Semigroups of Transformations Preserving an Equivalence Relation and a Cross-Section

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Abstract

For a set X, an equivalence relation ρ on X, and a cross-section R of the partition X/ρ induced by ρ, consider the semigroup $T(X,\rho,R)$ consisting of all mappings a from X to X such that a preserves both ρ (if $(x,y)\in\rho$ then $(xa,ya)\in\rho$) and R (if $r\in R$ then $ra\in R$). The semigroup $T(X,\rho,R)$ is the centralizer of the idempotent transformation with kernel ρ and image R. We determine the structure of $T(X,\rho,R)$ in terms of Green's relations, describe the regular elements of $T(X,\rho,R)$, and determine the following classes of the semigroups $T(X,\rho,R)$: regular, abundant, inverse, and completely regular.

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1 Introduction

Let X be an arbitrary nonempty set. The semigroup $T(X)$ of full transformations on X consists of the mappings from X to X with composition as the semigroup operation.

Let ρ be an equivalence relation on X and let R be a cross-section of the partition X/ρ induced by ρ. Consider the following subset of $T(X)$:

$$T(X,\rho,R)=\{a\in T(X): Ra\subseteq R \text{ and } (x,y)\in\rho\Rightarrow(xa,ya)\in\rho\}.$$

Clearly $T(X,\rho,R)$ is a subsemigroup of $T(X)$. The family of semigroups $T(X,\rho,R)$ includes the semigroup $T(X)$ ($T(X)=T(X,\Delta,X)$ where $\Delta=\{(x,x):x\in X\}$) and the semigroup $PT(X')$ of partial transformations on X' where X' is X with one element removed (if $X'=X-\{r\}$ then $PT(X')$ is isomorphic to $T(X,X\times X,\{r\})$).

Another way of describing the semigroups $T(X,\rho,R)$ is through the notion of the centralizer. Let S be a semigroup and $a\in S$. The *centralizer* $C(a)$ of a is defined as

$$C(a)=\{b\in S: ab=ba\}.$$

It is clear that $C(a)$ is a subsemigroup of S.

The full transformation semigroup $T(X)$ is the centralizer of the identity mapping id_X on X: $T(X) = C(id_X)$. More generally, the semigroups $T(X, \rho, R)$ are the centralizers of the idempotent transformations: $T(X, \rho, R)$ is the centralizer of the idempotent in $T(X)$ with kernel ρ and image R [1].

Centralizers in $T(X)$ for a finite set X have been studied by Higgins [4], Liskovec and Feĭnberg [10], [11], and Weaver [14]. The second author has studied centralizers in the semigroup $PT(X)$ of partial transformations on a finite set X [6], [7], [8]. In [1], the authors determined the automorphism group of $T(X, \rho, R)$.

In this paper, we study the structure and regularity of the semigroups $T(X, \rho, R)$ for an arbitrary set X. In Section 2, we determine Green's relations in $T(X, \rho, R)$. In particular, we find that, in general, the relations $\mathcal{D}$ and $\mathcal{J}$ are not the same in $T(X, \rho, R)$, and that the $\mathcal{J}$-classes of $T(X, \rho, R)$ do not form a chain. We characterize the relations ρ for which $\mathcal{D} = \mathcal{J}$ and the relations ρ for which the partially ordered set of $\mathcal{J}$-classes is a chain. In Section 3, we describe the regular elements of $T(X, \rho, R)$ and characterize the relations ρ for which $T(X, \rho, R)$ is a regular semigroup. In Section 4, we show that abundant semigroups $T(X, \rho, R)$ are precisely those that are regular. Finally, in Section 5, we determine that $T(X, \rho, R)$ is never an inverse semigroup (if $|X| \geq 3$) or a completely regular semigroup (if $|X| \geq 4$).

2 Green's Relations in $T(X, \rho, R)$

For $a \in T(X)$, we denote the kernel of a (the equivalence relation $\{(x, y) \in X \times X : xa = ya\}$) by $\mathrm{Ker}(a)$ and the image of a by ∇a. For $Y \subseteq X$, Ya will denote the image of Y under a, that is, $Ya = \{xa : x \in Y\}$. As customary in transformation semigroup theory, we write transformations on the right (that is, xa instead of $a(x)$).

