Long time behaviour and self-similarity in an addition model with slow input of monomers

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Abstract We consider a coagulation equation with constant coefficients and a time dependent power law input of monomers. We discuss the asymptotic behaviour of solutions as $t \to \infty$, and we prove solutions converge to a similarity profile along the non-characteristic direction.

1 Introduction

We study some aspects of the long time behaviour of a system with an infinite number of ordinary differential equations modelling the kinetics of particle coagulation; we consider a mean-field point island deposition growth process, with Becker-Döring type kinetics with critical island size $i = 1$. In [6] a different island growth model is considered, for which clusters of size j $(1 < j \leq i)$ do not arise.

The system we consider is composed of a large number of particles, each particle consisting of an integer number of monomers with mass 1, so that a j-cluster (a particle formed by j monomers) will have mass j. We assume these clusters can bind together to form larger clusters, and that we only have *binary* reactions, in the sense that we only consider aggregation of two clusters at a time, one of them being a monomer; we do not consider, for example, simultaneous aggregation of three clusters. The cluster interaction is assumed to follow the mass action law of chemical kinetics. Let $(c_j(t))_{j=1}^{\infty}$ be the sequence whose elements are the concentration of clusters of mass j at some time t, and we want to study the evolution of $c_j(t)$ as $t \to +\infty$, either pointwise in j (i.e., for each fixed j), or when j also converges

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to $+\infty$ in some way related to the convergence of t. The evolution of the cluster population can be described by the following coagulation kinetic equations

$$\begin{aligned} \dot{c}_1 &= -c_1^2 - c_1 \sum_{j=1}^{\infty} c_j \\ \dot{c}_j &= c_1(c_{j-1} - c_j), \quad j \geq 2. \end{aligned} \tag{1}$$

From the first equation in (1) it is clear that the number of monomers is decreasing; as described in more detail in [5], equations (1) are a special case of the Becker-Döring coagulation equations, corresponding to a situation where the only effective reactions are the ones involving monomers. Thus the special role played by monomers is expected to freeze the dynamics when we run out of monomers. In the context of aggregation models of cluster growth [3] we consider an "addition" model [7] where cluster growth can only occur by the addition of movable monomers to the immovable clusters [3]. We provide a source of monomers by adding a source term $J_1(t)$ to the right hand side of the c_1-equation in (1). One way to externally supply monomers is to define the input term $J_1(t)$ independently of the state of the system. This is a reasonable assumption in a number of applications, including in simple models of polymerization and of epitaxial growth [2]. The easiest hypothesis about $J_1(t)$, which turns out to be very useful in applications, is to make it a time independent constant. Another possible choice, quite interesting from a mathematical viewpoint, is to consider for J_1 a power law $J_1(t) = \alpha t^{\omega}$, with $\alpha > 0$ and $\omega \in \mathbb{R}$. The constant case was considered in [5], using an approach based on methods (Poincaré compactification and center manifold) that are not available for the general power law case; the case $\omega > -1/2$ was considered in [4]. For $\omega \leq -1/2$ partial results were obtained in [8]. In this paper we restrict ourselves to $\omega = -1/2$. A formal analysis was presented in [9], and we use the ansatz provided by [9] to rigorously analyse the addition model with a power law input of monomers $J_1(t) = \alpha t^{-1/2}$, namely

$$\begin{aligned} \dot{c}_1 &= \alpha t^{-1/2} - c_1^2 - c_1 \sum_{j=1}^{\infty} c_j \\ \dot{c}_j &= c_1(c_{j-1} - c_j) \ j \geq 2. \end{aligned} \tag{2}$$

We study two aspects of the dynamical behaviour of solutions to (2). First, we want to establish the componentwise behaviour of the solution as time $t \to +\infty$ and the behaviour of the total amount of clusters. The second aspect of the dynamics we are interested in is the occurrence of similarity behaviour. Our first step consists in transforming the infinite dimensional system (2) into a problem that is almost exactly solvable. Introducing the total number of clusters as a new macroscopic variable $c_0(t)$ defined by $c_0(t) = \sum_{j=1}^{\infty} c_j(t)$, and formally differentiating termwise, we conclude that c_0 satisfies the evolution equation

$$\dot{c}_0 = \alpha t^{-1/2} - c_0 c_1.$$

Using c_0, we can write system (2) as

$$\begin{aligned}\dot{c}_0 &= \alpha t^{-1/2} - c_0 c_1, \\ \dot{c}_1 &= \alpha t^{-1/2} - c_0 c_1 - c_1^2, \\ \dot{c}_j &= c_1(c_{j-1} - c_j), \quad j \geq 2.\end{aligned} \tag{3}$$

If $\sum_{j=1}^{\infty} c_j(0) < \infty$ then if $(c_0, c_1, c_j),\ j \geq 2$ is a solution of system (3) then $(c_1, c_j),\ j \geq 2$ is a solution of system (2). The proof can be done as in [5, Theorem 2.1].

The equations governing the dynamics of $c_0(t)$ and $c_1(t)$ actually define a nonautonomous bidimensional system

$$\begin{aligned}\dot{c}_0 &= \alpha t^{-1/2} - c_0 c_1 \\ \dot{c}_1 &= \alpha t^{-1/2} - c_0 c_1 - c_1^2,\end{aligned} \tag{4}$$

and we can now study the dynamics of (4) in a way totally independent of the remaining components of the infinite dimensional system. In order to solve this system we use an ansatz for a convenient change of variables suggested by [9, Table 2] and obtained via formal asymptotics. Based on [9, Table 2] we expect solutions (c_0, c_1) of system (4) to have the following behaviour as $t \to +\infty$

$$c_0(t) \sim \left(3\alpha^2\right)^{1/3} (\log t)^{1/3} \text{ and } c_1(t) \sim (\alpha/3)^{1/3} t^{-1/2} (\log t)^{-1/3}, \tag{5}$$

in the following sense

$$\lim_{t\to+\infty} c_0(t) \left(3\alpha^2 \log t\right)^{-1/3} = 1 \text{ and } \lim_{t\to+\infty} c_1(t) (\alpha/3)^{-1/3} t^{1/2} (\log t)^{1/3} = 1.$$

This suggests that defining functions $C_0(t)$ and $C_1(t)$ by

$$C_0(t) := \left(3\alpha^2\right)^{-1/3} (\log t)^{-1/3} c_0(t) \text{ and } C_1(t) := (\alpha/3)^{-1/3} t^{1/2} (\log t)^{1/3} c_1(t), \tag{6}$$

they might both be expected to converge to 1 as $t \to +\infty$, and reciprocally, if this happens then c_0 and c_1 will behave as stated in (5). To prove this convergence behaviour we need an equation for the evolution of (C_0, C_1). We begin by differentiating (6), and then replacing it into system (4). We then change the time scale $t \mapsto \tau$ by letting

$$\frac{\mathrm{d}\tau}{\mathrm{d}t} = \left(3\alpha^2\right)^{1/3} (\log t)^{1/3}. \tag{7}$$

Considering $t > 1$ we have a well defined change of variables, and defining

$$x(\tau) := C_1(t(\tau)) \text{ and } y(\tau) := C_0(t(\tau)),$$

and denoting $\frac{\mathrm{d}}{\mathrm{d}\tau}(\cdot)$ by $(\cdot)'$ we finally obtain an equation for (x, y):

$$\begin{aligned}x' &= 1 - xy - \hat{c}(\tau)x^2 + \hat{d}(\tau)x \\ y' &= \hat{c}(\tau)(1 - xy - \hat{c}(\tau)y),\end{aligned} \tag{8}$$

where

$$\hat{c}(\tau) = c\,(t(\tau)) := (9\alpha)^{-1/3}\,(t(\tau))^{-1/2}\,(\log t(\tau))^{-2/3}\,,$$

and

$$\hat{d}(\tau) = d\,(t(\tau)) := \hat{c}^2(\tau)(3/2\log t(\tau)+1).$$

In [4] we have seen that for $\omega > -1/2$, the change of variables corresponding to (7) can be explicitly solved; for $\omega = -1/2$ we do not have an explicit expression for t as a function of τ, and we will use some preliminary results to obtain what we need: the asymptotic relationship between the two time scales.

