# LOGARITHMIC DEFORMATIONS OF THE RATIONAL SUPERPOTENTIAL/LANDAU-GINZBURG CONSTRUCTION OF SOLUTIONS OF THE WDVV EQUATIONS

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Abstract. The superpotential in the Landau-Ginzburg construction of solutions to the Witten-Dijkgraaf-Verlinde-Verlinde (or WDVV) equations is modified to include logarithmic terms. This results in deformations - quadratic in the deformation parameters - of the normal prepotential solution of the WDVV equations. Such solution satisfy various pseudo-quasi-homogeneity conditions, on assigning a notional weight to the deformation parameters. This construction includes, as a special case, deformations which are polynomial in the flat coordinates, resulting in a new class of polynomial solutions of the WDVV equations.

## 1. Introduction

One of the most basic classes of Frobenius manifolds is comprised of those which are defined on orbit spaces $\mathbb{C}^n/W$ , $W$ being a finite Coxeter group [7]. Following from the observation of Arnold that the three polynomial solutions in 3-dimensions were related to the Coxeter numbers of the Platonic solids it was realized that the earlier Saito construction [18] provided a construction of Frobenius manifolds and that the prepotentials (solutions to the WDVV-equations - see below) were automatically polynomial with respect to a distinguished coordinate system, the so-called flat coordinates $\{t^i\}$ .

Such prepotentials are quasihomogeneous, a property that may be expressed in terms of an Euler vector field

$$E=\sum_i d^i t^i \frac{\partial}{\partial t^i}$$

as

$$\mathcal{L}_E F=(2h+2)F\,,$$

where the $d^i$ are the degrees of the basic $W$-invariant polynomials and $h$ is the Coxeter number of $W$ . Such solutions are semi-simple and it was conjectured by Dubrovin that all semi-simple polynomial solutions arise from this construction for some Coxeter group. This was later proved by Hertling [12].

In this paper we construct a new class of semi-simple polynomial solutions to the WDVV equations. This does not contradict the result of Hertling as the solution does not satisfy the full set of axioms of a Frobenius manifold, in particular the

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solutions are not quasi-homogeneous. These solutions may be regarded as a deformation of the $A_N$-polynomial solutions, in the sense that the prepotential takes the form

$$F(t^1, \ldots, t^N, b) = F^{(0)}(t^1, \ldots, t^N) + kF^{(1)}(t^1, \ldots, t^N, b)$$

where $F^{(0)}$ is the polynomial solutions defining the Frobenius manifold structure on the space $\mathbb{C}^N/A_N$ and $k$ is some deformation parameter. Such solutions satisfy a pseudo-quasi-homogeneity condition. With the Euler vector field

$$E = \sum_{i=1}^{N} (N+2-i) t^i \frac{\partial}{\partial t^i} + b\frac{\partial}{\partial b}$$

each part is separately quasi-homogeneous:

$$\begin{aligned} \mathcal{L}_E F^{(0)} &= (2N+4)F^{(0)}, \\ \mathcal{L}_E F^{(1)} &= (N+3)F^{(1)}. \end{aligned}$$

By assigning a fictitious scaling degree of $(N+1)$ to the deformation parameter $k$ the full solution may thought of a pseudo-quasi-homogeneous. These solutions will appear as a special case of a more general construction.

The Frobenius manifold structure on the orbit space $\mathbb{C}^N/A_N$ may also be derived [7, 13, 14] via a Landau-Ginzburg formalism as the structure on the parameter space of polynomials of the form

$$\lambda(p) = p^{N+1} + s_1 p^{N-1} + \ldots + s_N. \tag{1}$$

More explicitly, the metric

$$\eta(\partial_{s_i}, \partial_{s_j}) = -\sum \operatorname*{res}_{d\lambda=0} \left\{ \frac{\partial_{s_i}\lambda(p)\,\partial_{s_j}\lambda(p)}{\lambda'(p)}\, dp \right\} \tag{2}$$

is flat (though, in these variables, it does not have constant entries) and the tensor

$$c\,(\partial_{s_i}, \partial_{s_j}, \partial_{s_k}) = -\sum \operatorname*{res}_{d\lambda=0} \left\{ \frac{\partial_{s_i}\lambda(p)\,\partial_{s_j}\lambda(p)\,\partial_{s_k}\lambda(p)}{\lambda'(p)}\, dp \right\} \tag{3}$$

defines a totally symmetric $(3,0)$-tensor which further satisfies various potentiality conditions from which one may construct a so-called prepotential $F$ which satisfies the Witten-Dijkgraaf-Verlinde-Verlinde (or WDVV) equations of associativity

$$\frac{\partial^3 F}{\partial t^\alpha \partial t^\beta \partial t^\lambda} \eta^{\lambda\mu} \frac{\partial^3 F}{\partial t^\mu \partial t^\gamma \partial t^\delta} - \frac{\partial^3 F}{\partial t^\delta \partial t^\beta \partial t^\lambda} \eta^{\lambda\mu} \frac{\partial^3 F}{\partial t^\mu \partial t^\gamma \partial t^\alpha} = 0, \quad \alpha, \beta, \gamma, \delta = 1 \ldots, N$$

where the coordinates $\{t^i\}$ are a set of flat coordinates for the metric $\eta$ defined by (2). Geometrically, a solution defines a multiplication $\circ : TM \times TM \rightarrow TM$ of vector fields on the parameter space $M$, i.e.

$$\begin{aligned} \partial_{t^\alpha} \circ \partial_{t^\beta} &= \left( \frac{\partial^3 F}{\partial t^\alpha \partial t^\beta \partial t^\sigma} \eta^{\sigma\gamma} \right) \partial_{t^\gamma}, \\ &:= c^\gamma_{\alpha\beta}(t)\, \partial_{t^\gamma}, \end{aligned}$$

the metric $\eta$ being used to raise and lower indices.

**Example 1.** *With*

$$\lambda(p) = p^4 + s_1 p^2 + s_2 p + s_3$$

*the formula (2) gives the metric*[1]

$$\eta = \frac{1}{2} ds_1 ds_3 + \frac{1}{4} ds_2^2 - \frac{s_1}{8} ds_1^2 \,.$$

*While this metric is flat, the $s^i$ are not flat coordinates. With*

$$\begin{aligned} s_3 &= t_1 + \frac{1}{8} t_3^2 \,, \\ s_2 &= t_2 \,, \\ s_1 &= t_3 \end{aligned}$$

*one obtains a metric with constant coefficients. The tensor given by the formula (3) may then be used to construct the prepotential*

$$F = \frac{1}{8} t_1^2 t_3 + \frac{1}{8} t_1 t_2^2 - \frac{1}{64} t_2^2 t_3^2 + \frac{1}{3840} t_3^5 \,.$$

Such polynomial solution may be seen from a variety of different points of view (and part of the rich mathematical structure of Frobenius manifold arises as from the fact that it lies at the intersection of seemingly disconnected areas of mathematics):

(i) as a basic example of an orbit space construction. Here the manifold is $\mathbb{C}^n / A_N$ where $A_N$ is a Coxeter group;
(ii) as a topological Landau-Ginsburg field theory;
(iii) as a reduction of the dispersionless KP hierarchy.

The point of view that will be taken in this paper is last, i.e. that a solution to the WDVV equations may be obtained from a specific reduction of the dispersionless KP hierarchy [13, 14]. In particular it will be shown that the so-called water-bag reduction of the KP hierarchy [11] (see also [3]) also results in a solutions of the WDVV equations, though not, as in earlier examples, a full Frobenius manifold because of the non-existence of an Euler vector field. This builds on a recent preprint [5] where a 2-component system was studied.

