STUDIES IN FRACTIONAL POISSON MEASURES

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In this paper we investigate the quasi-invariance property of fractional Poisson measures
with respect to the diffeomorphism subgroup and we construct spaces of test and gener-
alized functions associated to the corresponding fractional Lebesgue-Poisson measures.

Keywords: Fractional Poisson measure; Diffeomorphism group; Test and generalized
functions; Mittag-Leffler function.

PACS numbers: 11.30.Rd, 05.40.-a

1. Introduction

In the recent years the fractional Poisson process and its associated measure has
become an object of study in finite dimensions, see for example Ref. 1, 2 and ref-
erences therein. The corresponding infinite dimensional version of the fractional
Poisson measure has been introduced in Ref. 3. In this paper we study further some
properties of the latter measure, namely, the quasi-invariance property with respect
to the diffeomorphism subgroup as well as the construction and description of a
nuclear triple associated to the fractional Lebesgue-Poisson measure.

2. Preliminaries

In this section we briefly recall the concepts and results of fractional Poisson mea-
sures shown in Ref. 3. For this proposal, we start by fixing an underlying manifold
M with the Borel σ-algebra $\mathcal{B}(M)$ and a non-atomic Radon measure σ which we as-
sume to be non-degenerate (i.e., $\sigma(O) > 0$ for all non-empty open sets $O \subset M$). We
always assume that M is a geodesically complete connected oriented (non-compact)

Riemannian C^∞-manifold. Having in mind the most interesting applications, we also assume that $\sigma(M) = \infty$. A typical example is $\mathbb{R}^d$ with σ being the Lebesgue measure.

2.1. *Fractional Poisson measures*

On the space $\mathcal{D} := \mathcal{D}(M)$ of C^∞-functions with compact support in M we consider the functional C_α given by

$$C_\alpha(\varphi) := E_\alpha \left(\int_M (e^{i\varphi(x)} - 1)\, d\sigma(x) \right), \quad \varphi \in \mathcal{D}, \tag{1}$$

where, for each $0 < \alpha < 1$ fixed, E_α is the Mittag-Leffler function of parameter α,

$$E_\alpha(z) := \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\alpha n + 1)}, \quad z \in \mathbb{C}.$$

For $\alpha = 1$ we see that the Mittag-Leffler function coincides with the exponential function and thus (1) is the characteristic functional of the classic Poisson measure on the distribution space $\mathcal{D}' := \mathcal{D}'(M)$ Ref. 4. For $0 < \alpha < 1$ we have the following result cf. Ref. 3.

Theorem 2.1. *For each $0 < \alpha < 1$ fixed, the functional C_α in (1) is the characteristic functional of a probability measure on $\mathcal{D}'$, denote by π_σ^α and called a fractional Poisson measure.*

It is well known that the Mittag-Leffler function E_α is completely monotonic which is equivalent to the existence of a probability measure [a] ν_α on $\mathbb{R}_0^+ := [0, \infty)$ such that

$$E_\alpha(-t) = \int_0^\infty e^{-t\tau}\, d\nu_\alpha(\tau), \quad \forall t > 0. \tag{2}$$

The same results hold for any complex $z \in \mathbb{C}$ such that $\Re(z) \geq 0$, cf. Lemma 7 in Ref. 3. Using the representation (2) we may rewrite (1) as

$$C_\alpha(\varphi) = \int_0^\infty \exp \left(\tau \int_M (e^{i\varphi(x)} - 1)\, d\sigma(x) \right) d\nu_\alpha(\tau)$$

showing that the integrand function corresponds to the characteristic function of the standard Poisson measure $\pi_{\tau\alpha}$, $\tau > 0$. Therefore, by uniqueness, we have

$$\pi_\sigma^\alpha = \int_0^\infty \pi_{\tau\sigma}\, d\nu_\alpha(\tau), \tag{3}$$

which means that π_σ^α is the mixture of standard Poisson measures $\pi_{\tau\sigma}$, $\tau > 0$. Since the standard Poisson measures are all supported on a subset of $\mathcal{D}'$, see for

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[a]The measure ν_α is absolutely continuous w.r.t. the Lebesgue measure and its density is related to the α-stable probability distribution, see Ref. 5 for more details.

example Ref. 4, property (3) then shows that π_σ^α has the same support, namely, the configuration space.

