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Dirac's Method Applied to the Time-Fractional Diffusion-Wave Equation

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Abstract We compute the fundamental solution for time-fractional diffusion Dirac-like equations, which arise from the factoriza-tion of the multidimensional time-fractional diffusion-wave equation using Dirac's factorization approach.

INTRODUCTION

The free Dirac equation arises from the factorization of the Klein-Gordon equation using matrix coefficients satisfying anticommutation relations. Here, we focus on the factorization of the multidimensional time-fractional diffusion-wave equation $({}^C\partial_t^\alpha - c^2\Delta_x)u(x,t) = 0$, with $0 < \alpha \leq 2$, applying Dirac's factorization method. The one-dimensional case has been previously studied by various authors (cf. [7, 8, 9] and references therein). Dirac's factorization method, as outlined in [3], says that for the sum of the square of two operators A and B, we can define an operator O such that $\mathsf{O} = \sqrt{A^2+B^2}$. This operator O can be written as $\mathsf{O} = \gamma_1 A + \gamma_2 B$, where γ_1 and γ_2 are such that $\gamma_1^2 + \gamma_2^2 = 1$ and $\gamma_1\gamma_2 + \gamma_2\gamma_1 = 0$. The Pauli's matrices

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad \sigma_i\sigma_j = i\varepsilon_{ijk}\sigma_k + \delta_{ij}I_2, \quad \{\sigma_i, \sigma_j\} = 2\delta_{ij}I_2,$$

where I_2 is the identity matrix of order 2, δ_{ij} represents the Kronecker's delta, and ε_{ijk} is the Levi-Civita symbol defined as totally antisymmetric in all three indices, satisfy the previously imposed condition for γ_1 and γ_2. In this sense, the operator O is expressed as $\mathsf{O} = \sigma_k A + \sigma_l B$, where $k,l \in \{1,2,3\}$ and $k \neq l$. Different choices of pairs (k,l) yields to different solutions for the problem under analysis.

PRELIMINARIES

Here, we review fundamental concepts related to fractional calculus, special functions, Clifford analysis, and integral transforms. The left Caputo fractional derivative of order $\alpha > 0$ over the interval $[a,b] \subset \mathbb{R}$ is defined as (see [6]):

$$\left({}^C D_{a^+}^\alpha f\right)(t) = \frac{1}{\Gamma(m-\alpha)} \int_a^t \frac{f^{(m)}(w)}{(t-w)^{\alpha-m+1}}\, dw, \quad t > a, \quad m = \lfloor \alpha \rfloor + 1 \tag{1}$$

The one and two-parameter Mittag-Leffler functions of a complex variable z are defined using power series expansions, as introduced in [5]:

$$E_{\beta_1}(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\beta_1 n + 1)}, \ \ \mathrm{Re}(\beta_1) > 0, \qquad E_{\beta_1,\beta_2}(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\beta_1 n + \beta_2)}, \quad \mathrm{Re}(\beta_1) > 0, \ \ \beta_2 \in \mathbb{C}. \tag{2}$$

As can be easily seen $E_{\beta_1,1}(z) = E_{\beta_1}(z)$.

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Let us now turn to the higher dimensional setting. Consider the standard basis of the Euclidean vector space in $\mathbb{R}^n$ given by $\{e_1,\cdots,e_n\}$. The associated Clifford algebra, which is denoted by $\mathbb{R}_{0,n}$, is the free algebra generated by $\mathbb{R}^n$ modulo $x^2=-\|x\|^2\,e_0$, where $x\in\mathbb{R}^n$ and e_0 is the identity element with respect to the multiplication operation in the Clifford algebra. The defining relation leads to the multiplication rules $e_ie_j+e_je_i=-2\delta_{ij}$. In particular, $e_i^2=-1$ for all $i=1,\ldots,n$. Hence, the standard basis vectors operate as imaginary units. A vector space basis for $\mathbb{R}_{0,n}$ is given by $\{e_A : A\subseteq\{1,\ldots,n\}\}$ with $e_A=e_{l_1}\ldots e_{l_r}$, where $1\leq l_1<\ldots<l_r\leq n$, $0\leq r\leq n$, and $e_\emptyset:=e_0:=1$. Thus, for $x\in\mathbb{R}_{0,n}$ we have $x=\sum_A x_Ae_A$ with $x_A\in\mathbb{R}$. The conjugation in $\mathbb{R}_{0,n}$ is given by $\overline{x}=\sum_A x_A\overline{e}_A$, with $\overline{e}_A=\overline{e}_{l_r}\ldots\overline{e}_{l_1}$, and $\overline{e}_j=-e_j$ for $j=1,\ldots,n$, $\overline{e}_0=e_0=1$. The multiplicative inverse of a non-zero vector $x\in\mathbb{R}^n$ is given by $\frac{\overline{x}}{|x|^2}$. An $\mathbb{R}_{0,n}$-valued function f over $\Omega\subseteq\mathbb{R}^n$ has the the following representation $f=\sum_A e_Af_A$ where the components are such that $f_A:\Omega\to\mathbb{R}_{0,n}$. Properties such as continuity or differentiability have to be understood componentwise. The Euclidean Dirac operator is given by $\partial_x=\sum_{j=1}^n e_j\partial_{x_j}$ and it is such that $\partial_x^2=-\Delta_x$, with Δ_x being the n-dimensional Euclidean Laplace operator. For more details about Clifford algebras we refer to [2].