For $t \in [1,+\infty[$ we have $\mathrm{d}\tau/\mathrm{d}t = \left(3\alpha^2\right)^{1/3}(\log t)^{1/3} > 0$; since $\lim_{t\to\infty}\mathrm{d}\tau/\mathrm{d}t = +\infty$, we can conclude that $\tau(t)$ (resp. $t(\tau)$) is a strictly increasing function of t (resp. τ). This allows us to conclude that $\tau \to +\infty$ (resp. $t \to +\infty$) as $t \to +\infty$ (resp. $\tau \to +\infty$). To get a better estimate on the asymptotic behaviour of $\tau(t)$, using integration by parts, we obtain from (7)

$$\tau(t) = t(3\alpha^2\log t)^{1/3}\,(1+\mathrm{o}(1))\ \text{ as } t \to +\infty.$$

This allows us to write $t(\tau) = \tau(3\alpha^2\log\tau)^{-1/3}\,(1+\mathrm{o}(1))$ as $\tau \to +\infty$.

We also have as $\tau, t \to +\infty$ that

$$\tau(t) = \mathrm{O}\left(t(\log t)^{1/3}\right) \text{ and } t(\tau) = \mathrm{O}\left(\tau(\log\tau)^{-1/3}\right),$$

and also

$$\hat{c}(\tau) = \mathrm{O}\left((\tau\log\tau)^{-1/2}\right) \text{ and } \hat{d}(\tau) = \mathrm{O}\left(\tau^{-1}\right).$$

In the next Section, we will study the bidimensional system (4); then in Section 3 we will study the long time behaviour of solutions, and in Section 4 we will study the existence of self-similar behaviour.

2 The bidimensional system

Since we are only interested in non-negative solutions to (4), by *solution* we shall mean *non-negative solution*. The main result of this section concerns the asymptotic behaviour of c_0 and c_1.

Theorem 1. *Let $\alpha > 0$, and (c_0, c_1) be any solution of (4). Then*

1. $(3\alpha^2)^{-1/3}\,(\log t)^{-1/3}\,c_0(t) \to 1$ *as* $t \to +\infty$,
2. $(\alpha/3)^{-1/3}t^{1/2}(\log t)^{1/3}c_1(t) \to 1$ *as* $t \to +\infty$,
3. $(3/\alpha\log t)^{2/3}t\left(\alpha t^{-1/2} - c_0c_1\right) \to 1$ *as* $t \to +\infty$.

To prove this theorem we use two propositions. These propositions follow closely what was done in a series of lemmas in [5] and [4], and the proofs differ mainly because now we have a log term and also, as mentioned already, because we do not have an explicit expression for the change of variables defined by (7). We start by showing that non-negative solutions to (8) remain non-negative as $\tau \to +\infty$, then we show how the x and y boundedness are closely related, and finally we show that every solution to (8) with positive initial data is bounded.

Proposition 1. *For the system of equations (4) the following holds*

1. *The first quadrant $\{x \geq 0,\ y \geq 0\}$ is positively invariant for (8).*
2. *y (resp. x) is bounded $\Longleftrightarrow$ x (resp. y) is bounded away from zero.*
3. *Every solution to (8) with positive initial data is bounded.*

An immediate consequence of Proposition 1 is that solutions to (8), with positive initial data, are bounded and bounded away from zero; we also have that the conclusions of Proposition 1 still hold if the initial condition is nonnegative. Proposition 1 also implies that every orbit of (8) is bounded and bounded away from zero. We are now ready to study the $\omega-$limit set of (8). We start by showing that the $\omega-$limit set of every orbit is contained in the hyperbola $\{xy = 1\}$, then we fully identify it by showing that both x and y converge to 1, and finally we establish the convergence rate of $x(\tau)y(\tau)$ as $\tau \to +\infty$.

Proposition 2. *For the system of equations (8) the following holds*

1. *Let (x,y) be any solution to (8) then $x(\tau)y(\tau) \to 1^-$ as $\tau \to +\infty$.*
2. *$\lim_{\tau\to+\infty} x(\tau) = 1$ and $\lim_{\tau\to+\infty} y(\tau) = 1$.*
3. *Let (x,y) be any solution to (8) then we have $\lim_{\tau\to+\infty} \frac{1-x(\tau)y(\tau)}{\hat{c}(\tau)} = 1$.*

Recalling the definition of x, y and $\hat{c}$, Theorem 1 follows from the last two statements in Proposition 2.

3 Long time behaviour of the system

Given a solution of (4), we introduce a new time scale

$$\varsigma(t) := \varsigma_0 + \int_{t_0}^{t} c_1(s)\mathrm{d}s, \tag{9}$$

where ς_0 is a positive constant, and we consider the new phase variables

$$\tilde{c}_j(\varsigma) := c_j(t(\varsigma)), \tag{10}$$

where $t(\varsigma)$ is the inverse function of $\varsigma(t)$. When $c_1(t) > 0$, these are well defined and ς is an increasing function of t. In these new variables, the equations for c_j in (3) now become

$$\tilde{c}_j{}' = \tilde{c}_{j-1} - \tilde{c}_j, \qquad j \geq 2,$$

where $(\cdot)' = \frac{\mathrm{d}}{\mathrm{d}\varsigma}(\cdot)$. This system of differential equations is a lower triangular linear system and thus can be explicitly solved in terms of the function $\tilde{c}_1(\varsigma)$ starting from the equation for $j = 2$ and applying the variation of constants formula recursively:

$$\tilde{c}_j(\varsigma) = \mathrm{e}^{-\varsigma} \sum_{k=2}^{j} \frac{\varsigma^{j-k}}{(j-k)!} c_k(0) + \frac{1}{(j-2)!} \int_0^{\varsigma} \tilde{c}_1(\varsigma - s) s^{j-2} \mathrm{e}^{-s} \mathrm{d}s. \tag{11}$$

From now on we will only consider the new time scale defined by (9).

We now establish convergence results similar to those of Theorem 1 but for all values of j, and in both time scales.

Proposition 3. *With c_j, ς and $\tilde{c}_j(\varsigma)$ as given by (9) and (10)*

1. $\varsigma(t) = (8\alpha/3)^{1/3} t^{1/2} (\log t)^{-1/3} (1 + \mathrm{o}(1))$, *as* $t \to +\infty$,
2. $(1/2)^{1/3} (3/\alpha)^{2/3} \varsigma (\log \varsigma)^{2/3} \tilde{c}_j(\varsigma) = 1 + \mathrm{o}(1), \forall j \geq 1$, *as* $\varsigma \to +\infty$,
3. $(\alpha/3)^{-1/3} t^{1/2} (\log t)^{1/3} c_j(t) \longrightarrow 1, \forall j \geq 1$, *as* $t \to +\infty$.

