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Abstract In this short paper, we study the Cauchy problem associated with the forced time-fractional relaxation-oscillation equation with distributed order. We employ the Laplace transform technique to derive the solution. Additionally, for the scenario without external forcing, we focus on density functions characterized by a single order, demonstrating that under these conditions, the solution can be expressed using two-parameter Mittag-Leffler functions.

INTRODUCTION

Relaxation and oscillation phenomena hold significant importance in the field of physics. Mathematically, the unforced relaxation-oscillation equation characterizes the behavior of a mass m connected to an elastic spring with a spring constant k, while being subjected to damping represented by the constant b. This equation is expressed as follows:

$$m\,u''(t)+b\,u'(t)+k\,u(t)=0, \quad m,k>0, \quad b\geq 0. \tag{1}$$

Three important cases result from the previous equation: (i) the *underdamped case* where the damping constant is small, i.e. $b<2\sqrt{mk}$; (ii) the *critically damped case* where the damping constant is equal to $2\sqrt{mk}$; (iii) the *overdamping case* where $b>2\sqrt{mk}$. The time-fractional relaxation-oscillation equation is obtained from (1) by replacing the derivatives of orders 1 and 2 by fractional derivatives of orders $\alpha\in]0,1]$ and $\beta\in]1,2]$. In the work by Mainardi (1996), an investigation into this subject matter was conducted, focusing on the one-dimensional scenario and specific instances of equation (1).

Distributed-order fractional calculus represents an important domain within fractional calculus, particularly significant for the modeling of complex systems. It extends the constant fractional operators by encompassing the integration of fractional kernels over an extended range of orders. The fractional differential operator of distributed order not exceeding two, is expressed as follows:

$$D^{\alpha}=\int_0^2 b(\alpha)\,\frac{d^{\alpha}}{dt^{\alpha}}\,d\alpha, \quad b(\alpha)\geq 0,$$

where $\frac{d^{\alpha}}{dt^{\alpha}}$ denotes a single-order fractional derivative, while the function $b(\alpha)$ corresponds to a non-negative weight function or generalized function (see [5, 7] and references therein indicated).

This work aims to combine these two topics and study the time-fractional relation-oscillation equation of distributed order.

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PRELIMINARIES

Given $a,b\in\mathbb{R}$, with $a<b$, the left Riemann-Liouville fractional integral $I^{\gamma}_{a^+}$ of order $\gamma>0$, is expressed by (see [3]):

$$\left(I^{\gamma}_{a^+}f\right)(x)=\frac{1}{\Gamma(\gamma)}\int_a^x\frac{f(t)}{(x-t)^{1-\gamma}}\,dt,\quad x>a.$$

Denote the left Caputo fractional derivative of order $\gamma>0$ on the interval $[a,b]\subset\mathbb{R}$ as ${}^{C}D^{\gamma}_{a^+}$. This derivative is defined by the following expression: (see [3])

$$\left({}^{C}D^{\gamma}_{a^+}f\right)(x)=\left(I^{m-\gamma}_{a^+}D^m f\right)(x)=\frac{1}{\Gamma(m-\gamma)}\int_a^x\frac{f^{(m)}(t)}{(x-t)^{\gamma-m+1}}\,dt,\quad x>a,\tag{2}$$

where $m=[\gamma]+1$ and $[\gamma]$ denotes the integer part of γ. The solution to the unforced fractional relaxation-oscillation equation in the single-order case is expressed using the three-parameter and bivariate Mittag-Leffler functions, which are defined through power series as follows (see [1, 2]):

$$E^{a_3}_{a_1,a_2}(z_1)=\sum_{l_1=0}^{\infty}\frac{(a_3)_{l_1}z_1^{l_1}}{l_1!\,\Gamma(a_1l_1+a_2)},\qquad E_{(a_1,a_2),b}(z_1,z_2)=\sum_{l_1=0}^{+\infty}\sum_{l_2=0}^{+\infty}\frac{(l_1+l_2)!}{l_1!\,l_2!}\frac{z_1^{l_1}z_2^{l_2}}{\Gamma(b+a_1l_1+a_2l_2)},\tag{3}$$

where a_1, a_2, b, z_1, and z_2 are complex numbers, $\operatorname{Re}(a_1),\operatorname{Re}(a_2)>0$, and $a_3>0$. The Laplace transform of a real valued function f is defined by (see [3])

$$\mathscr{L}\{f(t)\}(\mathbf{s})=\widetilde{f}(\mathbf{s})=\int_0^{+\infty}e^{-\mathbf{s}t}f(t)\,dt,\quad \operatorname{Re}(\mathbf{s})\in\mathbb{C},$$

and when it is applied to (2) leads to (see formula (5.3.3) in [3])

$$\mathscr{L}\left\{{}^{C}D^{\gamma}_{a^+}f(t)\right\}(\mathbf{s})=\mathbf{s}^{\gamma}\widetilde{f}(\mathbf{s})-\sum_{j=0}^{m-1}f^{(j)}(a)\,\mathbf{s}^{\gamma-j-1}.\tag{4}$$