Let ρ be an equivalence relation on X and R a cross-section of X/ρ. If $x \in X$ then there is exactly one $r \in R$ such that $x \,\rho\, r$, which will be denoted by r_x. Of course, for $s \in R$, we have $r_s = s$.

For the remainder of the paper, ρ will denote an equivalence relation on X and R will denote a cross-section of X/ρ.

If S is a semigroup and $a, b \in S$, we say that $a \,\mathcal{R}\, b$ if $aS^1 = bS^1$, $a \,\mathcal{L}\, b$ if $S^1a = S^1b$, and $a \,\mathcal{J}\, b$ if $S^1aS^1 = S^1bS^1$, where S^1 is the semigroup S with an identity adjoined, if necessary. We define $\mathcal{H}$ as the intersection of $\mathcal{L}$ and $\mathcal{R}$, and $\mathcal{D}$ as the join of $\mathcal{L}$ and $\mathcal{R}$, that is, the smallest equivalence relation on S containing both $\mathcal{L}$ and $\mathcal{R}$. These five equivalence relations on S are known as *Green's relations* [5, p. 45]. The relations $\mathcal{L}$ and $\mathcal{R}$ commute [5, Proposition 2.1.3], and consequently $\mathcal{D} = \mathcal{L} \circ \mathcal{R} = \mathcal{R} \circ \mathcal{L}$. If $\mathcal{T}$ is one of Green's relations and $a \in S$, we denote the equivalence class of a with respect to $\mathcal{T}$ by T_a. Since $\mathcal{R}$, $\mathcal{L}$, and $\mathcal{J}$ are defined in terms of principal ideals in S, which are partially ordered by inclusion, we have the induced partial orders in the sets of the equivalence classes of $\mathcal{R}$, $\mathcal{L}$, and $\mathcal{J}$: $R_a \leq R_b$ if $aS^1 \subseteq bS^1$, $L_a \leq L_b$ if $S^1a \subseteq S^1b$, and $J_a \leq J_b$ if $S^1aS^1 \subseteq S^1bS^1$.

Green's relations in the semigroup $T(X)$ are well known [5, Exercise 16, p. 63].

Lemma 2.1 *If $a, b \in T(X)$ then:*

(1) $a \,\mathcal{R}\, b \Leftrightarrow \mathrm{Ker}(a) = \mathrm{Ker}(b)$.

(2) $a\,\mathcal{L}\,b \Leftrightarrow \nabla a = \nabla b$.

(3) $a\,\mathcal{D}\,b \Leftrightarrow |\nabla a| = |\nabla b|$.

(4) $\mathcal{D} = \mathcal{J}$.

Our aim in this section is to describe Green's relations in the semigroups $T(X, \rho, R)$.

2.1 Relations $\mathcal{R}$ and $\mathcal{L}$

The relation $\mathcal{R}$ in $T(X, \rho, R)$ is simply the restriction of the relation $\mathcal{R}$ in $T(X)$ to $T(X, \rho, R) \times T(X, \rho, R)$. This result will follow from the following lemma.

Lemma 2.2 *Let $a, b \in T(X, \rho, R)$. Then $R_a \leq R_b$ if and only if* $\mathrm{Ker}(b) \subseteq \mathrm{Ker}(a)$.

Proof: Suppose $R_a \leq R_b$. Then there is $c \in T(X, \rho, R)$ such that $a = bc$, and so for all $x, y \in X$, $xb = yb$ implies $xa = (xb)c = (yb)c = ya$. Thus $\mathrm{Ker}(b) \subseteq \mathrm{Ker}(a)$.

