INDEFINITE NUMERICAL RANGE OF 3×3 MATRICES

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Abstract. The point equation of the associated curve of the indefinite numerical range is derived, following Fiedler's approach for definite inner product spaces. The classification of the associated curve is presented in the 3×3 indefinite case, using Newton's classification of cubic curves. Illustrative examples of all the different possibilities are given. The results obtained extend to Krein spaces results of Kippenhahn on the classical numerical range.

Keywords: indefinite numerical range, indefinite inner product space, plane algebraic curve

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1. Introduction

Let $J = I_r \oplus -I_{n-r}$ $(0 \leqslant r \leqslant n)$, where I_m denotes the identity matrix of order m. If $r \neq 0, n$, the matrix J endows $\mathbb{C}^n$ with the Krein structure defined by the indefinite inner product $\langle x, y\rangle_J = y^* J x$, $x, y \in \mathbb{C}^n$. For $A \in M_n$, the algebra of $n \times n$ complex matrices, consider the J-*Cartesian decomposition* $A = H^J + \mathrm{i}K^J$, where $H^J = (A + JA^*J)/2$ and $K^J = (A - JA^*J)/(2\mathrm{i})$ are J-*Hermitian* matrices, that is, $H^J = J(H^J)^*J$ and $K^J = J(K^J)^*J$. If $J = \pm I_n$, we obtain the well known *Cartesian decomposition* of A, where $H = H^J$ and $K = K^J$ are Hermitian matrices.

Let $F_A^J(u, v, w) = \det(uH^J + vK^J + wI_n)$ be the characteristic polynomial of the pencil $-(uH^J + vK^J)$. Our aim is to discuss the connection between $F_A^J(u, v, w)$ and the J-*numerical range* of A denoted and defined by

$$W_J(A) = \left\{ \frac{x^* J A x}{x^* J x} \colon x \in \mathbb{C}^n, x^* J x \neq 0 \right\}.$$

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If $J = \pm I_n$, then $W_J(A)$ reduces to the well-known *classical numerical range* or *field of values* of A, usually denoted by $W(A)$. For an arbitrary J, $W_J(A)$ coincides with the *numerical range of the pencil* $J\lambda - JA$, $\lambda \in \mathbb{C}$ [9], [14], [15]:

$$W(J\lambda - JA) = \{\mu \in \mathbb{C}\colon\ (x^*Jx)\mu - x^*JAx = 0 \text{ for some nonzero } x \in \mathbb{C}^n\}.$$

We briefly recall some known properties of $W_J(A)$. For any $A \in M_n$, $W(A)$ contains the spectrum of A, denoted by $\sigma(A)$, while for the J-numerical range the following inclusion holds: $\sigma_J^+(A) \cup \sigma_J^-(A) \subseteq W_J(A)$, where $\sigma_J^+(A)$ and $\sigma_J^-(A)$ correspond to the sets of eigenvalues of A with associated eigenvectors having positive and negative J-norms, respectively. By $\sigma_J^0(A)$ we represent the set of eigenvalues of A with isotropic eigenvectors, i.e., vectors x such that $x^*Jx = 0$. Note that if $\lambda \in \sigma_J^0(A)$, it may not belong to $W_J(A)$. The field of values $W(A)$ is a compact and convex set for $A \in M_n$ [7]. In contrast with the classical case, $W_J(A)$ may not be closed and is either unbounded or a singleton [10], [11]. For $\lambda \in \mathbb{C}$, $W_J(A) = \{\lambda\}$ if and only if $A = \lambda I_n$. On the other hand, $W_J(A)$ is not usually convex. However, it is the union of convex sets:

$$W_J(A) = W_J^+(A) \cup W_{-J}^+(A), \tag{1}$$

where

$$W_J^{\pm}(A) = \{x^*JAx\colon\ x \in \mathbb{C}^n,\ x^*Jx = \pm 1\}$$

and $W_{-J}^+(A) = -W_J^-(A)$ [11]. Moreover, $W_J(A)$ is *pseudo-convex* [11]; that is, for any pair of distinct points $x, y \in W_J(A)$, if x, y belong to the same convex set in (1), $W_J^+(A)$ or $W_{-J}^+(A)$, then $W_J(A)$ contains the closed line segment joining x and y; otherwise, $W_J(A)$ contains the two closed half-lines of the line defined by x and y with endpoints x and y.

A matrix $U \in M_n$ is called *J-unitary* of signature $(r, n-r)$, $0 \leqslant r \leqslant n$, if $U^{-1} = JU^*J$. These matrices form a group denoted by $\mathscr{U}_{r,n-r}$. For any $U \in \mathscr{U}_{r,n-r}$ we have $W_J(A) = W_J(U^{-1}AU)$. Also, the following identity holds:

$$W_J(\alpha I_n + \beta A) = \alpha + \beta\, W_J(A), \quad \alpha, \beta \in \mathbb{C}.$$

The well-known Elliptical Range Theorem for $A \in M_2$, obtained by Murnaghan [12], states that $W(A)$ is an elliptical disc (possibly degenerate) whose foci are the eigenvalues of A, α_1 and α_2. The major and minor axes are of length

$$\sqrt{\operatorname{tr}(A^*A) - 2\operatorname{Re}(\overline{\alpha_1}\alpha_2)} \quad \text{and} \quad \sqrt{\operatorname{tr}(A^*A) - |\alpha_1|^2 - |\alpha_2|^2},$$

respectively. In the indefinite case, for $A \in M_2$ and $J = \operatorname{diag}(1, -1)$, the Hyperbolical Range Theorem [2] states that $W_J(A)$ is bounded by the hyperbola (possibly degenerate) with foci at the eigenvalues of A, α_1 and α_2, and the transverse and non-transverse axes of length

$$\sqrt{\operatorname{tr}(JA^*JA) - 2\operatorname{Re}(\overline{\alpha_1}\alpha_2)} \quad \text{and} \quad \sqrt{|\alpha_1|^2 + |\alpha_2|^2 - \operatorname{tr}(JA^*JA)},$$

respectively. For the degenerate cases, $W_J(A)$ may be a singleton, a line, a subset of a line, the whole complex plane, or the complex plane except a line.

