The Redner–Ben-Avraham–Kahng coagulation system with constant coefficients: the finite dimensional case

F.P. da Costa, J.T. Pinto and R. Sasportes

Abstract. We study the behaviour as $t \to \infty$ of solutions $(c_j(t))$ to the Redner–Ben-Avraham–Kahng coagulation system with positive and compactly supported initial data, rigorously proving and slightly extending results originally established in [4] by means of formal arguments.

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1. Introduction

In a recent paper [2] we started the study of a coagulation model first considered in [3, 4] which we have called the Redner–Ben-Avraham–Kahng cluster system (RBK for short). This is the infinite-dimensional ODE system

$$\frac{dc_j}{dt} = \sum_{k=1}^{\infty} a_{j+k,k}c_{j+k}c_k - \sum_{k=1}^{\infty} a_{j,k}c_jc_k, \qquad j = 1, 2, \dots . \tag{1.1}$$

with symmetric positive coagulation coefficients $a_{j,k}$. As with the discrete Smoluchowski's coagulation system [1] this is a mean-field model describing the evolution of a system given at each instant by a sequence (c_j), such that c_j is the density of j-clusters for each integer j, undergoing a binary reaction described by a bilinear infinite-dimensional vector field. However, while in the Smoluchowski's coagulation model one k-cluster reacts with one j-cluster

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producing one $(j+k)$-cluster, in RBK the interaction between such clusters produce one $|k-j|$-cluster.

If we assume that there is no destruction of mass, in the former model it makes sense to think of j as the size, or mass, of each j-cluster. However in RBK the situation is different since with the same interpretation there would be a loss of mass in each reaction. Hence, it makes more sense to think of j as the size of the cluster 'active part', being the difference between $(j+k)$ and $|j-k|$ the size of the resulting cluster that becomes inactive for the reaction process. A pictorial illustration of this is presented in Figure 1.

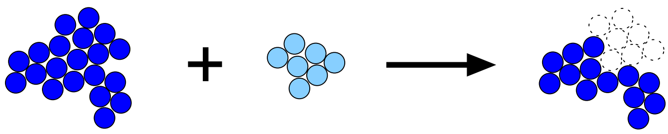

FIGURE 1. Schematic reaction in the RBK coagulation model

For more on the physical interpretation of (1.1) see [2, 3, 4].

The nonexistence of a mass conservation property in RBK model makes for one of the major differences with respect to the Smoluchowski's model. Also, unlike in this one, in RBK a j and a k-cluster react to produce a j'-cluster with $j' < \max\{j,k\}$, implying that to an initial condition with an upper bound N for the subscript values j for which $c_j(0) > 0$ there corresponds a solution with the same property for all instants $t \geqslant 0$. This is an invariance property rigorously stated on Proposition 7.1 in [2]. In this work we will consider such solutions for a finite prescribed upper bound $N \geqslant 3$ and j-independent coagulation coefficients $a_{j,k} = 1$, for all j,k. Then, if $c_j(0) = 0$, for all $j \geqslant N+1$, then $c_j(t) = 0$ for $t \geqslant 0$ and for the same values of j, while $(c_1(t), c_2(t), \ldots, c_N(t))$ satisfy the following N-dimensional ODE

$$\frac{dc_j}{dt} = \sum_{k=1}^{N-j} c_{j+k}c_k - c_j \sum_{k=1}^{N} c_k, \qquad j \in \mathbb{N} \cap [1,N], \tag{1.2}$$

where the first sum in the right-hand side is defined to be zero when $j = N$.

In this work we study system (1.2) for nonnegative initial conditions at $t = 0$, from the point of view of the asymptotic behaviour of each component, $c_j(t)$, $j = 1, \ldots, N$, as $t \to \infty$. This problem has already been addressed in [4], where the authors have used a formal approach. In Theorem 2.1, we obtain the result for the general case $c_j(0) \geqslant 0$, for $j = 1, 2, \ldots, N$, proving rigorously that the result in [4] is correct for initial conditions such that $c_N(0) > 0$ and the greater common divisor of the subscript values j for which $c_j(0) > 0$ is 1.