Conversely, suppose $\mathrm{Ker}(b) \subseteq \mathrm{Ker}(a)$. We shall construct $c \in T(X, \rho, R)$ such that $a = bc$. Consider an equivalence class $r\rho$ where $r \in R$. If $r \notin \nabla b$, define $yc = y$ for every $y \in r\rho$. Suppose $r \in \nabla b$. Then, since $b \in T(X, \rho, R)$, $r = tb$ for some $t \in R$. Since $a \in T(X, \rho, R)$, $ta = p$ for some $p \in R$. Let $y \in r\rho$. If $y = xb \in \nabla b$, define $yc = xa$; if $y \notin \nabla b$, define $yc = p$. Note that c is well defined since for all $x, x' \in X$, if $xb = x'b$ then $xa = x'a$ (since $\mathrm{Ker}(b) \subseteq \mathrm{Ker}(a)$) and so $(xb)c = xa = x'a = (x'b)c$. It is clear by the construction of c that $bc = a$. It remains to show that $c \in T(X, \rho, R)$.

If $r \notin \nabla b$ then $rc = r \in R$ and $(r\rho)c = r\rho$. Suppose $r \in \nabla b$. By the definition of c, $rc = (tb)c = ta = p \in R$. Next we show that $(r\rho)c \subseteq p\rho$. Let $y \in r\rho$. If $y \notin \nabla b$ then $yc = p \in p\rho$, and so $(r\rho)c \subseteq p\rho$, in this case. Let $y = xb \in \nabla b$. Then $x \in q\rho$ for some $q \in R$. Since $x \in q\rho$ and $xb \in r\rho$, $qb = r$. Since $\mathrm{Ker}(b) \subseteq \mathrm{Ker}(a)$, $tb = qb$ $(= r)$ implies $ta = qa$. Thus $qa = ta = p$, and so $(q\rho)a \subseteq p\rho$. Hence $yc = xa \in p\rho$. It follows that $c \in T(X, \rho, R)$, and so $R_a \leq R_b$. ■

Theorem 2.3 *Let $a, b \in T(X, \rho, R)$. Then $a\,\mathcal{R}\,b$ if and only if* $\mathrm{Ker}(a) = \mathrm{Ker}(b)$.

Proof: It follows immediately from Lemma 2.2. ■

Let $\mathcal{A}$ and $\mathcal{B}$ be families of sets. We write $\mathcal{A} \hookrightarrow \mathcal{B}$ if for every set $C \in \mathcal{A}$ there is a set $D \in \mathcal{B}$ such that $C \subseteq D$. If $\mathcal{A} \hookrightarrow \mathcal{B}$ and $\mathcal{B} \hookrightarrow \mathcal{A}$, we write $\mathcal{A} \leftrightarrow \mathcal{B}$.

Our characterization of the relation $\mathcal{L}$ in $T(X, \rho, R)$ will follow from the following lemma. For $a \in T(X, \rho, R)$, we denote by $\blacktriangledown a$ the family $\{(r\rho)a : r \in R\}$.

Lemma 2.4 *Let $a, b \in T(X, \rho, R)$. Then $L_a \leq L_b$ if and only if $\blacktriangledown a \hookrightarrow \blacktriangledown b$.*

Proof: Suppose $L_a \leq L_b$. Then there is $c \in T(X, \rho, R)$ such that $a = cb$. Let $A \in \blacktriangledown a$. Then $A = (r\rho)a = ((r\rho)c)b$ for some $r \in R$. Since $c \in T(X, \rho, R)$, $(r\rho)c \subseteq t\rho$ for some $t \in R$. Thus $A \subseteq (t\rho)b \in \blacktriangledown b$, and so $\blacktriangledown a \hookrightarrow \blacktriangledown b$.

Conversely, suppose $\blacktriangledown a \hookrightarrow \blacktriangledown b$. To construct $c \in T(X, \rho, R)$ such that $a = cb$, consider $r\rho$ $(r \in R)$. Since $\blacktriangledown a \hookrightarrow \blacktriangledown b$ and $b \in T(X, \rho, R)$, $(r\rho)a \subseteq (t\rho)b \subseteq p\rho$ for some $t, p \in R$. Thus, for every $x \in r\rho$, we can select $y_x \in t\rho$ such that $xa = y_x b$ (if $x = r$, we may assume that $y_x = t$ since $ra = tb = p$) and define $xc = y_x$. By the construction of c, $a = cb$ and $c \in T(X, \rho, R)$ (since $(r\rho)c \subseteq t\rho$ and $rc = t$). Thus $L_a \leq L_b$. ■