Regarding the inverse Laplace transform of functions involving a branch point, we can refer to the following theorem by Titchmarsh: (see [6]).

Theorem 1 *Let $\widetilde{f}(\mathbf{s})$ be an analytic function which has a branch cut on the real negative semiaxis, which has the following properties*

$$\widetilde{f}(\mathbf{s})=O(1),\quad |\mathbf{s}|\to+\infty,\qquad \widetilde{f}(\mathbf{s})=O\left(\frac{1}{|\mathbf{s}|}\right),\quad |\mathbf{s}|\to 0,$$

for any sector $|\arg(\mathbf{s})|<\pi-\eta$, where $0<\eta<\pi$. Then the inverse Laplace transform of $\widetilde{f}(\mathbf{s})$ is given by

$$f(t)=\mathscr{L}^{-1}\left\{\widetilde{f}(\mathbf{s})\right\}(t)=-\frac{1}{\pi}\int_0^{+\infty}e^{-rt}\operatorname{Im}\left(\widetilde{f}(re^{i\pi})\right)dr.$$

The convolution operator of two real valued functions defined on $[0,+\infty[$ is given by

$$(f*g)(t)=\int_0^t f(t-w)\,g(w)\,dw,\quad t\in\mathbb{R}^+\tag{5}$$

and the application of the Convolution Theorem to (5) leads to

$$\mathscr{L}\{(f*g)(t)\}(\mathbf{s})=\widetilde{f}(\mathbf{s})\,\widetilde{g}(\mathbf{s}).\tag{6}$$

DISTRIBUTED-ORDER RELAXATION-OSCILLATION EQUATION

Let's examine the time-fractional forced damped oscillator, described by the following equation within the context of distributed-order fractional calculus:

$$b\int_1^2 b_2(\beta)\left[{}^{C}D^{\beta}_{0^+}u(t)\right]d\beta+a\int_0^1 b_1(\alpha)\left[{}^{C}D^{\alpha}_{0^+}u(t)\right]d\alpha+d^2u(t)=q(t),\tag{7}$$

for a pair of positive order-density functions $b_2(\beta)$ and $b_1(\alpha)$, such that:

$$\int_1^2 b_2(\beta)\,d\beta = C_2, \quad \int_0^1 b_1(\alpha)\,d\alpha = C_1,$$

and subject to the following initial conditions

$$u(0) = k_1, \quad u'(0) = k_2, \quad k_1, k_2 \in \mathbb{R}, \tag{8}$$

where $(x,t) \in \mathbb{R}^n \times \mathbb{R}^+$, the time-fractional derivatives of orders $\beta \in]1,2]$ and $\alpha \in]0,1]$ are in the Caputo sense (see (2)), $b,d > 0$, $a \geq 0$, and $C_1, C_2 > 0$. The positive constants C_1 and C_2 can be taken as 1 if we want to assume the normalization condition for the integral. Applying to (7) the Laplace transform, and taking into account operational calculus together with (8), leads to

$$b\,\widetilde{u}(\mathbf{s}) \int_1^2 b_2(\beta)\,\mathbf{s}^\beta\,d\beta - b k_1 \int_1^2 b_2(\beta)\,\mathbf{s}^{\beta-1}\,d\beta - b k_2 \int_1^2 b_2(\beta)\,\mathbf{s}^{\beta-2}\,d\beta$$

$$+ a\,\widetilde{u}(\mathbf{s}) \int_0^1 b_1(\alpha)\,\mathbf{s}^\alpha\,d\alpha - a k_1 \int_0^1 b_1(\alpha)\,\mathbf{s}^{\alpha-1}\,d\alpha + d^2 \widetilde{u}(\mathbf{s}) = \widetilde{q}(\mathbf{s}) \tag{9}$$