2. The main result

Consider $N \geqslant 3$. We are concerned with nonnegative solutions of (1.2). By applying the results we have proved in [2] in the more general context refered above, we can deduce that, for a solution $c = (c_j)$ to (1.2), if $c_j(0) \geqslant 0$, for $j = 1, \dots, N$, then it is defined for all $t \in [0, \infty)$ and $c_j(t) \geqslant 0$, for $j = 1, \dots, N$, and all positive t. Let $P = \{j \in \mathbb{N} \cap [1, N] \,|\, c_j(0) > 0\}$ be the set of subscript values for which the components of the initial condition $c(0)$ are positive, and let $\gcd(P)$ be the greatest common divisor of the elements of P. In this paper we prove the following:

Theorem 2.1. *Let $c = (c_j)$ be a solution of (1.2) satisfying $c_j(0) \geqslant 0$ for all $j = 1, \dots, N$. If $m := \gcd(P)$ and $p := \sup P$, then, for each $j = m, 2m, \dots, p$, there exists $e_j : [0, \infty) \to \mathbb{R}$ such that $e_j(t) \to 0$ as $t \to \infty$, and*

$$c_j(t) = \frac{\widetilde{A}_j}{t(\log t)^{j/m-1}}(1 + e_j(t))$$

where

$$\widetilde{A}_j := \frac{(N-1)!}{(N-j/m)!}.$$

For all other $j \in \mathbb{N} \cap [1, N]$, $c_j(t) = 0$, for all $t \geqslant 0$.

We begin the proof of this theorem by reducing it to the case $m = 1$, $p = N$. Consider, for each $t \geqslant 0$, $\mathcal{J}(t) := \{j \in \mathbb{N} \cap [1, N] \,|\, c_j(t) > 0\}$, the set of subscript values for which the components of the solution are positive at instant t. Obviously, $P = \mathcal{J}(0)$. The case $\#P = 1$ is an immediate consequence of Proposition 7.3 in [2] and its proof. Consider now the case $\#P > 1$. Then, according to Proposition 7.2 in [2], $\mathcal{J}(t) = m\mathbb{N} \cap [1, p]$, for all $t > 0$. Let $\tilde{N} := p/m$ and, for $j = 1, 2, \dots, \tilde{N}$, let us write $\tilde{c}_j := c_{jm}$. Then it is straightforward to check that (1.2) is again satisfied with N and c_j, for $j = 1, 2, \dots, N$, replaced by $\tilde{N}$ and $\tilde{c}_j$, for $j = 1, 2, \dots, \tilde{N}$, respectively. From the definition of $\mathcal{J}(t)$, we also have that, for $j = 1, \dots, \tilde{N}$ and for all $t > 0$, $\tilde{c}_j(t) > 0$. For $j = 1, \dots, N$, if $j \notin m\mathbb{N} \cap [1, p]$, then $c_j(t) = 0$, for all $t \geqslant 0$. Hence, after having established the validity of Theorem 2.1 with the restrictions $m = 1$ and $p = N$, if we consider a solution $c(\cdot)$ with initial conditions for which $m > 1$, $p < N$ or both, we can apply that restricted version of the theorem to $\tilde{c}$ and then use the fact that, for $j = 1, \dots, p$, $c_j(t) = \tilde{c}_{j/m}(t)$. For the other subscript values, $c_j(t)$ identically vanishes.

In conclusion, it is sufficient to prove the above theorem for $m = 1$, $p = N$, in which case, as we have seen, $c_j(t) > 0$, for $j = 1, 2, \dots, N$, and all $t > 0$. This is done in next section.

3. Long time behaviour of strictly positive solutions

Consider a solution $c(\cdot) = (c_j(\cdot))$ to (1.2) such that $c_j(t) > 0$ for all $j = 1, \dots, N$ and all $t \geqslant 0$. By the above and the fact that the ODE is autonomous we will see that this does not imply a loss of generality. Let

$$\nu(t) := \sum_{j=1}^{N} c_j(t),$$

so that (1.2) can be rewritten as

$$\dot{c}_j(t) + c_j(t)\nu(t) = \sum_{k=1}^{N-j} c_{j+k}(t)c_k(t), \tag{3.1}$$

and, in particular,

$$\dot{c}_N(t) + c_N(t)\nu(t) = 0\,. \tag{3.2}$$

We start by following the procedure already used in [4] that consists in time rescaling (1.2) so that the resulting equations only retain the production terms. From (3.2)

$$c_N(t)/c_N(0) = \exp\left(-\int_0^t \nu(s)\,ds\right)\,.$$

Since $e^{\int_0^t \nu}$ is an integrating factor of (3.1), we conclude that

$$\frac{d}{dt}\left(\frac{c_j(t)}{c_N(t)}\right) = \frac{1}{c_N(t)}\sum_{k=1}^{N-j} c_{j+k}(t)c_k(t)\,. \tag{3.3}$$

Let $y(t) := \int_0^t c_N(s)\,ds$ and define functions $\phi_j(y)$, such that

$$c_j(t) = \phi_j(y(t))c_N(t), \tag{3.4}$$

for each $j = 1, \dots, N$, and $t \geqslant 0$. Then, for $j = 1, \dots, N-1$, $\phi_j(y)$ is defined and is strictly positive for $y \in [0, \omega)$, where $\omega := \int_0^\infty c_N \in (0, +\infty]$. Let us denote by $(\cdot)'$ the derivative with respect to y. Then, from (3.3) we obtain

$$\begin{aligned} \phi_j'(y) &= \sum_{k=1}^{N-j} \phi_{j+k}(y)\phi_k(y)\,, \quad j = 1, \dots, N-1, \\ \phi_N(y) &= 1\,, \end{aligned} \tag{3.5}$$

for $0 \leqslant y < \omega$. Conversely, if $(\phi_j(y))$ is a solution of (3.5) in its maximal positive interval $(0, \omega^*)$ and if $c_N(\cdot)$, and therefore $y(\cdot)$, is given, then $c_j(t) = c_N(t)\phi_j(y(t))$, for $j = 1, \dots, N$ solves (1.2) for $t \in [0, \infty)$, so that $\omega^* = \omega$.