A *supporting line* of a convex set $K \subseteq \mathbb{C}$ is a line that intersects K at least at one point and defines two half-planes, such that one of them does not contain any point of K. The supporting lines of $W_J(A)$ are, by definition, the supporting lines of the convex sets $W_J^+(A)$ and $W_{-J}^+(A)$. As proved in [2, Theorem 2.2] (cf. [15, Remark 3]), if $ux + vy + w = 0$ is the equation of a supporting line of $W_J(A)$, then

$$F_A^J(u, v, w) = \det(uH^J + vK^J + wI_n) = 0. \tag{2}$$

Since $F_A^J(u, v, w)$ is a homogeneous polynomial of degree n, (2) may be viewed as the line equation of an algebraic curve on the complex projective plane $P\mathbb{C}^2$. Considering the dual curve

$$\Gamma^* = \{(u, v, w) \in P\mathbb{C}^2 : \ F_A^J(u, v, w) = 0\},$$

we may determine by dualization

$$\Gamma = \{(x, y, z) \in P\mathbb{C}^2 : \ xu + yv + zw = 0 \text{ is a tangent of } \Gamma^*\},$$

whose real affine view

$$C_J(A) = \{(x, y) \in \mathbb{R}^2 : \ (x, y, 1) \in \Gamma\}$$

is called the *associated curve* of $W_J(A)$. If $J = \pm I_n$, then $C_J(A)$ is simply denoted by $C(A)$ and called the associated curve or *boundary generating curve* of $W(A)$. The curve $C_J(A)$ has class n, that is, through a general point in the plane there are n lines (may be complex) tangent to the curve. (For details on plane algebraic curves we refer to [4].)

For $J = \pm I_n$, Kippenhahn proved that the curve $C(A)$ generates $W(A)$ as its convex hull [8]. If $J \neq \pm I_n$, then $C_J(A)$ generates $W_J(A)$ as its pseudo-convex hull. Indeed, if there exists $\theta \in [0, 2\pi]$ such that the n eigenvalues $\lambda_1(\theta), \dots, \lambda_n(\theta)$ of $\cos\theta H^J + \sin\theta K^J$, $A = H^J + \mathrm{i}K^J \in M_n$, are real and have an associated basis of anisotropic eigenvectors $u_1(\theta), \dots, u_n(\theta)$, then

$$\frac{u_k^*(\theta) JA\, u_k(\theta)}{u_k^*(\theta) J u_k(\theta)} \in C_J(A), \quad k = 1, \dots, n.$$

Considering all the different directions θ satisfying the above requisites, the set $W_J(A)$ is the pseudo-convex hull of the points so obtained [3].

This paper is organized as follows. In Section 2, the point equation of $C_J(A)$ is derived, following the approach developed by Fiedler [5] for $W(A)$. In Section 3, $W_J(A)$ is characterized for J-Hermitian matrices of size 3. In Section 4, the associated curves $C_J(A)$ are classified in the 3×3 case using Newton's classification of cubics, and illustrative examples are provided. These results extend to Krein spaces results of Kippenhahn [8] on $W(A)$. In Section 5, some open problems are presented.

2. The point equation of $C_J(A)$

We recall that a point $P \neq (1, i, 0), (1, -i, 0)$, the circular points at infinity, is called a *focus* of an algebraic curve C if the lines l_1 and l_2 through P and $(1, i, 0)$ and through P and $(1, -i, 0)$, respectively, are tangent to C at points other than the circular points at infinity. It may be easily proved that the coefficients of the polynomial $F_A^J(u, v, w)$ are real. A curve of class n with real coefficients has n real foci, counting multiplicities, and $n^2 - n$ non-real foci [16]. Murnaghan [12] and Kippenhahn [8] independently proved that the real foci of the algebraic curve defined in homogeneous line coordinates by $\det(uH + vK + wI_n) = 0$ are the eigenvalues of $A = H + \mathrm{i}K$.

Proposition 1. *The n real foci of the algebraic curve defined in homogeneous line coordinates by the equation $F_A^J(u, v, w) = \det(uH^J + vK^J + wI_n) = 0$ are the eigenvalues $\alpha_1, \ldots, \alpha_n$ of the matrix $A = H^J + \mathrm{i}K^J$.*

P r o o f. Taking $u = 1$ and $v = \mathrm{i}$ in (2), we obtain $\det(H^J + \mathrm{i}K^J + wI_n) = \det(A + wI_n) = 0$, and w coincides with $-\alpha_j$, $j = 1, \ldots, n$. Repeating this procedure for $u = 1$ and $v = -\mathrm{i}$, we find $\det(H^J - \mathrm{i}K^J + wI_n) = \det(A^* + wI_n) = 0$. Thus, w coincides with $-\overline{\alpha_j}$, $j = 1, \ldots, n$. For $k, l = 1, \ldots, n$, let g_k and $\overline{g_l}$ be the lines given in line coordinates by $(1, \mathrm{i}, -\alpha_k)$ and $(1, -\mathrm{i}, -\overline{\alpha_l})$, respectively. The lines g_k pass through $(1, \mathrm{i}, 0)$, while the lines $\overline{g_l}$ pass through $(1, -\mathrm{i}, 0)$. Since g_k and $\overline{g_l}$ are solutions of the equation (2), they are the unique tangent lines to the curve that pass through the circular points at infinity. Easy calculations show that the intersection of g_k with $\overline{g_k}$ is given by $(\operatorname{Re}\alpha_k, \operatorname{Im}\alpha_k, 1)$, $k = 1, \ldots, n$. □

We recall that a common procedure to find the point equation of $C_J(A)$ is to eliminate one of the indeterminates, say u, from (2) and $ux + vy + w = 0$, dehomogenize the result by setting $w = 1$, and eliminate the remaining parameter v from the equations $F_A(v, x, y) = 0$ and $\partial F_A(v, x, y)/\partial v = 0$. In this section we present an alternative procedure, developed by Fiedler for the classical case.

For this purpose we recall some auxiliary results. The *second mixed compound* of two matrices $A = [a_{ij}]$ and $B = [b_{ij}]$ of the same size $m \times n$, denoted by $\mathscr{C}_2(A, B)$, is the $\binom{m}{2} \times \binom{n}{2}$ matrix with entries

$$\mathscr{C}_2(A, B)_{PQ} = \frac{1}{2}(a_{ir}b_{js} + a_{js}b_{ir} - a_{is}b_{jr} - a_{jr}b_{is}),$$

where $P = (i, j)$, $1 \leqslant i < j \leqslant m$, and $Q = (r, s)$, $1 \leqslant r < s \leqslant n$. In particular, the *second compound* of A is denoted and defined by $\mathscr{C}_2(A) = \mathscr{C}_2(A, A)$. From the definition it follows that

$$\mathscr{C}_2(A)_{PQ} = \det \begin{bmatrix} a_{ir} & a_{is} \\ a_{jr} & a_{js} \end{bmatrix}$$

for $P = (i, j)$, $1 \leqslant i < j \leqslant m$, and $Q = (r, s)$, $1 \leqslant r < s \leqslant n$. Therefore, $\mathscr{C}_2(A)$ is the array of all second order minors of A.

The following two lemmas were obtained in [5] and are used in the proof of Theorem 1.

Lemma 1. *For any matrices A, B, A_1, A_2 of the same size and for any complex numbers α_1, α_2 the following identities hold:*
(a) $\mathscr{C}_2(\alpha_1 A_1 + \alpha_2 A_2, B) = \alpha_1 \mathscr{C}_2(A_1, B) + \alpha_2 \mathscr{C}_2(A_2, B)$;
(b) $\mathscr{C}_2(A, B) = \mathscr{C}_2(B, A)$.