## 2. The dispersionless KP hierarchy

The dispersionless KP (or dKP) hierarchy is defined in terms of a Lax function

$$\lambda(p) = p + \sum_{n=1}^{\infty} u_n(x,t) p^{-n}$$

by the Lax equation

$$\partial_{T_n} \lambda(p) = \{ \lambda(p), [\lambda^n(p)_+] \}$$

where $\{f,g\} = f_x g_p - f_p g_x$ is the ordinary Poisson bracket and $[\,]_+$ denotes the projection onto non-negative powers of $p$. Various reduction of this infinite component hierarchy have been studied, the most fundamental being the $A_N$-reduction

$$\lambda(p) = \left[ p^{N+1} + s_1 p^{N-1} + \ldots + s_N \right]^{\frac{1}{N+1}}$$

and this leads to a Frobenius manifold structure, defined above, on the space of parameters $\{s_i\}$. More recently a so-called 'water-bag' reduction has been studied,

[1]In all examples indices are lowered for notational convenience

where one takes

$$\lambda(p) = p + \sum_{i=1}^{N} k_i \log\left(\frac{p-p_i}{p-\tilde{p}_i}\right) .$$

In a recent preprint Chang [5] showed that in the $N=1$ case one may construct a solution of the WDVV equation by analysing the recursion relations satisfied by the conservation laws of the associated 2-component dispersionless hierarchy. Here we generalise this setting and consider functions of the form

$$\lambda(p) = (\text{rational function})\,(p) + \sum_{i=1}^{M} k_i \log(p-b_i)$$

Formally one may expand this function for large $p$ as a series, but this will have terms of the form

$$\left(\sum_{i=1}^{M} k_i\right) \log p$$

and the constraint $\sum k_i = 0$ is often imposed. Here we show that one still gets a solution without such a constraint. To make $\lambda$ single valued one has to make various cuts on the complex plane. For simplicity we present proofs in the polynomial case, with

$$\lambda(p) = p^{N+1} + s_1 p^{N-1} + \ldots + s_N + \sum_{i=1}^{M} k_i \log(p-b_i) \tag{4}$$

and state the result for the rational case - no essential new features will be present in the rational case that are not already present in the polynomial case. Note that without this constraint the function is not technically a reduction of the dKP hierarchy, but one may associated a 'regularised' function

$$\lambda(p) \rightarrow \lambda(p) - \left(\sum_{i=1}^{M} k_i\right) \log p$$

which is [16]. For this reason we call the form (4) a generalised water-bag reduction. We denote the space of such superpotentials $\mathcal{M}^{(M,N)}$ or just $\mathcal{M}$.

## 3. Solutions of the WDVV equations from the generalised Water-bag reduction of the dispersionless KP hierarchy

We begin by proving that the formulae (2,3) with the function (4) define a commutative, associative, semi-simple multiplication on the tangent space to the manifold of parameters. This will be done using canonical coordinates - the critial values of $\lambda$ (i.e. $\lambda$ evaluated at its critical points). Since $\lambda(p)$ only involves logarithms its derivative is a rational function which may be written in the form

$$\lambda'(p) = \frac{(N+1)\prod_{i=1}^{M+N}(p-\xi_i)}{\prod_{j=1}^{M}(p-b_j)}$$

(we assume that we are considering the generic case, where the poles and zeros are all distinct). The canonical coordinates are then

$$u^i = \lambda(\xi_i)\,, \qquad\qquad i = 1\,\ldots\,, N+M$$

(for such a formula to be single-valued, various cuts have to be made in the complex plane). The proof follows [7], Lemma 4.5. From the formulae

$$\frac{\partial}{\partial u^i}\lambda(p)\Big|_{p=\xi_j} = \delta_{ij}\,, \qquad\qquad i = 1\,\ldots, N+M$$

and

$$\frac{\partial}{\partial u^i}\lambda(p) = \left\{\prod_{r=1}^{M}(p-b_r)\right\}^{-1} B_i(p)$$

(where $B_i$ is a polynomial of degree $N+M-1$) one obtains

$$B_i(\xi_j) = \begin{cases} 0\,, & i \neq j \\ \prod_{r=1}^{M}(\xi_i - b_r)\,, & i = j\,. \end{cases}$$

The Lagrange interpolation formula then gives

$$B_i(p) = \frac{\prod_{j\neq i}(p-\xi_j)\prod_{r=1}^{M}(\xi_i - b_r)}{\prod_{j\neq i}(\xi_i - \xi_j)}$$

and hence

$$\begin{aligned} \frac{\partial\lambda(p)}{\partial u^i} &= \frac{\prod_{j\neq i}(p-\xi_j)\prod_{r=1}^{M}(\xi_i - b_r)}{\prod_{j\neq i}(\xi_i - \xi_j)\prod_{r=1}^{M}(p - b_r)}\,, \\ &= \frac{1}{(p-\xi_i)}\lambda'(p)\left\{\frac{\prod_{r=1}^{M}(\xi_i - b_r)}{\prod_{j\neq i}(\xi_i-\xi_j)}\right\}\,, \\ &= \frac{1}{(p-\xi_i)}\frac{\lambda'(p)}{\lambda''(\xi_i)}\,. \qquad (5) \end{aligned}$$

Note that this is the same functional form as in the polynomial case. With this

$$\begin{aligned} \eta(\partial_{u_i},\partial_{u_j}) &= -\sum \operatorname*{res}_{d\lambda=0}\left\{\frac{1}{(p-\xi_i)(p-\xi_j)}\frac{\lambda'(p)}{\lambda''(\xi_i)\lambda''(\xi_j)}dp\right\}\,, \\ &= -\frac{1}{\lambda''(\xi_i)}\delta_{ij}\,. \end{aligned}$$

Note that while log-terms appear in $\lambda$, the metric formula involves derivatives of $\lambda$ and hence involves rational functions only.

Similarly

$$c(\partial_{u_i},\partial_{u_j},\partial_{u_k}) = \begin{cases} -\dfrac{1}{\lambda''(\xi_i)}\,, & i=j=k\,, \\ 0\,, & \text{otherwise}\,. \end{cases}$$

Collecting these results one arrives at the following:

**Lemma 2.** *The formulae (2) and (3) with $\lambda$ given by (4) define, at a generic point, a semi-simple, commutative, associative multiplication*

$$\frac{\partial}{\partial u^i}\circ\frac{\partial}{\partial u^j} = \delta_{ij}\frac{\partial}{\partial u_i}\,, \qquad (6)$$

*compatible with the metric*

$$\eta = -\sum_{r=1}^{M+N}\frac{du_i^2}{\lambda''(\xi_i)} \qquad (7)$$

This multiplication has an identity. Since $e(\lambda)=1$, where the vector field $e$ is defined to be

$$e=\frac{\partial}{\partial s^N}\,,$$

it is immediate from equations (2) and (3) that

$$c(\partial,\partial',e)=\eta(\partial,\partial')\,.$$

From this it follows that $e$ is the identity for the multiplication. In semi-simple coordinates it follows from the multiplication (6) that

$$e=\sum_{r=1}^{M+N}\frac{\partial}{\partial u^i}\,.$$

We prove next that the metric is flat and Ergoff. In the pure-polynomial case (or $A_N$-case) the flat coordinates are defined by an inverse series, using the so-called thermodynamic identity. The presence of the logarithms makes such an inversion problematical. However, it turns out that part of the flat-coordinates of the metric are exactly the same as in the polynomial case.

**Lemma 3.** *The formula (2) with $\lambda$ given by (4) gives the following:*

$$\eta(\partial_{s_i},\partial_{s_j})=-\sum\operatorname*{res}_{d\lambda_+=0}\left\{\frac{\partial_{s_i}\lambda_+(p)\partial_{s_j}\lambda_+(p)}{\lambda'_+(p)}dp\right\}\,,\qquad i\,,j=1\,,\ldots\,,N\,,$$

*where $\lambda_+(p)=p^{N+1}+s_1p^{N-1}+\ldots+s_N$ is a truncation of $\lambda$, and*

$$\begin{aligned}\eta(\partial_{b_r},\partial_{s_j})&=0\,,&& r=1\,,\ldots\,,M\,,j=1\,,\ldots\,,N\,,\\ \eta(\partial_{b_i},\partial_{b_j})&=k_i\delta_{ij}\,,&& i,j=1\,,\ldots\,,M\,.\end{aligned}$$

It follows from these formulae that the metric is flat.