Theorem 2.5 *Let $a, b \in T(X, \rho, R)$. Then $a \mathcal{L} b$ if and only if $\blacktriangledown a \leftrightarrow \blacktriangledown b$.*

Proof: It follows immediately from Lemma 2.4. ■

For $a \in T(X, \rho, R)$, denote by $m(\blacktriangledown a)$ the family of all sets maximal in $\blacktriangledown a$ (with respect to inclusion). Suppose X is finite. Then for every $A \in \blacktriangledown a$, there is $A' \in m(\blacktriangledown a)$ such that $A \subseteq A'$. (This is not necessarily true if X is infinite.) It easily follows that for all $a, b \in T(X, \rho, R)$, $\blacktriangledown a \leftrightarrow \blacktriangledown b$ if and only if $m(\blacktriangledown a) = m(\blacktriangledown b)$. Thus in the finite case, $a \mathcal{L} b$ if and only if $m(\blacktriangledown a) = m(\blacktriangledown b)$ [7].

2.2 Relations $\mathcal{D}$ and $\mathcal{J}$

Let $f : Y \to Z$ be a function from a set Y to a set Z. For a family $\mathcal{A}$ of subsets of Y, $f(\mathcal{A})$ denotes the family $\{f(A) : A \in \mathcal{A}\}$ of subsets of Z. The following theorem characterizes Green's $\mathcal{D}$-relation in $T(X, \rho, R)$.

Theorem 2.6 *Let $a, b \in T(X, \rho, R)$. Then $a \mathcal{D} b$ if and only if there is a bijection $f : \nabla a \to \nabla b$ such that $f(R \cap \nabla a) \subseteq R$ and $f(\blacktriangledown a) \leftrightarrow \blacktriangledown b$.*

Proof: Suppose $a \mathcal{D} b$. Since $\mathcal{D} = \mathcal{R} \circ \mathcal{L}$, there is $c \in T(X, \rho, R)$ such that $a \mathcal{R} c$ and $c \mathcal{L} b$. Then, by Theorem 2.3 and Theorem 2.5, $\mathrm{Ker}(a) = \mathrm{Ker}(c)$ and $\blacktriangledown c \leftrightarrow \blacktriangledown b$.

Next we shall construct a bijection $f : \nabla a \to \nabla c$ such that $f(R \cap \nabla a) \subseteq R$ and $f(\blacktriangledown a) = \blacktriangledown c$. For every $xa \in \nabla a$, define $f(xa) = xc$. For all $xa, x'a \in \nabla a$, $f(xa) = f(x'a) \Leftrightarrow xc = x'c \Leftrightarrow xa = x'a$ (since $\mathrm{Ker}(a) = \mathrm{Ker}(c)$). Thus f is well defined and one-to-one. It is obviously onto since for every $xc \in \nabla c$, $xc = f(xa)$. Suppose $r \in R \cap \nabla a$. Then there is $t \in R$ such that $r = ta$, and so $f(r) = f(ta) = tc \in R$. Thus $f(R \cap \nabla a) \subseteq R$. For every $(r\rho)a \in \blacktriangledown a$ $(r \in R)$, $f((r\rho)a) = \{f(xa) : x \in r\rho\} = \{xc : x \in r\rho\} = (r\rho)c$. It follows that $f(\blacktriangledown a) = \blacktriangledown c$, and so $f(\blacktriangledown a) \leftrightarrow \blacktriangledown b$.