By definition we have $\mathrm{d}\varsigma/\mathrm{d}t = c_1(t)$ and we already know from Theorem 1 the asymptotic behaviour of c_1, hence we have the following estimates

$$\begin{aligned} &\forall \varepsilon > 0, \exists T_\varepsilon : \forall t > T_\varepsilon, 1 - \varepsilon < t^{1/2} (3 \log t/\alpha)^{1/3} c_1(t) < 1 + \varepsilon \\ &\qquad \Longrightarrow t^{-1/2} (3 \log t/\alpha)^{-1/3} (1 - \varepsilon) < c_1(t) < t^{-1/2} (3 \log t/\alpha)^{-1/3} (1 + \varepsilon). \end{aligned}$$

We are thus naturally led to estimate the integral $\int_{t_0}^{t} s^{-1/2} (\log s)^{-1/3} \mathrm{d}s$, as $t \to +\infty$, to obtain, as $t \to +\infty$

$$\varsigma(t) = (8\alpha/3)^{\frac{1}{3}} t^{1/2} (\log t)^{-\frac{1}{3}} (1 + \mathrm{o}(1)). \tag{12}$$

Using equation (12) we obtain the following relation between the logarithms of $\varsigma(t)$ and $t(\varsigma)$

$$\log \varsigma(t) = \frac{1}{2} \log t(\varsigma)(1 + \mathrm{o}(1)),$$

and using this last equation,

$$t(\varsigma) = (4\alpha/3)^{-2/3} \varsigma^2 (\log \varsigma)^{2/3} (1 + \mathrm{o}(1)), \tag{13}$$

as $\varsigma \to \infty$. Using (13) we obtain the asymptotic behaviour of $\tilde{c}_1(\varsigma)$

$$\lim_{\varsigma \to +\infty} (1/2)^{1/3} (3/\alpha)^{2/3} \varsigma (\log \varsigma)^{2/3} \tilde{c}_1(\varsigma) = 1. \tag{14}$$

Using (14) and the representation of the $\tilde{c}_j$ given by (11) we can establish the behaviour of c_j in terms of the original t variable. To this end, letting

$$g(\varsigma) := (1/2)^{1/3} (3/\alpha)^{2/3} \varsigma (\log \varsigma)^{2/3}, \tag{15}$$

we can write $g(\varsigma)\tilde{c}_1(\varsigma) = 1 + \mathrm{o}(1)$, as $\varsigma \to +\infty$. Multiplying (11) by $g(\varsigma)$ we obtain

$$g(\varsigma)\tilde{c}_j(\varsigma) = g(\varsigma)\mathrm{e}^{-\varsigma} \sum_{k=2}^{j} \frac{\varsigma^{j-k}}{(j-k)!} c_k(0) + \frac{g(\varsigma)}{(j-2)!} \int_0^{\varsigma} \tilde{c}_1(\varsigma - s) s^{j-2} \mathrm{e}^{-s} \mathrm{d}s. \tag{16}$$

The first term on the right hand side of (16), corresponding to the non-monomeric initial data contribution, can be written as

$$\varsigma(\log\varsigma)^{2/3}\mathrm{e}^{-\varsigma}\sum_{k=2}^{j}\frac{\varsigma^{j-k}}{(j-k)!}c_k(0)=\mathrm{O}\left((\log\varsigma)^{2/3}\varsigma^{j-1}\mathrm{e}^{-\varsigma}\right)=\mathrm{o}\left(\mathrm{e}^{-\lambda\varsigma}\right)\ \text{as}\ \varsigma\to+\infty,$$

for every $\lambda<1$ and fixed j.

For the second term in the right hand side of (16) we start by changing integration variables $s\mapsto y=s/\varsigma$, which allows us to write the integral term as an integral over the fixed interval $[0,1]$. Defining the function

$$\psi(\cdot):=g(\cdot)\tilde{c}_1(\cdot), \tag{17}$$

we obtain

$$\begin{aligned}&\frac{g(\varsigma)}{(j-2)!}\int_0^{\varsigma}\tilde{c}_1(\varsigma-s)s^{j-2}\mathrm{e}^{-s}\mathrm{d}s\\&\qquad=\frac{\varsigma^{j-1}(\log\varsigma)^{2/3}}{(j-2)!}\int_0^1\frac{\psi(\varsigma(1-y))y^{j-2}}{(1-y)(\log\varsigma(1-y))^{2/3}}\mathrm{e}^{-\varsigma y}\mathrm{d}y.\end{aligned} \tag{18}$$

In order to evaluate the integral in (18) we split the interval of integration at the $y=1$ singularity as $(0,1-\varepsilon)$ and $(1-\varepsilon,1)$, for a fixed $\varepsilon\in(0,1)$. For the integral over $(1-\varepsilon,1)$ we know that since $\tilde{c}_1$ is continuous and goes to zero at infinity, by (14), there exists a positive constant M satisfying $0\le\tilde{c}_1(x)\le M$ for $x\in[0,+\infty[$ and hence

$$\begin{aligned}\varsigma^j(\log\varsigma)^{2/3}\int_{1-\varepsilon}^1\tilde{c}_1(\varsigma(1-y))y^{j-2}\mathrm{e}^{-\varsigma y}\mathrm{d}y&\le\varsigma^j(\log\varsigma)^{2/3}M\int_{1-\varepsilon}^1\mathrm{e}^{-\varsigma y}\mathrm{d}y\\&=M\varsigma^{j-1}(\log\varsigma)^{2/3}\mathrm{e}^{-\varsigma}\left(\exp(1-\varepsilon)-1\right),\end{aligned} \tag{19}$$

and this term is exponentially small when $\varsigma\to+\infty$.

For the integral over $(0,1-\varepsilon)$, we use the fact that $y<1-\varepsilon\Rightarrow\varsigma(1-y)>\varsigma\varepsilon\to+\infty$ as $\varsigma\to+\infty$, then for ς sufficiently large, we can use (14) and (17) to conclude that $\psi=1+\mathrm{o}(1)$ in the interval we are considering, and thus

$$\forall\delta_1>0,\exists T_1(\delta_1):\forall\varsigma>T_1(\delta_1),\psi(\varsigma(1-y))\in[1-\delta_1,1+\delta_1],$$

and hence as $\varsigma\to+\infty$ we have

$$(1-\delta_1)I_j(\varsigma)\le\int_0^{1-\varepsilon}\frac{(\log\varsigma)^{2/3}\psi(\varsigma(1-y))y^{j-2}}{(1-y)(\log\varsigma(1-y))^{2/3}}\mathrm{e}^{-\varsigma y}\mathrm{d}y\le(1+\delta_1)I_j(\varsigma), \tag{20}$$

where

$$I_j(\varsigma):=\int_0^{1-\varepsilon}\left(1+\frac{\log(1-y)}{\log\varsigma}\right)^{-2/3}\frac{y^{j-2}}{1-y}\mathrm{e}^{-\varsigma y}\mathrm{d}y.$$

For $y \in [0, 1-\varepsilon[$, we now have that

$$\forall \delta_2 > 0, \exists T_2(\delta_2) \colon \forall \varsigma > T_2(\delta_2), \left(1 + \frac{\log(1-y)}{\log \varsigma}\right)^{-2/3} \in [1-\delta_2, 1+\delta_2].$$

Hence for ς sufficiently large, it is enough to estimate the integral $\int_0^{1-\varepsilon} \frac{y^{j-2}}{1-y} \mathrm{e}^{-\varsigma y} \mathrm{d}y$, which, using Watson's lemma, is equal to

$$\int_0^{1-\varepsilon} \frac{y^{j-2}}{1-y} \mathrm{e}^{-\varsigma y} \mathrm{d}y = \frac{\Gamma(j-1)}{\varsigma^{j-1}} + \mathrm{O}\left(\frac{1}{\varsigma^j}\right), \text{ as } \varsigma \to +\infty.$$

Putting this last expression in (20) results in

$$\frac{\varsigma^{j-1}}{(j-2)!} \int_0^{1-\varepsilon} \frac{(\log \varsigma)^{2/3} \psi(\varsigma(1-y)) y^{j-2}}{(1-y)(\log \varsigma(1-y))^{2/3}} \mathrm{e}^{-\varsigma y} \mathrm{d}y = 1 + \mathrm{O}(\varsigma^{-1}) \text{ as } \varsigma \to +\infty. \quad (21)$$

Gathering (19) and (21), we have the following generalization of (14), as $\varsigma \to +\infty$

$$(1/2)^{1/3} (3/\alpha)^{2/3} \varsigma (\log \varsigma)^{2/3} \tilde{c}_j(\varsigma) = 1 + \mathrm{o}(1), \forall j \geq 1,$$

or in the original t variable (using (12)), as $t \to +\infty$

$$t^{1/2} (3 \log t/\alpha)^{1/3} c_j(t) \to 1, \forall j \geq 1.$$

This concludes the proof of Proposition 3.