The n-dimensional Fourier transform of a real-valued integrable function $f(x)$, where $x\in\mathbb{R}^n$, is defined as follows:

$$\mathscr{F}\{f(x)\}(\kappa)=\widehat{f}(\kappa)=\int_{\mathbb{R}^n}e^{i\kappa\cdot x}f(x)\,dx,\quad \kappa\in\mathbb{R}^n,$$

and the corresponding inverse Fourier transform is defined by

$$f(x)=\mathscr{F}^{-1}\left\{\widehat{f}(\kappa)\right\}(x)=\frac{1}{(2\pi)^n}\int_{\mathbb{R}^n}e^{-ix\cdot\kappa}\widehat{f}(\kappa)\,d\kappa.$$

The Fourier transform satisfies the Convolution Theorem

$$\mathscr{F}\{(f*_xg)(x)\}(\kappa)=\widehat{f}(\kappa)\,\widehat{g}(\kappa),$$

where the convolution $*_x$ is given by $(f*_xg)(x)=\int_{\mathbb{R}^n}f(x-z)\,g(z)\,dz$. We have the following Fourier pairs (see [1])

$$\mathscr{F}\{\Delta_xf(x)\}(\kappa)=-|\kappa|^2\,\widehat{f}(\kappa)\quad\text{and}\quad\mathscr{F}\{\partial_xf(x)\}(\kappa)=i\kappa\widehat{f}(\kappa).\tag{3}$$

TIME-FRACTIONAL DIFFUSION DIRAC-LIKE EQUATIONS

Starting from the time-fractional diffusion-wave equation in $\mathbb{R}^n\times\mathbb{R}^+$ with Caputo time-fractional partial derivative

$$\left({}^C\partial_t^\alpha-c^2\Delta_x\right)u(x,t)=0,\quad x\in\mathbb{R}^n,\ t>0,\ c\in\mathbb{R}^+,\ 0<\alpha\leq2,\ \Delta_x=\sum_{i=1}^n\partial^2_{x_ix_i}\tag{4}$$

and considering the Dirac factorization method, the square root of the time-fractional differential operator ${}^C\partial_t^\alpha-c^2\Delta_x$ yields to the following matrix fractional differential equation

$$\left(\sigma_j\,{}^C\partial_t^{\alpha/2}+c\,\sigma_k\partial_x\right)\Phi=0,\quad j\neq k,\tag{5}$$

where $\Phi(x,t)=[\phi_1(x,t)\quad\phi_2(x,t)]^T$ is a two component vector function, depending on x and t. Each component of $\Phi(x,t)$ also satisfies (4), if the semigroup property of the time-fractional is verified. Since $0<\alpha\leq2$ then $0<\alpha/2\leq1$, and the system (5) represents an interpolation between the Helmholtz equation ($\alpha=0$ as limit case), the diffusion case ($\alpha=1$) and the heat equation ($\alpha=2$). Solutions of these systems could model the diffusion of particles whose behavior depends on the space and time coordinates, as usual, but also on their internal structures (cf. [8, 9]).