Conversely, suppose there is a bijection $f : \nabla a \to \nabla b$ such that $f(R \cap \nabla a) \subseteq R$ and $f(\blacktriangledown a) \leftrightarrow \blacktriangledown b$. Define $c \in T(X)$ by $xc = f(xa)$. Let $r \in R$. Then $ra \in R$ and so $rc = f(ra) \in R$. Thus $Rc \subseteq R$. Moreover, $(r\rho)c = f((r\rho)a) \in f(\blacktriangledown a)$. Since $f(\blacktriangledown a) \leftrightarrow \blacktriangledown b$, there is $B \in \blacktriangledown b$ such that $(r\rho)c \subseteq B$. Since $B \in \blacktriangledown b$, $B \subseteq t\rho$ for some $t \in R$. Thus $(r\rho)c \subseteq t\rho$. It follows that $c \in T(X, \rho, R)$. For all $x, x' \in X$, $xc = x'c \Leftrightarrow f(xa) = f(x'a) \Leftrightarrow xa = x'a$ (since f is one-to-one). Thus $\mathrm{Ker}(a) = \mathrm{Ker}(c)$. Since for every $r \in R$, $(r\rho)c = f((r\rho)a)$, we have $\blacktriangledown c = f(\blacktriangledown a)$. Thus $\blacktriangledown c \leftrightarrow \blacktriangledown b$. Hence, by Theorem 2.3 and Theorem 2.5, $a \mathcal{R} c$ and $c \mathcal{L} b$, which gives $a \mathcal{D} b$. ■

Suppose X is finite and let $f : \nabla a \to \nabla b$ be as in the statement of Theorem 2.6. Since f is a bijection, $f(m(\blacktriangledown a)) = m(f(\blacktriangledown a))$. Thus, by the argument after Theorem 2.5, $f(\blacktriangledown a) \leftrightarrow \blacktriangledown b$ if and only if $f(m(\blacktriangledown a)) = m(\blacktriangledown b)$. Hence in the finite case, $a \mathcal{D} b$ if and only if there is a bijection $f : \nabla a \to \nabla b$ such that $f(R \cap \nabla a) \subseteq R$ and $f(m(\blacktriangledown a)) = m(\blacktriangledown b)$ [7].

In the semigroup $T(X)$, $\mathcal{D} = \mathcal{J}$ (Lemma 2.1). In general, this result is not true for $T(X, \rho, R)$. A characterization of the $\mathcal{J}$-relation in $T(X, \rho, R)$ is provided by Theorem 2.8. We start with the theorem that determines the partial order of the $\mathcal{J}$-classes in $T(X, \rho, R)$.

Theorem 2.7 *Let $a, b \in T(X, \rho, R)$. Then $J_a \le J_b$ if and only if there is a function $g : \nabla b \to \nabla a$ such that $g(R \cap \nabla b) \subseteq R$ and $\blacktriangledown a \hookrightarrow g(\blacktriangledown b) \hookrightarrow X/\rho$.*

Proof: Suppose $J_a \leq J_b$. Then $a = cbd$ for some $c, d \in T(X, \rho, R)$. Fix $r_0 \in \nabla a$ and define $g : \nabla b \to \nabla a$ by:

$$g(x) = \begin{cases} xd & \text{if } x \in \nabla(cb) \\ r_x d & \text{if } x \notin \nabla(cb) \text{ but } x\rho \cap \nabla(cb) \neq \emptyset \\ r_0 & \text{if } x\rho \cap \nabla(cb) = \emptyset. \end{cases}$$

Let $x \in \nabla b$. If $x \in \nabla(cb)$ then $xd \in \nabla a$ (since $a = cbd$). If $x \notin \nabla(cb)$ but $x\rho \cap \nabla(cb) \neq \emptyset$ then $r_x \in \nabla(cb)$, and so $r_x d \in \nabla a$. Thus g indeed maps ∇b to ∇a.

We have $g(R \cap \nabla b) \subseteq R$ since $Rd \subseteq R$. Let $C \in \blacktriangledown b$. Then $C = (r\rho)b$ for some $r \in R$. By the definition of g, either $g(C) = \{r_0\} \subseteq r_0\rho$ (if $x\rho \cap \nabla(cb) = \emptyset$) or $g(C) = (C \cap \nabla(cb))d \subseteq Cd = (r\rho)bd \subseteq q\rho$ for some $q \in R$ (since $b, d \in T(X, \rho, R)$). It follows that $g(\blacktriangledown b) \hookrightarrow X/\rho$.

