Polymer Measure: Varadhan's Renormalization Revisited

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Abstract

Through chaos decomposition we improve the Varadhan estimate for the rate of convergence of the centered approximate self-intersection local time of planar Brownian motion.

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1 Introduction

The Edwards model [2] for self-repelling or "weakly self-avoiding" d-dimensional Brownian motion, with applications in polymer physics and quantum field theory, is informally given by a Gibbs factor

$$G = \frac{1}{Z} \exp\left(-g \int_0^T ds \int_0^t dt \delta\left(B(s) - B(t)\ \right)\right)$$

with $g > 0$ and

$$Z = E\left(\exp\left(-g \int_0^T ds \int_0^t dt \delta\left(B(s) - B(t)\ \right)\right)\right).$$

Using

$$\delta_\varepsilon(x) := \frac{1}{(2\pi\varepsilon)^{d/2}} e^{-\frac{|x|^2}{2\varepsilon}}, \quad \varepsilon > 0,$$

one defines an approximate self-intersection local time by

$$L_\varepsilon := \int_0^T dt \int_0^t ds\, \delta_\varepsilon(B(t) - B(s)).$$

For $d \geq 2$

$$\lim_{\varepsilon \searrow 0} \mathbb{E}(L_\varepsilon) = \infty.$$

For the planar case $d = 2$, Varadhan [3] has shown that centering

$$L_{\varepsilon,c} \ := L_\varepsilon \ - \mathbb{E}(L_\varepsilon) \text{ and } L_c := \lim_{\varepsilon \searrow 0} L_{\varepsilon,c}$$

is sufficient to make the Gibbs factor $G = Z^{-1} \exp(-gL_c)$ well defined. An estimate for the rate of convergence

$$\|L_c \ - L_{\varepsilon,c}\|_2^2 \ \leq const.\varepsilon^\alpha$$

for all $\alpha < 1/2$, is in Varadhan's words, "the most difficult step of all and requires considerable estimation". In this note we shall use a multiple Wiener integral or chaos expansion for an alternate and comparatively straightforward argument, extending the estimate to all $\alpha < 1$.

2 Fock space representation of the local time

The Ito-Segal-Wiener isomorphism relates the L^2 space of planar Brownian motion with the Fock space

$$\mathfrak{F} = \left(\bigoplus_{n=0}^{\infty} \mathrm{Sym}\, L^2(\mathbb{R}^n, n! d^n x) \right)^{\otimes 2}.$$

We shall use the multi-index notation

$$\mathbf{n} = (n_1, n_2) \quad n = n_1 + n_2, \quad \mathbf{n}! = n_1! n_2!$$

The Fock space norm is then

$$\|F\|_2^2 = \sum_{n_i \geq 0} \mathbf{n}! \, \|F_{\mathbf{n}}\|_2^2 \,.$$

For $L_{\varepsilon,c}$ the kernel functions $F_{\mathbf{n}}$ were computed explicitly in [1]. For the planar case the result is

Proposition 1 *[1]: For $d = 2$ the kernel functions $F_{\mathbf{n}}$ of $L_{\varepsilon,c}(T)$ and $L_c(T)$ have their support on $[0,T]^n$ and are, with $\varepsilon > 0$, and $\varepsilon = 0$ respectively, for $n > 1$*

$$F_{2\mathbf{n},\varepsilon}(u_1, \ldots, u_{2n}) = \frac{1}{2\pi} \left(-\frac{1}{2} \right)^n \frac{1}{n(n-1)\, \mathbf{n}!}$$
$$\times \left(\frac{1}{(T+\varepsilon)^{n-1}} - \frac{1}{(v+\varepsilon)^{n-1}} - \frac{1}{(T-u+\varepsilon)^{n-1}} + \frac{1}{(v-u+\varepsilon)^{n-1}} \right),$$

where $v := \max_{1 \leq k \leq 2n} u_k \leq T$ and $u := \min_{1 \leq k \leq 2n} u_k \geq 0$. For $n = 1$

$$F_{2,\varepsilon}(u_1, u_2) = -\frac{1}{4\pi} \left(\ln(v+\varepsilon) + \ln(T-u+\varepsilon) - \ln(v-u+\varepsilon) - \ln(T+\varepsilon) \right).$$

All kernel functions $F_{\mathbf{n}}$ with odd n_i are zero.