In the next two lemmas we state some results about the asymptotic behaviour of $\phi(y)$.

Lemma 3.1. *Any solution of* (3.5)*, say* $\phi(y) = (\phi_1(y), \dots, \phi_{N-1}(y), 1)$*, satisfying* $\phi_j(0) > 0$*, for all* $j = 1, \dots, N$*, is defined for* $y \in [0, \omega)$ *where* $\omega > 0$ *is finite and moreover,*

(i) $\phi_j(y) \to +\infty$ *as* $y \to \omega$*, for all* $j = 1, 2, \dots, N-1$*;*
(ii) $\phi_j(y)/\phi_{j+1}(y) \to +\infty$ *as* $y \to \omega$*, for all* $j = 1, 2, \dots, N-1$*.*

Proof. Let $(\phi_j(y))$ be a solution of (3.5) in its positive maximal interval of existence $[0,\omega)$ satisfying the hypothesis of the lemma. Then, for all $j = 1,\dots,N$, $\phi_j(y) > 0$, for all $y \in [0,\omega)$. Since,

$$\phi_j'(y) \geqslant \phi_{j+1}(y)\phi_1(y)\,, \tag{3.6}$$

for $j = 1,\dots,N-1$ (with equality for $j = N-1$), and $\phi_N(y) = 1$, by defining $\tau(y) := \int_0^y \phi_1(s)\,ds$, and $\psi_j(\tau)$, such that $\phi_j(y) = \psi_j(\tau(y))$, we obtain,

$$\frac{d}{d\tau}\psi_j(\tau) \geqslant \psi_{j+1}(\tau), \tag{3.7}$$

for $j = 1,\dots,N-1$ (with equality for $j = N-1$), $\psi_N(\tau) = 1$, for $0 \leqslant \tau < \int_0^\omega \phi_1$. The $N-1$ equation gives,

$$\psi_{N-1}(\tau) = \tau + c_0.$$

Then by successively integrating (3.7) for $j = N-2,\ N-3,\dots,1$, and taking in account that $\psi_j(0) \geqslant 0$ for $j = 1,\dots,N$, we obtain

$$\psi_{N-k}(\tau) \geqslant \frac{\tau^k}{k!}, \quad k = 1,\dots,N-1\,.$$

In particular,

$$\psi_1(\tau) \geqslant \frac{\tau^{N-1}}{(N-1)!}\,,$$

which is equivalent to

$$\tau'(y) \geqslant \frac{\tau(y)^{N-1}}{(N-1)!}\,.$$

Since, by hypothesis, $N-1 > 1$, the last inequality means that $\tau(\cdot)$ blows up at a finite value of y, which implies that $\omega < +\infty$. By fundamental results in ODE theory, this in turn implies that, for our solution, we have $\|\phi(y)\| \to \infty$, as $y \to \omega$, where $\|\cdot\|$ is the euclidean norm in $\mathbb{R}^N$. This, together with the monotonicity property of each $\phi_j(y)$, implies that there is a $j^* \in \{1,\dots,N-1\}$ such that $\phi_{j^*}(y) \to +\infty$ as $y \to \omega$. We now prove the nontrivial fact that this is true for all $j = 1,\dots,N-1$. In order to derive such a conclusion we first prove that, for $j = 1,\dots,N-1$, $\phi_j(y)/\phi_{j+1}(y)$ is bounded away from zero for y sufficiently close to ω. Specifically, we prove that for $n = N-1, N-2,\dots,2,1$, there are $\eta > 0$, $Y \in [0,\omega)$ such that

$$\frac{\phi_j(y)}{\phi_{j+1}(y)} > \eta, \tag{3.8}$$

for $j = n, n+1,\dots,N-1$, and for all $y \in [Y,\omega)$.

Consider $n = N-1$. Then $\phi_{N-1}'(y) = \phi_1(y)$, so that $\phi_{N-1}(y)/\phi_N(y) = \phi_{N-1}(0) + \int_0^y \phi_1$ and, by the positivity of ϕ_1 the result is obvious with $\eta = \phi_{N-1}(Y)$ for any $Y \in (0,\omega)$.