Let $A \in \blacktriangledown a$. Then $A = (p\rho)a$ for some $p \in R$. Since $c \in T(X, \rho, R)$, $(p\rho)c \subseteq r\rho$ for some $r \in R$. Let $C = (r\rho)b$. Then $C \in \blacktriangledown b$ and $A = (p\rho)a = (p\rho)(cbd) \subseteq (C \cap \nabla(cb))d = g(C) \in g(\blacktriangledown b)$. Thus $\blacktriangledown a \hookrightarrow g(\blacktriangledown b)$

Conversely, suppose there is a function $g : \nabla b \to \nabla a$ such that $g(R \cap \nabla b) \subseteq R$ and $\blacktriangledown a \hookrightarrow g(\blacktriangledown b) \hookrightarrow X/\rho$. Let $A \in \blacktriangledown a$. Then there is a unique $q_A \in R$ such that $A \subseteq q_A\rho$. Since $\blacktriangledown a \hookrightarrow g(\blacktriangledown b)$, there is $C_A \in \blacktriangledown b$ such that $A \subseteq g(C_A)$. Since $C_A \in \blacktriangledown b$, there is $r_A \in R$ such that $C_A = (r_A\rho)b$. Let $t_A = r_A b$. Since $b \in T(X, \rho, R)$, $t_A \in R$ and $C_A = (r_A\rho)b \subseteq t_A\rho$. For every $z \in A$, select $u_z^A \in C_A$ such that $z = g(u_z^A)$ (since $g(t_A) = q_A$, we may assume that $u_z^A = t_A$ if $z = q_A$). Let $C'_A = \{u_z^A : z \in A\}$ and note that $C'_A \subseteq C_A$. For every $y \in C'_A$, select $w_y^A \in r_A\rho$ such that $y = w_y^A b$ (since $r_A b = t_A$, we may assume that $w_y^A = r_A$ if $y = t_A$).

We shall construct $c, d \in T(X, \rho, R)$ such that $a = cbd$. To construct c, let $x \in X$. Then there is a unique $p \in R$ such that $x \in p\rho$. Let $A = (p\rho)a$ and $z = xa$. Then $A \in \blacktriangledown a$ and $z \in A$. Let $y = u_z^A$ and note that $y \in C'_A$. Define $xc = w_y^A$. By the definition of c, we have $(p\rho)c \subseteq r_A\rho$. Since $pa = q_A$ (from the paragraph above), we have $pc = r_A$ (since $u_z^A = t_A$ for $z = q_A$ and $w_y^A = r_A$ for $y = t_A$). It follows that $c \in T(X, \rho, R)$.

To construct d, let $y \in X$. Then there is a unique $t \in R$ such that $y \in t\rho$. We define yd as follows:

(i) If $y \in C'_A$ for some $A \in \blacktriangledown a$, define $yd = g(y)$.

(ii) If $y \notin C'_B$ for every $B \in \blacktriangledown a$ but $C_A \subseteq t\rho$ for some $A \in \blacktriangledown a$, define $yd = q_A$.

(iii) If there is no $A \in \blacktriangledown a$ such that $C_A \subseteq t\rho$, define $yd = y$.

If $C_A, C_B \subseteq t\rho$ for some $A, B \in \blacktriangledown a$ then $q_A = g(t) = q_B$, and so the definition of d in (ii) does not depend on the choice of A. Thus d is well defined.

Let $t \in R$ and consider $t\rho$. Suppose $C_A \subseteq t\rho$ for some $A \in \blacktriangledown a$. Then, by (i), $td = g(t) = q_A$. Let $B \in \blacktriangledown a$ be such that $C_B \subseteq t\rho$. Then $q_A = g(t) = q_B$. It follows that $B \subseteq q_A\rho$ and so, by (i) and (ii), $(t\rho)d \subseteq q_A\rho$. If there is no $A \in \blacktriangledown a$ such that $C_A \subseteq t\rho$ then, by (iii), $td = t$ and $(t\rho)d = t\rho$. It follows that $d \in T(X, \rho, R)$.

