The fractional volatility model: No-arbitrage, leverage and completeness

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Abstract

Based on a criterion of mathematical simplicity and consistency with empirical market data, a stochastic volatility model has been obtained with the volatility process driven by fractional noise. Depending on whether the stochasticity generators of log-price and volatility are independent or are the same, two versions of the model are obtained with different leverage behavior. Here, the no-arbitrage and completeness properties of the models are studied.

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1 Introduction

In liquid markets the autocorrelation of price changes decays to negligible values in a few ticks, consistent with the absence of long term statistical arbitrage. Because innovations of a martingale are uncorrelated, this strongly suggests that it is a process of this type that should be used to model the stochastic part of the returns process. As a consequence, classical Mathematical Finance has, for a long time, been based on the assumption that the price process of market securities may be approximated by geometric Brownian motion

$$dS_t = \mu S_t dt + \sigma S_t dB(t) \tag{1}$$

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Geometric Brownian motion (GBM) models the absence of linear correlations, but otherwise has some serious shortcomings. It does not reproduce the empirical leptokurtosis nor does it explain why nonlinear functions of the returns exhibit significant positive autocorrelation. For example, there is volatility clustering, with large returns expected to be followed by large returns and small returns by small returns (of either sign). This, together with the fact that autocorrelations of volatility measures decline very slowly [1], [2], [3] has the clear implication that long memory effects should somehow be represented in the process and this is not included in the GBM hypothesis. The existence of an essential memory component is also clear from the failure of reconstruction of a Gibbs measure and the need to use chains with complete connections in the phenomenological reconstruction of the market process [4].

As pointed out by Engle [5], when the future is uncertain investors are less likely to invest. Therefore uncertainty (volatility) would have to be changing over time. The conclusion is that a dynamical model for volatility is needed and σ in Eq.(1), rather than being a constant, becomes itself a process. This idea led to many deterministic and stochastic models for the volatility ([6], [7] and references therein).

The stochastic volatility models that were proposed describe some partial features of the market data. For example leptokurtosis is easy to fit but the long memory effects are much harder. On the other hand, and in contrast with GBM, some of the phenomenological fittings of historical volatility lack the kind of nice mathematical properties needed to develop the tools of mathematical finance. In an attempt to obtain a model that is both consistent with the data and mathematically sound, a new approach was developed in [8]. Starting with some criteria of mathematical simplicity, the basic idea was to let the data itself tell what the processes should be.

The basic hypothesis for the model construction were:

(H1) The log-price process $\log S_t$ belongs to a probability product space $(\Omega_1 \times \Omega_2, P_1 \times P_2)$ of which the (Ω_1, P_1) is the Wiener space and the second one, (Ω_2, P_2), is a probability space to be reconstructed from the data. Denote by $\omega_1 \in \Omega_1$ and $\omega_2 \in \Omega_2$ the elements (sample paths) in Ω_1 and Ω_2 and by $\mathcal{F}_{1,t}$ and $\mathcal{F}_{2,t}$ the σ-algebras in Ω_1 and Ω_2 generated by the processes up to time t. Then, a particular realization of the log-price process is denoted

$$\log S_t \left(\omega_1, \omega_2\right)$$

This first hypothesis is really not limitative. Even if none of the non-trivial stochastic features of the log-price were to be captured by Brownian motion, that would simply mean that S_t was a trivial function in Ω_1.

(H2) The second hypothesis is stronger, although natural. It is assumed that for each fixed ω_2, $\log S_t(\cdot, \omega_2)$ is a P_1-square integrable random variable in Ω_1.

These principles and a careful analysis of the market data led, in an essentially unique way[1], to the following model:

$$dS_t = \mu_t S_t \, dt + \sigma_t S_t \, dB(t) \tag{2}$$

$$\log \sigma_t = \beta + \frac{k}{\delta} \{B_H(t) - B_H(t-\delta)\} \tag{3}$$

B_H being fractional Brownian motion with Hurst coefficient H. The data suggests [8] values of H in the range $0.8 - 0.9$. In this coupled stochastic system, in addition to a mean value, volatility is driven by fractional noise. Notice that this empirically based model is different from the usual stochastic volatility models which assume the volatility to follow an arithmetic or geometric Brownian process. Also in Comte and Renault [9] and Hu [10], it is fractional Brownian motion that drives the volatility, not its derivative (fractional noise). δ is the observation scale of the process. In the $\delta \to 0$ limit the driving process would be a distribution-valued process.