Suppose now that we have proved our claim for $n+1$, with $n \in \{1,\dots,N-1\}$, that is, there are $\eta > 0$, $Y \in [0,\omega)$ such that (3.8) is true, for $j =$

$n+1, n+2, \ldots, N-2$ and for $y \in [Y,\omega)$. We prove the same holds for n. Since, for $y \in [Y,\omega)$

$$\frac{\phi_n'(y)}{\phi_{n+1}'(y)} = \frac{\displaystyle\sum_{k=1}^{N-n} \phi_{k+n}(y)\phi_k(y)}{\displaystyle\sum_{k=1}^{N-n-1} \phi_{k+n+1}(y)\phi_k(y)} \geqslant \frac{\displaystyle\sum_{k=1}^{N-n-1} \phi_{k+n+1}(y)\phi_k(y)\cdot\frac{\phi_{k+n}(y)}{\phi_{k+n+1}(y)}}{\displaystyle\sum_{k=1}^{N-n-1} \phi_{k+n+1}(y)\phi_k(y)} \geqslant \eta,$$

and therefore

$$\phi_n'(y) \geq \eta\phi_{n+1}'(y),$$

by integration we obtain

$$\phi_n(y) - \phi_n(Y) \geqslant \eta(\phi_{n+1}(y) - \phi_{n+1}(Y))$$

or

$$\frac{\phi_n(y)}{\phi_{n+1}(y)} \geqslant \frac{\phi_n(Y)}{\phi_{n+1}(y)} + \eta\left(1 - \frac{\phi_{n+1}(Y)}{\phi_{n+1}(y)}\right).$$

Let $\tilde{Y} \in (Y,\omega)$. Then, for $y \in [\tilde{Y},\omega)$,

$$\phi_{n+1}(y) \geqslant \phi_{n+1}(\tilde{Y}) > \phi_{n+1}(Y),$$

and defining

$$\tilde{\eta} := \eta\left(1 - \frac{\phi_{n+1}(Y)}{\phi_{n+1}(\tilde{Y})}\right)$$

we conclude that, for $y \in [\tilde{Y},\omega)$,

$$\frac{\phi_n(y)}{\phi_{n+1}(y)} \geqslant \tilde{\eta}.$$

By redefining Y,η as $\tilde{Y},\tilde{\eta}$ we have proved (3.8) for n. This completes our induction argument.

Now let $K := \{j = 1,\ldots,N-1 \mid \phi_j(y) \to \infty \text{ as } y \to \omega\}$. We already know that $K \neq \emptyset$, so that we can define $J := \max K$. Then, from (3.8) we get

$$\phi_j(y) \to \infty \text{ as } y \to \omega, \quad \text{for all } j = 1,\ldots,J.$$

It is then sufficient to prove that, in fact, $J = N-1$. This is based on the integral version of (3.5), namely

$$\phi_j(y) - \phi_j(Y) = \int_Y^y \phi_{j+1}\phi_1 + \int_Y^y \phi_{j+2}\phi_2 + \ldots + \int_Y^y \phi_{N-j-1}\phi_{N-1} + \int_Y^y \phi_{N-j}, \quad (3.9)$$

for $j = 1,\ldots,N-1$. Now, in order to derive a contradiction, suppose that $J < N-1$. Then, for $j = J+1,\ldots,N-1$, $\phi_j(y)$ is bounded for $y \in [Y,\omega)$. But then, since (3.9) implies that

$$\phi_j(y) - \phi_j(Y) > \int_Y^y \phi_{N-j}, \quad (3.10)$$

we conclude that $\int_Y^y \phi_j$ must be bounded for $j = 1, 2, \ldots, N-J-1$ and $y \in [Y, \omega)$. Therefore, by the monotonicity of all the $\phi_j(\cdot)$, we get, for all $y \in [Y, \omega)$,

$$\begin{aligned}\phi_J(y) - \phi_J(Y) &\leqslant \phi_{J+1}(y)\int_Y^y \phi_1 + \phi_{J+2}(y)\int_Y^y \phi_2 + \ldots \\ &\qquad \ldots + \phi_{N-1}(y)\int_Y^y \phi_{N-J-1} + \int_Y^y \phi_{N-J} \\ &\leqslant M + \int_Y^y \phi_{N-J},\end{aligned}$$

for some positive constant M. Since $\phi_J(y) \to \infty$, as $y \to \omega$, this bound forces $\int_Y^y \phi_{N-J} \to \infty$ as $y \to \omega$. Now, again by (3.8), we have, for $y \in [Y, \omega)$,

$$\phi_1(y) \geqslant \eta\phi_2(y) \geqslant \eta^2\phi_3(y) \geqslant \ldots \geqslant \eta^{N-J-1}\phi_{N-J}(y)\,,$$

implying that, for all $j = 1, 2, \ldots, N-J-1$,

$$\int_Y^y \phi_j \geqslant \eta^{N-J-j}\int_Y^y \phi_{N-J},$$

contradicting the boundedness conclusion following inequality (3.10). This proves that $J = N-1$.