which is equivalent to

$$\widetilde{u}(\mathbf{s})\,[bB_2(\mathbf{s}) + aB_1(\mathbf{s})] = k_1 \left[\mathbf{s}^{-1}(bB_2(\mathbf{s}) - d^2) + a\mathbf{s}^{-1}B_1(\mathbf{s})\right] + k_2\,\mathbf{s}^{-2}\left[bB_2(\mathbf{s}) - d^2\right] + \widetilde{q}(\mathbf{s}), \tag{10}$$

with $B_2(s) = \int_1^2 b_2(\beta)\mathbf{s}^\beta\,d\beta + d^2$ and $B_1(s) = \int_0^1 b_1(\alpha)\mathbf{s}^\alpha\,d\alpha$. Hence, we have

$$\widetilde{u}(\mathbf{s}) = \frac{k_1\left[b\mathbf{s}B_2(\mathbf{s}) + a\mathbf{s}B_1(\mathbf{s}) - d^2\mathbf{s}\right] + k_2\left[bB_2(\mathbf{s}) - d^2\right]}{\mathbf{s}^2\left[bB_2(\mathbf{s}) + aB_1(\mathbf{s})\right]} + \frac{\widetilde{q}(\mathbf{s})}{bB_2(\mathbf{s}) + aB_1(\mathbf{s})} \tag{11}$$

$$= k_1\left[\frac{1}{s} - \frac{d^2}{\mathbf{s}[bB_2(\mathbf{s}) + aB_1(\mathbf{s})]}\right] + k_2\left[\frac{bB_2(\mathbf{s})}{\mathbf{s}^2(bB_2(\mathbf{s}) + aB_1(\mathbf{s}))} - \frac{d^2}{\mathbf{s}^2(bB_2(\mathbf{s}) + aB_1(\mathbf{s}))}\right] + \frac{\widetilde{q}(\mathbf{s})}{bB_2(\mathbf{s}) + aB_1(\mathbf{s})}. \tag{12}$$

Next, we proceed with the inversion of the Laplace transform to obtain our solution in the time domain. To do this, we introduce the following auxiliary functions in the Laplace domain:

$$\widetilde{u}_1(\mathbf{s}) = \frac{1}{s^p[bB_2(\mathbf{s}) + aB_1(\mathbf{s})]}, \ \ p = 0,1,2, \quad \text{and} \quad \widetilde{u}_2(\mathbf{s}) = \frac{bB_2(\mathbf{s})}{s^2[bB_2(\mathbf{s}) + aB_1(\mathbf{s})]}. \tag{13}$$

Applying Theorem 1 for the inverse Laplace transform, we have

$$u_j(t) = -\frac{1}{\pi}\int_0^{+\infty} e^{-rt}\,\mathrm{Im}[\widetilde{u}_j(re^{i\pi})]\,dr, \ \ j = 1,2. \tag{14}$$

In order to simplify (14), we need to evaluate the imaginary part of $\widetilde{u}_j(re^{i\pi})$, $j = 1,2$ along the $s = re^{i\pi}$ with $r > 0$. In this sense, by writing

$$B_j\left(re^{i\pi}\right) = \rho_j\left(\cos\left(\gamma_j\pi\right) + i\sin\left(\gamma_j\pi\right)\right) \Longrightarrow \begin{cases} \rho_j = \rho_j(r) = \left|B_j\left(re^{i\pi}\right)\right| \\ \gamma_j = \gamma_j(r) = \dfrac{1}{\pi}\arg\left(B_j\left(re^{i\pi}\right)\right), \end{cases} \quad j = 1,2 \tag{15}$$

and after straightforward calculations, we obtain the following expressions for the imaginary part of the functions $\widetilde{u}_j$:

$$\mathrm{Im}\left\{\widetilde{u}_1\left(re^{i\pi}\right)\right\} = D_1(p,r) = \frac{-B}{(-r)^p\left(A^2 + B^2\right)}, \ \ p = 0,1,2 \tag{16}$$

$$\mathrm{Im}\left\{\widetilde{u}_2\left(re^{i\pi}\right)\right\} = D_2(r) = \frac{A\rho_2\sin(\gamma_2\pi) - B\cos(\gamma_2\pi)}{r^2\left(A^2 + B^2\right)}, \tag{17}$$

where $A = b\rho_2\cos(\gamma_2\pi) + a\rho_1\cos(\gamma_1\pi)$ and $B = b\rho_2\sin(\gamma_2\pi) + a\rho_1\sin(\gamma_1\pi)$. Applying the inverse Laplace transform to (12) and taking into account (14), (16), and (17), we obtain

$$u(t) = k_1 + \frac{k_2 d^2}{\pi}\int_0^{+\infty} e^{-rt} D_1(1,r)dr - \frac{k_2}{\pi}\int_0^{+\infty} e^{-rt}\left[D_2(r) - d^2 D_1(2,r)\right]dr - \frac{q(t)}{\pi} * \int_0^{+\infty} e^{-rt} D_1(0,r)dr.$$