The configuration space $\Gamma := \Gamma_M$ over the manifold M is defined as the set of all locally finite subsets of M (simple configurations),

$$\Gamma := \{\gamma \subset M \,:\, |\gamma \cap K| < \infty \text{ for any compact } K \subset M\}, \tag{4}$$

where here (and below) $|A|$ denotes the cardinality of a set A. We identify each $\gamma \in \Gamma$ with a non-negative integer-valued Radon measure, $\Gamma \ni \gamma \mapsto \sum_{x \in \gamma} \delta_x \in \mathcal{M}(M)$, where δ_x is the Dirac measure with unit mass at x, $\sum_{x \in \emptyset} \delta_x := 0$ (zero measure), and $\mathcal{M}(M)$ denotes the set of all non-negative Radon measures on $\mathcal{B}(M)$. We may endow Γ with the relative topology as a subset of $\mathcal{M}(M)$ with the vague topology. This means, the weakest topology on Γ such that all mappings

$$\Gamma \ni \gamma \mapsto \langle \gamma, f \rangle := \int_M f(x)\, d\gamma(x) = \sum_{x \in \gamma} f(x)$$

are continuous for any $f \in C_0(M)$ (the set of all continuous functions on M with compact support). We denote the corresponding Borel σ-algebra on Γ by $\mathcal{B}(\Gamma)$.

2.2. *Fractional Lebesgue-Poisson measures*

In order to introduce fractional Lebesgue-Poisson measures let us consider the space of finite configurations over M, Γ_0, defined as the disjoint union

$$\Gamma_0 := \bigsqcup_{n=0}^\infty \Gamma_M^{(n)}$$

of the spaces of n-point configurations $\Gamma_M^{(n)} := \{\gamma \in \Gamma |\ |\gamma \cap M| = n\}$, $\Gamma_M^{(0)} := \{\emptyset\}$. One may endow Γ_0 with the disjoint union of the topological spaces $\Gamma_M^{(n)}$ and consider the corresponding Borel σ-algebra denoted by $\mathcal{B}(\Gamma_0)$. In fact, each $\Gamma_M^{(n)}$ has a topological structure introduced by the symmetric mapping $\mathrm{sym}_M^n(x_1, \ldots, x_n) := \{x_1, \ldots, x_n\}$ from $\widetilde{M^n}$ (the set of all $(x_1, \ldots, x_n) \in M^n$ such that $x_i \neq x_j$ if $i \neq j$) to $\Gamma_M^{(n)}$. For more details see Ref. 6.

For $\alpha = 1$, the correlation measure corresponding to the Poisson measure π_σ is the so-called Lebesgue-Poisson measure defined on $\mathcal{B}(\Gamma_0)$ by

$$\lambda_\sigma^1 := \sum_{n=0}^\infty \frac{1}{n!} \sigma^{(n)},$$

where $\sigma^{(n)} := \sigma^{\otimes n} \circ (\mathrm{sym}_M^n)^{-1}$, see Ref. 6 for more details.

For $0 < \alpha < 1$ we have the following result, cf. Ref. 3 for more details.

Theorem 2.2. *For each $0 < \alpha < 1$, the correlation measure corresponding to the fractional Poisson measure π_σ^α is the measure on $(\Gamma_0, \mathcal{B}(\Gamma_0))$ given by*

$$\lambda_\sigma^\alpha := \sum_{n=0}^\infty \frac{1}{\Gamma(\alpha n + 1)} \sigma^{(n)}, \quad \sigma^{(n)} := \sigma^{\otimes n} \circ (\mathrm{sym}_M^n)^{-1}.$$

In other words, for

$$E_\alpha^{(n)}(0) = \frac{n!}{\Gamma(\alpha n + 1)}, \quad n \in \mathbb{N}_0$$

denoting the n-th derivative of the Mittag-Leffler function E_α at the origin, one has $d\lambda_\sigma^\alpha = E_\alpha^{(|\cdot|)}(0)\, d\lambda_\sigma^1$. The measure λ_σ^α is called the fractional Lebesgue-Poisson measure.