We consider two choices of couples (j,k), namely $(j,k)=(1,2)$ and $(j,k)=(3,2)$. In the first case, the matrix equation (5) originates the following two independent equations

$$\left({}^C\partial_t^{\alpha/2}-ic\,\partial_x\right)\phi_2(x,t)=0,\qquad\left({}^C\partial_t^{\alpha/2}+ic\,\partial_x\right)\phi_1(x,t)=0.\tag{6}$$

In the second case, $(j,k)=(3,2)$ we obtain a system of coupled equations

$${}^C\partial_t^{\alpha/2}\phi_1^*(x,t)-ic\,\partial_x\phi_2^*(x,t)=0,\qquad{}^C\partial_t^{\alpha/2}\phi_2^*(x,t)-ic\,\partial_x\phi_1^*(x,t)=0.\tag{7}$$

The solutions of (7) are related with the solutions of (6). In fact, if ϕ_1 and ϕ_2 are solutions of (6) then

$$\phi_1^* = (\phi_1 + \phi_2)/2 \quad \text{and} \quad \phi_2^* = (\phi_2 - \phi_1)/2 \tag{8}$$

are solutions of (7). This is based on the fact that we can find matrices A and B such that $A\sigma_1 B = \sigma_3$ and $A\sigma_2 B = \sigma_2$.

Now, we construct the fundamental solution of the equation $\left({}^C\partial_t^{\alpha/2} - ic\,\partial_x\right)\phi_2(x,t) = 0$ subject to the initial condition $\phi_2(x,0) = \delta(x) = \prod_{j=1}^n \delta(x_j)$. Applying the Fourier transform with respect to x and taking into account (3) and the initial condition $\widehat{\phi_2}(\kappa,0) = 1$, we obtain the time-fractional differential equation

$$\left({}^C\partial_t^{\alpha/2} + c\kappa\right)\widehat{\phi}_2(\kappa,t) = 0, \tag{9}$$

whose solution in the Fourier domain is given by $\widehat{\phi_2}(\kappa,t) = E_{\alpha/2}(-c\,t^{\alpha/2}\kappa)$ (see [4]). Taking into account the series representation of the one-parameter Mittag-Leffler function (see (2)), and the relations $\kappa^{2j} = (-1)^j|\kappa|^{2j}$ and $\kappa^{2j+1} = (-1)^j|\kappa|^{2j}\kappa$ valid for all $j \in \mathbb{N}_0$, we obtain the splitting formula

$$\begin{aligned} E_{\alpha/2}\left(-c\,t^{\alpha/2}\kappa\right) &= \sum_{j=0}^{\infty} \frac{\left(-c\,t^{\alpha/2}\right)^{2j}\kappa^{2j}}{\Gamma\left(\frac{\alpha}{2}2j+1\right)} + \sum_{j=0}^{\infty} \frac{\left(-c\,t^{\alpha/2}\right)^{2j+1}\kappa^{2j+1}}{\Gamma\left(\frac{\alpha}{2}(2j+1)+1\right)} \\ &= \sum_{j=0}^{\infty} \frac{\left(c^2 t^{\alpha}\right)^j(-1)^j|\kappa|^{2j}}{\Gamma(\alpha j+1)} - c\,t^{\alpha/2}\kappa \sum_{j=0}^{\infty} \frac{(c^2 t^{\alpha})^j(-1)^j|\kappa|^{2j}}{\Gamma\left(\alpha j + \frac{\alpha}{2}+1\right)} \\ &= E_{\alpha}\left(-c^2 t^{\alpha}|\kappa|^2\right) - c\,t^{\alpha/2}\kappa\, E_{\alpha,\alpha/2+1}\left(-c^2 t^{\alpha}|\kappa|^2\right). \end{aligned} \tag{10}$$

By Proposition 3.6 and Theorem 4.3 in [5] we can ensure that the Mittag-Leffler functions appearing in (10) belong to the space $L_1(\mathbb{R}^n)$ for each t fixed. Therefore, applying the inverse Fourier transform and using the convolution theorem together with the relation $\mathscr{F}^{-1}\{\kappa\}(x) = -i\partial_x\delta(x)$ derived from (3), we get

$$\phi_2(x,t) = \mathscr{F}^{-1}\left\{E_{\alpha}\left(-c^2 t^{\alpha}|\kappa|^2\right)\right\}(x,t) + i\,c\,t^{\alpha/2}\int_{\mathbb{R}^n} \partial_y\delta(y)\mathscr{F}^{-1}\left\{E_{\alpha,\alpha/2+1}\left(-c^2 t^{\alpha}|\kappa|^2\right)\right\}(x-y,t)\,dy. \tag{11}$$