4 Self-similar behaviour

We can now turn to the results concerning convergence of solutions to self-similar profiles.

Da Costa and Sasportes [4] showed that when the input of monomers is given by $J_1(t) = \alpha t^\omega$, with $\omega > -1/2$ we have a similarity profile $\Phi_{1,\omega} : \mathbb{R}^+ \setminus \{1\} \to \mathbb{R}$ defined by

$$\Phi_{1,\omega}(\eta) := \begin{cases} (1-\eta)^{\frac{\omega-1}{\omega+2}} & \text{if } 0 < \eta < 1 \\ 0 & \text{if } \eta > 1. \end{cases}$$

The following result states that choosing $\omega = -1/2$, the function $\Phi_{1,-1/2}$ is still a similarity profile for the solutions to (2) along non-characteristic directions.

Theorem 2. *Let (c_j) be any non-negative solution of (2) with initial data satisfying $\exists \rho > 0, \mu > 1 : \forall j, c_j(0) \leq \rho / j^\mu$. Let $\varsigma(t)$ and $\tilde{c}_j(\varsigma)$ be as in (9) and (10), respectively. Then for $\eta = j/\varsigma$ fixed and $0 < \eta \neq 1$, we have*

$$\lim_{j,\varsigma \to +\infty} (1/2)^{1/3} (3/\alpha)^{2/3} \varsigma (\log \varsigma)^{2/3} \tilde{c}_j(\varsigma) = \Phi_{1,-1/2}(\eta).$$

Monomeric initial data

For monomeric initial data, the representation formula for $\tilde{c}_j$ (given by (11)) shows that we only have the integral term, and multiplying (11) by $g(\varsigma)$ we have

$$g(\varsigma)\tilde{c}_j(\varsigma) = \frac{g(\varsigma)}{(j-2)!}\int_0^{\varsigma} \tilde{c}_1(\varsigma - s)s^{j-2}\mathrm{e}^{-s}\mathrm{d}s. \tag{22}$$

In order to evaluate the right hand side of (22) we replace the discrete variable j by a continuous one x, allowing us to use Stirling's asymptotic formula for the Γ function. Let φ_1 on $[2,\infty)\times[0,\infty)$ be defined by

$$\varphi_1(x,\varsigma) := \frac{g(\varsigma)}{\Gamma(x-1)}\int_0^{\varsigma} \tilde{c}_1(\varsigma - s)s^{x-2}\mathrm{e}^{-s}\mathrm{d}s.$$

When $x \geq 2$ is an integer, the function φ_1 clearly satisfies $\varphi_1(x,\varsigma) = g(\varsigma)\tilde{c}_x(\varsigma)$, and we shall use φ_1 instead of the definition of $\tilde{c}_j$. Using Stirling's asymptotic formula $\Gamma(x) = \mathrm{e}^{-x}x^{x-1/2}\sqrt{2\pi}\left(1+\mathrm{O}(x^{-1})\right)$ as $x\to\infty$, the recursive relation $\Gamma(x-1) = \Gamma(x)/(x-1)$, letting $\eta := x/\varsigma$, and changing variable $s \mapsto y = s/\varsigma$, we can write,

$$\begin{aligned}\varphi_1(\varsigma\eta,\varsigma) = \frac{1}{\sqrt{2\pi}}\left(\frac{9}{2\alpha^2}\right)^{1/3}\eta^{3/2-\eta\varsigma}\varsigma^{1/2}\varsigma(\log\varsigma)^{2/3}\times \\ \times\left(1+\mathrm{O}\left(\varsigma^{-1}\right)\right)\int_0^1 \tilde{c}_1(\varsigma(1-y))\frac{\exp(\varsigma(\eta\log y - y + \eta))}{y^2}\mathrm{d}y.\end{aligned} \tag{23}$$

In order to make clear the asymptotic behaviour of $\tilde{c}_1(\varsigma)$ we multiply (and divide) inside the previous integral by $g(\varsigma(1-y))$, as defined in (15) and (17), and we obtain

$$\begin{aligned}\varphi_1(\varsigma\eta,\varsigma) = \frac{1}{\sqrt{2\pi}}\left(\frac{9}{2\alpha^2}\right)^{1/3}\eta^{\frac{3}{2}-\eta\varsigma}\varsigma^{1/2}\varsigma(\log\varsigma)^{2/3}\left(\frac{9}{2\alpha^2}\right)^{-1/3}\varsigma^{-1}\times \\ \times\left(1+\mathrm{O}\left(\varsigma^{-1}\right)\right)\int_0^1 \psi(\varsigma(1-y))\frac{\exp(\varsigma(\eta\log y - y + \eta))}{(\log(\varsigma(1-y)))^{2/3}y^2(1-y)}\mathrm{d}y.\end{aligned}$$

Simplifying and grouping the logarithmic terms we obtain

$$\begin{aligned}\varphi_1(\varsigma\eta,\varsigma) = \frac{1}{\sqrt{2\pi}}\eta^{\frac{3}{2}-\eta\varsigma}\varsigma^{1/2}\left(1+\mathrm{O}\left(\varsigma^{-1}\right)\right)\times \\ \times\int_0^1 \psi(\varsigma(1-y))\left(1+\frac{\log(1-y)}{\log\varsigma}\right)^{-2/3}\frac{\exp(\varsigma(\eta\log y - y + \eta))}{y^2(1-y)}\mathrm{d}y.\end{aligned} \tag{24}$$

Rearranging the last expression, the proof reduces to the asymptotic evaluation as $\varsigma\to+\infty$ of the function $I(\eta,\varsigma)$ defined by

$$I(\eta,\varsigma) := \varsigma^{1/2}\eta^{-\eta\varsigma}\mathrm{e}^{\varsigma\eta}\times$$

$$\times \int_0^1 \psi(\varsigma(1-y)) \left(1+\frac{\log(1-y)}{\log \varsigma}\right)^{-2/3} \frac{\exp(\varsigma(\eta \log y - y))}{y^2(1-y)} \mathrm{d}y. \tag{25}$$

We start by showing that for $\eta > 1$ we have $I(\eta, \varsigma) \to 0$, as $\varsigma \to +\infty$. In order to study the behaviour of φ_1 we first split the interval of integration as $(0, 1-\varepsilon)$ and $(1-\varepsilon, 1)$, for a fixed $\varepsilon \in (0,1)$.

In $(0, 1-\varepsilon)$ both $\psi(\varsigma(1-y))$ and $\left(1+\frac{\log(1-y)}{\log\varsigma}\right)^{-1}$ are $1+\mathrm{o}(1)$ for large values of ς, and hence to evaluate (25) it is enough to estimate

$$\begin{aligned}
\varsigma^{1/2}\eta^{-\eta\varsigma}\mathrm{e}^{\varsigma\eta} & \int_0^{1-\varepsilon} \frac{\exp(\varsigma(\eta \log y - y))}{y^2(1-y)} \mathrm{d}y \\
&= \varsigma^{1/2}\eta^{-\eta\varsigma} \exp(\varsigma\eta) \int_0^{1-\varepsilon} \frac{y^{-2}\exp(\varsigma(\eta\log y - y))}{(1-y)} \mathrm{d}y \\
&\leq \varsigma^{1/2}\eta^{-\eta\varsigma} \exp(\varsigma\eta) \exp\left(\max_{t\in[0,1-\varepsilon]} g_1(t)\right) \int_0^{1-\varepsilon} \frac{1}{1-y} \mathrm{d}y \\
&= \varsigma^{1/2}\eta^{-\eta\varsigma} \exp(\varsigma\eta) \exp\left(\max_{t\in[0,1-\varepsilon]} g_1(t)\right) \log \varepsilon^{-1},
\end{aligned} \tag{26}$$