Lemma 2. *For $A, B \in M_n$, the form $\det(xA + yB)$ (which is a product of linear complex factors $\alpha_i x + \beta_j y$) is either identically zero, or has a multiple linear factor if and only if*

$$\det \begin{bmatrix} \mathscr{C}_2(A) & \mathscr{C}_2(A, B) \\ \mathscr{C}_2(A, B) & \mathscr{C}_2(B) \end{bmatrix} = 0.$$

As observed by Fiedler, the next theorem is not practical for large matrices. However, it can be used for small size matrices and for theoretical purposes. The proof is essentially Fiedler's proof and it is presented for the sake of completeness.

Theorem 1. *Let $A = H^J + \mathrm{i}K^J \in M_n$, where H^J and K^J are J-Hermitian. If $C_J(A)$, given in line coordinates by (2), is irreducible, then its point equation is given by the non-linear part of the equation*

$$\det \begin{bmatrix} \mathscr{C}_2(H^J - xI_n) & \mathscr{C}_2(H^J - xI_n, K^J - yI_n) \\ \mathscr{C}_2(H^J - xI_n, K^J - yI_n) & \mathscr{C}_2(K^J - yI_n) \end{bmatrix} = 0 \tag{3}$$

or, equivalently, by the non-linear part of the equation

$$
\det\begin{bmatrix} \mathscr{C}_2(H^J) & \mathscr{C}_2(H^J,K^J) & \mathscr{C}_2(H^J,I_n) & xI_{\binom{n}{2}} \\ \mathscr{C}_2(H^J,K^J) & \mathscr{C}_2(K^J) & \mathscr{C}_2(K^J,I_n) & yI_{\binom{n}{2}} \\ \mathscr{C}_2(H^J,I_n) & \mathscr{C}_2(K^J,I_n) & I_{\binom{n}{2}} & I_{\binom{n}{2}} \\ xI_{\binom{n}{2}} & yI_{\binom{n}{2}} & I_{\binom{n}{2}} & 0 \end{bmatrix} = 0. \tag{4}
$$

The linear factors of equations (3) *and* (4) *correspond to flexional tangents or to multiple tangents (at real or complex points, finite or infinite points) of the complex algebraic curve that contains* $C_J(A)$.

P r o o f. By elementary operations with the blocks of the matrix in (4), Laplace Theorem and Lemma 1, we easily conclude that the left hand sides of equations (3) and (4) are equal if $\binom{n}{2}$ is even, or symmetric if $\binom{n}{2}$ is odd. Therefore, the equations (3) and (4) are equivalent. On the other hand, (3) is invariant under the transformation $A \mapsto A \pm (x + \mathrm{i}y)I_n$, and $C_J(A)$ is obtained from $C_J(A \pm (x + \mathrm{i}y)I_n)$ by a translation. Hence, it is sufficient to prove that the result holds for $(0, 0)$. The origin satisfies (4) if and only if

$$
\det\begin{bmatrix} \mathscr{C}_2(H^J) & \mathscr{C}_2(H^J,K^J) \\ \mathscr{C}_2(H^J,K^J) & \mathscr{C}_2(K^J) \end{bmatrix} = 0.
$$

By Lemma 2, $\det(uH^J + vK^J)$ is identically zero or has a multiple linear factor. The first case implies that $\det(uH^J + vK^J + wI_n)$ is divisible by w and, consequently, $C_J(A)$ is reducible. Thus, the form $\det(uH^J + vK^J)$ has a multiple linear factor.

Now, the proof proceeds by duality arguments. Consider the dual curve of $C_J(A)$ given in homogeneous point coordinates by (2). Taking $w = 0$ in (2), we obtain the intersection points of the dual curve with the line of infinity. Since $\det(uH^J + vK^J)$ has a multiple linear factor, this implies that the line of infinity has two intersections with the dual curve at a certain point. If that point is singular, it corresponds to a flexional tangent or to a multiple tangent of the curve whose real part is $C_J(A)$. Otherwise, the line of infinity must be tangent to the dual curve at that point and this implies that the origin belongs to $C_J(A)$. Consequently, the equations (3) and (4) give the points of $C_J(A)$ and the flexional and multiple tangents of the complex curve that contains $C_J(A)$. □

Remark 1. From (4) we conclude that the order of $C_J(A)$ is given by $n(n-1)$ minus the number of flexional and multiple tangents counted according to multiplicities, which agrees with Plücker's formulae (cf. [4]).

3. Characterization of $W_J(A)$ for a J-Hermitian matrix A of size 3

In this Section we present auxiliary results which will be used in Section 4. Let $A \in M_n$ be a J-Hermitian matrix. It is known that ([13], [9], [14], [15])

$$W_J^+(A) = \{x \in \mathbb{R}\colon\ t(x+\mathrm{i}) \in W(JA+\mathrm{i}J) \ \text{ for some } 0 < t \leqslant 1\}\,; \tag{5}$$

$$W_{-J}^+(A) = \{x \in \mathbb{R}\colon\ t(-x-\mathrm{i}) \in W(JA+\mathrm{i}J) \ \text{for some } 0 < t \leqslant 1\}\,. \tag{6}$$

From (5) and (6) it follows that $W_J^+(A)$ is a right half-line $[m_1,+\infty[$, or $]m_1,+\infty[$, for a certain $m_1 \in \mathbb{R}$, if and only if $W_{-J}^+(A)$ is a left half-line $]-\infty,m_2]$, or $]-\infty,m_2[$, for some $m_2 \in \mathbb{R}$. The endpoints of these half-lines are eigenvalues of A [13]. On the other hand, $W_J^+(A) = \mathbb{R}$ if and only if $W_{-J}^+(A) = \mathbb{R}$. The following lemma plays an important role in our investigation.

Lemma 3 [13]**.** *Let $A \in M_n$ be a non-scalar J-Hermitian matrix with $J \neq \pm I_n$. Then*

(a) *$W_J^+(A)$ is an open or closed half-line of $\mathbb{R}$ if and only if $0 \in \partial W(JA+\mathrm{i}J)$ or $0 \notin W(JA+\mathrm{i}J)$;*

(b) *$W_J^+(A) = \mathbb{R}$ if and only if 0 is an interior point of $W(JA+\mathrm{i}J)$.*

In the sequel $J = \mathrm{diag}(1,1,-1)$. For $A \in M_3$ a J-Hermitian matrix, the equation $\det(uJA + vJ + wI_3) = 0$ has the (simple) root $(0,1,1)$ and the (double) root $(0,1,-1)$. Thus, $y=-1$ is a simple tangent of $C(JA+\mathrm{i}J)$, while $y=1$ is a double tangent. Since $W(J) = [-1,1]$, it is also clear that $y = \pm 1$ are supporting lines of $W(JA+\mathrm{i}J)$. Using Kippenhahn's classification of the boundary generating curve [8], we arrive at the following possibilities for $C(JA+\mathrm{i}J)$:

I. the curve reduces to three points such that one belongs to $y=-1$ and the other two to the line $y=1$;

II. the curve is the union of a point in $y=1$ and an ellipse with $y=\pm1$ as tangent lines;

III. the curve is irreducible with $y=1$ as a double tangent. Therefore, it must be a cardioid contained in the strip limited by the lines $y=\pm1$.