Equation (3) leads to

$$\sigma_t = \theta e^{\frac{k}{\delta}\{B_H(t) - B_H(t-\delta)\} - \frac{1}{2}\left(\frac{k}{\delta}\right)^2 \delta^{2H}} \tag{4}$$

with $E[\sigma_t] = \theta > 0$.

The model has been shown [8] to describe well the statistics of price returns for a large δ-range and a new option pricing formula, with "smile" deviations from Black-Scholes, was also obtained. An agent-based interpretation [11] also led to the conclusion that the statistics generated by the model was consistent with the limit order book price setting mechanism.

In the past, several authors tried to describe the long memory effect by replacing in the price process Brownian motion by fractional Brownian motion with $H > 1/2$. However it was soon realized [12], [13], [14], [15] that this replacement implied the existence of arbitrage. These results might be avoided either by restricting the class of trading strategies [16], introducing

[1] Essentially unique in the sense that the empiricaly reconstructed volatility process is the simplest one, consistent with the scaling properties of the data.

transaction costs [17] or replacing pathwise integration by a different type of integration [18],[19]. However this is not free of problems because the Skorohod integral approach requires the use of a Wick product either on the portfolio or on the self-financing condition, leading to unreasonable situations from the economic point of view (for example positive portfolio with negative Wick value, etc.) [20].

The fractional volatility model in Eqs.(2-3) is not affected by these considerations, because it is the volatility process that is driven by fractional noise, not the price process. In fact a no-arbitrage result may be proven. This is no surprise because our requirement (H2) that, for each sample path $\omega_2 \in \Omega_2$, $\log S_t(\cdot, \omega_2)$ is a square integrable random variable in Ω_1 already implies that $\int \sigma_t dB_t$ is a martingale. The square integrability is also essential to guarantee the possibility of reconstruction of the σ process from the data, because it implies [21]

$$\frac{dS_t}{S_t}(\cdot, \omega_2) \quad = \quad \mu_t(\cdot, \omega_2)\, dt + \sigma_t(\cdot, \omega_2)\, dB_t \tag{5}$$

The empirical sucess of this fractional volatility model was already documented in Ref.[8]. The purpose of the present paper is to give a solid mathematical construction of the fractional volatility model, discussing existence questions, arbitrage and market completeness.

2 No-arbitrage and incompleteness

Let $(\Omega_1, \mathcal{F}_1, P_1)$ be the complete filtered Wiener probability space, carrying a standard Brownian motion $B = (B_t)_{0 \leq t < \infty}$ and a filtration $\mathbb{F}_1 = (\mathcal{F}_{1,t})_{0 \leq t < \infty}$. Let also $(\Omega_2, \mathcal{F}_2, P_2)$ be another probability space associated to a fractional Brownian motion B_H with Hurst parameter $H \in (0,1)$ and a filtration $\mathbb{F}_2 = (\mathcal{F}_{2,t})_{0 \leq t < \infty}$ generated by B_H. We will denote by $\mathbb{E}_1$ and $\mathbb{E}_2$ the expectations with respect to P_1 and P_2 respectively.

Let us now embed these two probability spaces in a product space $(\overline{\Omega}, \overline{\mathcal{F}}, \overline{P})$, where $\overline{\Omega}$ is the Cartesian product $\Omega_1 \times \Omega_2$ and $\overline{P}$ is the product measure $P_1 \otimes P_2$. We also introduce π_1 and π_2, the projections of $\overline{\Omega}$ onto Ω_1 and Ω_2, as well as the $\sigma-$algebra $\mathcal{N}$ generated by all null sets from the product $\sigma-$algebra $\mathcal{F}_1 \otimes \mathcal{F}_2$, that is,

$$\mathcal{N} = \sigma\left(\{F \subseteq \Omega_1 \times \Omega_2 | \exists G \in \mathcal{F}_1 \otimes \mathcal{F}_2 \text{ such that } F \subseteq G \text{ and } (P_1 \otimes P_2)(G) = 0\}\right).$$