*Proof.* These formulae just involve the use of basic ideas from complex variable theory.

$$\begin{aligned}\eta(\partial_{s_i},\partial_{s_j})&=-\sum\operatorname*{res}_{d\lambda=0}\left\{\frac{p^{2N-i-j}}{\lambda'(p)}dp\right\}\,,\\ &=\operatorname*{res}_{p=\infty}\left\{\frac{p^{2N-i-j}}{\lambda'(p)}dp\right\}\,.\end{aligned}$$

Now

$$\begin{aligned}\lambda'(p)&=\lambda'_+(p)+\sum_{r=1}^{M}\frac{k_i}{(p-b_i)}\,,\\ &=\lambda'_+(p)\left\{1+\frac{1}{\lambda'_+(p)}\sum_{r=1}^{M}\frac{k_i}{(p-b_i)}\right\}\,.\end{aligned}$$

Hence

$$\begin{aligned}
\eta(\partial_{s_i},\partial_{s_j}) &= \operatorname*{res}_{p=\infty}\left\{\frac{p^{2n-i-j}}{\lambda'_+(p)}\left[1+\frac{1}{\lambda'_+(p)}\sum_{r=1}^{M}\frac{k_i}{(p-b_i)}\right]^{-1}dp\right\},\\
&= -\operatorname*{res}_{\tilde{p}=0}\left\{\frac{\tilde{p}^{i+j-N-2}}{\mu(\tilde{p})}\left[1+\frac{\tilde{p}^{N+1}}{\mu(\tilde{p})}\sum_{r=1}^{M}\frac{k_i}{1-\tilde{p}b_i}\right]^{-1}d\tilde{p}\right\},\\
&= -\operatorname*{res}_{\tilde{p}=0}\left\{\frac{\tilde{p}^{i+j-N-2}}{\mu(\tilde{p})}d\tilde{p}\right\},
\end{aligned}$$

where $\tilde{p}=p^{-1}$ and $\lambda'_+(p)=\tilde{p}^{-N}\mu(\tilde{p})$. Reversing the argument yields the result. Similarly,

$$\begin{aligned}
\eta(\partial_{s_i},\partial_{b_r}) &= \sum\operatorname*{res}_{d\lambda=0}\left\{\frac{p^{N-i}}{\lambda'(p)}\frac{k_r}{(p-b_r)}dp\right\},\\
&= -\frac{1}{N+1}\operatorname*{res}_{p=\infty}\left\{\frac{k_r p^{N-i}\prod_{r\neq i}(p-b_r)}{\prod_{j=1}^{M+N}(p-\xi_j)}dp\right\},\\
&= \frac{1}{N+1}\operatorname*{res}_{\tilde{p}=0}\left\{k_r p^{i-1}\frac{\prod_{r\neq i}(1-b_r\tilde{p})}{\prod_{j=1}^{M+N}(1-\xi_j\tilde{p})}dp\right\},\\
&= 0\,.
\end{aligned}$$

Finally,

$$\eta(\partial_{b_i},\partial_{b_j}) = -\frac{1}{N+1}\sum\operatorname*{res}_{d\lambda=0}\left\{\frac{k_i}{(p-b_i)}\frac{k_j}{(p-b_j)}\frac{\prod_{r=1}^{M}(p-b_r)}{\prod_{k=1}^{M+N}(p-\xi_k)}dp\right\}.$$

For $i\neq j$ this, on deforming the contour around the Riemann sphere, gives zero: there is no pole at infinity, and the simple poles cancel. For $i=j$,

$$\begin{aligned}
\eta(\partial_{b_i},\partial_{b_i}) &= -k_i^2\sum\operatorname*{res}_{d\lambda=0}\left\{\frac{1}{(p-b_i)^2}\frac{1}{\lambda'(p)}dp\right\},\\
&= k_i^2\frac{1}{N+1}\frac{\prod_{k\neq i}(b_i-b_k)}{\prod_i(b_i-\xi_i)}\,.
\end{aligned}$$

On evaluating the residue at the poles using the two different formulae for $\lambda'(p)$,

$$(N+1)p^N+(N-1)s_1p^{N-2}+\ldots s_1+\sum_{r=1}^{M}\frac{k_i}{(p-b_r)} = (N+1)\frac{\prod_{i=1}^{M+N}(p-\xi_i)}{\prod_{j=1}^{M}(p-b_j)}$$

one obtains

$$k_i=(N+1)\frac{\prod_i(b_i-\xi_i)}{\prod_{k\neq i}(b_i-b_k)}$$

from which the final formulae follows. □

*Proof.* ('Thermodynamical identity' -type proof of flat coordinates)

Following the polynomial case in [7], invert $\lambda_+(p)$ as

$$p_+(k)=k+\frac{1}{N+1}\left(\frac{t^N}{k}+\frac{t^{N-1}}{k^2}+\ldots+\frac{t^1}{k^N}\right)+O\left(\frac{1}{k^{N+1}}\right),$$

where $\lambda_+ = k^{N+1}$. Then

$$\begin{aligned}\lambda(p_+(k,t),t,b) &= \lambda_+(p_+(k,t),t) + \sum_{i=1}^{M} k_i \log(p_+ - b_i)\,,\\ &= k^{N+1} + \sum_{i=1}^{M} k_i \log(p_+ - b_i)\,.\end{aligned}$$

Differentiating with respect to $t^\alpha$ gives

$$\begin{aligned}\left.\frac{d\lambda}{dp}\right|_{p=p_+(k)} \frac{\partial p_+}{\partial t^\alpha} + \frac{\partial \lambda}{\partial t^\alpha} &= \sum_{i=1}^{M} \frac{k_i}{p_+ - b_i}\frac{\partial p_+}{\partial t^\alpha}\,.\\ &= O\left(\frac{1}{k^{N+2-\alpha}}\right)\,.\end{aligned}$$

So we have as our thermodynamical identity in this case

$$\frac{\partial}{\partial t^\alpha}(\lambda dp) + \frac{\partial}{\partial t^\alpha}(p_+ d\lambda) = O\left(\frac{1}{k^{N+1-\alpha}}\right) dk\,.$$

Although the right hand side is not zero as it is for polynomial $\lambda$, this identity is sufficient to give

$$\frac{\partial}{\partial t^\alpha}(\lambda dp) = -k^{\alpha-1}dk + O\left(\frac{1}{k}\right) dk$$

(eqn. (4.68) in [7]), from which it follows, using

$$d\lambda = d\lambda_+ + O\left(\frac{1}{k}\right)\, dk\,,$$

that

$$\eta(\partial_{t^\alpha},\partial_{t^\beta}) = -\frac{\delta_{\alpha+\beta,N+1}}{N+1}\,.$$

□

The flat coordinates are therefore

$$\{t^i\,, i=1\,,\ldots\,,N\,; b_j\,, j=1\,,\ldots\,,M\}$$

where the $t^i$ are defined by the inverse series for the truncated function $\lambda_+ = \lambda_+(p)$, expanded as a Puiseaux series as $\lambda \to \infty$ ,

$$p(k) = k + \frac{1}{N+1}\left(\frac{t^N}{k} + \frac{t^{N-1}}{k^2} + \ldots + \frac{t^1}{k^N}\right) + O\left(\frac{1}{k^{N+1}}\right) \tag{8}$$

where $k = (\lambda_+)^{\frac{1}{N+1}}$ , in the standard way [7]. Note that each $t^i$ is a polynomial in the $s_i$ and vice versa.

Consider the diagonal metric (7). Its rotation coefficients $\beta_{ij}$ are defined by the formula

$$\beta_{ij} = \frac{\partial_{u_i} H_j}{H_i}\,, \qquad H_i^2 = \frac{1}{\lambda''(\xi_i)}\,.$$

Such a metric is said to be Egoroff if the rotation coefficients are symmetric. This then implies that the metric may be written in terms of a single potential function $V(u)$ ,

$$\eta = \sum_{i=1}^{M+N} \frac{\partial V}{\partial u^i} \left(du^i\right)^2\,.$$

**Lemma 4.** *The metric (7) is Egoroff.*

*Proof.* In canonical coordinates $\eta$ is diagonal with $i^{th}$ entry

$$-\frac{1}{\lambda''(\xi_i)}\,.$$

From (5)

$$\begin{aligned}\frac{\partial\lambda}{\partial u^i} &= \frac{1}{p-\xi_i}\frac{\lambda'(p)}{\lambda''(\xi_i)}\,,\\ &= \frac{N+1}{\lambda''(\xi_i)}\frac{\prod_{r\neq i}(p-\xi_r)}{\prod_{s=1}^{M}(p-b_s)}\,,\end{aligned}$$

so we have

$$\frac{\partial\lambda}{\partial u^i}\prod_{s=1}^{m}(p-b_s)=\frac{N+1}{\lambda''(\xi_i)}\prod_{r\neq i}(p-\xi_i)$$

where each side is a polynomial of degree $N+M-1$.