2.1 The rate of convergence

Theorem 2 *Given $T > 0$. Then for any $\alpha < 1$ there is a constant $C_{T,\alpha} > 0$ such that for all $\varepsilon > 0$*

$$\|L_{\varepsilon,c}(T) - L_c(T)\|_2^2 \leq C_{T,\alpha} \varepsilon^{\alpha}.$$

Proof: From Proposition 1

$$\|F_{2\mathbf{n},0} - F_{2\mathbf{n},\varepsilon}\|_2^2 = \Big(n(n-1)2\pi 2^n \mathbf{n}!\Big)^{-2} \int_0^T d^{2n}u_k K_\varepsilon^2(u,v,T)$$

where for $n > 1$

$$\begin{aligned} K_\varepsilon(u,v,T) &= \Big(T^{-n+1} - (T+\varepsilon)^{-n+1}\Big) - \Big(v^{-n+1} - (v+\varepsilon)^{-n+1}\Big) \\ &\quad -\Big((T-u)^{-n+1} - (T-u+\varepsilon)^{-n+1}\Big) + \Big((v-u)^{-n+1} - (v-u+\varepsilon)^{-n+1}\Big). \end{aligned}$$

Since $K_\varepsilon(u,v,T)$ does not depend on $2n-2$ of the u_k-variables, we may integrate them out:

$$\|F_{2\mathbf{n},0} - F_{2\mathbf{n},\varepsilon}\|_2^2 = \Big(n(n-1)2\pi 2^n \mathbf{n}!\Big)^{-2} 2n(2n-1) \int_0^T dv \int_0^v du (v-u)^{2n-2} K_\varepsilon^2(u,v,T)\,.$$

Of the four terms in K_ε, the last one is dominant so that

$$\begin{aligned} \|F_{2\mathbf{n},0} - F_{2\mathbf{n},\varepsilon}\|_2^2 &\leq 16 \frac{2n(2n-1)}{\Big(n(n-1)2\pi 2^n \mathbf{n}!\Big)^2} \int_0^T dv \int_0^v du \left(1 - \left(\frac{v-u}{v-u+\varepsilon}\right)^{n-1}\right)^2 \\ &= 16 \frac{2n(2n-1)}{\Big(n(n-1)2\pi 2^n \mathbf{n}!\Big)^2} \int_0^T d\tau \int_0^{T-\tau} du \left(1 - \left(\frac{\tau}{\tau+\varepsilon}\right)^{n-1}\right)^2 \\ &\leq 16 \frac{2n(2n-1)T}{\Big(n(n-1)2\pi 2^n \mathbf{n}!\Big)^2} \int_0^T d\tau \left(1 - \left(\frac{\tau}{\tau+\varepsilon}\right)^{n-1}\right)^2 \\ &= 16 \frac{2n(2n-1)T}{\Big(n(n-1)2\pi 2^n \mathbf{n}!\Big)^2} \int_0^T d\tau \left(\tau^{n-1}(n-1) \int_0^\varepsilon \frac{dx}{(x+\tau)^n}\right)^2 (1) \end{aligned}$$

By Hölder's inequality

$$\int_0^\varepsilon \frac{dx}{(x+\tau)^n} \leq \varepsilon^{1/q} \left(\int_0^\infty \frac{dx}{(x+\tau)^{np}}\right)^{1/p} = \varepsilon^{1/q} \left(\frac{1}{np-1}\tau^{1-np}\right)^{1/p}$$

if $\frac{1}{q} + \frac{1}{p} = 1$. Insertion of this estimate into (1) produces

$$\sum_{\mathbf{n}:n>1} (2\mathbf{n})!\, \|F_{2\mathbf{n},0} - F_{2\mathbf{n},\varepsilon}\|_2^2 \leq \frac{4p}{(2-p)\pi^2} T^{\frac{2}{p}} \varepsilon^{\frac{2}{q}} \sum_{\mathbf{n}:n>1} (2\mathbf{n})! \frac{2n(2n-1)}{(n2^n\mathbf{n}!)^2} \frac{1}{(pn-1)^{\frac{2}{p}}}$$

which is convergent if $\frac{2}{p} > 1$, i.e., $q > 2$. For the $n = 1$ term an $\varepsilon^{\frac{2}{q}}$ estimate is likewise obtained via Hölder's inequality.

Hence, for any $\alpha < 1$,

$$\|L_c(T) - L_{\varepsilon,c}(T)\|_2^2 = \sum_{\mathbf{n}:n\geq 1} (2\mathbf{n})! \, \|F_{2\mathbf{n},0} - F_{2\mathbf{n},\varepsilon}\|_2^2 \leq C(T,\alpha)\varepsilon^{\alpha}, \quad \forall \varepsilon > 0.$$

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