where $g_1(t) := (\varsigma\eta - 2)\log t - \varsigma t$. For $\varsigma > 2/(\eta-1)$ and $t \leq 1$, the function g_1 satisfies $g_1'(t) = (\varsigma\eta-2)/t - \varsigma \geq (\varsigma\eta - 2) - \varsigma = \varsigma(\eta-1) - 2 > 0$, and hence $g_1(t) \leq g_1(1-\varepsilon) = -\varsigma(1-\varepsilon-\eta\log(1-\varepsilon)) - 2\log(1-\varepsilon)$. Plugging this result back in (26) we have

$$\begin{aligned}
\varsigma^{1/2}\eta^{-\eta\varsigma}\mathrm{e}^{\varsigma\eta} \; & \exp\left(\max_{t\in[0,1-\varepsilon]} g_1(t)\right) \log \varepsilon^{-1} \\
&= \frac{\varsigma^{1/2}\log\varepsilon^{-1}}{(1-\varepsilon)^2} \exp\left(-\varsigma\big(\eta\log\eta - \eta + (1-\varepsilon) - \eta\log(1-\varepsilon)\big)\right),
\end{aligned}$$

and so it is enough to check that we have $\eta\log\eta - \eta + (1-\varepsilon) - \eta\log(1-\varepsilon) > 0$ for $\varsigma > 2/(\eta-1)$ and $\eta > 1$. But

$$\begin{aligned}
\eta\log\eta - \eta + (1-\varepsilon) - \eta\log(1-\varepsilon) > 0 &\Leftrightarrow (1-\varepsilon) - \eta > \eta \log\frac{1-\varepsilon}{\eta} \\
&\Leftrightarrow \frac{1-\varepsilon}{\eta} - 1 > \log\frac{1-\varepsilon}{\eta},
\end{aligned}$$

and, letting $z = (1-\varepsilon)/\eta$, this last inequality amounts to $z > \log z + 1$ which holds for all $z \neq 1$, and that is the case since $\eta > 1 \Rightarrow \eta \neq 1-\varepsilon$. This concludes the proof in the interval $(0, 1-\varepsilon)$.

We now show that the integral over $(1-\varepsilon, 1)$ also goes to 0 as $\varsigma \to +\infty$. Since $\tilde{c}_1$ is continuous and goes to 0 as $\varsigma \to +\infty$ it is bounded in $[1-\varepsilon, 1]$, and so there is a constant $M > 0$ such that $\tilde{c}_1(\varsigma(1-y)) < M, \forall y \in [1-\varepsilon, 1]$. Now we have to estimate

$$\begin{aligned}
&\eta^{-\eta\varsigma}\varsigma^{3/2}(\log\varsigma)^{2/3}\int_{1-\varepsilon}^{1}\frac{\exp\left(\varsigma(\eta\log y-y+\eta)\right)}{y^2}\mathrm{d}y\\
&\quad=\varsigma^{3/2}(\log\varsigma)^{2/3}\int_{1-\varepsilon}^{1}\frac{\exp\left(-\varsigma(\eta\log\eta-\eta\log y+y-\eta)\right)}{y^2}\mathrm{d}y\\
&\quad=\varsigma^{3/2}(\log\varsigma)^{2/3}\int_{1-\varepsilon}^{1}\frac{\exp(-\varsigma h(y))}{y^2}\mathrm{d}y\\
&\quad<\varsigma^{3/2}(\log\varsigma)^{2/3}\exp\left(-\varsigma\min_{t\in[1-\varepsilon,1]}h(t)\right)\int_{1-\varepsilon}^{1}\frac{1}{y^2}\mathrm{d}y\\
&\quad=\varsigma^{3/2}(\log\varsigma)^{2/3}\frac{\varepsilon}{1-\varepsilon}\exp\left(-\varsigma h(1)\right),
\end{aligned}\tag{27}$$

where $h(t):=\eta\log\eta-\eta\log t+t-\eta$ has a unique minimum at $t=1$, and since $h(1)=\eta\log\eta+1-\eta$, and $\eta\log\eta+1-\eta>0$ for $\eta\neq 1$ the expression in (27) is exponentially small as $\varsigma\to+\infty$. This concludes the proof for $\eta>1$.

For $\eta<1$ we use a similar approach, but the situation being slightly more delicate, since now the (unique) maximum of $\eta\log y-y$ is attained at an interior point ($1>\eta\in(0,1)$), we need to split the integral by writing it as a sum of integrals over $(0,\varepsilon)$, $(\varepsilon,1-\varepsilon)$ and $(1-\varepsilon,1)$. With g and ψ defined as above, for every $\varepsilon>0$ we split the integral over $[0,1]$ as the sum of three integrals: I_1 over $(0,\varepsilon)$, I_2 over $(\varepsilon,1-\varepsilon)$ and I_3 over $(1-\varepsilon,1)$. We will show that both I_1 and I_3 go to zero, and that the only non zero contribution comes from the integral over $(\varepsilon,1-\varepsilon)$. Given $\eta<1$, we choose $\varepsilon>0$ in such a way that $\eta\in(\varepsilon,1-\varepsilon)$, for instance $\varepsilon<\min\{\eta/a,1-\eta\}$, with $a>1$.

For the integral over I_1, we now have that both $\psi(\varsigma(1-y))$ and $\left(1+\frac{\log(1-y)}{\log\varsigma}\right)^{-1}$ are $1+\mathrm{o}(1)$ when estimating the integral for large values of ς; and hence to evaluate the integral over I_1 we can use an argument similar to the one we already used in the $\eta>1$ case. To evaluate I_1 it is then enough to estimate, as $\varsigma\to+\infty$, the value of

$$\varsigma^{1/2}\eta^{-\eta\varsigma}\mathrm{e}^{\varsigma\eta}\int_0^{\varepsilon}\frac{\exp(\varsigma(\eta\log y-y))}{y^2(1-y)}\mathrm{d}y.$$

As in $(0,1-\varepsilon)$, using $g_1(t)=(\varsigma\eta-2)\log t-\varsigma t$, we have $0<t<\varepsilon<\eta/a<\eta$ and hence for $\varsigma>2/(1-1/a)\eta$, we can conclude that g_1' satisfies $tg_1'(t)=\varsigma(\eta-t)-2>0$, since $\varsigma>2/(\eta-\eta/a)>2/(\eta-t)$ and hence $g_1(t)\le g_1(\varepsilon)=-\varsigma(\varepsilon-\eta\log\varepsilon)-2\log\varepsilon$. We then have the following estimates

$$\begin{aligned}
\varsigma^{1/2}\eta^{-\eta\varsigma}\mathrm{e}^{\varsigma\eta}\int_0^{\varepsilon}\frac{\exp(\varsigma(\eta\log y-y))}{y^2(1-y)}\mathrm{d}y&=\varsigma^{1/2}\eta^{-\eta\varsigma}\mathrm{e}^{\varsigma\eta}\int_0^{\varepsilon}\frac{\exp(g_1(y))}{1-y}\mathrm{d}y\\
&\le\varsigma^{1/2}\eta^{-\eta\varsigma}\mathrm{e}^{\varsigma\eta}\exp\big(\max_{t\in[0,\varepsilon]}g_1(t)\big)\int_0^{\varepsilon}\frac{1}{1-y}\mathrm{d}y\\
&=\varsigma^{1/2}\eta^{-\eta\varsigma}\exp(\varsigma\eta+g_1(\varepsilon)\log(1-\varepsilon)^{-1}\\
&=\varsigma^{1/2}\varepsilon^{-2}\log(1-\varepsilon)^{-1}\exp(-\varsigma(\eta\log\eta-\eta+\varepsilon-\eta\log\varepsilon)).
\end{aligned}$$

And so we only need to check that $\eta\log\eta-\eta+\varepsilon-\eta\log\varepsilon>0$, which is true since this last expression is always positive, except for $\eta=\varepsilon$ where it is zero, and we chose $\varepsilon<\eta$. Hence $I_1\to 0$, as $\varsigma\to+\infty$.