It remains to prove assertion (ii). For $j = N-1$ it is trivial, since

$$\frac{\phi_{N-1}(y)}{\phi_N(y)} = \phi_{N-1}(y) \to +\infty \quad \text{as} \quad y \to \omega,$$

as we have seen before. Suppose we have proved (ii) for $j = N-1, N-2, \ldots, n+1$ for some $n \in \{1, 2, \ldots, N-2\}$. We prove that the same holds for $j = n$. We consider again, for y close to ω, the quotient

$$\begin{aligned}\frac{\phi_n'(y)}{\phi_{n+1}'(y)} &= \frac{\displaystyle\sum_{k=1}^{N-n}\phi_{k+n}(y)\phi_k(y)}{\displaystyle\sum_{k=1}^{N-n-1}\phi_{k+n+1}(y)\phi_k(y)} = \frac{\displaystyle\sum_{k=1}^{N-n}\frac{\phi_{k+n}(y)}{\phi_{2+n}(y)}\cdot\frac{\phi_k(y)}{\phi_1(y)}}{1+\displaystyle\sum_{k=2}^{N-n-1}\frac{\phi_{k+n+1}(y)}{\phi_{2+n}(y)}\cdot\frac{\phi_k(y)}{\phi_1(y)}} \\ &> \frac{\phi_{1+n}(y)}{\phi_{2+n}(y)}\left(1+\sum_{k=2}^{N-n-1}\eta^{-k+1}\frac{\phi_{k+n+1}(y)}{\phi_{2+n}(y)}\right)^{-1} \to +\infty,\end{aligned}$$

as $y \to \omega$. Then, we know by Cauchy's rule that

$$\lim_{y\to\omega}\frac{\phi_n(y)}{\phi_{n+1}(y)} = \lim_{y\to\omega}\frac{\phi_n'(y)}{\phi_{n+1}'(y)} = +\infty,$$

and our induction argument is complete. □

Lemma 3.2. *In the conditions of the previous lemma, for each* $j = 1, \ldots, N-1$*, there is* $\rho_j : [0, \omega) \to \mathbb{R}$ *such that* $\rho_j(y) \to 0$ *as* $y \to \omega$*, and*

$$\phi_j(y) = \frac{A_j}{(\omega - y)^{\alpha_j}}(1 + \rho_j(y))\,,$$

where

$$\alpha_j := \frac{N-j}{N-2}, \qquad A_j := \frac{1}{(N-j)!}\left(\frac{(N-1)!}{N-2}\right)^{\alpha_j}.$$

Proof. By (ii) of the previous lemma, we know that, for $j = 1, \ldots, N-1$,

$$\frac{\sum_{k=1}^{N-j} \phi_{j+k}(y)\phi_k(y)}{\phi_{j+1}(y)\phi_1(y)} = 1 + \sum_{k=2}^{N-j} \frac{\phi_{j+k}(y)}{\phi_{j+1}(y)} \cdot \frac{\phi_k(y)}{\phi_1(y)} \to 1 \quad \text{as} \quad y \to \omega\,.$$

Hence, we can write, for $j = 1, \ldots, N-1$, and $y \in (0, \omega)$

$$\phi_j'(y) = \phi_{1+j}(y)\phi_1(y)(1 + r_j(y)) \tag{3.11}$$

such that $r_j(y) \to 0$, as $y \to \omega$. We now perform the same change of variables as in the beginning of the proof of the previous lemma, this time giving, for $\tau \geqslant 0$,

$$\frac{d}{d\tau}\psi_j(\tau) = \psi_{j+1}(\tau)(1 + \hat{r}_j(\tau)), \tag{3.12}$$

such that $\hat{r}_j(\tau) \to 0$, as $\tau \to \infty$. We now prove that, for $j = 1, \ldots, N-1$,

$$\psi_j(\tau) = \frac{\tau^{N-j}}{(N-j)!}(1 + \hat{\rho}_j(\tau)) \tag{3.13}$$

where $\hat{\rho}_j(\tau) \to 0$ as $\tau \to \infty$. For $j = N-1$, taking into account that $\hat{r}_{N-1}(\tau) \equiv 0$, the result easily follows:

$$\psi_{N-1}(\tau) = \tau + c_0 = \tau(1 + c_0\tau^{-1})\,.$$

Now suppose we have verified (3.13) for $j = n+1$, for some $n = 1, \ldots, N-2$. We prove the same holds for $j = n$. Defining $\delta(\tau)$ by