Also

$$\frac{\partial\lambda}{\partial u^i}=\frac{\partial s_1}{\partial u^i}p^{N-1}+\frac{\partial s_2}{\partial u^i}p^{N-2}+\cdots+\frac{\partial s_N}{\partial u^i}-\sum_{r=1}^{M}\frac{k_r}{p-b_r}\frac{\partial b_r}{\partial u^i}\,,$$

so

$$\frac{\partial\lambda}{\partial u^i}\prod_{s=1}^{m}(p-b_s)=\left(\frac{\partial s_1}{\partial u^i}p^{N-1}+\cdots+\frac{\partial s_N}{\partial u^i}\right)\prod_{s=1}^{M}(p-b_s)-\sum_{r=1}^{M}k_r\frac{\partial b_r}{\partial u^i}\prod_{s\neq r}(p-b_s)\,.$$

Comparing coefficients of $p^{N+M-1}$ in $\frac{\partial\lambda}{\partial u^i}\prod_{s=1}^{M}(p-b_s)$ in these two expressions gives

$$\frac{N+1}{\lambda''(\xi_i)}=\frac{\partial s_1}{\partial u^i}\,.$$

Hence

$$\eta(\frac{\partial}{\partial u^i},\frac{\partial}{\partial u^i})=-\frac{1}{\lambda''(\xi_i)}=\frac{\partial}{\partial u^i}\left(-\frac{1}{N+1}s_1\right)\,.$$

□

This Egoroff property is equivalent to a potentiality condition on the $(3,0)$-tensor $c$, namely that the tensor $\nabla c$ is totally symmetric. Since the metric is flat one may, in flat-coordinates, integrate by Poincaré's lemma and express everything in terms of a prepotential $F$ which satisfies the WDVV equations. Collecting these results together one obtains:

**Proposition 5.** *The flat metric (2) and totally symmetric* $(3,0)$ *tensor (3), with* $\lambda$ *given by*

$$\lambda=p^{N+1}+s_1p^{N-1}+\ldots+s_N+\sum_{i=1}^{M}k_i\log(p-b_i)\,,\quad k_i\text{ constant}$$

*define, on the space* $\mathcal{M}^{(M,N)}$ *a solution to the WDVV equations. Geometrically they define a semi-simple, associative, commutative algebra with unity on the tangent space* $T\mathcal{M}$ *compatible with the flat metric.*

Before giving some examples, it must be remarked that we do not have a Frobenius manifold, just a solution to the WDVV equations. As was remarked in one of the earliest papers on water-bag reductions, such reductions do have have a scaling symmetry and this fact manifests itself in the non-existence of an Euler vector field, the existence of which is part of the definition of a Frobenius manifold (though it should be remarked that some authors do not require such a field in their definition, denoting manifolds with such a field as a conformal Frobenius manifold).

**Example 6.** $N=0\,, M=2\,.$ *In the above proofs it has been assumed that* $N\neq 0\,.$ *However one may adapt these proofs to deal with this case. In particular, the identity field, normally associated to the variable* $s_N$*, has to be carefully defined. With*

$$\lambda(p) = p + k_1 \log\left[p-(t_1+t_2)\right] + k_2 \log\left[p-(t_1-t_2)\right]$$

*one obtains the prepotential*

$$F = \frac{1}{6}\left\{k_1(t_1+t_2)^3 + k_2(t_1-t_2)^3\right\} + 2k_1k_2\, t_2^2 \,\log t_2\,.$$

Note that if the condition $k_1+k_2=0$ is imposed, one obtains, after some rescalings, the solution obtained by Chang. This example was the original motivation of this work.

**Lemma 7.**

$$\begin{aligned}
c\left(\frac{\partial}{\partial b_\alpha},\frac{\partial}{\partial b_\beta},\frac{\partial}{\partial b_\gamma}\right) &= 0\,, \qquad \alpha,\beta,\gamma \textit{ distinct}\,,\\
c\left(\frac{\partial}{\partial b_\alpha},\frac{\partial}{\partial b_\alpha},\frac{\partial}{\partial b_\beta}\right) &= \frac{k_\alpha k_\beta}{b_\beta-b_\alpha}\,, \qquad \alpha\neq\beta\,,\\
c\left(\frac{\partial}{\partial b_\alpha},\frac{\partial}{\partial b_\alpha},\frac{\partial}{\partial b_\alpha}\right) &= k_\alpha \lambda_+'(b_\alpha) + \sum_{r\neq\alpha}\frac{k_\alpha k_r}{b_\alpha-b_r}\,,
\end{aligned}$$

$$\begin{aligned}
c\left(\frac{\partial}{\partial b_\alpha},\frac{\partial}{\partial b_\beta},\frac{\partial}{\partial s_\gamma}\right) &= 0\,, \qquad \alpha\neq\beta\,,\\
c\left(\frac{\partial}{\partial b_\alpha},\frac{\partial}{\partial b_\alpha},\frac{\partial}{\partial s_\gamma}\right) &= k_\alpha (b_\alpha)^{N-\gamma}\,,
\end{aligned}$$

$$c\left(\frac{\partial}{\partial b_\alpha},\frac{\partial}{\partial s_\beta},\frac{\partial}{\partial s_\gamma}\right) = k_\alpha S_{\beta+\gamma}(s_1,\ldots,s_N,b_\alpha)\,,$$

$$c\left(\frac{\partial}{\partial s_\alpha},\frac{\partial}{\partial s_\beta},\frac{\partial}{\partial s_\gamma}\right) = R^{(0)}_{\alpha+\beta+\gamma}(s_1,\ldots,s_N) + \sum_{j=1}^M k_j R^{(1)}_{\alpha+\beta+\gamma}(s_1,\ldots,s_N,b_j)$$

*where* $S_\sigma$, $R^{(0)}_\sigma$ *and* $R^{(1)}_\sigma$ *are polynomial functions of their respective variables, and independent of all* $k_i$*'s.*

*In particular, the term independent of* $k_j$, $R^{(0)}_{\alpha+\beta+\gamma}(s_1,\ldots,s_N)$, *is precisely the value of* $c(\partial_{s_\alpha},\partial_{s_\beta},\partial_{s_\gamma})$ *found from (3) using the polynomial* $\lambda_+(p)$ *as the Landau-Ginzburg potential (1).*

*Proof.* Here we write

$$\lambda'(p) = \frac{\nu(p)}{\prod_{j=1}^M (p-b_j)}$$

where

$$\begin{aligned}\nu(p) &= \lambda'_+(p)\prod_{j=1}^M (p-b_j) + \sum_{j=1}^M k_j \prod_{k\neq j}(p-b_k)\,,\\ &= (N+1)\prod_{j=1}^M (p-\xi_j)\,.\end{aligned}$$

After the substitution $p \to 1/\tilde{p}$ we will have cause to refer to the polynomial

$$\mu(\tilde{p}) = \tilde{p}^N \lambda'_+\left(\frac{1}{\tilde{p}}\right) = (N+1) + (N-1)s_1\tilde{p}^2 + (N-2)s_2\tilde{p}^3 + \cdots + s_{N-1}\tilde{p}^N\,.$$

(**bbb**) From the definition (3),

$$c\left(\frac{\partial}{\partial b_\alpha},\frac{\partial}{\partial b_\beta},\frac{\partial}{\partial b_\gamma}\right) = \sum \operatorname*{res}_{\nu=0} \frac{k_\alpha k_\beta k_\gamma}{(p-b_\alpha)(p-b_\beta)(p-b_\gamma)}\frac{\prod_{j=1}^M (p-b_j)}{\nu(p)}dp\,.$$

This is evaluated by deforming the contour to encompass the poles at $p=\infty$ and possibly at $p=b_\alpha$ if there is repetition in the $b$'s. The residue at infinity is zero, and so in particular $c(\partial_{b_\alpha},\partial_{b_\beta},\partial_{b_\gamma})=0$ for $\alpha,\beta,\gamma$ distinct.

For the case $(\alpha,\alpha,\beta)$, the pole at $p=b_\alpha$ is simple, and the result follows immediately, noting that $\nu(b_\alpha) = k_\alpha \prod_{k\neq\alpha}(b_\alpha - b_k)$.