Moreover, we let $\overline{\mathcal{F}} = (\mathcal{F}_1 \otimes \mathcal{F}_2) \vee \mathcal{N}$, the $\sigma-$algebra generated by the union of the $\sigma-$algebras $\mathcal{F}_1 \otimes \mathcal{F}_2$ and $\mathcal{N}$. Then $\overline{\mathbb{F}} = \left(\overline{\mathcal{F}}_t\right)_{0 \leq t < \infty}$ is the filtration for $\overline{\mathcal{F}}_t = (\mathcal{F}_{1,t} \otimes \mathcal{F}_{2,t}) \vee \mathcal{N}$.

Furthermore, we extend B and B_H to $\overline{\mathbb{F}}-$adapted processes on $\left(\overline{\Omega}, \overline{\mathcal{F}}, \overline{P}\right)$ by $\overline{B}(\omega_1, \omega_2) = (B \circ \pi_1)(\omega_1, \omega_2)$ and $\overline{B}_H(\omega_1, \omega_2) = (B_H \circ \pi_2)(\omega_1, \omega_2)$ for $(\omega_1, \omega_2) \in \overline{\Omega}$. Then, it is easy to prove that $\overline{B}$ and $\overline{B}_H$ are Brownian and fractional Brownian motions with respect to $\overline{P}$ and are independent. For notational simplicity, hereafter B and B_H will stand for $\overline{B}$ and $\overline{B}_H$.

We now consider a market with a risk-free asset with dynamics given by

$$dA_t = rA_t \, dt \quad A_0 = 1 \tag{6}$$

with $r > 0$ constant and a risky asset with price process S_t given by Eqs.(2)-(3), with μ_t a $\overline{\mathbb{F}}$-adapted process with continuous paths, k the volatility intensity parameter and δ the observation time scale of the process.

The volatility σ_t is a measurable and an $\overline{\mathbb{F}}-$adapted process satisfying for all $0 \leq t < \infty$

$$\begin{aligned}
\mathbb{E}_{\overline{P}}\left[\int_0^t \sigma_s^2 ds\right] &= \int_0^t \theta^2 e^{-\left(\frac{k}{\delta}\right)^2 \delta^{2H}} \mathbb{E}_{\overline{P}}\left[e^{\frac{2k}{\delta}\{B_H(s) - B_H(s-\delta)\}}\right] ds \\
&= \theta^2 \exp\left\{\left(\frac{k}{\delta}\right)^2 \delta^{2H}\right\} t < \infty
\end{aligned}$$

by Fubini's theorem and the moment generating function of the Gaussian random variable $B_H(s) - B_H(s-\delta)$.

Moreover $\int_0^t |\mu_s| \, ds$ being finite $\overline{P}-$almost surely for $0 \leq t < \infty$, an application of Itô's formula yields

$$S_t = S_0 \exp\left\{\int_0^t \left(\mu_s - \frac{1}{2}\sigma_s^2\right) ds + \int_0^t \sigma_s dB_s\right\}$$

Additionally, we assume that investors are allowed to trade only up to some fixed finite planning horizon $T > 0$.

Lemma 2.1. *Consider the measurable process defined by*

$$\gamma_t = \frac{r - \mu_t}{\sigma_t}, \quad 0 \leq t < \infty \tag{7}$$

with $\mu \in L^\infty\left([0, T] \times \overline{\Omega}\right)$. *Then, for* a continuous version of the B_H process

$$\exp\left[\frac{1}{2}\int_0^T \gamma_s^2(\omega_2) \, ds\right] < A(\omega_2) < \infty \tag{8}$$

P_2-almost all $\omega_2 \in \Omega_2$.