For I_3, the integral over $[1-\varepsilon,1]$, we have $0\le\varsigma(1-y)\le\varsigma\varepsilon$, and we use equation (23), which involves $\tilde{c}_1$, and we have to evaluate, as $\varsigma\to+\infty$,

$$\eta^{-\eta\varsigma}\varsigma^{3/2}(\log\varsigma)^{2/3}\int_{1-\varepsilon}^{1}\frac{\exp(\varsigma(\eta\log y-y+\eta))}{y^2}\mathrm{d}y.$$

This can be done as before, by showing that the function $h(y):=\eta\log\eta-\eta-\eta\log y+y$ is always positive for $y\in[1-\varepsilon,1]$, remembering that we picked $\varepsilon<1-\eta$, and hence $y>1-\varepsilon>\eta$, when evaluating I_3. And so recalling that $h(y)\ge 0$, and $h(y)>0$ for $y\neq\eta$, we conclude that I_3 is also exponentially small as $\varsigma\to+\infty$.

For the integral I_2, we use again the fact that for $y\in(\varepsilon,1-\varepsilon)$, we have $\left(1+\frac{\log(1-y)}{\log\varsigma}\right)^{-2/3}\to 1$ as $\varsigma\to+\infty$, and so we rewrite (23) as

$$\begin{aligned}
\sqrt{2\pi}\,\varphi_1(\varsigma\eta,\varsigma) &= \left(2\alpha^2/9\right)^{-1/3}\eta^{3/2-\eta\varsigma}\varsigma^{1/2}\varsigma(\log\varsigma)^{2/3}\times\\
&\times\left(1+\mathrm{O}\left(\varsigma^{-1}\right)\right)\int_{\varepsilon}^{1-\varepsilon}\tilde{c}_1(\varsigma(1-y))\frac{\exp(\varsigma(\eta\log y-y+\eta))}{y^2}\mathrm{d}y\\
&=\eta^{3/2-\eta\varsigma}\varsigma^{1/2}\times\\
&\times\left(1+\mathrm{O}\left(\varsigma^{-1}\right)\right)\int_{\varepsilon}^{1-\varepsilon}\psi(\varsigma(1-y))\frac{(\log\varsigma)^{2/3}\exp(\varsigma(\eta\log y-y+\eta))}{y^2(1-y)(\log\varsigma(1-y))^{2/3}}\mathrm{d}y\\
&=\eta^{3/2-\eta\varsigma}\varsigma^{1/2}\left(1+\mathrm{O}\left(\varsigma^{-1}\right)\right)\times\\
&\times\int_{\varepsilon}^{1-\varepsilon}\psi(\varsigma(1-y))\left(\frac{\log\varsigma}{\log\varsigma(1-y)}\right)^{2/3}\frac{\exp(\varsigma(\eta\log y-y+\eta))}{y^2(1-y)}\mathrm{d}y.
\end{aligned}$$

Since in this case $\psi(\varsigma(1-y))$ and $\left(\frac{\log\varsigma}{\log\varsigma(1-y)}\right)^{2/3}$ are $1+\mathrm{o}(1)$ when $\varsigma\to\infty$, it holds that

$$\forall\delta>0,\exists T(\delta)\colon\forall\varsigma>T(\delta),\psi(\varsigma(1-y))\left(\frac{\log\varsigma}{\log\varsigma(1-y)}\right)^{2/3}\in[1-\delta,1+\delta].$$

It is then enough to study the limit, as $\varsigma\to+\infty$, of the function

$$J(\eta,\varsigma):=\eta^{3/2-\eta\varsigma}\varsigma^{1/2}\int_{\varepsilon}^{1-\varepsilon}\frac{\exp(\varsigma(\eta\log y-y+\eta))}{y^2(1-y)}\mathrm{d}y,$$

since we can write $(1-\delta)J(\eta,\varsigma)\le I_2\le(1+\delta)J(\eta,\varsigma)$, for ς sufficiently large.

Applying Laplace's method for the asymptotic evaluation of integrals [1, pg 431] to the integral

$$\int_{\varepsilon}^{1-\varepsilon}\frac{\exp(-\varsigma(y-\eta\log y-\eta))}{y^2(1-y)}\mathrm{d}y=\int_{\varepsilon}^{1-\varepsilon}\frac{\exp(-\varsigma\phi(y))}{y^2(1-y)}\mathrm{d}y,$$

where $\phi : (0,1) \to \mathbb{R}$ defined by $\phi(y) := y - \eta \log y - \eta$, is smooth and has a unique minimum, attained at $y = \eta \in (\varepsilon, 1-\varepsilon)$ with value $\phi(\eta) = -\eta \log \eta$ and $\phi''(\eta) = \eta^{-1}$, we obtain, as $\varsigma \to +\infty$,

$$\int_{\varepsilon}^{1-\varepsilon} \frac{\exp(-\varsigma(y - \eta \log y - \eta))}{y^2(1-y)} \mathrm{d}y = \\ = \frac{\sqrt{2\pi\eta/\varsigma}\exp(\varsigma\eta\log\eta)}{\eta^2(1-\eta)} + \mathrm{O}\Big(\frac{\exp(\varsigma\eta\log\eta)}{\varsigma^{3/2}}\Big). \tag{28}$$

Now from (23) and (28), we obtain for $\eta < 1$, as $\varsigma \to +\infty$

$$\begin{aligned} \varphi_1(\varsigma\eta, \varsigma) &= \frac{1}{\sqrt{2\pi}} \eta^{3/2 - \eta\varsigma} \varsigma^{1/2} \exp(\varsigma\eta\log\eta) \frac{1}{\eta^2(1-\eta)} \sqrt{2\pi\eta/\varsigma} + \mathrm{O}\left(\varsigma^{-1}\right) \\ &= \frac{1}{1-\eta}\left(1 + \mathrm{o}(1)\right). \end{aligned}$$

This concludes the proof in the monomeric case.

Non monomeric initial data

If the initial condition is not monomeric we have the contribution from the sum term in the right hand side of (11). Multiplying it by $g(\varsigma)$ we now have to prove that

$$\lim_{j,\varsigma \to +\infty} g(\varsigma) \mathrm{e}^{-\varsigma} \sum_{k=2}^{j} \frac{\varsigma^{j-k}}{(j-k)!} c_k(0) = 0, \eta = j/\varsigma \text{ fixed, and } \eta \neq 1.$$

Since we want to show the limit is zero, we will drop the constants in the definition of g, and so we only consider the terms $\varsigma(\log\varsigma)^{2/3}$. The proof is based on the same argument used in [5, Section 5.2]. Defining $\nu := \eta^{-1}$, letting $\varsigma = j\nu$, and using the assumption on the initial condition in Theorem 2, namely $c_j(0) \leq \rho/j^{\mu}$, we then have

$$\begin{aligned} \varsigma(\log\varsigma)^{2/3} \mathrm{e}^{-\varsigma} \sum_{k=2}^{j} \frac{\varsigma^{j-k}}{(j-k)!} c_k(0) \;&\leq\; \rho j\nu(\log j\nu)^{2/3} \exp(-j\nu) \sum_{k=2}^{j} \frac{(j\nu)^{j-k}}{(j-k)! k^{\mu}} \\ &:= \rho \varphi_2(\nu, j). \end{aligned}$$

Our goal is to prove that $\varphi_2(\nu, j) \to 0$ as $j \to \infty$, for all positive $\nu \neq 1$. We can adapt the results in the study of φ_2 presented in [5, Section 5.2], noticing that we only need to multiply all the estimates in [5, Section 5.2] by $\sqrt{j\nu}(\log j\nu)^{2/3}$. The estimates show that now in order for φ_2 to converge to zero we need to consider initial data satisfying $c_j(0) \leq \rho/j^{\mu}$, but in this case with $\mu > 1$. The $\log j$ term growing much slower than $\sqrt{j}$ has no influence on the convergence of φ_2 to zero. This completes the proof of the theorem.