The explicit formulas for the inverse Fourier transforms of the Mitagg-Leffler functions in (11) can be obtained from the general formula (cf. [4] for the special case $\beta_2 = 1$)

$$\begin{aligned} \mathscr{F}^{-1}\left\{E_{\beta_1,\beta_2}\left(-\lambda|\kappa|^2\right)\right\}(x) &= \frac{1}{2\pi^{n/2}|x|^n}\frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} \frac{\Gamma\left(1-\frac{s}{2}\right)\Gamma\left(\frac{n-s}{2}\right)}{\Gamma(\beta_2-\beta_1 s)}\left(\frac{2\sqrt{\lambda}}{|x|}\right)^{-s} ds \qquad (12)\\ &= \frac{1}{2\pi^{n/2}|x|^n} H_{2,1}^{0,2}\left[\frac{2\sqrt{\lambda}}{|x|}\,\middle|\,\begin{matrix}\left(0,\frac{1}{2}\right),\left(1-\frac{n}{2},\frac{1}{2}\right)\\ \left(1-\beta_2,\frac{\beta_1}{2}\right)\end{matrix}\right], \qquad (13) \end{aligned}$$

where $H_{m,n}^{p,q}(z)$ is the Fox H-function of one complex variable (see [6]). Using the duality relation $\int_{\mathbb{R}^n}\partial_y\delta(y)\varphi(y)\,dy = -\int_{\mathbb{R}^n}\delta(y)\partial_y\varphi(y)\,dy$ and (12) in (11) we obtain, after straightforward computations, the following theorem:

Theorem 1 *The fundamental solution of the time-fractional diffusion Dirac-like equation* $\left({}^C\partial_t^{\alpha/2} - ic\,\partial_x\right)\phi_2(x,t) = 0$ *subject to the initial condition* $\phi_2(x,0) = \delta(x)$ *is given by*

$$\begin{aligned} \phi_2(x,t) &= \frac{1}{2\pi^{n/2}|x|^n}\frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty}\frac{\Gamma\left(1-\frac{s}{2}\right)\Gamma\left(\frac{n-s}{2}\right)}{\Gamma\left(1-\frac{\alpha}{2}s\right)}\left(\frac{2ct^{\alpha/2}}{|x|}\right)^{-s} ds - \frac{ict^{\alpha/2}}{\pi^{n/2}}\frac{x}{|x|^{n+2}}\frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty}\frac{\Gamma\left(1-\frac{s}{2}\right)\Gamma\left(1+\frac{n-s}{2}\right)}{\Gamma\left(1+\frac{\alpha}{2}-\frac{\alpha}{2}s\right)}\left(\frac{2ct^{\alpha/2}}{|x|}\right)^{-s} ds \\ &= \frac{1}{2\pi^{n/2}|x|^n} H_{2,1}^{0,2}\left[\frac{2ct^{\alpha/2}}{|x|}\,\middle|\,\begin{matrix}\left(0,\frac{1}{2}\right),\left(1-\frac{n}{2},\frac{1}{2}\right)\\ \left(0,\frac{\alpha}{2}\right)\end{matrix}\right] - i\frac{ct^{\alpha/2}}{\pi^{n/2}}\frac{x}{|x|^{n+2}} H_{2,1}^{0,2}\left[\frac{2ct^{\alpha/2}}{|x|}\,\middle|\,\begin{matrix}\left(0,\frac{1}{2}\right),\left(-\frac{n}{2},\frac{1}{2}\right)\\ \left(-\frac{\alpha}{2},\frac{\alpha}{2}\right)\end{matrix}\right]. \end{aligned} \tag{14}$$

We give a direct proof of our main theorem.

Proof: First we compute the term ${}^C\partial_t^{\alpha/2}\phi_2(x,t)$. Using the Mellin-Barnes representation of ϕ_2 (see (14)) we first make the change of variables $s \mapsto -s$ and then we use the differentiation formulas

$${}^C\partial_t^{\alpha/2}\left(t^{\frac{\alpha}{2}s}\right) = \frac{\Gamma\left(1+\frac{\alpha}{2}s\right)}{\Gamma\left(1+\frac{\alpha}{2}s-\frac{\alpha}{2}\right)}\,t^{\frac{\alpha}{2}s-\frac{\alpha}{2}} \qquad \text{and} \qquad {}^C\partial_t^{\alpha/2}\left(t^{\frac{\alpha}{2}s+\frac{\alpha}{2}}\right) = \frac{\Gamma\left(1+\frac{\alpha}{2}s+\frac{\alpha}{2}\right)}{\Gamma\left(1+\frac{\alpha}{2}s\right)}\,t^{\frac{\alpha}{2}s}$$