We use the fact that P_2-almost surely the paths of a continuous version of fractional Brownian motion are Hölder continuous of any order $\alpha \geq 0$ strictly less than H, that is, there is a random variable $C_\alpha > 0$ such that for P_2-almost all $\omega_2 \in \Omega_2$ $|B_H(t) - B_H(s)| \leq C_\alpha(\omega_2)|t-s|^\alpha$ for every $t, s \in [0, \infty)$

$$\begin{aligned}
& \exp\left[\frac{1}{2}\int_0^T \gamma_s^2(\omega_2)\,ds\right] \\
\leq\ & \exp\left[\frac{e^{k^2\delta^{2H-2}}}{2\theta^2}\int_0^T (r+|\mu_s|)^2 e^{-2\frac{k}{\delta}(B_H(s)-B_H(s-\delta))}ds\right] \\
\leq\ & \exp\left[\frac{(r+\|\mu_s\|_\infty)^2}{2\theta^2}e^{k^2\delta^{2H-2}}\int_0^T e^{-2\frac{k}{\delta}(B_H(s)-B_H(s-\delta))}ds\right] \\
\leq\ & \exp\left\{\frac{T(r+\|\mu_s\|_\infty)^2}{2\theta^2}e^{k^2\delta^{2H-2}+2kC_\alpha(\omega_2)\delta^{\alpha-1}}\right\} < A(\omega_2) < \infty
\end{aligned}$$

■

Proposition 2.2. *The market defined by (2), (3) and (6) is free of arbitrage*

Proof: Restricting the process to a particular path ω_2 of the B_H-process, we construct the stochastic exponential of $\int_0^t \gamma_s(\omega_2)\,dB_s$, that is

$$\eta_t(\omega_2) = \exp\left\{\int_0^t \gamma_s(\omega_2)\,dB_s - \frac{1}{2}\int_0^t \gamma_s^2(\omega_2)\,ds\right\}$$

The bound proved on Lemma 2.1 is the Kallianpur condition [24] that insures that[2]

$$\mathbb{E}_{P_1}[\eta_t(\omega_2)] = 1 \qquad \omega_2 - a.s. \tag{9}$$

Hence, we are in the framework of Girsanov theorem and each nonnegative continuous supermartingale $\eta_t(\omega_2)$ in (**??**) is a true P_1-martingale. Hence we can define for each $0 \leq T < \infty$ a new probability measure $Q_T(\omega_2)$ on $\mathcal{F}_1$ by

$$\frac{dQ_T(\omega_2)}{dP_1} = \eta_T(\omega_2), \quad P_1 - a.s. \tag{10}$$

[2]Notice that this is different from $\mathbb{E}_{\overline{P}}[\eta_t] = 1$ but what is needed for the construction of the equivalent martingale measure by Girsanov is Eq.(9) and not the latter stronger condition.

Then, by the Cameron-Martin-Girsanov theorem, for each fixed $T \in [0, \infty)$, the process

$$B_t^* = B_t - \int_0^t \frac{r - \mu_s}{\sigma_s(\omega_2)} ds \quad 0 \le t \le T \tag{11}$$

is a Brownian motion on the probability space $(\Omega, \mathcal{F}_1, Q_T(\omega_2))$.

Consider now the discounted price process

$$Z_t = \frac{S_t}{A_t} \quad 0 \le t \le T$$

Under the new probability measure $Q_T(\omega_2)$, equivalent to P_1 on $\mathcal{F}_1$, its dynamics is given by

$$Z_t(\omega_2) = Z_0 + \int_0^t \sigma_s(\omega_2) Z_s(\omega_2)\, dB_s^* \tag{12}$$

and is a martingale in the probability space $(\Omega_1, \mathcal{F}_1, Q_T(\omega_2))$ with respect to the filtration $(\mathcal{F}_{1,t})_{0 \le t < T}$. By the fundamental theorem of asset pricing, the existence of an equivalent martingale measure for Z_t implies that there are no arbitrages, that is, $\mathbb{E}_{Q_T(\omega_2)}[Z_t(\omega_2)|\mathcal{F}_{1,s}] = Z_s(\omega_2)$ for $0 \le s < t \le T$.