Let $x \in X$. Then there is a unique $p \in R$ such that $x \in p\rho$. Let $A = (p\rho)a$ and $z = xa$. Then $A \in \blacktriangledown a$ and $z \in A$. Let $y = u_z^A$ and note that $y \in C'_A$. By the definition of c and d, $xc = w_y^A$ (recall that w_y^A was selected so that $w_y^A b = y$) and $yd = g(y) = g(u_z^A) = z = xa$. Thus $x(cbd) = w_y^A(bd) = yd = xa$. Hence $a = cbd$ and so $J_a \leq J_b$. ■

Theorem 2.8 *Let $a, b \in T(X, \rho, R)$. Then $a \mathcal{J} b$ if and only if there are functions $f : \nabla a \to \nabla b$ and $g : \nabla b \to \nabla a$ such that $f(R \cap \nabla a) \subseteq R$, $\blacktriangledown b \hookrightarrow f(\blacktriangledown a) \hookrightarrow X/\rho$, $g(R \cap \nabla b) \subseteq R$, and $\blacktriangledown a \hookrightarrow g(\blacktriangledown b) \hookrightarrow X/\rho$.*

Proof: It is immediate by Theorem 2.7. ■

Let $a, b \in T(X, \rho, R)$ with $a \mathcal{J} b$, and let f and g be functions as in the statement of Theorem 2.8. We make the following observations.

(1) The functions f and g are onto.

(2) $|\nabla a| = |\nabla b|$.

(3) For every $r \in R$, there are $s, t \in R$ such that $f(r\rho \cap \nabla a) \subseteq s\rho$ and $g(r\rho \cap \nabla b) \subseteq t\rho$.

To illustrate Theorems 2.6 and 2.8, consider $T(X, \rho, R)$, where $X = \{0, 1, 2, 3, 4, \ldots\}$,

$$X/\rho = \{\{0, 2, 4\}, \{6, 8\}, \{10, 12\}, \{14, 16\}, \ldots, \{1, 3\}, \{5, 7\}, \{9, 11\}, \ldots\},$$

and $R = \{0, 6, 10, 14, \ldots, 1, 5, 9, \ldots\}$. Let $a, b \in T(X, \rho, R)$ be such that

$$\begin{aligned} \blacktriangledown a &= \{\{0, 2\}, \{0, 4\}, \{6, 8\}, \{10, 12\}, \{14, 16\}, \ldots\}, \\ \blacktriangledown b &= \{\{1, 3\}, \{5, 7\}, \{9, 11\}, \{13, 15\}, \{17, 19\}, \ldots\}. \end{aligned}$$

It is clear that such a and b can be defined. It is also clear that we can define $f : \nabla a \to \nabla b$ and $g : \nabla b \to \nabla a$ such that $f(R \cap \nabla a) \subseteq R$, $g(R \cap \nabla b) \subseteq R$, and

$$\begin{aligned} &f(\{0, 2\}) = \{1, 3\},\ f(\{0, 4\}) = \{1, 3\},\ f(\{6, 8\}) = \{5, 7\},\ f(\{10, 12\}) = \{9, 11\}, \ldots \\ &g(\{1, 3\}) = \{0, 2\},\ g(\{5, 7\}) = \{0, 4\},\ g(\{9, 11\}) = \{6, 8\},\ g(\{13, 15\}) = \{10, 12\}, \ldots \end{aligned}$$

Then f and g satisfy the conditions given in Theorem 2.8 (in fact, $\blacktriangledown b = f(\blacktriangledown a)$ and $\blacktriangledown a = g(\blacktriangledown b)$), and so $a \mathcal{J} b$. However, there is no bijection $f : \nabla a \to \nabla b$ such that $f(R \cap \nabla a) \subseteq R$ and $f(\blacktriangledown a) \leftrightarrow \blacktriangledown b$ since if such an f existed, there would have to be $C \in \blacktriangledown b$ such that $f(\{0, 2\}) \subseteq C$ and $f(\{0, 4\}) \subseteq C$, which is impossible because every $C \in \blacktriangledown b$ has 2 elements. Thus, by Theorem 2.6, a and b are not in the same $\mathcal{D}$-class of $T(X, \rho, R)$.

The above example shows that, in general, $\mathcal{D} \neq \mathcal{J}$ in $T(X, \rho, R)$. This is in contrast with the semigroups $T(X)$ and $PT(X)$ of, respectively, full and partial transformations on X, in which $\mathcal{D} = \mathcal{J}$ [5, Exercises 16 and 17, p. 63]. We shall characterize the equivalence relations ρ on X for which $\mathcal{D} = \mathcal{J}$ in $T(X, \rho, R)$.