3. Quasi-invariance property of the fractional Poisson measure

We denote by $\mathrm{Diff}(M)$ the group of all diffeomorphisms of M and by $\mathrm{Diff}_0(M)$ the subgroup of all diffeomorphisms $\phi : M \longrightarrow M$ with compact support, i.e., $\phi(x) = x$ for all x outside a compact set (depending on ϕ). Any $\phi \in \mathrm{Diff}_0(M)$ defines a measurable mapping

$$\Gamma \ni \gamma \mapsto \phi(\gamma) := \{\phi(x),\, x \in \gamma\} \in \Gamma.$$

For any $\gamma \in \Gamma$, the configuration $\phi(\gamma)$ coincides with γ for all but a finite number of points. The measurability of this mapping allows us to define the image $\phi^*\pi_\sigma^\alpha$ of the fractional Poisson measure π_σ^α under ϕ by $\phi^*\pi_\sigma^\alpha = \pi_\sigma^\alpha \circ \phi^{-1}$, more explicitly,

$$(\phi^*\pi_\sigma^\alpha)(B) := \pi_\sigma^\alpha(\phi^{-1}(B)), \quad B \in \mathcal{B}(\Gamma).$$

The next proposition shows that the image measure $\phi^*\pi_\sigma^\alpha$ is again a fractional Poisson measure but with a different underlying measure.

Proposition 3.1. *For any $\phi \in \mathrm{Diff}_0(M)$, the measure $\phi^*\pi_\sigma^\alpha$ is a fractional Poisson measure with intensity measure $\phi^*\sigma$, i.e.,*

$$\phi^*\pi_\sigma^\alpha = \pi_{\phi^*\sigma}^\alpha.$$

Proof. It is sufficient to compute the Laplace transform of the measure $\phi^*\pi_\sigma^\alpha$. For any $f \in C_0(M)$

$$\int_\Gamma e^{\langle \gamma, f\rangle}\, d(\phi^*\pi_\sigma^\alpha)(\gamma) = \int_\Gamma e^{\langle \phi(\gamma), f\rangle}\, d\pi_\sigma^\alpha(\gamma)$$

$$= E_\alpha\left(\int_{\mathbb{R}^d}\left(e^{(f\circ\phi)(x)} - 1\right) d\sigma(x)\right)$$

$$= E_\alpha\left(\int_{\mathbb{R}^d}\left(e^{f(x)} - 1\right) d(\phi^*\sigma)(x)\right)$$

$$= \int_\Gamma e^{\langle \gamma, f\rangle}\, d\pi_{\phi^*\sigma}^\alpha(\gamma). \qquad \square$$

Theorem 3.1. *The fractional Poisson measure π_σ^α is quasi-invariant with respect to the group $\mathrm{Diff}_0(M)$, and for any $\phi \in \mathrm{Diff}_0(M)$ the generalized Radon-Nikodym derivative is given by*

$$\Phi_\sigma^\alpha(\gamma) := \frac{d(\phi^*\pi_\sigma^\alpha)}{d\pi_\sigma}(\gamma) = \int_0^\infty \frac{d(\phi^*\pi_{\tau\sigma})}{d\pi_\sigma}(\gamma)\, d\nu_\alpha(\tau).$$

Proof. Having in mind the previous proposition it is enough to show that

$$\int_\Gamma e^{\langle\gamma,\varphi\rangle}\Phi_\sigma^\alpha(\gamma)\,d\pi_\sigma(\gamma) = \int_\Gamma e^{\langle\gamma,\varphi\rangle}\,d\pi_{\phi^*\sigma}^\alpha(\gamma), \quad \varphi\in\mathcal{D}.$$

To show this we proceed as follows:

$$\int_\Gamma e^{\langle\gamma,\varphi\rangle}\Phi_\sigma^\alpha(\gamma)\,d\pi_\sigma(\gamma) = \int_\Gamma e^{\langle\gamma,\varphi\rangle}\int_0^\infty \frac{d(\phi^*\pi_{\tau\sigma})}{d\pi_\sigma}(\gamma)\,d\nu_\alpha(\tau)\,d\pi_\sigma(\gamma)$$
$$= \int_0^\infty \int_\Gamma e^{\langle\gamma,\varphi\rangle}\frac{d(\phi^*\pi_{\tau\sigma})}{d\pi_\sigma}(\gamma)\,d\pi_\sigma(\gamma)\,d\nu_\alpha(\tau).$$