For the case $\alpha=\beta=\gamma$, the pole is second order, and is evaluated directly as

$$\begin{aligned}c\left(\frac{\partial}{\partial b_\alpha},\frac{\partial}{\partial b_\alpha},\frac{\partial}{\partial b_\alpha}\right) &= -\operatorname*{res}_{p=b_\alpha}\frac{k_\alpha^3}{(p-b_\alpha)^2}\frac{\prod_{k\neq\alpha}(p-b_k)}{\nu(p)}dp\,,\\ &= -k_\alpha^3 \left.\frac{d}{dp}\right|_{p=b_\alpha}\frac{\prod_{k\neq\alpha}(p-b_k)}{\nu(p)}\,.\end{aligned}$$

**(bbs)**

$$\begin{aligned}c\left(\frac{\partial}{\partial b_\alpha},\frac{\partial}{\partial b_\beta},\frac{\partial}{\partial s_\gamma}\right) &= -\sum\operatorname*{res}_{\nu=0}\frac{k_\alpha k_\beta}{(p-b_\alpha)(p-b_\beta)}\frac{p^{N-\gamma}\prod_{j=1}^M(p-b_j)}{\nu(p)}dp\,,\\ &= \left(\operatorname*{res}_{p=\infty}+\operatorname*{res}_{p=b_\alpha}+\operatorname*{res}_{p=b_\beta}\right)\frac{k_\alpha k_\beta}{(p-b_\alpha)(p-b_\beta)}\frac{p^{N-\gamma}\prod_{j=1}^M(p-b_j)}{\nu(p)}dp\,.\end{aligned}$$

Once again there is no pole at infinity, and there exists a (simple) pole at $p=b_\alpha$ only if $\alpha=\beta$. The result again follows from $\nu(b_\alpha) = k_\alpha\prod_{j\neq\alpha}(b_\alpha-b_j)$.

**(sss)**

$$\begin{aligned}c\left(\frac{\partial}{\partial s_\alpha},\frac{\partial}{\partial s_\beta},\frac{\partial}{\partial s_\gamma}\right) &= \operatorname*{res}_{p=\infty}\frac{p^{3N-\alpha-\beta-\gamma}\prod_{j=1}^M(p-b_j)}{\lambda'_+(p)\prod_{j=1}^M(p-b_j)+\sum_{j=1}^M k_j\prod_{k\neq j}(p-b_k)}dp\,,\\ &= \operatorname*{res}_{p=\infty}\frac{p^{3N-\alpha-\beta-\gamma}}{\lambda'_+(p)}\left[1+\sum_{j=1}^M\frac{k_j}{\lambda'_+(p)(p-b_j)}\right]^{-1}dp\,.\end{aligned}$$

This is expanded as a Taylor series in $x=\sum k_j/\lambda'_+(p)(p-b_j)$ to give a series of terms

$$c\left(\frac{\partial}{\partial s_\alpha},\frac{\partial}{\partial s_\beta},\frac{\partial}{\partial s_\gamma}\right)=\sum_{i=0}^{\infty}\tilde{R}^{(i)}_{\alpha+\beta+\gamma}$$

where

$$\tilde{R}^{(i)}_\sigma=(-1)^{i+1}\operatorname*{res}_{p=\infty}\frac{p^{3N-\sigma}}{\lambda'_+(p)}\left[\frac{1}{\lambda'_+(p)}\sum_{j=1}^{M}\frac{k_j}{p-b_j}\right]^i dp\,.$$

So, in particular, $R^{(0)}_{\alpha+\beta+\gamma}:=\tilde{R}^{(0)}_{\alpha+\beta+\gamma}=\operatorname*{res}_{p=\infty}\frac{\partial_{s_\alpha}\lambda_+\partial_{s_\beta}\lambda_+\partial_{s_\gamma}\lambda_+}{\lambda'_+}dp$ is $c_{\alpha\beta\gamma}$ from the $A_N$ orbit space corresponding to $\lambda_+$.

$\tilde{R}^{(1)}_\sigma(s_1,\ldots,s_N,b_1,\ldots,b_M)$ can be decomposed as $\sum_{i=1}^M k_i R^{(1)}_\sigma(s_1,\ldots,s_N,b_i)$ where

$$\begin{aligned}R^{(1)}_\sigma(s_1,\ldots,s_N,b) &= -\operatorname*{res}_{p=\infty}\frac{p^{3N-\sigma}}{(p-b)(\lambda'_+(p))^2}dp\,,\\ &= \operatorname*{res}_{\tilde{p}=0}\frac{1}{(1-b\tilde{p})(\mu(\tilde{p}))^2}\tilde{p}^{\sigma-N-1}d\tilde{p}\,.\end{aligned}$$

This is seen to be zero for $\sigma\geq N+1$, and $1/(N+1)^2$ for $\sigma=N$. For $\sigma<N$ it is a pole of order $N+1-\sigma$ and can be evaluated as

$$\frac{1}{(N-\sigma)!}\left(\frac{d}{d\tilde{p}}\right)^{N-\sigma}\Bigg|_{\tilde{p}=0}\frac{1}{(1-b\tilde{p})(\mu(\tilde{p}))^2}\,. \tag{9}$$

Clearly this evaluates to a polynomial in $\{s_1,\ldots,s_N,b\}$. Finally, by making the substitution $p\rightarrow 1/\tilde{p}$ it can be seen that $\tilde{R}^{(i)}_\sigma=0$ for $i\geq 2$.

**(bss)** Proceeding as in the (sss) case, we are led to

$$c\left(\frac{\partial}{\partial b_\alpha},\frac{\partial}{\partial s_\beta},\frac{\partial}{\partial s_\gamma}\right)=k_\alpha\sum_{i=1}^{M}S^{(i)}_{\beta+\gamma}$$

where

$$\begin{aligned}S^{(i)}_\sigma &= (-1)^{i+1}\operatorname*{res}_{p=\infty}\frac{p^{2N-\sigma}}{p-b_\alpha}\frac{1}{(\lambda'_+(p))^{i+1}}\left[\sum_{j=1}^{M}\frac{c_j}{p-b_j}\right]^i dp\,,\\ &= (-1)^i\operatorname*{res}_{\tilde{p}=0}\frac{\tilde{p}^{\sigma-N-1+i(N+1)}}{(1-b_\alpha\tilde{p})(\mu(\tilde{p}))^{i+1}}\left[\frac{c_j}{1-b_j\tilde{p}}\right]^i d\tilde{p}\,.\end{aligned}$$

From this we can see that $S^{(i)}_\sigma=0$ for $i\geq 1$. This leaves only

$$S_\sigma:=S^{(0)}_\sigma=\operatorname*{res}_{\tilde{p}=0}\frac{\tilde{p}^{\sigma-N-1}}{(1-b_\alpha\tilde{p})\mu(\tilde{p})}d\tilde{p}\,,$$

which is zero for $\sigma\geq N+1$, and $1/(N-1)$ for $\sigma=N$, whilst for $\sigma\leq N-1$ we evaluate as

$$\frac{1}{(N-\sigma)!}\left(\frac{d}{d\tilde{p}}\right)^{N-\sigma}\Bigg|_{\tilde{p}=0}\frac{1}{(1-b_\alpha\tilde{p})\mu(\tilde{p})}\,. \tag{10}$$

□

For the Frobenius structure on the space of polynomials

$$\lambda(p) = p^{N+1} + s_1 p^{N-1} + \ldots + s_N ,$$

the variables $s_i$ inherit a scaling symmetry from the scaling of the polynomial. Namely if $p \to \epsilon p$ and we ask $\lambda \to \epsilon^{N+1}\lambda$, then we require $s_i \to \epsilon^{i+1} s_i$. Thus we conclude $s_i$ has degree $i+1$.

For the water-bag reduction

$$\lambda(p) = p^{N+1} + s_1 p^{N-1} + \cdots + s_N + \sum_{i=1}^{M} k_i \log(p - b_i),$$

the same degrees may be attached to the coefficients $\{s_i\}$, and to preserve homogeneity of the arguments of the logarithms, each $b_i$ is assigned degree 1. If, in addition, an non-geometrically justified degree of $N+1$ is assigned to each $k_i$, then the regularised function $\lambda(p) - \sum k_i \log p$ is homogeneous of degree $N+1$.

**Lemma 8.** *Under the rescalings*

$$\begin{array}{rclr} s_i & \to & \epsilon^{i+1} s_i & i = 1 \ldots N \, , \\ b_i & \to & \epsilon b_i & i = 1 \ldots M \, , \\ k_i & \to & \epsilon^{N+1} k_i & i = 1 \ldots M \end{array}$$

*the prepotential $F$ associated to the water-bag reduction(4) is homogeneous of degree $2N+4$.*

*Proof.* This may be verified from the explicit expressions for the components of the tensor $c(\partial, \partial', \partial'')$ obtained in Lemma 7, remembering to add the degrees lost from differentiating along $\partial, \partial', \partial''$.