The field of values $W(JA+\mathrm{i}J)$ is the convex hull of $C(JA+\mathrm{i}J)$. Using (5) and (6), we characterize $W_J^+(A)$ and $W_{-J}^+(A)$, and so $W_J(A)$.

Lemma 4. *If $A = [a_{ij}] \in M_3$ is a J-Hermitian matrix and $\mathrm{tr}(A) = \alpha$, then there exist $U \in \mathscr{U}_{2,1}$ and $b \in \mathbb{R}$, $c, d \geqslant 0$ such that*

$$U^{-1}AU = \begin{bmatrix} \alpha - b - a_{33} & c & 0 \\ c & b & d \\ 0 & -d & a_{33} \end{bmatrix}.$$

P r o o f. Considering the diagonal matrix $U_1 = \operatorname{diag}(\mathrm{e}^{\mathrm{i}\beta}, \mathrm{e}^{\mathrm{i}\gamma}, 1)$, $\beta, \gamma \in \mathbb{R}$, we may obtain a matrix $B = [b_{ij}] = U_1^{-1} A U_1$ such that $b_{13} = -b_{31} \geqslant 0$, $b_{23} = -b_{32} \geqslant 0$. Next, applying the J-unitary transformation associated with the matrix

$$U_2 = \begin{bmatrix} p & q & 0 \\ -q & p & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

with $p, q \in \mathbb{R}$ and $p^2 + q^2 = 1$, we may consider $C = [c_{ij}] = U_2^{-1} B U_2$ such that $c_{13} = c_{31} = 0$. Finally, by the J-unitary transformation induced by $U_3 = \operatorname{diag}(\mathrm{e}^{\mathrm{i}\beta}, 1, 1)$, $\beta \in \mathbb{R}$, we get $D = [d_{ij}] = U_3^{-1} C U_3$ such that $d_{12}, d_{21} \geqslant 0$.

If $a_{13} \neq 0$, take $U = U_1 U_2 U_3$. Otherwise, it is sufficient to consider $U = U_1 U_3$. We notice that a_{33} is invariant under these operations. $\square$

We denote by $\operatorname{tr} \mathscr{C}_2(B)$ the sum of the 2×2 principal minors of a matrix B. Easy calculations yield

Lemma 5. *For $A = H^J + \mathrm{i}K^J \in M_3$ and $J = I_r \oplus -I_{3-r}$ $(0 \leqslant r \leqslant 3)$ we have*

$$\begin{aligned} F_A^J(u, v, w) = {} & w^3 + \det(H^J)u^3 + \det(K^J)v^3 + \operatorname{Re} \operatorname{tr}(A)uw^2 + \operatorname{Im} \operatorname{tr}(A)vw^2 \\ & + \operatorname{Im} \operatorname{tr} \mathscr{C}_2(A)uvw + \operatorname{tr} \mathscr{C}_2(H^J)u^2 w + \operatorname{tr} \mathscr{C}_2(K^J)v^2 w \\ & + [\det(H^J) - \operatorname{Re} \det(A)]uv^2 + [\det(K^J) + \operatorname{Im} \det(A)]u^2 v. \end{aligned}$$

A matrix $A \in M_n$ is called *nilpotent* if there exists $k \in \mathbb{N}$ such that $A^k = 0$. The smallest k satisfying $A^k = 0$ is the *nilpotency index* of A. If $k = 1$, then $A = 0$ and $W_J(A) = W_J^+(A) = W_{-J}^+(A) = \{0\}$.

Theorem 2. *Let $0 \neq A = [a_{ij}] \in M_3$ be a nilpotent J-Hermitian matrix with nilpotency index $k \leqslant 3$. The following assertions hold:*

(a) *if $k = 2$ and $a_{33} > 0$ $(a_{33} < 0)$, then $W_J^+(A) =]-\infty, 0]$, $W_{-J}^+(A) =]0, +\infty[$ $(W_J^+(A) = [0, +\infty[$, $W_{-J}^+(A) =]-\infty, 0[)$;*

(b) *if $k = 3$, then $W_J^+(A) = W_{-J}^+(A) = \mathbb{R}$.*

P r o o f. Since A is J-Hermitian and has zero trace, by the J-unitary similarity invariance of $W_J(A)$ and Lemma 4 we may write A in the form

$$A = \begin{bmatrix} -a-b & c & 0 \\ c & b & d \\ 0 & -d & a \end{bmatrix}, \tag{7}$$

where $a = a_{33}$ and $b \in \mathbb{R}$, $c, d \geqslant 0$. Notice that $d \neq 0$, because $A^3 = 0$ and $A \neq 0$. Since c is a real number and $A^3 = 0$, considering the $(2,3)$-th and $(3,3)$-th entries of

A^3 it is easy to prove that $d^2 - a^2 \geqslant 0$. Furthermore, $-(d^2 - a^2)$ coincides with the $(3,3)$-th entry of A^2. Thus, if $d \neq |a|$, that is, $d > |a|$, the nilpotency index k of A is equal to 3. If $d = |a|$ then analyzing the entries of A^3 and A^2 we easily conclude that $k = 2$. Having in mind that $\operatorname{tr}(A) = \det(A) = \operatorname{tr}\mathscr{C}_2(A) = 0$, from Lemma 5 we obtain

$$\det(uJA + vJ + wI_3) = (w+v)(w^2 - v^2 - 2auw) - 2(d^2 - a^2)u^2 w.$$

The associated curve $C(JA + \mathrm{i}J)$ is reducible if and only if $d = |a|$. In this case it is the union of the point $(0,1,1)$ and the ellipse

$$\frac{(x+a)^2}{a^2} + y^2 = 1, \tag{8}$$

which is the dual of the conic $w^2 - v^2 - 2auw = 0$. The ellipse passes through the origin, the imaginary axis being the tangent of the curve at that point. Consequently, $W_J^+(A) =]-\infty, 0]$ and $W_{-J}^+(A) =]0, +\infty[$ if $a > 0$, or $W_J^+(A) = [0, +\infty[$ and $W_{-J}^+(A) =]-\infty, 0[$, if $a < 0$.