The inner integral is solved as in Ref. 4 and is equal to

$$\int_\Gamma e^{\langle\gamma,\varphi\rangle}\frac{d(\phi^*\pi_{\tau\sigma})}{d\pi_\sigma}(\gamma)\,d\pi_\sigma(\gamma) = \int_\Gamma e^{\langle\gamma,\varphi\rangle}\,d\pi_{\tau\phi^*\sigma}(\gamma)$$

Substituting we obtain

$$\int_0^\infty \int_\Gamma e^{\langle\gamma,\varphi\rangle}\,d\pi_{\tau\phi^*\sigma}(\gamma)\,d\nu_\alpha(\tau) = \int_\Gamma e^{\langle\gamma,\varphi\rangle}\int_0^\infty d\pi_{\tau\phi^*\sigma}(\gamma)\,d\nu_\alpha(\tau)$$
$$= \int_\Gamma e^{\langle\gamma,\varphi\rangle}\,d\pi_{\phi^*\sigma}^\alpha(\gamma). \qquad \square$$

4. Test and generalized functions on the fractional Lebesgue-Poisson space

For $\alpha=1$, the description of the elements of the space $L^2(\Gamma,\pi_\sigma)$ of complex-valued square integrable functions by the corresponding chaos decomposition provides a unitary isomorphism between the spaces $L^2(\Gamma,\pi_\sigma)$ and $L^2(\Gamma_0,\lambda_\sigma^1)$. This unitary isomorphism is at the origin of the spaces of test and generalized functions on the Lebesgue-Poisson space $(\Gamma_0,\mathcal{B}(\Gamma_0),\lambda_\sigma^1)$ introduced in Ref. 7. For a generic $0<\alpha<1$, it has been constructed in Ref. 3 a unitary isomorphism I_α between the spaces $L^2(\Gamma_0,\lambda_\sigma^1)$ and $L^2(\Gamma_0,\lambda_\sigma^\alpha)$,

$$I_\alpha(G) = \frac{G}{\sqrt{E_\alpha^{(|\cdot|)}(0)}}, \quad G\in L^2(\Gamma_0,\lambda_\sigma^1).$$

As a result, the unitary isomorphism I_α leads to a natural construction of spaces of test and generalized functions on the fractional Lebesgue-Poisson space $(\Gamma_0,\mathcal{B}(\Gamma_0),\lambda_\sigma^\alpha)$. To carry out this construction, first we briefly recall the definition of the spaces of test and generalized functions for the case $\alpha=1$.

The starting point is the space $L^2_{\mathrm{Re}}(M,\sigma)$ of all real-valued square integrable functions with respect to the underlying measure σ and a nuclear triple

$$\mathrm{pr}\lim_{p\in\mathbb{N}}\mathcal{H}_p = \mathcal{N} \subset L^2_{Re}(M,\sigma) \subset \mathcal{N}' = \mathrm{ind}\lim_{p\in\mathbb{N}}\mathcal{H}_{-p}$$

such that

$$\mathcal{D}\subset\mathcal{N}\subset L^2_{Re}(M,\sigma)\subset\mathcal{N}'\subset\mathcal{D}'.$$

More precisely, $\mathcal{N}$ is a nuclear space endowed with the projective limit topology of a family of Hilbert spaces $\mathcal{H}_p$ with the Hilbertian norm denote by $|\cdot|_p$, $p \in \mathbb{N}$, and $\mathcal{N}'$ is the dual space of $\mathcal{N}$ with respect to $L^2_{Re}(M, \sigma)$ endowed with the inductive topology of the family of (Hilbert) dual spaces $\mathcal{H}_{-p}$ of $\mathcal{H}_p$, $p \in \mathbb{N}$, with respect to $L^2_{Re}(M, \sigma)$. We denote by $|\cdot|_{-p}$ the corresponding (Hilbertian) norm on the space $\mathcal{H}_{-p}$, $p \in \mathbb{N}$. Since there is no risk of confusion we preserve the notation $|\cdot|_p$ for the norm on the symmetric tensor power $(\mathcal{H}_{p,\mathbb{C}})^{\hat{\otimes}n}$ of the complexified space $\mathcal{H}_{p,\mathbb{C}}$, and denoting by $(\mathcal{H}_{-p,\mathbb{C}})^{\hat{\otimes}n}$ the corresponding dual space with respect to the space $(L^2(M, \sigma))^{\hat{\otimes}n}$ of complex-valued functions, we also preserve the notation $|\cdot|_{-p}$ for the norm on $(\mathcal{H}_{-p,\mathbb{C}})^{\hat{\otimes}n}$, $n \in \mathbb{N}$.