In particular, for $c(\partial_{b_\alpha}, \partial_{b_\alpha}, \partial_{b_\alpha}) = k_\alpha \lambda'_+(b_\alpha) + \sum_{r \neq \alpha} \frac{k_\alpha k_r}{b_\alpha - b_r}$ we note that $\lambda'_+(b_\alpha) = (N+1)(b_\alpha)^N + (N-1)s_1(b_\alpha)^{N-1} + \cdots + s_{N-1}$ has degree $N$.

The degrees of the polynomials $R^{(0)}_\sigma$, $R^{(1)}_\sigma$ and $S_\sigma$, when they are not zero or constant, can de determined from the differential expressions (9), (10) and the corresponding expression for $R^{(0)}_\sigma$, which is

$$R^{(0)}_\sigma = \left\{ \begin{array}{cl} 0 & \sigma \geq 2N+2 \\ -1/(N+1) & \sigma = 2N+1 \\ \frac{1}{(2N+1-\sigma)!} \left(\frac{d}{d\tilde{p}}\right)^{2N+1-\sigma} \Big|_{\tilde{p}=0} \frac{1}{\mu(\tilde{p})} & \sigma \leq 2N \end{array} \right. .$$

In this the degree of zero is undetermined, whilst for the middle case, the degree of a constant is 0. Integrating with respect to $s_\alpha$,$s_\beta$ and $s_\gamma$ adds to this degree $(\alpha+1)+(\beta+1)+(\gamma+1) = \sigma+3 = 2N+4$. In the final case, if $\tilde{p} = 1/p$ is considered to have degree $-1$, then $\mu(\tilde{p})$ has degree zero. Thus on differentiation we obtain the quotient of two homogeneous polynomials with relative degrees $2N+1-\sigma$. Evaluation at $\tilde{p} = 0$ merely makes this the ratio of constant terms, so that $R^{(0)}_\sigma$ has degree $2N+1-\sigma$. Integrating will add $\sigma+3$ to this, making $2N+4$ as required. $S_\sigma$ and $R^{(1)}_\sigma$ proceed similarly. □

The degrees of the flat coordinates $\{t^i, i = 1 \ldots N\}$ are inherited from the polynomial transformations rules relating them to the $s_i$. They can also be deduced from the Puiseaux series (8), in which we require both $p$ and $k$ to scale with degree 1, so that the degree of $t^i$ is $N+2-i$.

We now draw together some simple observations, which follow immediately from lemmas 3, 7 and 8.

**Proposition 9.** *The prepotential is at most quadratic in the parameters* $k_i$ *, that is, up to quadratic terms in the flat coordinates:*

$$\begin{aligned} F(t^1,\ldots,t^N,b^1,\ldots,b^M) &= F^{(0)}(t^1,\ldots,t^N) \\ &+\sum_i k_i F^{(1)}(t^1,\ldots,t^N,b^i) \\ &+\sum_{i\neq j} k_i k_j F^{(2)}(b^i,b^j) \end{aligned}$$

*where* $F^{(0)}$ , $F^{(1)}$ , $F^{(2)}$ *are independent of the parameters* $k_i$. $F^{(0)}$ *is the prepotential for the* $\mathbb{C}^N/A_N$ *orbit space with* $\lambda_+$ *as the Landau-Ginzburg potential, and as such is a polynomial in the flat coordinates* $\{t^1,\ldots,t^N\}$. $F^{(1)}$ *is also a polynomial, and*

$$F^{(2)}(b^i,b^j) = \frac{1}{8}(b^i-b^j)^2\log(b^i-b^j)^2\,.$$

*In place of quasi-homogeneity we have*

$$\begin{aligned} \deg\left(F^{(0)}\right) &= 2N+4\,, \\ \deg\left(F^{(1)}\right) &= N+3\,, \\ \deg\left(F^{(2)}\right) &= 2\,, \qquad (\textit{modulo quadratic terms})\,. \end{aligned}$$

*The structure functions for the Frobenius algebra are always at most linear in the parameters* $k_i$ *, that is:*

$$c_{\alpha\beta}{}^{\gamma} = c^{(0)}_{\alpha\beta}{}^{\gamma} + \sum_i k_i\, c^{(i)}_{\alpha\beta}{}^{\gamma}\,.$$

*where the* $c^{(0)}_{\alpha\beta}{}^{\gamma}$ *and* $c^{(i)}_{\alpha\beta}{}^{\gamma}$ *are independent of the parameters.*

An important class of solutions are polynomial in the flat coordinates.

**Corollary 10.** *For* $M=1$*, the prepotential on the space of functions*

$$\lambda(p) = p^{N+1} + s_1 p^{N-1} + \cdots + s_N + k\log(p-b)$$

*is polynomial in the flat coordinates* $\{t^i,b\}$ *Conversely, if the prepotential is polynomial in the flat coordinates then* $M=1$ *(or* $M=0$*).*

*Proof.* This is an immediate consequence of the decomposition of $F$ given in Proposition 9: the component $F^{(2)}$ contains all non-polynomial terms appearing in $F$, and is present if and only if $M\geq 2$.

□

We finish this main section with two simple examples.

**Example 11.**

- $\mathbf{N=2\,,M=1}$.

*With*

$$\lambda(p) = p^3 + t_2 p + t_1 + k\log(p-t_3)$$

*one obtains the prepotential*

$$F = \frac{1}{6}t_1^2 t_2 - \frac{1}{2}k\, t_1 t_3^2 - \frac{1}{216}t_2^4 - \frac{1}{6}k\,(t_2^2 t_3 + t_2 t_3^3) - \frac{1}{20}k\, t_3^5\,.$$

- $\mathbf{N = 1}\,, \mathbf{M}$ **arbitrary.**

*In this case one has*

$$\lambda(p) = p^2 + t_1 + \sum_{i=1}^{M} k_i \log(p - b_i)\,.$$

*With this, lemmas 3 and 7 give, on integrating, the following prepotential:*

$$F = -\frac{1}{12}t_1^3 + \sum_{i=1}^{M} k_i \left\{ \frac{t_1 b_i^2}{2} + \frac{b_i^4}{12} \right\} + \frac{1}{8}\sum_{i\neq j} k_i k_j (b_i - b_j)^2 \log(b_i - b_j)^2\,.$$

We note that the $z^2 \log z$-type terms have appeared in the WDVV-literature before (see, for example, [10, 15]) but one normally considers these are being derived as examples of dual Frobenius manifolds [8]. Their functional form suggests the type of term that may be present in a construction of deformed solutions for other Coxeter group orbit spaces.

## 4. Geometric and Algebraic Properties

In this section we study certain geometric and algebraic properties of the manifold.

4.1. **Geometric Properties.** An important addition structure on a Frobenius manifold is an addition flat metric known as the intersection form, It plays a vital role in the understanding of various properties of the manifold, such as the monodromy properties of the Gauss-Manin connection and associated bi-Hamiltonian structures. Following this, we defined a second metric on manifold; while this is not flat, it shares many properties of the intersection form of a genuine Frobenius manifold.