On the self-similar behaviour along the characteristic direction

In the case with input αt^{ω} with $\omega > -1/2$, we have seen [4] that for values of $\omega < 1$ the singularity of the self-similar solution $\Phi_{1,\omega}$ can be dealt with by considering a different similarity variable and a different time-scaling, allowing us a sort of inner expansion for the characteristic direction $\eta = 1$, and we obtained a function $\Phi_{2,\omega}$ satisfying

$$\tilde{c}_j(\varsigma) \sim \varsigma^{(\omega-1)/(2\omega+4)} \Phi_{2,\omega}\left((j-\varsigma)/\sqrt{\varsigma}\right).$$

It is worth noticing that for $\omega > -1/2$ the similarity variable was independent of ω, and the exponent of the time scaling variable, although $\omega-$dependent was always *half* the exponent used for $\Phi_{1,\omega}$. Now we also have a singularity at $\eta = 1$ and so it is natural to check if this similarity variable also gives rise to a solution, and if that is the case, one for which $\eta = 1$ is no longer a singularity.

Choosing the similarity variable $(j-\varsigma)/\sqrt{\varsigma}$ and replacing ς by $\varsigma^{1/2}$ in the expression in the limit in Theorem 2

$$\varsigma^{1/2}\left(\log(\varsigma)^{1/2}\right)^{2/3} = (1/2)^{2/3}\varsigma^{1/2}(\log\varsigma)^{2/3},$$

and letting $\Phi_{2,-1/2} : \mathbb{R} \to \mathbb{R}$ be defined by

$$\Phi_{2,-1/2}(\xi) := \mathrm{e}^{-\xi^2/2}\int_0^{+\infty} y^{-1}\mathrm{e}^{-\xi y^2 - y^4/2}\mathrm{d}y,$$

we hope it is equal to the limit for $\xi = (j-\varsigma)/\sqrt{\varsigma}$ fixed and $\xi \in \mathbb{R}$, of

$$\lim_{j,\varsigma\to+\infty} C_\alpha \varsigma^{1/2}(\log\varsigma)^{2/3}\tilde{c}_j(\varsigma), \tag{29}$$

where $C_\alpha > 0$ is a constant that only depends on α. We now show that this limit does not exist.

Following a strategy similar to the one we used in [5] for $\omega > -1/2$, for monomeric initial data we have to estimate

$$(\log\varsigma)^{2/3}\varsigma^{1/2}\tilde{c}_j(\varsigma) = \frac{(\log\varsigma)^{2/3}\varsigma^{1/2}}{\Gamma(j-1)}\int_0^{\varsigma}\tilde{c}_1(\varsigma-s)s^{j-2}\mathrm{e}^{-s}\mathrm{d}s,$$

as $\varsigma \to +\infty, j \to +\infty, (j-\varsigma)/\sqrt{\varsigma}$ fixed.

We define the function φ_3 in $[2,\infty)\times[0,\infty)$ by

$$\varphi_3(x,\varsigma) := \frac{(\log\varsigma)^{2/3}\varsigma^{1/2}}{\Gamma(x-1)}\int_0^{\varsigma}\tilde{c}_1(\varsigma-s)s^{j-2}\mathrm{e}^{-s}\mathrm{d}s,$$

and using the similarity variable $\xi = (j-\varsigma)/\sqrt{\varsigma}(= (x-\varsigma)/\sqrt{\varsigma})$ we rewrite φ_3 as

$$\varphi_3(\varsigma+\xi\sqrt{\varsigma},\varsigma) = \frac{(\log\varsigma)^{2/3}\varsigma^{1/2}}{\Gamma(\varsigma+\xi\sqrt{\varsigma}-1)}\int_0^{\varsigma}\tilde{c}_1(\varsigma-s)s^{\varsigma+\xi\sqrt{\varsigma}-2}\mathrm{e}^{-s}\mathrm{d}s.$$

If $2 \leq x = j \in \mathbb{N}$, we have $\varphi_3(j,\varsigma) = (\log\varsigma)^{2/3}\,\varsigma^{1/2}\tilde{c}_j(\varsigma)$, and hence we need to evaluate the limit

$$\lim_{\varsigma\to+\infty} \varphi_3(\varsigma+\xi\sqrt{\varsigma},\varsigma). \tag{30}$$

Changing variables $s \mapsto w := \sqrt{\sqrt{\varsigma} - s/\sqrt{\varsigma}}$, in such a way that $\varsigma - s = \sqrt{\varsigma}w^2$ and $\mathrm{d}s = -2\sqrt{\varsigma}w\mathrm{d}w$, we obtain

$$\begin{aligned}\varphi_3(\varsigma+\xi\sqrt{\varsigma},\varsigma) &= \frac{(\log\varsigma)^{2/3}\,\varsigma^{1/2}}{\Gamma(\varsigma+\xi\sqrt{\varsigma}-1)}\times \\ &\times\int_0^{\varsigma^{1/4}} \tilde{c}_1(\sqrt{\varsigma}w^2)(\varsigma-\sqrt{\varsigma}w^2)^{\varsigma+\xi\sqrt{\varsigma}-2}\exp(-\varsigma+\sqrt{\varsigma}w^2)2\sqrt{\varsigma}w\mathrm{d}w. \end{aligned}\tag{31}$$

Using (17), and letting $D = 2(\alpha/3)^{2/3}$, we rewrite (31) as

$$\begin{aligned}\varphi_3(\varsigma+\xi\sqrt{\varsigma},\varsigma) &= D\frac{(\log\varsigma)^{2/3}\,\varsigma^{1/2}}{\Gamma(\varsigma+\xi\sqrt{\varsigma}-1)}\int_0^{\varsigma^{1/4}}\left(\log(\sqrt{\varsigma}w^2)\right)^{-2/3}(\sqrt{\varsigma}w^2)^{-1}\times \\ &\qquad\times\psi(\sqrt{\varsigma}w^2)(\varsigma-\sqrt{\varsigma}w^2)^{\varsigma+\xi\sqrt{\varsigma}-2}\exp(-\varsigma+\sqrt{\varsigma}w^2)\sqrt{\varsigma}w\mathrm{d}w \\ &= D\frac{\varsigma^{\varsigma+\xi\sqrt{\varsigma}-3/2}\mathrm{e}^{-\varsigma}}{\Gamma(\varsigma+\xi\sqrt{\varsigma}-1)}\int_0^{\varsigma^{1/4}}\left(1+\frac{4\log w}{\log\varsigma}\right)^{-2/3}\psi(\sqrt{\varsigma}w^2)\times \\ &\qquad\times\left(1-\frac{w^2}{\sqrt{\varsigma}}\right)^{\varsigma+\xi\sqrt{\varsigma}-2}\exp(\sqrt{\varsigma}w^2)\frac{1}{w}\mathrm{d}w.\end{aligned}$$

Using Stirling's asymptotic expansion for the Gamma function we can write

$$\frac{1}{\Gamma(\varsigma+\xi\sqrt{\varsigma}-1)} = \frac{1}{\sqrt{2\pi}}\frac{\exp(\varsigma+\xi\sqrt{\varsigma})}{(\varsigma+\xi\sqrt{\varsigma})^{\varsigma+\xi\sqrt{\varsigma}-3/2}}(1+\mathrm{o}(1)),$$

as $\varsigma\to+\infty$, and hence φ_3 can be written, as $\varsigma\to+\infty$,

$$\begin{aligned}\varphi_3(\varsigma+\xi\sqrt{\varsigma},\varsigma) &= \frac{D}{\sqrt{2\pi}}\frac{\varsigma^{\varsigma+\xi\sqrt{\varsigma}-3/2}\exp(\xi\sqrt{\varsigma})}{(\varsigma+\xi\sqrt{\varsigma})^{\varsigma+\xi\sqrt{\varsigma}-3/2}}(1+\mathrm{o}(1))\times \\ &\times\int_0^{\varsigma^{1/4}}\left(1+\frac{4\log w}{\log\varsigma}\right)^{-2/3}\psi(\sqrt{\varsigma}w^2)\left(1-\frac{w^2}{\sqrt{\varsigma}}\right)^{\varsigma+\xi\sqrt{\varsigma}-2}\exp(\sqrt{\varsigma}w^2)\frac{1}{w}\mathrm{d}w.\end{aligned}\tag{32}$$