Suppose now that $d > |a|$. Since $C(JA + \mathrm{i}J)$ is irreducible, the associated curve is a cardioid with the double tangent $y = 1$. Consider the dual curve defined in homogeneous point coordinates by $\det(xJA + yJ + zI_3) = 0$. Take the affine view $x = 1$ and define

$$f(y,z) = \det(JA + yJ + zI_3) = (z+y)(z^2 - y^2 - 2az) - 2(d^2 - a^2)z.$$

Evaluating the first partial derivatives of f at $(0,0)$, we conclude that $(1,0,0)$ is a simple point of the curve, whose tangent has homogeneous line coordinates $(0,0,1)$. Analyzing the second order partial derivatives of f, we can easily prove that this line has more than two intersections with the dual curve. Therefore, the point $(1,0,0)$ is a flex. By dual considerations, we conclude that $(0,0,1)$ is a cuspid of $C(JA + \mathrm{i}J)$ and the tangent is the line $(1,0,0)$, i.e., the imaginary axis. If the origin of the affine plane coincides with the cuspid of the cardioid, then it lies in the interior of its convex hull. Hence, by Lemma 3, $W_J^+(A) = W_{-J}^+(A) = \mathbb{R}$. $\square$

Theorem 3. *Let $A = [a_{ij}] \in M_3$ be a J-Hermitian matrix with the eigenvalues 0 (double) and $\alpha > 0$ (simple). Then the following possibilities may occur:*

(a) *A is J-unitarily diagonalizable and either $\alpha \in \sigma_J^-(A)$, $W_J^+(A) =]-\infty, 0]$, $W_{-J}^+(A) = [\alpha, +\infty[$ or $\alpha \in \sigma_J^+(A)$, $W_J^+(A) = [0, +\infty[$, $W_{-J}^+(A) =]-\infty, 0]$.*

(b) *A is J-unitarily reducible to the form*

$$\begin{bmatrix} \alpha & 0 & 0 \\ 0 & -a & a \\ 0 & -a & a \end{bmatrix}, \ a \neq 0,$$

and either $a > 0$, $W_J(A) = W_J^+(A) = W_{-J}^+(A) = \mathbb{R}$ *or* $a < 0$, $W_J^+(A) =]0, +\infty[$, $W_{-J}^+(A) =]-\infty, 0[$ *and* $W_J(A) = \mathbb{R} \setminus \{0\}$.

Proof. Since α is a simple eigenvalue, either $\alpha \in \sigma_J^+(A)$ or $\alpha \in \sigma_J^-(A)$.

(a) Suppose that $D = U^{-1}AU$, where $U \in \mathscr{U}_{2,1}$ and D is a diagonal matrix. If $\alpha \in \sigma_J^-(A)$, then $D = \mathrm{diag}(0, 0, \alpha)$, and $W(JD + \mathrm{i}J)$ is the line segment joining the points $(0, 1, 1)$ and $(-\alpha, -1, 1)$. Otherwise, $D = \mathrm{diag}(\alpha, 0, 0)$ or $D = \mathrm{diag}(0, \alpha, 0)$, and $W(JD + \mathrm{i}J)$ is the triangle with vertices $(0, 1, 1)$, $(0, -1, 1)$ and $(\alpha, 1, 1)$. The result follows easily from (5) and (6).

(b) Suppose that A is not J-unitarily diagonalizable. Taking into account that $\alpha \notin \sigma_J^0(A)$, A must be a J-decomposable matrix. Since $\det(A) = 0$, there exists $V \in \mathscr{U}_{2,1}$ such that

$$B = V^{-1}AV = \begin{bmatrix} \alpha & 0 & 0 \\ 0 & -a & a \\ 0 & -a & a \end{bmatrix}, \ a \neq 0.$$

In this case, α necessarily belongs to $\sigma_J^+(A)$, otherwise A would be diagonalizable. Using the proof technique of the last theorem, we deduce the expression for $\det(uJB + vJ + wI_3)$ and, by dual considerations, we find that $C(JB + \mathrm{i}J)$ is the union of the point $(\alpha, 1, 1)$ and the ellipse (8). If $a > 0$, then the point and the ellipse are in the right and left half-planes, respectively, and 0 is an interior point of $W(JB + \mathrm{i}J)$. Otherwise, they are both in the right half-plane and $0 \in \partial W(JB + \mathrm{i}J)$. The sets $W_J^+(A) = W_J^+(B)$ and $W_{-J}^+(A) = W_{-J}^+(B)$ are given by (5) and (6). $\square$

Remark 2. Noting that $W_J^+(-A) = -W_J^+(A)$, we see that the case $\alpha < 0$ is covered by Theorem 3.

Remark 3. Let $A = [a_{ij}] \in M_3$ be an arbitrary J-Hermitian matrix with real spectrum. The above theorems allow the characterization of $W_J(A)$ according to the multiplicity of its eigenvalues.

(a) Suppose that A has a triple eigenvalue λ. Then $A - \lambda I_3$ is nilpotent with nilpotency index $k \leqslant 3$. If A is a scalar matrix, then $W_J(A) = \{\lambda\}$. Otherwise, by Theorem 2, $W_J(A) = \mathbb{R}$ ($W_J^+(A)$ and $W_{-J}^+(A)$ are also obtained from Theorem 2).

(b) Suppose that A has a simple eigenvalue α and a double eigenvalue $\lambda \neq \alpha$. Thus, $\alpha - \lambda$ and 0 are a simple and a double eigenvalue of the J-Hermitian matrix $A - \lambda I_3$, respectively, and Theorem 3 characterizes $W_J(A)$, $W_J^+(A)$ and $W_{-J}^+(A)$.

(c) Finally, suppose that A has three distinct real eigenvalues, $\lambda_1 < \lambda_2 < \lambda_3$. If $\lambda_1, \lambda_3 \in \sigma_J^+(A), \lambda_2 \in \sigma_J^-(A)$, then $W_J(A) = \mathbb{R}$. Otherwise, $W_J(A)$ is the real line except for the open line segment joining two of the eigenvalues.

Additionally, if A has non-real eigenvalues, then it is known that $W_J(A) = W_J^+(A) = W_{-J}^+(A) = \mathbb{R}$ [3].

4. Characterization of $W_J(A)$ for $A \in M_3$

We characterize $W_J(A)$ for $J = \operatorname{diag}(1,1,-1)$ and for an arbitrary $A \in M_3$, following Kippenhahn's approach in the classical case [8]. That is, we classify the associated curve $C_J(A)$, considering the factorizability of the polynomial $F_A^J(u,v,w)$. The following possibilities may occur.

1st Case: Suppose that $A \in M_3$ is a J-decomposable matrix, i.e., there exists a J-unitary matrix $U \in \mathscr{U}_{2,1}$ such that

$$U^{-1}AU = \begin{bmatrix} b & 0 \\ 0 & B \end{bmatrix}, \tag{9}$$

or

$$U^{-1}AU = \begin{bmatrix} B & 0 \\ 0 & b \end{bmatrix}, \tag{10}$$

where $b \in \mathbb{C}$ and $B \in M_2$. Since $W_J(A) = W_J(U^{-1}AU)$, without loss of generality we may consider that A is a block diagonal matrix of the form (9) or (10). If A is of the form (9), then $F_A^J(u,v,w) = (\operatorname{Re} b\, u + \operatorname{Im} b\, v + w) F_B^{J_1}(u,v,w)$, where $J_1 = \operatorname{diag}(1,-1)$. The linear factor of $F_A^J(u,v,w)$ corresponds to the eigenvalue $b \in \sigma_J^+(A)$. It follows that $W_J(A)$ is the pseudo-convex hull of b and $C_{J_1}(B)$, which is by the Hyperbolical Range Theorem a hyperbola (possibly degenerate).