to get

$$ {}^{C}\partial_t^{\alpha/2}\,\phi_2(x,t) = \frac{1}{2\pi^{n/2}t^{\alpha/2}|x|^n}\frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty}\frac{\Gamma\left(1-\frac{s}{2}\right)\Gamma\left(\frac{n-s}{2}\right)}{\Gamma\left(1-\frac{\alpha}{2}-\frac{\alpha}{2}s\right)}\left(\frac{2ct^{\alpha/2}}{|x|}\right)^{-s}ds $$

$$ -\frac{ic}{\pi^{n/2}}\frac{x}{|x|^{n+2}}\frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty}\frac{\Gamma\left(1-\frac{s}{2}\right)\Gamma\left(1+\frac{n-s}{2}\right)}{\Gamma\left(1-\frac{\alpha}{2}s\right)}\left(\frac{2ct^{\alpha/2}}{|x|}\right)^{-s}ds, $$

where we have changed $-s\mapsto s$ in the end of the calculations. Now we compute the term $\partial_x\phi_2(x,t)$. Since

$$ \partial_x(|x|^{s-n}) = (s-n)\,x\,|x|^{s-n-2} \quad \text{and} \quad \partial_x(x|x|^{s-n-2}) = \partial_x(x)|x|^{s-n-2} + x\,\partial_x(|x|^{s-n-2}|) = (2-s)|x|^{s-n-2} $$

we get

$$ \partial_x\phi_2(x,t) = -\frac{1}{\pi^{n/2}}\frac{x}{|x|^{n+2}}\frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty}\frac{\Gamma\left(1-\frac{s}{2}\right)\Gamma\left(1+\frac{n-s}{2}\right)}{\Gamma\left(1-\frac{\alpha}{2}s\right)}\left(\frac{2ct^{\alpha/2}}{|x|}\right)^{-s}ds $$

$$ -i\,\frac{2ct^{\alpha/2}}{\pi^{n/2}|x|^{n+2}}\frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty}\frac{\Gamma\left(2-\frac{s}{2}\right)\Gamma\left(1+\frac{n-s}{2}\right)}{\Gamma\left(1+\frac{\alpha}{2}-\frac{\alpha}{2}s\right)}\left(\frac{2ct^{\alpha/2}}{|x|}\right)^{-s}ds. \qquad (15) $$

Making the change of variables $s\mapsto s+2$ in the second integral of (15), it becomes equal to

$$ -\frac{i}{2\pi^{n/2}ct^{\alpha/2}|x|^n}\frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty}\frac{\Gamma\left(1-\frac{s}{2}\right)\Gamma\left(\frac{n-s}{2}\right)}{\Gamma\left(1-\frac{\alpha}{2}-\frac{\alpha}{2}s\right)}\left(\frac{2ct^{\alpha/2}}{|x|}\right)^{-s}ds. $$

By the terms calculated it follows immediately that the equation $\left({}^{C}\partial_t^{\alpha/2} - ic\,\partial_x\right)\phi_2(x,t) = 0$ is fulfilled.

■

Corollary 2 *The fundamental solution of the time-fractional diffusion Dirac-like equation* $\left({}^{C}\partial_t^{\alpha/2} + ic\,\partial_x\right)\phi_1(x,t) = 0$ *subject to the initial condition* $\phi_1(x,0) = \delta(x)$ *is given by*

$$ \phi_1(x,t) = \frac{1}{2\pi^{n/2}|x|^n}H_{2,1}^{0,2}\left[\frac{2ct^{\alpha/2}}{|x|}\,\middle|\, \begin{matrix}\left(0,\frac{1}{2}\right),\left(1-\frac{n}{2},\frac{1}{2}\right)\\ \left(0,\frac{\alpha}{2}\right)\end{matrix}\right] + i\,\frac{ct^{\alpha/2}}{\pi^{n/2}}\frac{x}{|x|^{n+2}}H_{2,1}^{0,2}\left[\frac{2ct^{\alpha/2}}{|x|}\,\middle|\, \begin{matrix}\left(0,\frac{1}{2}\right),\left(-\frac{n}{2},\frac{1}{2}\right)\\ \left(-\frac{\alpha}{2},\frac{\alpha}{2}\right)\end{matrix}\right]. \qquad (16) $$

Moreover, the fundamental solutions of the coupled equations (7) *are obtained by putting* (14) *and* (16) *in* (8).

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