Before this, we normalise the Euler vector field, so

$$E = \frac{1}{N+1}\sum_{i=1}^{N}(N+2-i)t^i \frac{\partial}{\partial t^i} + \frac{1}{N+1}\sum_{j=1}^{M} b^j \frac{\partial}{\partial b^j}\,. \tag{11}$$

**Definition 12.** *The metric $g$ on $\mathcal{M}$ is defined as:*

$$g^{-1}(\omega_1, \omega_2) = i_E(\omega_1, \omega_2)\,.$$

It follows immediately from this that

$$g(E \circ u, v) = \eta(u, v)$$

and, in components,

$$g^{ij} = c^{ij}_k E^k\,.$$

To understand the scaling properties of this metric we introduce an extended Lie derivative $\mathcal{L}_X^{ext}$ ,

$$\mathcal{L}_X^{ext} = \mathcal{L}_X + \sum_{r=1}^{M} k^r \frac{\partial}{\partial k^r}\,,$$

so, for an arbitrary tensor $\omega^{i\ldots j}_{a\ldots b}$,

$$\left(\mathcal{L}^{ext}_X\omega\right)^{i\ldots j}_{a\ldots b} = \left(\mathcal{L}_X\omega\right)^{i\ldots j}_{a\ldots b} + \sum_{r=1}^{M} k^r \frac{\partial}{\partial k^r}\omega^{i\ldots j}_{a\ldots b}\,.$$

This may be used to clarify the pseudo-quasi-homogeneity properties of the various structures, for example

$$\mathcal{L}^{ext}_E\,F = (3-d)F\,, \qquad\qquad d = \frac{N-1}{N+1}\,.$$

Similarly the metrics $g$ and $\eta$ have various pseudo-quasi-homogeneity properties:

**Lemma 13.** *The following equations hold:*

$$\begin{array}{rclrcl} [e,E] & = & e\,, & & & \\ \mathcal{L}^{ext}_E g^{-1} & = & (d-1)g^{-1}\,, & \mathcal{L}^{ext}_E \eta^{-1} & = & (d-2)\eta^{-1}\,, \\ \mathcal{L}^{ext}_e g^{-1} & = & \eta^{-1}\,, & \mathcal{L}^{ext}_e \eta^{-1} & = & 0\,. \end{array}$$

However, the metric $g$ is not flat, and moreover, despite being linear in $t^1$ the pencil $g^{-1}_\Lambda = g^{-1} + \Lambda\eta^{-1}$ does not define an almost compatible pencil (the tensor $E\circ : T\mathcal{M} \rightarrow T\mathcal{M}$ fails to satisfy the Nijenhuis condition [6]), let alone a compatible pencil. The role of this second metric is therefore unclear. Given the origin of these structures in reductions of the dKP hierarchy one would expect bi-Hamiltonian structures of differential-geometric type. One possibility is the metric

$$g = \sum \frac{1}{u_i \lambda''(\xi_i)} du_i^2\,.$$

This does define a non-local bi-Hamiltonian structure [17] but finding its form in the flat-coordinate system for the metric $\eta$ is problematical. A related problem is to relate the Euler vector field (11) with the vector field

$$E' = \sum_{i=1}^{M+N} u^i \frac{\partial}{\partial u^i}\,,$$

the two being equal in the undeformed case.

The various structures on the manifold may be encoded in the deformed (or Dubrovin) connection

$${}^D\nabla_X Y = \nabla_X Y + z\, X\circ Y\,, \qquad z \in \mathbb{P}^1\,.$$

For this connection to be torsion free and flat one requires commutativity and associativity of the multiplication, flatness of the Levi-Civita connection $\nabla$ and potentiality, and visa-versa. Solutions of the system ${}^D\nabla_\alpha \zeta_\beta$ are automatically gradients, $\zeta_\alpha = \partial_\alpha \tilde{t}$. Expanding $\tilde{t} = \sum_n \psi^{(n)} z^n$ yields the recursion relation

$$\frac{\partial^2\psi^{(n)}}{\partial t^i \partial t^j} + c^k_{ij}\frac{\partial \psi^{(n-1)}}{\partial t^k} = 0\,.$$

Starting with the seed solutions $\psi^{(0)} = t^i\,, i = 1\,, \ldots \dim\mathcal{M}$ one may construct a fundamental system of solutions.

4.2. **Algebraic deformation theory.** In this section we examine the linearity of the structure functions of the Frobenius algebra with respect to the parameters $k^i$ from the point of view of deformation theory (we follow the notation of [4]). Let

$$M^k(V) = \{m : \underbrace{V \times \ldots \times V}_{k} | m \text{ linear in each arguement}\}$$

Recall that a bilinear map $c \in M^2(V)$ defines an associative structure if and only if

$$[c, c]_{\mathcal{G}} = 0\,,$$

where $[\cdot,\cdot]_{\mathcal{G}}$ is the Gerstenhaber bracket. Owing to the super-Jacobi identity one has $\delta_c^2 = 0$ , where

$$\delta_c = [c, \cdot]_{\mathcal{G}} \quad : M^{\bullet}(V) \to M^{\bullet+1}(V)$$

and this gives rise to the Hochschild complex of $(V, c)$ .

From proposition 9 we have the following structure

$$c(k) = c^{(0)} + \sum_i k_i c^{(i)}\,,$$

that is, linearity of the structure functions of the associative algebra. Decomposing the condition $[c(k), c(k)]_{\mathcal{G}} = 0$ for all $k$ one obtains the following conditions:

$$\begin{aligned} [c^{(0)}, c^{(0)}]_{\mathcal{G}} &= 0\,, \\ [c^{(0)}, c^{(i)}]_{\mathcal{G}} &= 0\,, \quad i = 1\,, \ldots\,, M\,, \\ [c^{(i)}, c^{(j)}]_{\mathcal{G}} &= 0\,, \quad i, j = 1\,, \ldots\,, M\,. \end{aligned}$$

Thus each $c^{(i)}\,, i = 0\,, 1\,, \ldots\,, M$ separately defines and associative structure on $T\mathcal{M}$ . Each of these define a map $\delta_{c^{(i)}}$ and each $c^{(i)}$ is a cocycle with respect to each cohomology map $\delta_{c^{(j)}}$ , that is:

$$\begin{aligned} [c^{(i)}, c^{(i)}]_{\mathcal{G}} &= 0\,, \quad i = 0\,, 1\,, \ldots\,, M\,, \\ \delta_{c^{(i)}} c^{(j)} &= 0\,, \quad i, j = 0\,, 1\,, \ldots\,, M\,. \end{aligned}$$

It is also interesting to note that the pair $(\circ, E)$ satisfy the conditions

$$\mathcal{L}_{X \circ Y}(\circ) = X \circ \mathcal{L}_Y(\circ) + Y \circ \mathcal{L}_X(\circ)$$

(following from the semi-simplicity of the multiplication) and the pseudo-scaling condition

$$\mathcal{L}_E^{ext}(\circ) = d\, \circ\,.$$

If one had $\mathcal{L}_E(\circ) = d\circ$ then one would have a $F$-manifold [12]. Here on has a modified version, where the scaling condition is replaced by the pseudo-scaling condition. One could also regard the multiplication as defining a deformation of the $F$-manifold based on the orbit space $\mathbb{C}^N/A_N$ .

## 5. Further Results

An immediate question these result raise is whether or not the ideas may be applied to other classes of Frobenius manifolds, the obvious potential generalization being to other Coxeter orbit spaces $\mathbb{C}^n/W$, for an arbitrary Coxeter group $W$ . By this we mean is there a prepotential schematically of the form

$$F(\mathbf{t}, \mathbf{b}) = F_W(\mathbf{t}) + kF^{(1)}(\mathbf{t}, \mathbf{b}) + k^2 F^{(2)}(\mathbf{t}, \mathbf{b})$$

based on the $\mathbb{C}^n/W$ prepotential $F_W$ which is pseudo-quasi-homogeneous with respect to some suitable Euler field.

For the group $W = B_n$ this is immediate, using the idea originally due to Zuber [19], of embedding the group $B_n$ as a subgroup of $A_{2n+1}$ , or geometrically, as the $B_n$ Frobenius manifold as the induced manifold on certain hyperplanes submanifolds in the $A_{2n+1}$ Frobenius manifold. This idea generalizes to water-bag type type reductions and this will be presented in section 5.1.

Another possible generalization, already alluded to above, is to replace the polynomial part of $\lambda$ by an arbitrary rational function, generalizing the construction of [1, 2]. The Frobenius manifold structure on the space of such rational functions has been much studied and these results can be generalized to include logarithmic terms. These results are presented in section 5.2

5.1. **$B_N$-type Reductions.** The $B_n$ Frobenius manifold may be regarded as a submanifold in the $A_n$ Frobenius manifold [19]. This idea generalizes to water-bag type potentials.