Suppose now that A is of the form (10). We have $b \in \sigma_J^-(A)$, $C(B)$ is characterized by the Elliptical Range Theorem, and so $W_J(A)$ is the pseudo-convex hull of a point and an ellipse (possibly degenerate).

2nd Case: The matrix A is J-indecomposable, but the polynomial $F_A^J(u,v,w)$ factorizes into a linear and a quadratic factor or into three linear factors (possibly not distinct). The linear factors correspond to the foci of the algebraic curve, and consequently, to the eigenvalues of A. If there exists a quadratic factor, it corresponds to a hyperbola or to an ellipse (the conic cannot be a parabola, because one of its real foci is a point of infinity and this contradicts Proposition 1). Therefore, $C_J(A)$ consists of three points (counting multiplicities) or of one point and an ellipse or a hyperbola.

3rd Case: Finally, suppose that the polynomial $F_A^J(u,v,w)$ is irreducible. By Newton's classification of cubics [1] and by dual considerations there are the following possibilities for the associated curve:

C1. $C_J(A)$ is a sextic with three cusps and at least one oval component;

C2. $C_J(A)$ is a quartic with three cusps and an ordinary double tangent (at two complex points);

C3. $C_J(A)$ is a quartic with one cusp and an ordinary double tangent at two of its points;

C4. $C_J(A)$ is a cubic with one cusp and one flex;

C5. $C_J(A)$ is a sextic with three cusps and contains neither oval components nor ordinary double tangents.

In the second and third cases, we may determine $W_J(A)$ as follows. The investigation of the projections of $W_J^+(A)$ $(W_{-J}^+(A))$ on lines that pass through the origin and defining an angle θ with the real axis is crucial. These projections are given by $W_J^+(\cos\theta H^J + \sin\theta K^J)$ $(W_{-J}^+(\cos\theta H^J + \sin\theta K^J))$, and we use the characterization of the numerical range of J-Hermitian matrices from the previous section. If there exists a single direction $\theta \in \mathbb{R}$ such that $W_J^+(\cos\theta H^J + \sin\theta K^J)$ is a half-line, then $W_J^+(A)$ is a half-plane (possibly open) perpendicular to the direction θ. When the same happens for several directions, then $W_J^+(A)$ is the intersection of the corresponding half-planes. The boundaries of the half-planes are supporting lines of $W_J^+(A)$ and tangents to the associated curve. In this case, the intersections of the half-planes coincide with the pseudo-convex hull of the associated curve. If no such direction exists, $W_J^+(A)$ is the complex plane.

In the examples presented here, it is enough to determine the projections $W_J(H^J)$ of $W_J(A)$ on the real axis, because the matrices are real and so the J-numerical range is symmetric with respect to to the real axis. The figures have been produced with *Mathematica 5.1*, and Theorem 1 is used to determine the point equation of $C_J(A)$, while the line equation is given by Lemma 5. Not only the associated curve is represented, but also the eigenvalues of each matrix. The boundaries of $W_J^+(A)$ and $W_{-J}^+(A)$ are represented by thick lines.

Example 1. It can be easily seen that the matrix

$$A_1 = \begin{bmatrix} 1 & 0 & 0 \\ -1 & -1 & 0 \\ 0 & -1 & 1 \end{bmatrix}$$

is J-indecomposable, the Kippenhahn polynomial is reducible, $F_{A_1}^J(u,v,w) = -(u-w)(u+w)^2$ and $C_J(A_1) = \{(-1,0,1),(1,0,1)\}$. Since A_1 is not a J-Hermitian matrix, $W_J(A_1)$ cannot be a subset of the real line. To characterize $W_J(A_1)$, we determine its projection on the real axis and so we consider the J-Hermitian component matrix of the J-Cartesian decomposition of A_1,

$$H_1^J = \begin{bmatrix} 1 & -\frac{1}{2} & 0 \\ -\frac{1}{2} & -1 & \frac{1}{2} \\ 0 & -\frac{1}{2} & 1 \end{bmatrix}.$$

The eigenvalues of H_1^J are 1 (double) and -1. By Remark 3(b) we conclude that $W_J^+(H_1^J) = W_{-J}^+(H_1^J) = \mathbb{R}$, and so $W_J(A_1) = W_J^+(A_1) = W_{-J}^+(A_1) = \mathbb{C}$.

In the following examples, the polynomial $F_A^J(u,v,w)$ is irreducible.

Example 2. Let

$$A_2 = \begin{bmatrix} 0 & -\frac{1}{2} & 0 \\ \frac{1}{2} & 0 & -\frac{1}{2} \\ 0 & \frac{1}{2} & \sqrt{2} \end{bmatrix} \quad \text{with } H_2^J = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -\frac{1}{2} \\ 0 & \frac{1}{2} & \sqrt{2} \end{bmatrix}.$$

The line equation of $C_J(A_2)$ is $4 + 4\sqrt{2}u + u^2 - v^2 - \sqrt{2}uv^2 = 0$ (cf. Figure 1). The eigenvalues of H_2^J are $0, (\sqrt{2}-1)/2 \in \sigma_J^+(H_2^J)$ and $(\sqrt{2}+1)/2 \in \sigma_J^-(H_2^J)$. Therefore, by Remark 3(c),

$$W_J^+(H_2^J) =]-\infty, (\sqrt{2}-1)/2] \quad \text{and} \quad W_{-J}^+(H_2^J) = [(\sqrt{2}+1)/2, +\infty[.$$

Then $W_J^+(A_2)$ is contained in the half-plane $x \leqslant (\sqrt{2}-1)/2$ and $W_{-J}^+(A_2)$ is contained in the half-plane $x \geqslant (\sqrt{2}+1)/2$. Moreover, $W_J^+(A_2)$ and $W_{-J}^+(A_2)$ are bounded by the outer branches of $C_J(A_2)$ contained in those half-planes.

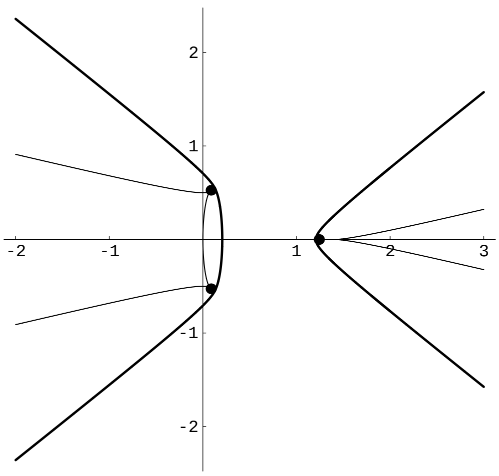

Figure 1. Curve of C1 type with two oval components and two components with cusps.