$$\delta(\tau) = (1 + \hat{\rho}_{n+1}(\tau))(1 + \hat{r}_n(\tau)) - 1\,,$$

we have $\delta(\tau) \to 0$ as $\tau \to \infty$, and by (3.12) and (3.13),

$$\frac{d}{d\tau}\psi_n(\tau) = \frac{\tau^{N-n-1}}{(N-n-1)!}(1 + \delta(\tau))\,,$$

and therefore, upon integration,

$$\psi_n(\tau) - \psi_n(0) = \frac{\tau^{N-n}}{(N-n)!} + \frac{1}{(N-n-1)!}\int_0^\tau s^{N-n-1}\delta(s)\,ds,$$

which can be written as

$$\psi_n(\tau) = \frac{\tau^{N-n}}{(N-n)!}(1 + \hat{\rho}_n(\tau))$$

where

$$\hat{\rho}_n(\tau) := \frac{(N-n)!\psi_n(0)}{\tau^{N-n}} + \frac{N-n}{\tau^{N-n}}\int_0^\tau s^{N-n-1}\delta(s)\,ds\,.$$

If the integral in the right hand side stays bounded for $\tau \geqslant 0$, then the last term converges to 0 as $\tau \to \infty$. If it is unbounded, since its integrand is positive then the integral tends to $+\infty$, as $\tau \to \infty$. In this case we can apply Cauchy's rule since

$$\frac{\left(\int_0^\tau s^{N-n-1}\delta(s)\,ds\right)'}{(\tau^{N-n})'} = \frac{\delta(\tau)}{N-n} \to 0, \quad \text{as} \quad \tau \to \infty\,,$$

thus proving that also in this case, the last term converges to 0 as $\tau \to \infty$. Either way we have $\hat{\rho}_n(\tau) \to 0$ as $\tau \to \infty$, thus proving assertion (3.13) for $j = n$. Our induction argument is complete.

In particular,

$$\psi_1(\tau) = \frac{\tau^{N-1}}{(N-1)!}(1 + \hat{\rho}_1(\tau))$$

which is equivalent to

$$\tau'(y) = \frac{\tau(y)^{N-1}}{(N-1)!}(1 + \hat{\rho}_1(\tau(y)))$$

for $y \in (0, \omega)$.

Let $0 < y < y_1 < \omega$. Then, the integration of the previous equality in $[y, y_1]$ yields

$$\tau(y)^{2-N} - \tau(y_1)^{2-N} = \frac{N-2}{(N-1)!}\left(y_1 - y + \int_y^{y_1} \hat{\rho}_1(\tau(s))\,ds\right)\,.$$

Define $\hat{R}(y, y_1) := \frac{1}{y_1 - y}\int_y^{y_1} \hat{\rho}_1(\tau(s))\,ds$. Then,

$$\tau(y) = \left[\tau(y_1)^{2-N} + \frac{N-2}{(N-1)!}(y_1 - y)(1 + \hat{R}(y, y_1))\right]^{-\frac{1}{N-2}}\,. \tag{3.14}$$

Now, observe that $\tau(y_1)^{2-N} \to 0$, as $y_1 \to \omega$. Also, by fixing $y \in (0, \omega)$, for $y_1 \in [y + \eta, \omega)$ with $\eta > 0$ small, $y_1 \mapsto \hat{R}(y, y_1)$ is bounded. Therefore we can define $R_0(y) := \lim_{y_1 \to \omega} \hat{R}(y, y_1)$. Then by making $y_1 \to \omega$ in (3.14) we obtain

$$\tau(y) = \left[\frac{N-2}{(N-1)!}(\omega - y)(1 + R_0(y))\right]^{-\frac{1}{N-2}}\,. \tag{3.15}$$

with

$$R_0(y) = \frac{1}{\omega - y}\int_y^\omega \hat{\rho}_1(\tau(s))\,ds \to 0 \quad \text{as} \quad y \to \omega,$$

by Cauchy rule and the fact that $\hat{\rho}_1(\tau(y)) \to 0$ as $y \to \omega$.

For $j = 1, \ldots, N-1$, define

$$\rho_j(y) := (1 + R_0(y))^{-\frac{N-j}{N-2}}\,(1 + \hat{\rho}_j(\tau(y))) - 1\,.$$

so that $\rho_j(y) \to 0$, as $y \to \omega$. By (3.13) and (3.15), for $j = 1, \ldots, N-1$ and $y \in (0, \omega)$,

$$\phi_j(y) = \psi_j(\tau(y)) = \frac{1}{(N-j)!}\left(\frac{(N-1)!}{N-2}\right)^{\frac{N-j}{N-2}}(\omega - y)^{-\frac{N-j}{N-2}}(1 + \rho_j(y))$$

and the proof is complete. □

The following lemma is a weaker version of Theorem 2.1 which will be used to complete the proof of the full result:

Lemma 3.3. *If $c_j(0) > 0$, for $j = 1, \ldots, N$, then, for each such j, there exists $e_j : [0, \infty) \to \mathbb{R}$ such that $e_j(t) \to 0$ as $t \to \infty$, and*

$$c_j(t) = \frac{\widetilde{A}_j}{t(\log t)^{j-1}}(1 + e_j(t))$$

where

$$\widetilde{A}_j := \frac{(N-1)!}{(N-j)!}.$$

Proof. It was proved in [2] that

$$\nu_{\mathrm{odd}}(t) := \sum_{\substack{j=1 \\ j \text{ odd}}}^{N} c_j(t)$$

satisfies the differential equation $\dot{\nu}_{\mathrm{odd}} = -\nu_{\mathrm{odd}}^2$, and thus

$$\nu_{\mathrm{odd}}(t) = \frac{1}{(\nu_{\mathrm{odd}}(0))^{-1} + t}.$$

It follows that

$$\nu_{\mathrm{odd}}(t) = \frac{1}{t}(1 + o(1)) \qquad \text{as } t \to \infty.$$

Defining $\nu_{\mathrm{even}}(t) = \sum_{j=2, j \text{ even}}^{N} c_j(t)$ we have

$$\frac{\nu_{\mathrm{even}}(t)}{\nu_{\mathrm{odd}}(t)} = \frac{\frac{c_2}{c_1} + \frac{c_4}{c_1} + \cdots + \frac{c_{2\lfloor N/2 \rfloor}}{c_1}}{1 + \frac{c_3}{c_1} + \cdots + \frac{c_{2\lfloor (N-1)/2 \rfloor + 1}}{c_1}} = o(1), \qquad \text{as } t \to \infty,$$

since by Lemma 3.1(ii) we conclude $\frac{c_1(t)}{c_i(t)} \to \infty$ as $t \to \infty$, for all $1 < i \leqslant N$. It follows that, as $t \to \infty$,

$$\nu(t) = \nu_{\mathrm{odd}}(t)\left(1 + \frac{\nu_{\mathrm{even}}(t)}{\nu_{\mathrm{odd}}(t)}\right) = \nu_{\mathrm{odd}}(t)(1 + o(1)) = \frac{1}{t}(1 + o(1)). \tag{3.16}$$

On the other hand, again by Lemma 3.1(ii) and (3.4), we conclude that, as $t \to \infty$,

$$\nu(t) = \sum_{j=1}^{N} c_j(t) = c_1(t)\left(1 + \sum_{j=2}^{N} \frac{c_j(t)}{c_1(t)}\right) = c_1(t)(1 + o(1)). \tag{3.17}$$

From (3.16) and (3.17) we conclude that

$$tc_1(t) \to 1, \qquad \text{as } t \to \infty.$$

By (3.4) with $j = 1$, we can write $c_1(t) = \phi_1(y(t))c_N(t)$, and thus

$$t\phi_1(y(t))c_N(t) \to 1, \qquad \text{as } t \to \infty. \tag{3.18}$$

When $j = 1$, Lemma 3.2 reduces to

$$\phi_1(y) = \frac{A_1}{(\omega - y)^{\frac{N-1}{N-2}}}(1 + o(1)), \qquad \text{as } y \to \omega. \tag{3.19}$$

From (3.15) we have $\omega - y = \frac{(N-1)!}{N-2}\tau(y)^{2-N}(1+o(1))$, as $y \to \omega$, where $\tau(y)$ was defined by $\tau(y) = \int_0^y \phi_1(\tilde{y})d\tilde{y}$ in the beginning of the proof of Lemma 3.1, and hence

$$\tau(y(t)) = \int_0^{y(t)} \phi_1(\tilde{y})d\tilde{y} = \int_0^t \phi_1(y(s))c_N(s)ds = \int_0^t c_1(s)ds.$$

Since

$$\frac{(\tau(y(t)))'}{(\log t)'} = \frac{c_1(t)}{1/t} = tc_1(t) \to 1, \qquad \text{as } t \to \infty,$$

using Cauchy's rule we have $\tau(y(t)) = (\log t)(1 + o(1))$, as $t \to \infty$, so that

$$\omega - y(t) = \frac{(N-1)!}{N-2}(\log t)^{2-N}(1 + o(1)), \qquad \text{as } t \to \infty, \tag{3.20}$$

and by (3.19)

$$\phi_1(y(t)) = A_1\left(\frac{N-2}{(N-1)!}\right)^{\frac{N-1}{N-2}}(\log t)^{N-1}(1 + o(1)), \qquad \text{as } t \to \infty.$$

