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On the origin of photon mass, momentum, and energy in a dielectric medium [Invited]

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Abstract: The debate and controversy concerning the momentum of light in a dielectric medium (Abraham vs Minkowski) is well-known and still not fully resolved. In this paper, we investigate the origin of both momenta in the frame of special relativity by considering photons in media as relativistic quasiparticles. We demonstrate for the first time to the best of our knowledge that the Minkowski form of the photon mass, momentum, and energy follows directly from the relativistic energy conservation law. We introduce a new expression for the momentum of light in a dispersive medium, consistent with the experimentally observed propagation of photons at the group velocity. Finally, the effect of light-induced optical stretching is discussed, which can be used for experimental verification of the existing expressions for the photon momentum.

1. Introduction

There was a century-long debate concerning the momentum of light and the electromagnetic energy-momentum tensor in a dielectric medium related to two famous German scientists: Max Abraham and Hermann Minkowski. Many excellent reviews have been devoted to this problem [1–4], but the controversy is still not fully resolved, as can be seen from recent publications [5–13]. Theoretical papers devoted to derivations of components of the energy-momentum tensor based on different formulations of the electromagnetic field theory continue to appear. For example, it has been shown that the Abraham energy-momentum tensor contradicts special relativity, whereas the Minkowski tensor is fully consistent with it [14–18].

The debate about the momentum of light has gained additional importance, following recent advances in the development of single- and few-photon sources and detectors for quantum technologies (see a recent review [19]). Investigations of the discrete nature of light, single photon experiments including temporal and spatial control, manipulation, and structuring of photons, all these achievements are opening a new era of quantum photonics. That is why it is important to know what happens to a single photon in a dielectric medium, where, due to the excitation of dipole oscillations of bound electrons and polarization wave, the photon (together with the excited dipoles) forms a relativistic quasiparticle (photon in medium, which will be called *m-photon*) propagating at the speed of light $v = c/n$ in nondispersive dielectrics. Concerning a full classification of the existing photonic quasiparticles, a recent review [20] can be recommended.

In many papers and textbooks, without mentioning the above controversy, people are simply using the Minkowski momentum for a photon in a dielectric medium $p_M = n\hbar\omega/c$, where n is the refractive index, ω is the frequency of light, and c is the speed of light in free space (vacuum). We all agree, at least many of us, that this expression is correct and is broadly used in photonics, laser physics, and nonlinear optics. However, to obtain and verify this expression from classical electrodynamics, the components of the energy-momentum tensor must be derived. Even for the case of plane monochromatic waves, this derivation is quite complex and can only be followed by experts in the field. Moreover, using different formulations of electrodynamics results in different

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expressions for the energy-momentum tensor and photon momentum – the reason for the ongoing Abraham/Minkowski debate.

In this paper, we present a simple derivation of the photon momentum, by considering photons in media as relativistic quasiparticles (m-photons). In contrast to the complex derivations and discussions of the energy-momentum tensor components, our derivations can be easily reproduced by a very broad scientific audience including students. In Sec. 2, we first introduce the well-known formulas for the momentum, mass, and energy of the m-photon and discuss arguments for and against the Abraham and Minkowski momenta. Afterwards in Sec. 3, we demonstrate that the Minkowski form of the photon momentum follows directly from the relativistic energy conservation law for the m-photon quasiparticle. In Sec. 4, the presented theory is directly applied to derive a new expression for the photon momentum in dispersive media. In the final part, we discuss the effect of light-induced optical stretching and its potential application for measurement of photon momenta in dispersive media.

2. Mass, momentum, and energy of the m-photon (Abraham vs. Minkowski)

In this section, we introduce the well-known formulas for the momentum, mass, and energy of the m-photon following Abraham and Minkowski. For simplicity, we consider first a uniform, non-dispersive, non-absorbing, linear dielectric medium with the refractive index $n \geq 1$. In such a medium the photon phase and group velocities coincide and are determined by $\mathbf{v} = c/n \times \mathbf{k}/k$, where $\mathbf{k}$ is the wave vector. Both, the speed of light c/n and its wavelength λ/n, where λ is the wavelength of light in free space, are reduced. The frequency of light ω remains unchanged. In this paper, the refractive index and light frequency are defined for a medium at rest with anti-reflection coatings and the geometry shown in Fig. 1. The medium at rest coincides with the laboratory coordinate frame.