Example 3. Let

$$A_3 = \begin{bmatrix} 3 & 2 & 1 \\ -2 & -5 & 0 \\ 1 & 0 & 0 \end{bmatrix} \quad \text{with } H_3^J = \begin{bmatrix} 3 & 0 & 0 \\ 0 & -5 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

The line equation of $C_J(A_3)$ is $1 - 2u - 3(5u^2 + v^2) - 5uv^2 = 0$ (cf. Figure 2). The eigenvalues of H_3^J are $-5, 3 \in \sigma_J^+(H_3^J)$ and $0 \in \sigma_J^-(H_3^J)$. Having in mind Remark 3(c), we obtain $W_J^+(H_3^J) = W_{-J}^+(H_3^J) = \mathbb{R}$. Then $W_J(A_3) = W_J^+(A_3) = W_{-J}^+(A_3) = \mathbb{C}$.

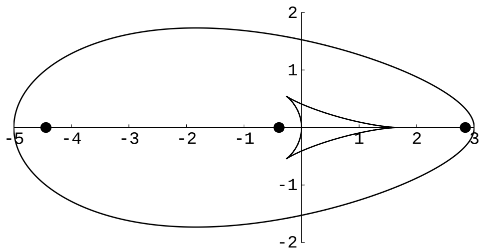

Figure 2. Curve of C1 type with one closed oval and a deltoid.

Example 4. For

$$A_4 = \begin{bmatrix} 0 & 2\sqrt{2} & 0 \\ -2\sqrt{2} & 1 & \sqrt{5} \\ 0 & \sqrt{5} & 1 \end{bmatrix} \quad \text{with } H_4^J = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

the line equation of $C_J(A_4)$ is given by $1 + 2u + u^2 - 3v^2 - 8uv^2 = 0$ (cf. Figure 3). Since $H_4^J = \mathrm{diag}(0, 1, 1)$, by Remark 3(b), we conclude that

$$W_J^+(H_4^J) =]-\infty, 1] \quad \text{and} \quad W_{-J}^+(H_4^J) = [1, +\infty[.$$

Therefore, $W_J^+(A_4)$ is the half-plane $x \leqslant 1$ and $W_{-J}^+(A_4)$ is the half-plane $x \geqslant 1$, and $W_J(A_4) = \mathbb{C}$.

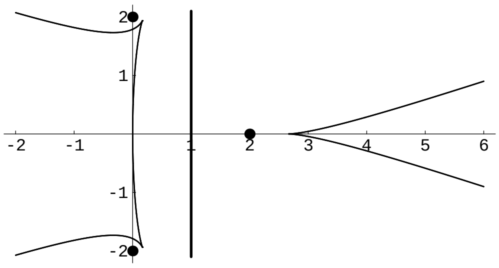

Figure 3. Curve of C2 type with two components with cusps and a double tangent (at complex points) of equation $x = 1$.

Example 5. Let

$$A_5 = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & -1 \\ 0 & -\frac{3}{2} & \frac{1}{2} \end{bmatrix} \quad \text{with } H_5^J = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & \frac{1}{4} \\ 0 & -\frac{1}{4} & \frac{1}{2} \end{bmatrix}.$$

The line equation of $C_J(A_5)$ is $16 + u^2 + 9v^2 - 8u(v^2 - 1) = 0$ (cf. Figure 4). Since H_5^J has the eigenvalues 0 and 1/4 (double), Remark 3(b) implies that

$$W_J^+(H_5^J) =]-\infty, 1/4[\quad \text{and} \quad W_{-J}^+(H_5^J) =]1/4, +\infty[.$$

Consequently, $W_J^+(A_5)$ is the half-plane $x < 1/4$ and $W_{-J}^+(A_5)$ is the half-plane $x > 1/4$. Finally, $W_J(A_5) = \mathbb{C} \setminus \{z \in \mathbb{C}\colon \operatorname{Re} z = 1/4\}$.

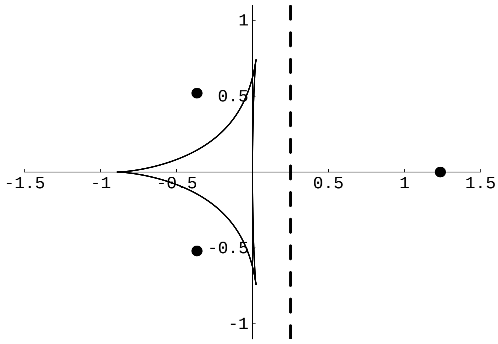

Figure 4. Curve of C2 type with a deltoid and a double tangent (at complex points) of equation $x = 1/4$.

Example 6. Consider now

$$A_6 = \begin{bmatrix} \frac{3}{16} & -\frac{7}{4} & \frac{3}{4} \\ \frac{9}{4} & \frac{1}{3} & -\frac{3}{2} \\ -\frac{3}{4} & -\frac{7}{2} & -3 \end{bmatrix} \quad \text{with } H_6^J = \begin{bmatrix} \frac{3}{16} & \frac{1}{4} & \frac{3}{4} \\ \frac{1}{4} & \frac{1}{3} & 1 \\ -\frac{3}{4} & -1 & -3 \end{bmatrix}.$$

The associated curve $C_J(A_6)$ is represented in Figure 5 and its line equation is $4 + 9v^2 + 7u(-68 + 567v^2)/48 = 0$. The eigenvalues of H_6^J are $-119/48$ and 0, where 0 has algebraic multiplicity 2. By Remark 3 (b) it follows that

$$W_J^+(H_6^J) = [0, +\infty[\quad \text{and} \quad W_{-J}^+(H_6^J) =]-\infty, -119/48].$$

The set $W_J^+(A_6)$ is contained in the half-plane $x \geqslant 0$ and it is the convex hull of the branch of $C_J(A_6)$ in the right closed half-plane. Analogously, $W_{-J}^+(A_6)$ is contained in the half-plane $x \leqslant -119/48$, being the convex hull of the associated curve contained in this region.

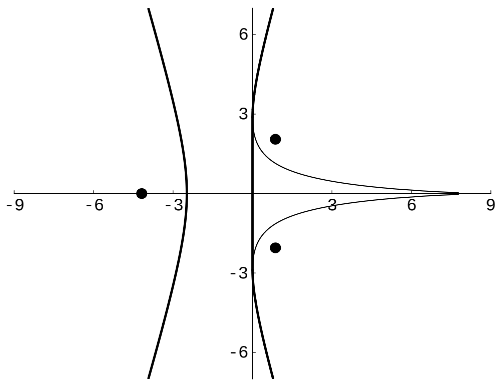

Figure 5. Curve of C3 type with a cusp in the real axis and the imaginary axis as a double tangent.