**Proposition 14.** *On the space of functions*

$$\lambda(p) = p^{2N+2} + s_1 p^{2N} + s_3 p^{2N-2} + \cdots + s_{2N+1} + \sum_{i=1}^{M} k_i \log(p^2 - b_i^2)$$

*the formulas (2) and (3) define a pseudo-quasi-homogeneous solution of the WDVV equations.*

*Proof.* The function $\lambda$ above is obtained from the following waterbag deformation of the $A_{2N+1}$ superpotential:

$$\begin{aligned} \lambda_A(p) &= p^{2N+2} + s_1 p^{2N} + s_2 p^{2N-1} + s_3 p^{2N-2} + \cdots + s_{2N+1} \\ &\quad + \sum_{i=1}^{M} k_i \log(p - b_i) + \sum_{i=1}^{M} k_i \log(p - b_{i+M}) \,. \end{aligned}$$

We restrict this to the submanifold

$$\begin{aligned} s_r &= 0 \text{ for } r \text{ even}, \\ b_{i+M} &= -b_i \text{ for } 1 \leq i \leq M \,. \end{aligned}$$

The restriction of the $s_r$ may be achieved in flat coordinates by setting all $t^i$ of odd degree (i.e. even $i$) to zero. We introduce new flat coordinates $\tilde{b}_i = b_i$ and $\tilde{d}_i = b_i + b_{i+M}$ $(i = 1, \ldots, M)$, and restrict to $\tilde{d}_i = 0$. We check the following components of the multiplication tensor restrict to zero on this hyperplane:

$$\begin{array}{rr} c^{\tilde{d}_k}_{\tilde{b}_i \tilde{b}_j}\,, & c^{t^r}_{\tilde{b}_i \tilde{b}_j} \text{ for } r \text{ even}, \\ c^{\tilde{d}_k}_{\tilde{b}_i t^r} \text{ for } r \text{ odd}, & c^{t^s}_{\tilde{b}_i t^r} \text{ for } r \text{ odd}, s \text{ even}, \\ c^{\tilde{d}_k}_{t^r t^s} \text{ for } r, s \text{ odd}, & c^{t^u}_{t^r t^s} \text{ for } r, s \text{ odd}, u \text{ even}. \end{array}$$

Polynomial terms arising in these components can be seen to vanish from consideration of their degrees; all polynomials in $\{t^1, \ldots, t^{2N+1}\}$ of odd degree must vanish when all $t^i$ of odd degree vanish, whereas polynomials in $\{t^i\}$ of even degree are always multiplied by (at least) a factor of $b_i + b_{i+M}$ for some $i$, and hence vanish on $d_i = 0$. Non-polynomial terms are given explicitly in Lemma 7.

□

It would be of interest to see if these ideas can be applied to arbitrary Coxeter group orbit space.

5.2. **Rational Water-bag Potentials.**

**Proposition 15.** *On the space of functions*

$$\begin{aligned}\lambda(p) &= p^{N+1}+s_1p^{N-1}+\cdots+s_N\\ &+\sum_{i=1}^{K}\left[\frac{v_{i,1}}{(p-s_i)}+\cdots+\frac{v_{i,L_i}}{(p-s_i)^{L_i}}\right]\\ &+\sum_{i=1}^{M}k_i\log(p-b_i)\,,\end{aligned}$$

*the formulas (2) and (3) define a solution of the WDVV equations.*

*Proof.* Canonical coordinates are found as in Lemma 2.

The flat coordinates are $\{b_1,\ldots,b_M\}$ together with those obtained for the purely rational case [1],[**?**]. Namely invert $\lambda_+(p)=p^{N+1}+s_1p^{N-1}+\cdots+s_N$ about $p=\infty$ using the Puiseaux series (8), and invert

$$\lambda_{-i}(p)=\frac{v_{i,1}}{(p-s_i)}+\cdots+\frac{v_{L_i}}{(p-s_i)^{L_i}}$$

for $p\sim s_i$ as

$$p=\frac{1}{L_i}\left(x_{i,L_i+1}+\frac{x_{i,L_i}}{w}+\cdots+\frac{x_{i,1}}{w^{L_i}}\right)\,,$$

where $\lambda_{-i}=w^{L_i}$, and $x_{i,L_i+1}=L_is$. The flat coordinates are then $\{t^\alpha,x_{\beta,\gamma},b_\delta\}$. In these coordinates the metric has only the following non-zero components:

$$\begin{aligned}\eta\left(\frac{\partial}{\partial t^\alpha},\frac{\partial}{\partial t^\beta}\right) &= -\frac{1}{N+1}\delta_{\alpha+\beta,N+1}\,,\\ \eta\left(\frac{\partial}{\partial x_{i,j}},\frac{\partial}{\partial x_{i,k}}\right) &= -\frac{1}{L_i}\delta_{j+k,L_i+2}\,,\\ \eta\left(\frac{\partial}{\partial b_\alpha},\frac{\partial}{\partial b_\beta}\right) &= k_\alpha\delta_{\alpha\beta}\,.\end{aligned}$$

□

Note one may combine the results from the last to sections and consider $B_n$-type reductions of the rational case, where the superpotential, including logarithmic terms, is an even function.

In the above proposition the location of the poles $\{s_i\}$ and the logarithmic poles $\{b^i\}$ were taken to be distinct. However, a modified of the above proposition may be formulated which takes into account possible coincidences in these sets. Rather than state this we give an example.

**Example 16.** *The superpotential*

$$\lambda(p)=p^2+t_1+\frac{t_2}{(p-t_3)}+k\log(p-t_3)$$

*leads to the following solution of the WDVV equation*

$$F=\frac{1}{12}t_1^3+t_1t_2t_3-\frac{1}{2}k\,t_1t_3^2-\frac{3}{4}t_2^2+\frac{1}{2}t_2^2\log t_2+\frac{1}{3}t_2t_3^3-\frac{1}{12}k\,t_3^4\,. \tag{12}$$

This produces an interesting class of solutions, as no extra variables have had to be introduced, so in a sense they are true deformations of the original solution. The single pole case - generalizations of the above example - are isomorphic to deformations of the extended-affine-Weyl orbit space [9], since

$$H_{0,N+L+1}(N+1,L) \cong \mathbb{C}^{N+L+1}/\widetilde{W}^{(L)}(A_{N+L})\,.$$

Explicitly this is given by a Legendre transformation (which acts on solutions of the WDVV equations, not just to those solutions which define Frobenius manifolds).

**Example 17.** *Applying the Legendre transformation $S_2$ (using the notation of [7]) to the solution (12) yields the solution*

$$\hat{F} = \frac{1}{4}\hat{t}_1 + \frac{1}{2}\hat{t}_2^2\hat{t}_3 - \frac{1}{2}k\,\hat{t}_2\hat{t}_3^2 - \frac{1}{96}\hat{t}_1^4 + \hat{t}_1 e^{\hat{t}_3} - k\left(\frac{1}{4}\hat{t}_1^2\hat{t}_3 + \frac{1}{2}\hat{t}_2\hat{t}_3^2\right) + \frac{1}{6}k^2\hat{t}_3^3\,.$$

*This defines a deformation of the extended-affine-Weyl space $\mathbb{C}^3/\widetilde{W}^1(A_2)$.*

One would expect that the associated dispersionless integrable systems would be related to water-bag type-reductions of the dispersionless Toda equations and their generalizations [3].

## 6. Open problems

Some open problems have already been outlined above; here we draw them together and raise some other open problems, potential generalizations and applications.

- Can the construction be applied, independent of the Landau/Ginzburg construction, directly to an arbitrary Coxeter group orbit space, or more generally, to other orbit spaces? By this we mean, is there a Saito-type construction of these solutions? The absence of a flat 'intersection form' would seem problematical. A related question is whether one can formulate axiomatically a theory of pseudo-quasi-homogeneous solutions of the WDVV equations.
- The Frobenius manifold structure on the space of rational functions may be generalized to the space of branched coverings of an arbitrary Riemann surface (i.e. a Hurwitz space). All that is required for the direct calculation of the residues (2) and (3) is the meromorphicity of the *derivatives* of $\lambda$ rather than the meromorphicity of $\lambda$ itself. This suggests that one should look at generalizations where $\lambda$ lies in some extension of the field of meromorphic functions.
- In a semi-simple Frobenius manifold there exists interesting submanifolds: discriminants and caustics [17]. What are the properties of such structures in the present case?
- What are the properties of the dispersionless integrable systems associated to such solutions of the WDVV equations, i.e. the water-bag reductions of the dKP hierarchy itself, and how are they encoded in the geometry of these pseudo-quasi-homogeneous manifolds? In particular, the (non-local) bi-Hamiltonian structure, especially in the flat coordinates system for the metric $\eta$ is unknown in general. Can these dispersionless systems be deformed, and how do the form of such deformations follow from the geoemtry of the undeformed systems [5].

- Finally, is there an algebraic description, say of the $A_n$-deformations, in terms of a deformed Milnor ring? Are there field theoretic interpretation of the results in terms of a topological quantum field theory [13, 14].

We hope to address some of these problems in the future.

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