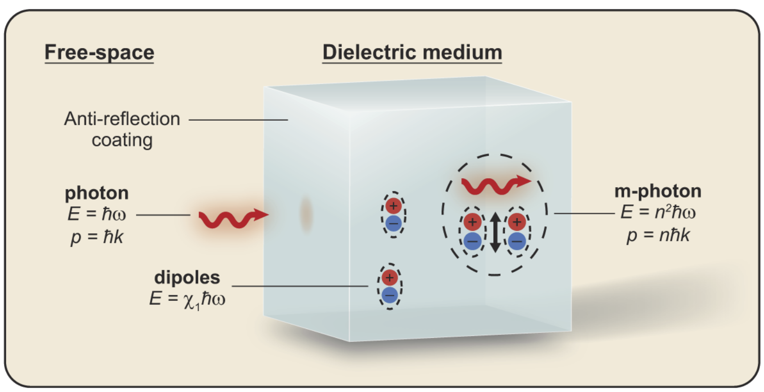

Fig. 1. Illustration of a photon entering a dielectric medium with antireflection coatings at normal incidence, exciting dipole oscillations, and generating a "photon in medium" quasiparticle (m-photon).

The speed of light and its wavelength are directly proportional to each other. For the electrical field of a plane wave propagating along the z axis, we can write

$$\mathbf{E} = \mathbf{E}_0\, e^{i\omega(t-zn/c)} = \mathbf{E}_0\, e^{i(\omega t-kz)}\,, \tag{1}$$

where $k = n\omega/c = 2\pi n/\lambda$ is the wavenumber. Using the definition of momentum $\mathbf{p} = \hbar\mathbf{k}$, we get the absolute value of the Minkowski momentum $p_M = n\hbar\omega/c$, which is also called the canonical momentum [5,21,22]. On the other hand, in special relativity the particle momentum $\mathbf{p}$ and energy E are connected by the equation $\mathbf{p} = \mathbf{v}E/c^2$, where $\mathbf{v}$ is the particle velocity. Introducing

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the photon energy $E = \hbar\omega$ and photon speed $v = c/n$, we get from $p = vE/c^2$ the absolute value of the Abraham momentum $p_A = \hbar\omega/nc$, which is also called the kinetic momentum [5,21,22]. Both momenta coincide in vacuum, but differ in a dielectric medium with $p_M = n^2 p_A$. Some people say that the Abraham momentum emphasises the particle nature of a photon and the Minkowski momentum corresponds to its wave nature. It will be shown in the next section, that the Minkowski momentum also corresponds to the particle nature of a photon. Note that there are several important arguments in favour of the Minkowski momentum following from the Snell's law [23], Doppler and Cherenkov effects [22,24–26], and diffraction [27].

Below we provide expressions for the m-photon mass and energy, which follow from the Abraham and Minkowski momenta. A photon has a zero mass in vacuum. In a dielectric medium it becomes a quasiparticle (m-photon) with non-zero mass, which can be found by equating the relativistic momentum to that of Abraham or Minkowski. From $v = c/n$, the relativistic Lorentz factor $\gamma = 1/\sqrt{1 - v^2/c^2} = n/\sqrt{n^2 - 1}$, and the expression $\gamma\, m_{A,M}\, \mathbf{v} = \mathbf{p}_{A,M}$, we obtain two different rest masses for Abraham and Minkowski m-photons, which are related by the following expression:

$$m_M = n^2\, m_A = \frac{\hbar\omega}{c^2}\, n\sqrt{n^2 - 1}\,. \tag{2}$$

Remember that here we consider the case of $n \geq 1$. Introducing these masses into the relativistic expression for the quasiparticle energy $E_{A,M} = \gamma\, m_{A,M}\, c^2$, we get $E_A = \hbar\omega$ and $E_M = n^2\, \hbar\omega$, respectively. In both cases the relativistic energy-momentum relation is fulfilled:

$$E^2_{A,M}/c^2 = \mathbf{p}^2_{A,M} + m^2_{A,M} c^2\,. \tag{3}$$

Moreover, $(E_{A,M}/c,\ \mathbf{p}_{A,M}) = p^i_{A,M}$ is the four-momentum of the m-photon and adheres to Lorentz transformations. For example, Lorentz transformation from the laboratory coordinate frame to the m-photon rest frame is obtained by:

$$E^R_{A,M} = \gamma\,(E^L_{A,M} - v p^L_{A,M}), \quad p^R_{A,M} = \gamma\,(p^L_{A,M} - \frac{v}{c^2} E^L_{A,M})\,, \tag{4}$$

and results correctly in $E^R_{A,M} = m_{A,M}\, c^2$ and $p^R_{A,M} = 0$.

Now we turn to the discussion of the m-photon energy. In Abraham's case, the m-photon energy $E_A = \hbar\omega$ is equal to the photon energy in vacuum and at first glance everything appears perfect, since the energy conservation is fulfilled. In case of Minkowski, the m-photon energy $E_M = n^2\, \hbar\omega$ is n^2 times larger than the energy of the photon in vacuum. To fulfil the energy conservation law for the Minkowski m-photon, an additional term $U = (n^2 - 1)\hbar\omega$ must be introduced:

$$\hbar\omega = n^2\, \hbar\omega - U. \tag{5}$$

The appearance of this term and its physical meaning has so far not been fully understood. Some authors attribute it to a negative potential energy of the photon inside the medium [28], giving no explanation for the origin of this energy. More recent publications [8,9], have suggested that U represents the mass energy, which is transferred by a mass density wave (MDW) inside the medium. The authors justified this assumption by numerical simulations of elastic and optical waves in dielectric media. However, their MDW theory would imply that the atoms inside the dielectric medium are periodically displaced, which has so far not been experimentally verified.

In the next section, we will use relativistic energy conservation principles to present a direct derivation of Eq. (5), without the reliance on additional assumptions. Based on this, we will explain the physical origin of the U term and arrive at the Minkowski form for the momentum of the m-photon.

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3. Energy conservation for a photon crossing boundary with a dielectric medium

In this section, we demonstrate for the first time to the best of our knowledge that the Minkowski form of the photon momentum follows directly from the relativistic energy conservation law for the m-photon quasiparticle. The first statement that we make, is that a low energy (infrared, visible, ultraviolet) photon with a zero-mass in vacuum is not creating a new mass when entering into the dielectric medium. The low energy photon is not able to start a nuclear reaction and/or to convert its energy to the birth of a new particle. In a non-absorbing medium the photon energy can be transferred only to kinetic energy, defined as the total relativistic energy minus the rest energy:

$$\hbar\omega = \gamma mc^2 - mc^2, \tag{6}$$

where m is the total rest mass. Using $\gamma = n/\sqrt{n^2-1}$, this equation allows to define the following masses:

$$m = \frac{\hbar\omega}{c^2}\left(n\sqrt{n^2-1} + n^2 - 1\right) = m_M + m_d, \tag{7}$$

$$m_M = \frac{\hbar\omega}{c^2}\, n\sqrt{n^2-1} \quad \text{and} \quad m_d = \frac{\hbar\omega}{c^2}\left(n^2-1\right). \tag{8}$$

Now we can provide the physical meaning of these two rest masses: m_M is the rest mass of the m-photon quasiparticle, which coincides with the Minkowski mass given by Eq. (2), and m_d can be considered as the effective rest mass of dipole oscillations of bound electrons induced by the photon. The rest mass of the m-photon is determined in the moving frame. The rest mass of the oscillating dipoles is determined in the laboratory frame. Taking into account that in a non-absorbing medium $n^2 = 1 + \chi^{(1)}$ in SI units, where $\chi^{(1)}$ is the first-order linear susceptibility defining refraction as a two-photon process shown in Fig. 2, we can write $m_d = \chi^{(1)}\hbar\omega/c^2$. Since $\chi^{(1)}$ determines the medium polarization, it justifies the interpretation of m_d as the effective dipole rest mass.