Example 7. Let

$$A_7 = \begin{bmatrix} 0 & 4 & \sqrt{2} \\ 0 & 0 & \sqrt{2} \\ 0 & 0 & 0 \end{bmatrix}, \quad \text{with } H_7^J = \begin{bmatrix} 0 & 2 & \frac{\sqrt{2}}{2} \\ 2 & 0 & \frac{\sqrt{2}}{2} \\ -\frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} & 0 \end{bmatrix}.$$

The line equation of $C_J(A_7)$ is $1 - 3(u^2 + v^2) - 2u(u^2 + v^2) = 0$ (cf. Figure 6).

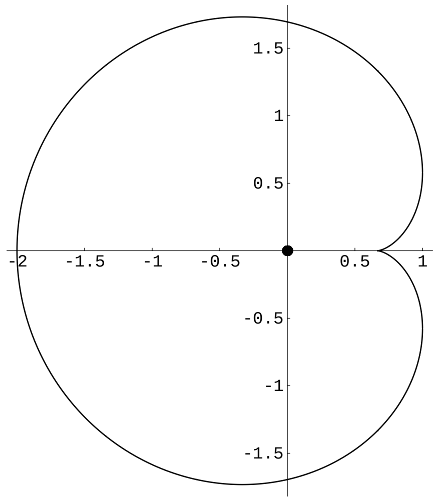

Figure 6. Curve of C3 type: a cardioid with a cusp in the real axis and $x = 1$ as a double tangent.

The eigenvalues of H_7^J are -2 and 1, the latter having algebraic multiplicity equal to 2. By Remark 3(b), $W_J^+(H_7^J) = W_{-J}^+(H_7^J) = \mathbb{R}$. Then $W_J(A_7) = W_J^+(A_7) = W_{-J}^+(A_7) = \mathbb{C}$.

Example 8. The line equation of $C_J(A_8)$, where

$$A_8 = \begin{bmatrix} \sqrt{2} & 0 & 2 \\ 0 & -\sqrt{2} & 4 \\ 0 & 2 & 0 \end{bmatrix} \quad \text{and} \quad H_8^J = \begin{bmatrix} \sqrt{2} & 0 & 1 \\ 0 & -\sqrt{2} & 1 \\ -1 & -1 & 0 \end{bmatrix},$$

is given by $1 + 10v^2 + 8\sqrt{2}uv^2 = 0$ (Figure 7).

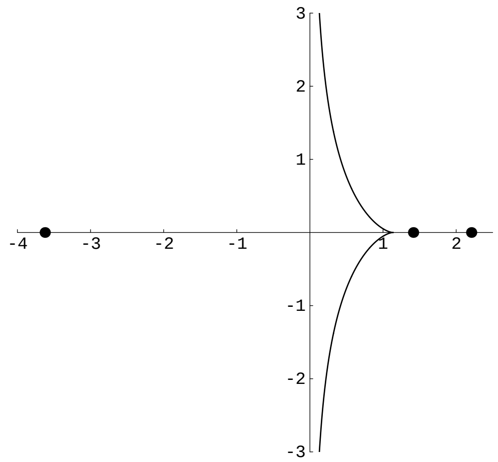

Figure 7. Curve of C4 type (Cissoid of Diocles) with a cusp in the real axis and a flex in the line of infinity. The imaginary axis is the flexional tangent and it corresponds to an assymptote of the curve.

The matrix H_8^J is nilpotent with nilpotency index $k = 3$. By Theorem 2(b), $W_J^+(H_8^J) = W_{-J}^+(H_8^J) = \mathbb{R}$ and so $W_J(A_8) = W_J^+(A_8) = W_{-J}^+(A_8) = \mathbb{C}$.

Example 9. Let

$$A_9 = \begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \quad \text{with } H_9^J = \begin{bmatrix} 0 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & 0 & \frac{1}{2} \\ -\frac{1}{2} & -\frac{1}{2} & 0 \end{bmatrix}.$$

The associated curve $C_J(A_9)$ is represented in Figure 8, and its line equation is given given by $4 + (u^2 + v^2) - u(u^2 + v^2) = 0$. The matrix H_9^J has complex eigenvalues $(1 \pm \sqrt{7}\mathrm{i})/4$. Consequently, $W_J^+(H_9^J) = W_{-J}^+(H_9^J) = \mathbb{R}$ and $W_J(A_9) = W_J^+(A_9) = W_{-J}^+(A_9) = \mathbb{C}$.

Example 10. Finally, let

$$A_{10} = \begin{bmatrix} 0 & 2 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \quad \text{with } H_{10}^J = \begin{bmatrix} 0 & 1 & \frac{1}{2} \\ 1 & 0 & \frac{1}{2} \\ -\frac{1}{2} & -\frac{1}{2} & 0 \end{bmatrix}.$$

The line equation of $C_J(A_{10})$ is $2 - (u^2 + v^2) - u(u^2 + v^2) = 0$ (cf. Figure 9).

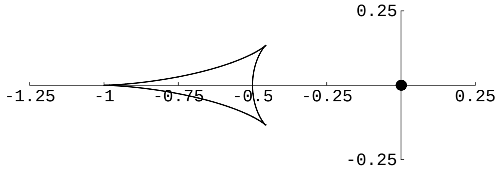

Figure 8. Curve of C5 type: a sextic reduced to one component with three cusps.

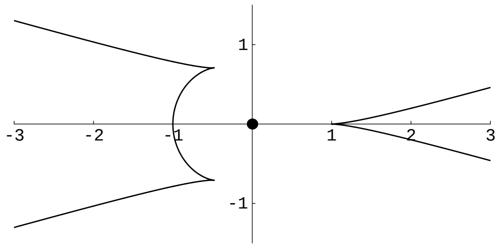

Figure 9. Curve of C5 type with two components.

Since $(1 \pm \mathrm{i})/2$ are (complex) eigenvalues of H_{10}^J, $W_J^+(H_{10}^J) = W_{-J}^+(H_{10}^J) = \mathbb{R}$ and $W_J(A_{10}) = W_J^+(A_{10}) = W_{-J}^+(A_{10}) = \mathbb{C}$.

5. Open problems

Kippenhahn [8] proved that the cases C2, C4 and C5 cannot occur in the classical case. In the indefinite case, the examples of the previous section show that all the 5 types of curves may occur. However, the associated curves of types C2, C4 and C5 have led to degenerate cases, where $W_J(A)$ coincides with $\mathbb{C}$ or with $\mathbb{C}$ except for a line. It is an open problem to prove (or disprove) that this property is valid in general.

In the classical case, Kippenhahn [8] also showed that the curves of C1 and C3 types correspond to a closed oval with a deltoid in its interior and to a cardioid, respectively. In the preceding section, we presented non-degenerate examples for unbounded associated curves $C_J(A)$ of types C1 and C3 (cf. Figures 1 and 5). We also obtained degenerate examples when those curves are bounded (cf. Figures 2 and 6). It is also an open problem to determine whether this is true in general.

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