We have proved that there are no arbitrages for P_2-almost all ω_2 trajectories of the B_H process. But because this process is independent from the B process in (2), it follows that the no-arbitrage result is also valid in the probability product space. ■

Another important concept is market completeness. We note that, in this financial model, trading takes place only in the stock and in the money market and, as a consequence, volatility risk cannot be hedged. Hence, since there are more sources of risk than tradable assets, in this model, the market is incomplete, as proved in the next proposition.

Proposition 2.3. *The market defined by (2),(3) and (6) is incomplete*

*Proof***:** Here we use an integral representation for the fractional Brownian motion [22], [23]

$$B_H(t) = \int_0^t K_H(t, s)\, dW_s \tag{13}$$

W_t being a Brownian motion independent from B_t and K is the square integrable kernel

$$K_H(t, s) = C_H s^{\frac{1}{2}-H} \int_s^t (u - s)^{H-\frac{3}{2}} u^{H-\frac{1}{2}}\, du, \quad s < t$$

$(H > 1/2)$. Then the process

$$\eta'_t = \exp\left(W_t - \frac{1}{2}t\right) \tag{14}$$

is a square-integrable P_2-martingale. Then, defining a standard bi-dimensional Brownian motion,

$$W_t^* = (B_t, W_t)$$

the process $\eta_t^*(\omega_2) = \eta_t \eta_t^{'}(\omega_2)$

$$\eta_t^*(\omega_2) = \exp\left\{\int_0^t \Gamma_s(\omega_2) \bullet dW_t^* - \frac{1}{2}\int_0^t \|\Gamma_s(\omega_2)\|^2 ds\right\}$$

where, by Lemma 2.1, $\Gamma(\omega_2) = (\gamma(\omega_2), 1)$ satisfies the Novikov condition, is also a P_1-martingale. Then, by the Cameron-Martin-Girsanov theorem, the process

$$\widetilde{W}_t^* = \left(\widetilde{W}_t^{*(1)}, \widetilde{W}_t^{*(2)}\right)$$

defined by

$$\begin{aligned} \widetilde{W}_t^{*(1)} &= B_t - \int_0^t \gamma_s(\omega_2)\, ds \\ \widetilde{W}_t^{*(2)} &= W_t - t \end{aligned}$$

is a bi-dimensional Brownian motion on the probability space $(\Omega_1, \mathcal{F}_1, Q_T^*(\omega_2))$, where $Q_T^*(\omega_2)$ is the probability measure

$$\frac{dQ_T^*(\omega_2)}{dP_1} = \eta_T^*(\omega_2) \tag{15}$$

Moreover, the discounted price process Z remains a martingale with respect to the new measure $Q_T^*(\omega_2)$. $Q_T^*(\omega_2)$ being an equivalent martingale measure distinct from $Q_T(\omega_2)$, the market is incomplete.

■

As stated above, incompleteness of the market is a reflection of the fact that there are two different sources of risk and only one of the risks is tradable. A choice of measure is how one fixes the volatility risk premium.

3 Leverage and completeness

The following nonlinear correlation of the returns

$$L(\tau) = \left\langle |r(t+\tau)|^2 r(t) \right\rangle - \left\langle |r(t+\tau)|^2 \right\rangle \langle r(t) \rangle \tag{16}$$

is called *leverage* and the *leverage effect* is the fact that, for $\tau > 0$, $L(\tau)$ starts from a negative value whose modulus decays to zero whereas for $\tau < 0$ it has almost negligible values. In the form of Eqs.(2)(3) the volatility process σ_t affects the log-price, but is not affected by it. Therefore, in its simplest form the fractional volatility model contains no leverage effect.

Leverage may, however, be implemented in the model in a simple way. Using a standard representation for fractional Brownian motion [23] as a stochastic integral over Brownian motion and identifying the random generator of the log-price process with the stochastic integrator of the volatility, at least a part of the leverage effect is taken into account [25]. Here, for the generator of the volatility process we use a truncated version of a representation of Fractional Brownian motion [23],

$$\mathcal{H}(t) = \Pi^{(M)} \left[C_H \left\{ \int_{-\infty}^{0} \left((t-u)^{H-\frac{1}{2}} - (-u)^{H-\frac{1}{2}} \right) dW_u + \int_0^t (t-u)^{H-\frac{1}{2}} dW_u \right\} \right] \tag{17}$$

$\Pi^{(M)}$ meaning the truncation of the representation to an interval $[-M, M]$ with M arbitrarily large.

