# On Idempotent Generated Semigroups

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**Abstract**

We provide short and direct proofs for some classical theorems proved by Howie, Levi and McFadden concerning idempotent generated semigroups of transformations on a finite set.

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Let $n$ be a natural number and $[n] = \{1, \ldots, n\}$. Let $T_n, S_n$ be, respectively, the transformation semigroup and the symmetric group on $[n]$. Let $P = (A_i)_{i\in[k]}$ be a partition of $[n]$ and let $C = \{x_1, \ldots, x_k\}$ be a cross-section of $P$ (say $x_i \in A_i$). Then we represent $A_i$ by $[x_i]_P$ and the pair $(P, C)$ induces an idempotent mapping defined by $[x_i]_P e = \{x_i\}$. Conversely, every idempotent can be so constructed. To save space instead of $e = \begin{pmatrix} [x_1]_P & \ldots & [x_k]_P \\ x_1 & \ldots & x_k \end{pmatrix}$ we write $e = ([x_1]_P, \ldots, [x_k]_P)$. This notation extends to $e = ([\underline{x_1}, y]_P, [x_2]_P, \ldots, [x_k]_P)$ when $y \in [x_1]_P$ and $[x_i]_P e = \{x_i\}$. By $([x_1], \ldots, [\underline{x_i}, y], \ldots, [x_k])$ we denote the set of all idempotents $e$ with image $\{x_1, \ldots, x_n\}$ and such that the $Ker(e)$-class of $x_i$ contains (at least) two elements: $x_i$ and $y$, where the underlined element (in this case $x_i$) is the image of the class under $e$. For $a \in T_n$, we denote the image of $a$ by $\nabla a$.

**Lemma 1** *Let $a \in I_n = T_n \setminus S_n$, $rank(a) = k$, and $(xy) \in Sym(X)$. Then $a(xy) = ab$, where $b = 1$ or $b = e_1e_2e_3$, with $e_i^2 = e_i$ and $rank(e_i) = k$.*

*Proof.* Let $\nabla a = \{a_i \mid i \in [k]\}$. If $x, y \notin \nabla a$, then $a(xy) = a$. If, say, $x = a_1$ and $y \notin \nabla a$, then $ae_1 = a(a_1y)$, for all $e_1 \in ([a_1, \underline{y}], [a_2], \ldots, [a_k])$. Finally, if $x, y \in \nabla a$, without loss of generality, we can assume that $x = a_1$ and $y = a_2$. Then $a(a_1\ a_2) = ae_2e_3e_4$, for $e_2 \in ([a_1], [a_2, \underline{u}], [a_3], \ldots, [a_k])$, $e_3 \in ([u], [a_1, \underline{a_2}], [a_3], \ldots, [a_k])$ and $e_4 \in ([u, \underline{a_1}], [a_2], \ldots, [a_k])$. ■

**Theorem 2** *[1] Every ideal of $I_n$ is generated by its own idempotents.*

*Proof.* Let $a \in I_n$. Then $a = eg$ for some $e = e^2 \in T_n$ and $g \in S_n$. Therefore $a = e(x_1y_1)\dots(x_my_m)$ and hence, applying $m$ times Lemma 1, $a$ can be obtained as a product of idempotents of the same rank as $a$. ■

Let $t \in I_n$ and $g \in S_n$. Denote $g^{-1}tg$ by $t^g$ and let $C_t = \{t^g \mid g \in S_n\}$ and $t^{S_n} = \langle \{t\} \cup S_n \rangle \setminus S_n$. For $f = ([x_1, \underline{w}]_Q, [x_2]_Q, \dots, [x_k]_Q)$ and $g \in S_n$, it is easy to check that we have $f^g = ([x_1g, \underline{wg}]_{Qg}, \dots, [x_kg]_{Qg}) \in ([x_1g, \underline{wg}], \dots, [x_kg])$.

**Lemma 3** *Let $a \in I_n$, $rank(a) = k$, let $f = ([x_1, \underline{w}]_Q, [x_2]_Q, \dots, [x_k]_Q)$ and let $(xy) \in S_n$. Then $a(xy) = ab$, where $b = 1$ or $b = e_1e_2e_3$, with $e_i \in C_f$.*

*Proof.* As in Lemma 1, one only has to show that the sets $([a_1, \underline{y}], [a_2], \dots, [a_k])$, $([a_1], [a_2, \underline{u}], \dots, [a_k])$, $([u], [a_1, \underline{a_2}], \dots, [a_k])$ and $([\underline{a_1}, u], [a_2], \dots, [a_k])$ intersect $C_f$. Let $g \in S_n$ such that $x_1g = a_1, wg = y$ and $x_ig = a_i$ $(2 \le i \le k)$. Thus $f^g = ([x_1g, \underline{wg}]_{Qg}, [x_2g]_{Qg}, \dots, [x_kg]_{Qg}) \in ([a_1, \underline{y}], [a_2], \dots, [a_k])$. The proof that $C_f$ intersects the remaining three sets is similar. ■

Repeating the arguments of Theorem 1 together with Lemma 3 we have

**Corollary 4** *Let $eg \in I_n$ (for some $e = e^2$, $g \in S_n$) and let $f^2 = f$ with $rank(eg) = rank(f)$. Then $eg \in e\langle C_f \rangle$ and, in particular, $eg \in \langle C_e \rangle$.*

From now on let $a = eg \in I_n$ (for some $e = e^2$, $g \in S_n$).

**Corollary 5** $\langle C_e \rangle = e^{S_n} = (eg)^{S_n} = a^{S_n}$.

*Proof.* The only non-trivial inclusion is $e^{S_n} \subseteq \langle C_e \rangle$. As $e^{S_n}$ is generated by $\{geh \mid g, h \in S_n\}$, let $g, h \in S_n$. By Corollary 4, $he = (heh^{-1})h \in \langle C_{heh^{-1}} \rangle$ and $eg \in \langle C_e \rangle$. Since $heh^{-1} \in \langle C_e \rangle$, it follows that $\langle C_{heh^{-1}} \rangle \le \langle C_e \rangle$. Thus $heg = (he)(eg) \in \langle C_e \rangle$. It is proved that $e^{S_n} \subseteq \langle C_e \rangle$. ■

**Theorem 6** *[2] $\langle C_a \rangle = \langle C_e \rangle = a^{S_n}$ and hence is idempotent generated.*

*Proof.* Since $a = eg$, then $e = ag^{-1}$ and it is easy to check that we have $(ag^{-1})^n = aa^ga^{g^2}\dots a^{g^{n-1}}g^{-n}$. But for some natural $m$ we have $g^{-m} = 1$. Thus $(ag^{-1})^m = aa^ga^{g^2}\dots a^{g^{m-1}} \in \langle C_a \rangle$ so that $e = e^m = (ag^{-1})^m \in \langle C_a \rangle$ and hence, by Corollary 5, it follows that $a^{S_n} = \langle C_e \rangle \le \langle C_a \rangle \le a^{S_n}$. (Cf. [3]). ■

# References


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