To estimate the multiplicative prefactor in (32) as $\varsigma\to+\infty$ we can write it as

$$\frac{\varsigma^{\varsigma+\xi\sqrt{\varsigma}-3/2}\exp(\xi\sqrt{\varsigma})}{(\varsigma+\xi\sqrt{\varsigma})^{\varsigma+\xi\sqrt{\varsigma}-3/2}} = \exp\left(-\frac{\xi^2}{2}\right)(1+\mathrm{o}(1)), \tag{33}$$

where in (33) to compute the limit as $\varsigma\to+\infty$ we use the change of variables $\varsigma\mapsto x := \xi/\sqrt{\varsigma}$ to obtain $\left(\mathrm{e}(1+x)^{-1/x}\right)^{\xi^2/x}$ and then we apply L'Hôpital's rule twice. Using this last expression we can write (32) as $\varsigma\to+\infty$ in the following way

$$\varphi_3(\varsigma+\xi\sqrt{\varsigma},\varsigma)=\frac{D}{\sqrt{2\pi}}\exp(-\xi^2/2)(1+\mathrm{o}(1))\times$$
$$\times\int_0^{\varsigma^{1/4}}\left(1+\frac{4\log w}{\log\varsigma}\right)^{-2/3}\psi(\sqrt{\varsigma}w^2)\left(1-\frac{w^2}{\sqrt{\varsigma}}\right)^{\varsigma+\xi\sqrt{\varsigma}-2}\exp(\sqrt{\varsigma}w^2)\frac{1}{w}\mathrm{d}w. \tag{34}$$

In the case where $\omega>-1/2$ in [5] the proof continued with a study of the integral term in (34) by considering first w (and also $\sqrt{\varsigma}w^2$) close to zero, and then w away from zero, and showing that the integral, for small values of w, could be made arbitrarily small, while the remaining integral, for w away from zero converged as $\varsigma\to+\infty$. In the case at hand we no longer have convergence, essentially due to the singularity arising from $1/w$ in the integrand of (34). We now consider $\varepsilon>0$ arbitrarily small and $1<T<\varsigma^{1/4}$ and we show that the integral over $[\varepsilon,T]$ can be made arbitrarily large. We start by splitting the integral over $[0,\varsigma^{1/4}]$ in (34) as a sum over 3 intervals $[0,\varepsilon],[\varepsilon,T]$ and $[T,\varsigma^{1/4}]$. The integrals over $[0,\varepsilon]$ and $[T,\varsigma^{1/4}]$ are both non negative and for $w\in[\varepsilon,T]$ we have $\sqrt{\varsigma}w^2\geq\sqrt{\varsigma}\varepsilon^2\to+\infty$ as $\varsigma\to+\infty$, and so as by (14), (15), and (17), it follows that $\left(1+\frac{4\log w}{\log\varsigma}\right)^{-2/3}\psi(\sqrt{\varsigma}w^2)=1+\mathrm{o}(1)$ as $\varsigma\to+\infty$.

This means that for $w\in[\varepsilon,T]$ the integral we want to evaluate is asymptotically equal to

$$(1+\mathrm{o}(1))\int_\varepsilon^T\left(1-\frac{w^2}{\sqrt{\varsigma}}\right)^{\varsigma+\xi\sqrt{\varsigma}-2}\exp(\sqrt{\varsigma}w^2)\frac{1}{w}\mathrm{d}w.$$

To estimate this last integral we have as $\varsigma\to+\infty$

$$\left(1-\frac{w^2}{\sqrt{\varsigma}}\right)^{\varsigma+\xi\sqrt{\varsigma}-2}\exp(\sqrt{\varsigma}w^2)=\exp\Big(-\frac{w^4}{2}-\xi w^2\Big)\big(1+\mathrm{o}(1)\big), \tag{35}$$

where (35) is obtained by first changing variables $\varsigma\mapsto x=1/\sqrt{\varsigma}$ and then applying L'Hôpital's rule. From (35) we conclude that there exists a continuous function $g(w,\varsigma)$ defined for $\varsigma^{1/4}>\varepsilon$ and $w\in[\varepsilon,T]$ and satisfying $1+g(w,\varsigma)\geq 0$ and $g(w,\varsigma)\to 0$ as $\varsigma\to+\infty$ for each fixed w, such that

$$\left(1-\frac{w^2}{\sqrt{\varsigma}}\right)^{\varsigma+\xi\sqrt{\varsigma}-2}\exp(\sqrt{\varsigma}w^2)=\exp(-w^4/2-\xi w^2)\big(1+g(w,\varsigma)\big). \tag{36}$$

We now estimate the integral

$$\int_\varepsilon^T\exp(-w^4/2-\xi w^2)\big(1+g(w,\varsigma)\big)\frac{1}{w}\mathrm{d}w. \tag{37}$$

Using (36)

$$1+g(w,\varsigma)=\left(1-\frac{w^2}{\sqrt{\varsigma}}\right)^{\varsigma+\xi\sqrt{\varsigma}-2}\exp(\sqrt{\varsigma}w^2+w^4/2+\xi w^2),$$

which implies, as $\varsigma \to +\infty$, $\left(1-\frac{w^2}{\sqrt{\varsigma}}\right)^{\xi\sqrt{\varsigma}} \geq \frac{1}{2}\exp(-w^2\xi)$ and

$$\left(\mathrm{e}^{w^2}\left(1-\frac{w^2}{\sqrt{\varsigma}}\right)^{\sqrt{\varsigma}}\right)^{\sqrt{\varsigma}} \geq \frac{1}{2}\exp(-w^4/2).$$

Since $\left(1-\frac{T^2}{\sqrt{\varsigma}}\right)^{-2} > 1$ we have

$$1+g(w,\varsigma) = \left(1-\frac{w^2}{\sqrt{\varsigma}}\right)^{\varsigma+\xi\sqrt{\varsigma}-2}\exp(\sqrt{\varsigma}w^2+(w^4/2)+\xi w^2) \geq \frac{1}{4}\exp(-w^4/2),$$

and hence the integral in (37) can be estimated as

$$\int_{\varepsilon}^{T}\exp(-w^4/2-\xi w^2)\big(1+g(w,\varsigma)\big)\frac{1}{w}\mathrm{d}w \geq \frac{1}{4}\int_{\varepsilon}^{T}\exp(-w^4-\xi w^2)\frac{1}{w}\mathrm{d}w$$
$$> L_1\int_{\varepsilon}^{T}\frac{1}{w}\mathrm{d}w, \qquad (38)$$

where $L_1 = L_1(\xi,T) := \exp(-T^4-\xi T^2)/4$. The integral in (38) can be made arbitrarily large, since we can choose $\varepsilon > 0$ suitably small, and hence since the integral in (34) is (strictly) larger than the integral in (37), this concludes the proof that the limit in (30) [and hence in (29)] does not exist.

5 Concluding remarks

We studied the addition model with input $J_1 = \alpha t^{-1/2}$ and showed the existence of self-similar behaviour along non-characteristic directions.

Along the characteristic direction we considered a different similarity variable, $\xi = (j-\varsigma)/\sqrt{\varsigma}$. This new "layer" variable of width $\sqrt{\varsigma}$ around $j=\varsigma$ provides a kind of expansion of the singularity of the scaling transformation $\Phi_{1,-1/2}$. For this similarity variable, we concluded that $\Phi_{2,-1/2}$ does not scale like $\varsigma^{1/2}(\log\varsigma)^{2/3}$. Whether there is some similarity variable and some constants a and b such that the similarity function scales like $\varsigma^{a}(\log\varsigma)^{b}$ remains an open question.

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