strong approximation when G is isotropic at S), the rightmost term is isomorphic by Lemma 2.3 to $\mathrm{Pic}(\mathcal{O}_S)/2 \times {}_2\mathrm{Br}(\mathcal{O}_S)$. □

Example 7.2. Let C be the projective line defined over $\mathbb{F}_3$ and $S = \{t, t^{-1}\}$. Then $K = \mathbb{F}_3(t)$ and $\mathcal{O}_S = \mathbb{F}_3[t, t^{-1}]$. For the quaternion $\mathcal{O}_S$-algebra $A = (i^2 = -1, j^2 = -t)_{\mathcal{O}_S}$ we get:

$$\forall\, x, y, z, w \in \mathcal{O}_S : \ \mathrm{Nrd}(x + yi + zj + wk) = x^2 + y^2 + t(z^2 + w^2)$$

which shows that $\mathrm{Nrd}(A^\times) = \mathcal{O}_S^\times = \mathbb{F}_3^\times \cdot t^n, n \in \mathbb{Z}$. As $\mathcal{O}_S$ is a UFD, the Hasse principle holds for $\underline{G} = \underline{\mathbf{SL}}_1(A)$, though its generic fiber $G \cong \mathrm{Spin}_q$ for $q(x, y, z) = x^2 + y^2 + tz^2$ is anisotropic at S (cf. [19, Lemma 6]). We have two distinct classes in $\mathbf{Twist}(\underline{G})$, namely, $[\underline{G}]$ and $[\underline{G}^{\text{op}}]$. For $A = (-1, -1)_{\mathcal{O}_S}$, however, we get:

$$\mathrm{Nrd}(x + yi + zj + wk) = x^2 + y^2 + z^2 + w^2$$

which does not surject on $\mathcal{O}_S^\times$ as $t \notin \mathrm{Nrd}(A^\times)$, so the Hasse principle does not hold for $\underline{\mathbf{SL}}_1(A)$.

Similarly, applying étale cohomology to the exact sequence of smooth $\mathcal{O}_S$-groups:

$$1 \to \underline{\mathbf{SU}}(B,\sigma) \to \underline{\mathbf{U}}(B,\sigma) \xrightarrow{\mathrm{Nrd}} \underline{\mathbf{GL}}_1(R) \to 1$$

induces the exactness of:

$$1 \to R^\times / \mathrm{Nrd}(\underline{\mathbf{U}}(B,\sigma)(\mathcal{O}_S)) \to H^1_{\text{ét}}(\mathcal{O}_S, \underline{\mathbf{SU}}(B,\sigma)) \\ \to H^1_{\text{ét}}(\mathcal{O}_S, \underline{\mathbf{U}}(B,\sigma)) \xrightarrow{\mathrm{Nrd}_*} H^1_{\text{ét}}(\mathcal{O}_S, \mathbf{Aut}(R)).$$

Let $A = D(B,\sigma)$ be the discriminant algebra. If R splits, namely, $R \cong \mathcal{O}_S \times \mathcal{O}_S$, then $B \cong A \times A^{\mathrm{op}}$ and σ is the exchange involution. Then $\underline{\mathbf{U}}(B,\sigma) \cong \underline{\mathbf{GL}}_1(A)$ and $\underline{\mathbf{SU}}(B,\sigma) \cong \underline{\mathbf{SL}}_1(A)$, so we are back in the previous situation.

Corollary 7.3. *If $\underline{\mathbf{U}}(B,\sigma)(\mathcal{O}_S) \xrightarrow{\mathrm{Nrd}} R^\times$ is not surjective then the Hasse-principle does not hold for $\underline{\mathbf{SU}}(B,\sigma)$.*