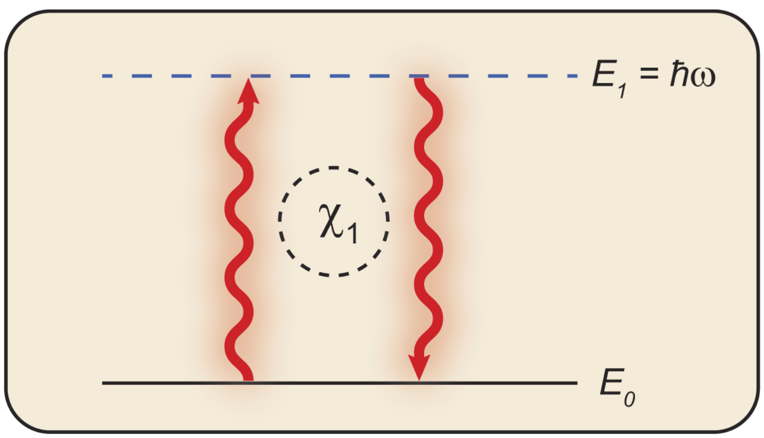

Fig. 2. Illustration of $\chi^{(1)}$ processes responsible for refraction and absorption.

The excited dipole oscillations of bound electrons and moving polarization wave are the integral part of the m-photon quasiparticle. The masses m_M and m_d are connected to each other by $m_M = m_d\gamma$, i.e. the so-called relativistic mass of oscillating dipoles $m_d\gamma$ is equal to the rest mass of the m-photon. Taking this into account, Eq. (6) can be rewritten in the form equivalent to Eq. (5) :

$$\hbar\omega = \gamma m_M c^2 - m_d c^2 = n^2\,\hbar\omega - U\,. \tag{9}$$

The additional term U in Eqs. (5), (9) corresponds to the rest energy of oscillating dipoles induced by the photon: $U = m_d c^2 = (n^2 - 1)\hbar\omega = \chi^{(1)}\hbar\omega$. The derived above energy conservation Eq. (9) rewritten in the form $\hbar\omega + m_d c^2 = \gamma m_M c^2$ tells us that the photon energy plus the dipole rest energy are equal to the total energy of the m-photon. The m-photon quasiparticle with the mass m_M is moving at the velocity $v = c/n$ and its momentum coincides with the Minkowski expression $p_M = n\hbar\omega/c$. The total energy of the m-photon is related to the dipole rest energy by the following equation $E_M = \gamma m_M c^2 = \gamma^2 m_d c^2 = n^2\hbar\omega$.

Now we turn to the discussion of the derived energy conservation Eq. (9) in its final, most important form:

$$\hbar\omega = \gamma m_M c^2 - m_M c^2/\gamma, \tag{10}$$

where the last term on the right hand side represents the Minkowski rest mass reduced by the γ factor. Recall that this energy conservation law follows directly from Eq. (6). According to the relativistic generalization of the virial theorem for a bounded system of charged particles interacting via the Coulomb law, the term $m_M c^2/\gamma$ represents the total average energy of such a system [29], which is smaller than the rest energy due to the negative potential energy of this bounded system. In the non-relativistic limit $v = c/n << c$, corresponding to a medium with very high refractive index, from Eq. (10) using Taylor expansion we get:

$$\hbar\omega \simeq m_M c^2 + m_M v^2/2 - m_M c^2 + m_M v^2/2 = m_M v^2, \tag{11}$$

where $m_M = n^2\hbar\omega/c^2$ is the classical mechanics mass of the m-photon quasiparticle. Note that in classical mechanics the energy conservation Eq. (11) can be obtained by considering the m-photon as a field oscillator. According to the virial theorem for an oscillator [30], its average kinetic and potential energies are equal giving for the total m-photon energy $E = m_M v^2 = \hbar\omega$.