The identification of the two Brownian processes means that now, instead of two, there is only one source of risk. Hence it is probable that in this case completeness of the market might be achieved.

The new fractional volatility model would be

$$\begin{aligned} dS_t &= \mu_t S_t dt + \sigma_t S_t dW_t \\ \log \sigma_t &= \beta + \frac{k'}{\delta} \{\mathcal{H}(t) - \mathcal{H}(t-\delta)\} \end{aligned} \tag{18}$$

Proposition 3.1. *The market defined by (18), (17) and (6) is free of arbitrage and complete.*

Proof: In this case because the two processes are not independent we cannot use the same argument as before to obtain the Kallianpur condition. However with the truncation in (17) the Hölder condition is trivially verified for all the truncated paths of σ_t and the construction of an equivalent martingale measure follows the same steps as in Proposition 2.2. Hence we have a P_1-martingale with respect to $(\mathcal{F}_{1,t})_{0\leq t<T}$

$$\eta_t = \exp\left\{\int_0^t \frac{r-\mu_s}{\sigma_s} dW_s - \frac{1}{2}\int_o^t \left(\frac{r-\mu_s}{\sigma_s}\right)^2 ds\right\}$$

and the probability measure Q_T, defined by $\frac{dQ_T}{dP_1} = \eta_T$ is an equivalent martingale measure.

The set of equivalent local martingale measures for the market being nonempty, let Q^* be an element in this set. Then, recalling that $(\mathcal{F}_{1,t})_{0\leq t<T}$ is the augmentation of the natural filtration of the Brownian motion W_t, by the Girsanov converse [26] [27] there is a $(\mathcal{F}_{1,t})_{0\leq t<T}$ progressively measurable $\mathbb{R}$-valued process ϕ such that the Radon-Nikodym density of Q^* with respect to P_1 equals

$$\frac{dQ^*_T}{dP_1} = \exp\left\{\int_0^T \phi_s dW_s - \frac{1}{2}\int_0^T \phi_s^2 ds\right\}$$

Moreover the process W^*_t given by

$$W^*_t = W_t - \int_0^t \phi_s ds$$

is a standard Q^*-Brownian motion and the discounted price process Z satisfies the following stochastic differential equation

$$dZ_t = (\mu_t - r + \sigma_t\phi_t)Z_t dt + \sigma_t Z_t dW^*_t$$

Because Z_t is a Q^*-martingale, then it must be hold $\mu(t,\omega)-r+\sigma(t,\omega)\phi(t,\omega) = 0$ almost everywhere w.r.t. $dt \times P$ in $[0,T]\times\Omega$. It implies

$$\phi(t,\omega) = \frac{r-\mu(t,\omega)}{\sigma(t,\omega)}$$

a. e. $(t,\omega)\in[0,T]\times\Omega_1$. Hence $Q^*_T = Q_T$, that is, Q_T is the unique equivalent martingale measure. This market model is complete.

■

4 Remarks and conclusions

1) Partially reconstructed from empirical data, the fractional volatility model describes well the statistics of returns. The fact that, once the parameters are adjusted by the data for a particular observation time scale δ, it describes well different time lags is related to the fact that the volatility is driven not by fractional Brownian motion but its increments.

Specific trader strategies and psychology should play a role on market crisis and bubbles. However, the fact that in the fractional volatility model the same set of parameters would describe very different markets [8] seems to imply that the market statistical behavior (in normal days) is more influenced by the nature of the financial institutions (the double auction process) than by the traders strategies [11]. Therefore some kind of universality of the statistical behavior of the bulk data throughout different markets would not be surprising.

The identification of the Brownian process of the log-price with the one that generates the fractional noise driving the volatility, introduces an asymmetric coupling between σ_t and S_t that is also exhibited by the market data.

2) In this paper, mathematical consistency of the fractional volatility model has been established. This and its better consistency with the experimental data, makes it a candidate to replace geometrical Brownian as the standard market model.

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