Within this framework, on the Lebesgue-Poisson space one defines a space of test functions $(\mathcal{N})^1_1$ by the projective limit of the Hilbert spaces

$$(\mathcal{H}_p)^1_{q,1} := \left\{ G \in L^2(\Gamma_0, \lambda^1_\sigma) : \|G\|_{p,q,1} < \infty \right\}, \quad p, q \in \mathbb{N},$$

where, for each $G \in L^2(\Gamma_0, \lambda^1_\sigma)$,

$$\|G\|^2_{p,q,1} := \sum_{n=0}^\infty 2^{nq} |G^{(n)}|^2_p, \quad G^{(n)} := G\restriction_{\Gamma^{(n)}_M}.$$

To introduce spaces of test and generalized functions in the fractional Lebesgue-Poisson case, for each $0 < \alpha < 1$ fixed, let us consider the family of Hilbert spaces $(\mathcal{H}_p)^1_{q,\alpha} := I_\alpha((\mathcal{H}_p)^1_{q,1})$ with the natural Hilbertian norms given by

$$\|G\|^2_{p,q,\alpha} := \sum_{n=0}^\infty 2^{nq} E^{(n)}_\alpha(0) |G^{(n)}|^2_p = \sum_{n=0}^\infty \frac{2^{nq} n!}{\Gamma(\alpha n + 1)} |G^{(n)}|^2_p, \quad p, q \in \mathbb{N}.$$

This means, in particular,

$$(\mathcal{H}_p)^1_{q,\alpha} = \left\{ G \in L^2(\Gamma_0, \lambda^\alpha_\sigma) : \|G\|_{p,q,\alpha} < \infty \right\}, \quad p, q \in \mathbb{N}.$$

The space of test functions $(\mathcal{N})^1_\alpha$ is then defined as the projective limit of $(\mathcal{H}_p)^1_{q,\alpha}$, $p, q \in \mathbb{N}$. By the general duality theory the dual space $(\mathcal{N})^{-1}_\alpha$ of $(\mathcal{N})^1_\alpha$ (with respect to $L^2(\Gamma_0, \lambda^\alpha_\sigma)$) is given by the inductive limit of the dual spaces $(\mathcal{H}_{-p})^{-1}_{-q,\alpha}$ of $(\mathcal{H}_p)^1_{q,\alpha}$ (with respect to $L^2(\Gamma_0, \lambda^\alpha_\sigma)$). In this way we have defined the nuclear triple

$$(\mathcal{N})^1_\alpha \subset L^2(\Gamma_0, \lambda^\alpha_\sigma) \subset (\mathcal{N})^{-1}_\alpha,$$

where the (bilinear) dual pairing $\langle\!\langle \cdot, \cdot \rangle\!\rangle_\alpha$ between $(\mathcal{N})^{-1}_\alpha$ and $(\mathcal{N})^1_\alpha$ is realized as an extension of the (sesquilinear) inner product on $L^2(\Gamma_0, \lambda^\alpha_\sigma)$:

$$\langle\!\langle G_1, G_2 \rangle\!\rangle_\alpha = (G_1, \bar{G}_2)_{L^2(\Gamma_0, \lambda^\alpha_\sigma)}, \quad G_1 \in (\mathcal{N})^1_\alpha, \ G_2 \in L^2(\Gamma_0, \lambda^\alpha_\sigma)$$

with $\bar{G}_2$ being the complex conjugate function of G_2.