The Abraham mass satisfies the following equation $\hbar\omega = \gamma m_A c^2$, which assumes that the Abraham photon is generating a new mass in the dielectric medium, since in this case the photon energy is spent not only on the kinetic energy, but also on the rest energy, which is not physically justified. In contrast, the presented derivation of Eq. (10) and above discussions provide a straightforward physical justification for the Minkowski momentum $n\hbar\omega/c$, which in this form is broadly used in scientific literature as the photon momentum in a dielectric medium.

4. Dispersive medium

A single photon radiated by an atom or a quantum dot has a certain time duration and frequency bandwidth satisfying the Heisenberg uncertainty relation $\Delta\omega\Delta t \sim 2\pi$. Therefore, it is reasonable to expect that the behaviour of such a photon in a dispersive medium with a frequency dependent refractive index will change.

In a dispersive medium, due to the difference between phase and group velocities, the planes of constant phase and constant amplitude do not in general coincide and can be tilted against each other. In such a medium, the propagation of light is determined by the group velocity $v_g = d\omega/dk = c/n_g$, where $n_g = n + \omega\, dn/d\omega$ is the refractive index for the group velocity (the *group index*) calculated at the carrier frequency ω_0. The group index is constrained by causality to be greater than unity regardless of the sign of the medium refractive index [14,31]. In a uniform isotropic medium the following relation between the group and phase velocities is fulfilled, $\mathbf{v}_g n_g = \mathbf{v}_p n = c\ \mathrm{sign}(n)\mathbf{k}/k$, where the phase velocity, wavevector, and the refractive index are defined at the carrier frequency. This relation is valid also in case of $n<0$ when the phase and group velocities are antiparallel with the phase and group fronts propagating in the opposite directions [14,31].

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It has been experimentally demonstrated that single photons propagate in different media with the group velocity [32–34]. For the m-photon propagating with the group velocity, we can use the energy conservation Eq. (6) with the Lorentz factor determined by the group velocity $\gamma = 1/\sqrt{1 - v_g^2/c^2} = n_g/\sqrt{n_g^2 - 1}$. We will get the same Eqs. (7)–(10), with n replaced by n_g and the m-photon momentum determined by $p = p_M = n_g \hbar\omega/c$. This expression corresponds to the m-photon propagating with the group velocity as relativistic quasiparticle with the rest mass $m_M = (\hbar\omega/c^2)\, n_g \sqrt{n_g^2 - 1}$ and the total energy $E_M = n_g^2 \hbar\omega$.

Using $\hbar\omega = \gamma m_A c^2$, it is straightforward to show that the Abraham momentum in a dispersive medium is determined by $p_A = p_M/n_g^2 = \hbar\omega/cn_g$. All papers agree with this expression for the Abraham momentum, whereas several different expressions have been proposed for the Minkowski momentum in a dispersive medium. The first one is $p_{M1} = n^2\hbar\omega/cn_g$ [21,22,26] and another $p_{M2} = n\hbar\omega/c$ [2,5]. Recall that according to the relativistic quasiparticle approach, as it is shown above, $p_M = n_g\hbar\omega/c$. Thus, there are three different expressions for the m-photon momentum. This could be a good motivation for experimentalists to look at what happens to the light momentum in a dispersive medium and which expression is correct. There is only one relatively old experiment [35], where an attempt to measure the light momentum in a dispersive medium has been performed with sufficient accuracy, with the result that this momentum is equal to $p = n\hbar\omega/c$. However, recent theoretical simulations of this experiment leave a lot of questions and uncertainties [36].