Multiplying by $tc_N(t)$ and recalling (3.18) we have

$$A_1\left(\frac{N-2}{(N-1)!}\right)^{\frac{N-1}{N-2}}(\log t)^{N-1}tc_N(t)(1 + o(1)) \to 1, \qquad \text{as } t \to \infty,$$

and since $A_1\left(\frac{N-2}{(N-1)!}\right)^{\frac{N-1}{N-2}} = \frac{1}{(N-1)!}$, we obtain

$$\frac{t(\log t)^{N-1}}{(N-1)!}c_N(t)(1 + o(1)) \to 1, \qquad \text{as } t \to \infty,$$

and it follows that, as $t \to \infty$,

$$c_N(t) = ((N-1)!)\frac{1}{t(\log t)^{N-1}}(1 + o(1)). \tag{3.21}$$

Now we can use (3.4), Lemma 3.2, and (3.21) to obtain

$$c_j(t) = \frac{A_j}{(\omega - y(t))^{\alpha_j}}((N-1)!)\frac{1}{t(\log t)^{N-1}}(1 + o(1)) \qquad \text{as } t \to \infty,$$

and from this, using (3.20) and the definitions of α_j and A_j in the statement of Lemma 3.2, it follows that

$$c_j(t) = \frac{(N-1)!}{(N-j)!}\frac{1}{t(\log t)^{j-1}}(1 + o(1)) \qquad \text{as } t \to \infty, \tag{3.22}$$

as we wanted to prove. □

Now, consider the case $c_j(0) \geqslant 0$, for $j = 1, \dots, N$, with $m = \gcd(P) = 1$ and $p = \sup P = N$. By Proposition 7.2 in [2] this implies that $\mathcal{J}(t) = \mathbb{N} \cap [1, N]$ for all $t > 0$, which means that, for all $t > 0$ and $j \in \mathbb{N} \cap [1, N]$, all components $c_j(t)$ of $c(t)$ are strictly positive. Hence, since (1.2) is an autonomous ODE, given a small $\varepsilon > 0$, for $t \geqslant \varepsilon$, $c(t) = c_\varepsilon(t-\varepsilon)$, where $c_\varepsilon(\cdot)$ is the solution of (1.2) satisfying the initial condition $c_\varepsilon(0) = c(\varepsilon)$. Therefore, the conditions of Lemma 3.3 apply to $c_\varepsilon(\cdot)$. Then, it is easy to see that the asymptotic results that we conclude with respect to $c_\varepsilon(t)$ also apply to $c(t)$, allowing us to state the following:

Lemma 3.4. *Let $c = (c_j)$ be a solution satisfying $c_j(0) \geqslant 0$, with $m = 1$ and $p = N$. Then the conclusions of Lemma* 3.3 *hold.*

This is, in fact, the particular case of Theorem 2.1 from which the full case follows as stated at the end of section 2.

4. Final remarks

A natural question to ask is: what is the asymptotic behaviour of the solutions of (1.2) in the infinite dimensional case ($N = \infty$)? It is clear that Theorem 2.1 by itself is unsufficient to answer this question since the passage to the limit, $N \to \infty$, is not allowed without results on the uniformity of the various limits involved, which seems to be a hard task. Also it is far from clear how to rebuild the proofs of the lemmas in section 3 in this more general case since they heavily rely on the fact that there is a 'last equation', the N-component equation, that can be integrated by the reduction method we have used, being the asymptotic behaviour of the other components deduced in a 'backwards' manner. Such procedure is obviously impossible in an infinite dimensional setting. In fact, that the situation can be very different for $N = \infty$ from the one displayed by Theorem 2.1 is shown by the existence of the self-similar solutions given by,

$$c_j(t) = (\kappa + t)^{-1}(1-\alpha^2)\alpha^{j-1}, \quad j = 1, 2, \dots, \quad t \geqslant 0,$$

with constants $\kappa > 0$ and $\alpha \in (0, 1)$ (see [2]), in which case, $tc_j(t) \to (1-\alpha^2)\alpha^{j-1}$, as $t \to \infty$, for $j = 1, 2, \dots$. Further work will be devoted to fully understand this problem.

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F.P. da Costa
Departamento de Ciências e Tecnologia, Universidade Aberta, Lisboa, Portugal, and Centro de Análise Matemática, Geometria e Sistemas Dinâmicos, Instituto Superior Técnico, Universidade de Lisboa, Lisboa, Portugal
e-mail: fcosta@uab.pt

J.T. Pinto
Departamento de Matemática and Centro de Análise Matemática Geometria e Sistemas Dinâmicos, Instituto Superior Técnico, Universidade de Lisboa, Lisboa, Portugal
e-mail: jpinto@math.tecnico.ulisboa.pt

R. Sasportes
Departamento de Ciências e Tecnologia, Universidade Aberta, Lisboa, Portugal, and Centro de Análise Matemática, Geometria e Sistemas Dinâmicos, Instituto Superior Técnico, Universidade de Lisboa, Lisboa, Portugal
e-mail: rafael@uab.pt