Observe that as a direct consequence of this construction, for any $p, q \in \mathbb{N}$, $I_\alpha : (\mathcal{H}_p)^1_{q,1} \to (\mathcal{H}_p)^1_{q,\alpha}$ is a unitary isomorphism. Therefore, the mapping I_α can be extended to the dual space $(\mathcal{H}_{-p})^{-1}_{-q,1}$ of $(\mathcal{H}_p)^1_{q,1}$ (with respect to $L^2(\Gamma_0, \lambda^1_\sigma)$) and the extended mapping, also denoted by I_α, maps $(\mathcal{H}_{-p})^{-1}_{-q,1}$ onto $(\mathcal{H}_{-p})^{-1}_{-q,\alpha}$ as

follows. For each $\Psi \in (\mathcal{H}_{-p})^{-1}_{-q,1}$, $I_\alpha(\Psi)$ is the unique element of $(\mathcal{H}_{-p})^{-1}_{-q,\alpha}$ such that

$$\langle\!\langle I_\alpha(\Psi), G \rangle\!\rangle_\alpha = \langle\!\langle \Psi, I_\alpha^{-1}(G) \rangle\!\rangle_1, \quad \forall\, G \in (\mathcal{H}_p)^1_{q,\alpha},$$

where $\langle\!\langle \cdot, \cdot \rangle\!\rangle_1$ denotes the dual pairing between $(\mathcal{N})^{-1}_1$ and $(\mathcal{N})^1_1$. Moreover, for any $p, q \in \mathbb{N}$, $I_\alpha : (\mathcal{H}_{-p})^{-1}_{-q,1} \to (\mathcal{H}_{-p})^{-1}_{-q,\alpha}$ is a unitary isomorphism. Hence, one may use these unitary isomorphims to transport the internal description of generalized functions on the Lebesgue-Poisson space to those on the fractional Lebesgue-Poisson space. As shown in Ref. 7, given the dual space $(\mathcal{N})^{-1}_1$ of the space $(\mathcal{N})^1_1$ of test functions (with respect to $L^2(\Gamma_0, \lambda^1_\sigma)$), for generalized functions $\Psi \in (\mathcal{N})^{-1}_1$ there is a natural decomposition $(\Psi^{(n)})^\infty_{n=0}$, $\Psi^{(n)} \in \mathcal{D}'^{\hat\otimes n}_{\mathbb{C}}$, $n \in \mathbb{N}_0$, such that for all $G \in (\mathcal{N})^1_1$,

$$\langle\!\langle \Psi, G \rangle\!\rangle_1 = \sum_{n=0}^\infty \frac{1}{n!} \langle \Psi^{(n)}, G^{(n)} \rangle, \tag{5}$$

and each Hilbert space $(\mathcal{H}_{-p})^{-1}_{-q,1}$ can be described as the set of elements $\Psi = (\Psi^{(n)})^\infty_{n=0} \in (\mathcal{N})^{-1}_1$ so that

$$\|\Psi\|^2_{-p,-q,1} = \sum_{n=0}^\infty \frac{2^{-nq}}{(n!)^2} |\Psi^{(n)}|^2_{-p} < \infty, \tag{6}$$

where $\|\cdot\|_{-p,-q,1}$ denotes the Hilbertian norm on $(\mathcal{H}_{-p})^{-1}_{-q,1}$. Thus, in accordance with (5) and (6), for each $0 < \alpha < 1$ fixed, each generalized funtion $\Phi \in (\mathcal{N})^{-1}_\alpha$ has a decomposition of the form $(\Phi^{(n)})^\infty_{n=0}$, $\Phi^{(n)} \in \mathcal{D}'^{\hat\otimes n}_{\mathbb{C}}$, $n \in \mathbb{N}_0$, such that for all $G \in (\mathcal{N})^1_\alpha$,

$$\langle\!\langle \Phi, G \rangle\!\rangle_\alpha = \sum_{n=0}^\infty \frac{1}{\Gamma(\alpha n + 1)} \langle \Phi^{(n)}, G^{(n)} \rangle,$$

and each Hilbert space $(\mathcal{H}_{-p})^{-1}_{-q,\alpha}$, $p, q \in \mathbb{N}$, can be described by

$$(\mathcal{H}_{-p})^{-1}_{-q,\alpha} = \left\{ \Phi \in (\mathcal{N})^{-1}_\alpha : \|\Phi\|_{-p,-q,\alpha} < \infty \right\}$$

with

$$\|\Phi\|^2_{-p,-q,\alpha} = \sum_{n=0}^\infty \frac{2^{-nq}}{n!\,\Gamma(\alpha n + 1)} |\Phi^{(n)}|^2_{-p}, \quad \Phi = (\Phi^{(n)})^\infty_{n=0} \in (\mathcal{N})^{-1}_\alpha.$$

Acknowledgements

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