8. In the Zariski topology

A $\underline{G}$-torsor P is *Zariski*, if the twisted form ${}^P\underline{G}$ is generically and locally everywhere away of S isomorphic to $\underline{G}$, i.e., if it belongs to the principal genus of $\underline{G}$ (see Section 2). Let $\underline{G}_0$ be a quasi-split semisimple $\mathcal{O}_S$-group with an almost-simple generic fiber. The continuous morphism between the categories of open subsets of $\mathcal{O}_S$: $(\mathcal{O}_S)_{\text{ét}} \to (\mathcal{O}_S)_{\mathrm{Zar}}$ results, given a variety X defined over $\mathcal{O}_S$, in the opposite inclusion of cohomology sets $H^r_{\mathrm{Zar}}(\mathcal{O}_S, X) \subseteq H^r_{\text{ét}}(\mathcal{O}_S, X)$ for all $r > 0$. The restriction of the decomposition (3.6)

$$\text{(8.1)} \qquad \mathbf{Twist}(\underline{G}_0) \cong H^1_{\text{ét}}(\mathcal{O}_S, \mathrm{Aut}(\underline{G}_0)) \cong \coprod_{[P]} H^1_{\text{ét}}(\mathcal{O}_S, {}^P(\underline{G}_0^{\mathrm{ad}}))/\Theta(\mathcal{O}_S)$$

to Zariski torsors gives (compare with [18, p. 181]):

$$\text{(8.2)} \qquad \mathbf{Twist}_{\mathrm{Zar}}(\underline{G}_0) \cong H^1_{\mathrm{Zar}}(\mathcal{O}_S, \mathrm{Aut}(\underline{G}_0)) \cong H^1_{\mathrm{Zar}}(\mathcal{O}_S, \underline{G}_0^{\mathrm{ad}})/\Theta(\mathcal{O}_S).$$

But as aforementioned, $H^1_{\mathrm{Zar}}(\mathcal{O}_S, \underline{G}_0^{\mathrm{ad}})$ is equal to the principal genus of $\underline{G}_0^{\mathrm{ad}}$ on which the action of $\Theta(\mathcal{O}_S)$ is trivial, hence (8.2) refines to:

$$\text{(8.3)} \qquad \mathbf{Twist}_{\mathrm{Zar}}(\underline{G}_0) \cong H^1_{\mathrm{Zar}}(\mathcal{O}_S, \underline{G}_0^{\mathrm{ad}}).$$

Moreover, restricting the bijection $H^1_{\text{ét}}(\mathcal{O}_S, \underline{G}_0^{\mathrm{ad}}) \cong H^2_{\text{ét}}(\mathcal{O}_S, F(\underline{G}_0^{\mathrm{ad}}))$ (Lemma 2.2) to the Zariski topology, $H^1_{\mathrm{Zar}}(\mathcal{O}_S, \underline{G}_0^{\mathrm{ad}})$ can be replaced with $H^2_{\mathrm{Zar}}(\mathcal{O}_S, F(\underline{G}_0^{\mathrm{ad}}))$. All twisted forms of $\underline{G}_0$ in the Zariski topology being K-isomorphic are isotropic, so this time this includes groups of type A. Suppose $F(\underline{G}_0^{\mathrm{ad}})$ is admissible, splitting over an étale extension R of $\mathcal{O}_S$. Then $H^1_{\mathrm{Zar}}(\mathcal{O}_S, \underline{\mathbb{G}}_m) \cong \mathrm{Pic}(\mathcal{O}_S)$ while as R is locally factorial $H^2_{\mathrm{Zar}}(R, \underline{\mathbb{G}}_m)$ is trivial ([8, Rem. 3.5.1]) thus $i(F(\underline{G}_0))$ as well (see Definition 2). Hence similarly as was done for $H^2_{\text{ét}}(\mathcal{O}_S, F(\underline{G}_0^{\text{ad}}))$ in Lemma 2.3, we get that $H^2_{\text{Zar}}(\mathcal{O}_S, F(\underline{G}_0^{\text{ad}})) \cong j(F(\underline{G}_0))$.

Corollary 8.1. *Let $\underline{G}_0$ be a semisimple $\mathcal{O}_S$-group with an almost-simple generic fiber and an admissible fundamental group. Then:* $\mathbf{Twist}_{\text{Zar}}(\underline{G}_0) \cong j(F(\underline{G}_0))$.

Acknowledgements. The authors thank P. Gille, B. Kunyavskiĭ and A. Quéguiner-Mathieu for valuable discussions concerning the topics of the present article. They would like also to thank the anonymous referee for a careful reading and many constructive remarks.

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