Let us briefly discuss the case of $0<n<1$, which naturally occurs at high frequencies, when the refractive index is determined by $n = \sqrt{1 - \omega_p^2/\omega^2}$, where ω_p is a model parameter (plasma frequency). In this case the phase velocity $v = c/n$ is larger than the speed of light and cannot be used in the relativistic energy conservation Eqs. (6), (10). These equations can be used only with the group velocity. It is easy to verify that the group velocity refractive index is given by $n_g = 1/n$, corresponding to the group velocity $v_g = cn$. According to the expressions derived above, we get in such a medium for the m-photon Minkowski momentum $p_M = n_g\hbar\omega/c = \hbar\omega/cn$. Note, that the Abraham momentum will be defined by $p_A = \hbar\omega/cn_g = n\hbar\omega/c$.

In a medium with negative refraction index, the light momentum defined by $p = n\hbar\omega/c$ is oppositely directed to the group velocity and corresponds in case of total reflection to the attractive (negative) light pressure [14,31]. This scenario is difficult to combine with the behaviour of single photon quasiparticles. According to the above discussions, their momentum is determined by $p = n_g\hbar\omega/c$ and they can generate only the positive light pressure. This conclusion is in agreement with recent numerical simulations examining optical forces in negative refractive index materials and demonstrating the positive light pressure [37].

5. Optical stretching

The effect of light-induced optical stretching can be used for experimental verification of the existing expressions for the photon momentum. According to Minkowski, for a photon entering a dielectric medium at normal incidence from free space (see Fig. 1), the momentum increases and the change of momentum $\Delta p = (n-1)\hbar\omega/c = F\Delta t$ produces a stretching force F acting at the interface in the opposite direction to the propagation of the photon. If we have a laser pulse with the pulse duration $\Delta t = \tau$ and the total number of photons N, the stretching force acting at the interface will be given by $F = (n-1)\hbar\omega N/c\tau = (n-1)P/c$, where P is the laser pulse power. When the laser pulse leaves the medium at the opposite end, the same stretching force is produced at the interface acting now in the direction of the laser pulse. The total stretching force acting at the dielectric medium (assuming that the medium length fulfils the condition $l<<c\tau$) $F = 2(n-1)P/c$. If the dielectric material is elastic, as for example polydimethylsiloxan (PDMS), the light-induced changes in the material length can be measured by interferometric

techniques. Choosing a medium with a strong dispersion, one should be able to verify the value of the momentum $p = n_d\hbar\omega/c$ in the dispersive medium and the corresponding expression for the stretching force $F = 2(n_d - 1)P/c$, where there are 3 possibilities for the n_d value: n, n^2/n_g, and n_g.

6. Conclusion

In this paper the origin of differences between the Abraham and Minkowski expressions for the photon mass, momentum, and energy in a dielectric medium has been clarified by considering photons in media as relativistic quasiparticles. The Abraham expressions can be easily derived from the equation $\hbar\omega = \gamma mc^2$. For the first time, to the best of our knowledge, we have shown how the Minkowski expressions can be derived from the following energy conservation equation $\hbar\omega = \gamma mc^2 - mc^2 = \gamma m_M c^2 - m_M c^2/\gamma$. In this case, it is considered that the photon energy is transferred into the kinetic energy of a "photon in medium" quasiparticle (m-photon) and is not used for the creation of a new mass (as in the Abraham case). Therefore, from the relativistic energy conservation law the Minkowski expressions for the photon mass, given by Eq. (2), momentum $p = n\hbar\omega/c$, and energy $E = n^2\hbar\omega$ in a dielectric medium are correct and fully justified.

Using the same relativistic approach for quasiparticles moving with the group velocity, a new expression for the photon momentum in a dispersive dielectric medium $p = n_g\hbar\omega/c$, where n_g is the group index, has been derived. Additionally, we have discussed the effect of light induced optical stretching, which may be exploited for an experimental verification of the existing expressions for the photon momentum in dispersive media.

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Data availability. Data underlying the results presented in this paper are not publicly available at this time but may be obtained from the authors upon reasonable request.

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