SINGLE ORDER CASE

Now, we analyze the unforced time-fractional relaxation-oscillation equation with single-order fractional derivatives. Considering

$$b_1(\alpha) = \delta(\alpha - \alpha_1),\ \ 0 < \alpha_1 \leq 1, \qquad b_2(\beta) = \delta(\beta - \beta_1),\ \ 1 < \beta \leq 2$$

in (7), we get $B_1(\mathbf{s}) = \mathbf{s}^{\alpha_1}$ and $B_2(\mathbf{s}) = \mathbf{s}^{\beta_1} + d^2$. In these conditions (12) becomes

$$\widetilde{u}(\mathbf{s}) = k_1 \frac{\mathbf{s}^{\beta_1 - 1}}{b\mathbf{s}^{\beta_1} + a\mathbf{s}^{\alpha_1} + d^2} + a k_1 \frac{\mathbf{s}^{\alpha_1 - 1}}{b\mathbf{s}^{\beta_1} + a\mathbf{s}^{\alpha_1} + d^2} + k_2 \frac{\mathbf{s}^{\beta_1 - 2}}{b\mathbf{s}^{\beta_1} + a\mathbf{s}^{\alpha_1} + d^2}. \tag{18}$$

Taking into account the following relation (see formula (17.6) in [2])

$$\mathscr{L}\left\{ t^{\alpha-\gamma} \sum_{p=0}^{+\infty} \left(-a t^{\alpha-\beta}\right)^p E^{p+1}_{\alpha,\alpha+(\alpha-\beta)p-\gamma+1}\left(-\lambda t^{\alpha}\right) \right\}(\mathbf{s}) = \frac{\mathbf{s}^{\gamma-1}}{\mathbf{s}^{\alpha} + a\mathbf{s}^{\beta} + \lambda}, \tag{19}$$

the application of the inverse Laplace transform results in

$$\begin{aligned} u(t) = {} & \frac{k_1}{b} \sum_{p=0}^{+\infty} \left(-\frac{a}{b} t^{\beta_1 - \alpha_1}\right)^p E^{p+1}_{\beta_1,(\beta_1-\alpha_1)p+1}\left(-\frac{d^2}{b} t^{\beta_1}\right) + \frac{a k_1}{b} t^{\beta_1-\alpha_1} \sum_{p=0}^{+\infty} \left(-\frac{a}{b} t^{\beta_1 - \alpha_1}\right)^p E^{p+1}_{\beta_1,\beta_1+(\beta_1-\alpha_1)p-\alpha_1+1}\left(-\frac{d^2}{b} t^{\beta_1}\right) \\ & + \frac{k_2}{b} t \sum_{p=0}^{+\infty} \left(-\frac{a}{b} t^{\beta_1 - \alpha_1}\right)^p E^{p+1}_{\beta_1,(\beta_1-\alpha_1)p+2}\left(-\frac{d^2}{b} t^{\beta_1}\right). \end{aligned} \tag{20}$$

By using the definition of the bivariate Mittag-Leffler function (see (3)), we can express the previous expression as follows

$$\begin{aligned} u(t) = {} & \frac{k_1}{b} E_{(\beta_1,\beta_1-\alpha_1),1}\left(-\frac{d^2}{b} t^{\beta_1}, -\frac{a}{b} t^{\beta_1-\alpha_1}\right) + \frac{a k_1}{b} t^{\beta_1-\alpha_1} E_{(\beta_1,\beta_1-\alpha_1),\beta_1-\alpha_1+1}\left(-\frac{d^2}{b} t^{\beta_1}, -\frac{a}{b} t^{\beta_1-\alpha_1}\right) \\ & + \frac{k_2}{b} t\, E_{(\beta_1,\beta_1-\alpha_1),2}\left(-\frac{d^2}{b} t^{\beta_1}, -\frac{a}{b} t^{\beta_1-\alpha_1}\right). \end{aligned} \tag{21}$$

Hence, the solution can be represented using a series of three-parameter Mittag-Leffler functions of a single variable or bivariate Mittag-Leffler functions. We end this work observing that expressions (20) and (21) are particular cases of the corresponding equations derived in [8], where we deal with ψ-Hilfer derivatives.

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