Lemma 2.9 *Let ρ be a relation such that exactly one ρ-class has size at least 2. Then for all $a, b \in T(X, \rho, R)$, if $a \mathcal{J} b$ then $a \mathcal{D} b$.*

Proof: Let $a, b \in T(X, \rho, R)$ be such that $a \mathcal{J} b$. Then, by Theorem 2.8, there are functions $f : \nabla a \to \nabla b$ and $g : \nabla b \to \nabla a$ such that $f(R \cap \nabla a) \subseteq R$, $\blacktriangledown b \hookrightarrow f(\blacktriangledown a) \hookrightarrow X/\rho$, $g(R \cap \nabla b) \subseteq R$, and $\blacktriangledown a \hookrightarrow g(\blacktriangledown b) \hookrightarrow X/\rho$. Let $r\rho$ be the ρ-class of size at least 2 ($r \in R$).

Suppose $\nabla a \subseteq R$. Then every element of $\blacktriangledown a$ has size 1, and so, since $\blacktriangledown b \hookrightarrow f(\blacktriangledown a)$, every element of $\blacktriangledown b$ also has size 1 and $\nabla b \subseteq R$. Since $|\nabla a| = |\nabla b|$ (see observation (2) after Theorem 2.8), there is a bijection $h : \nabla a \to \nabla b$. Since $\nabla a, \nabla b \subseteq R$, we clearly have $h(R \cap \nabla a) \subseteq R$. Since $\blacktriangledown a = \{\{s\} : s \in \nabla a\}$ and $\blacktriangledown b = \{\{t\} : t \in \nabla b\}$, we have $h(\blacktriangledown a) = \blacktriangledown b$. Thus $a\,\mathcal{D}\,b$ by Theorem 2.6.

Suppose $\nabla a \not\subseteq R$. This can only happen when $|r\rho \cap \nabla a| \geq 2$ (since $r\rho$ is the only ρ-class of size at least 2). We claim that $|r\rho \cap \nabla a| = |r\rho \cap \nabla b|$ and $|\nabla a - r\rho| = |\nabla b - r\rho|$.

To see that this claim is true, note that since $g(R \cap \nabla b) \subseteq R$, g is onto, and g preserves ρ-classes (see observations (1) and (3) after Theorem 2.8), we have that $|r\rho \cap \nabla a| \geq 2$ implies $r\rho \cap \nabla a = g(r\rho \cap \nabla b)$. Thus $|r\rho \cap \nabla a| \leq |r\rho \cap \nabla b|$. By a similar argument, $|r\rho \cap \nabla b| \leq |r\rho \cap \nabla a|$. Now $r\rho \cap \nabla a = g(r\rho \cap \nabla b)$ and $\nabla a = g(\nabla b)$ imply $\nabla a - r\rho = g(\nabla b - r\rho)$. Thus $|\nabla a - r\rho| \leq |\nabla b - r\rho|$. By a similar argument, $|\nabla b - r\rho| \leq |\nabla a - r\rho|$.

The claim has been proved. Thus there is a bijection $h : \nabla a \to \nabla b$ such that $h(r) = r$, $h(r\rho \cap \nabla a) = r\rho \cap \nabla b$, and $h(\nabla a - r\rho) = \nabla b - r\rho$. Note that if $s \in (R \cap \nabla a) - \{r\}$ then $s \in \nabla a - r\rho$, and so $h(s) \in \nabla b - r\rho$. Since $\nabla b - r\rho \subseteq R$, it follows that $h(R \cap \nabla a) \subseteq R$. Since $\blacktriangledown a = \{\{r\rho \cap \nabla a\}\} \cup \{\{s\} : s \in \nabla a - \{r\}\}$ and $\blacktriangledown b = \{\{r\rho \cap \nabla b\}\} \cup \{\{t\} : t \in \nabla b - \{r\}\}$, we have $h(\blacktriangledown a) = \blacktriangledown b$. Thus $a\,\mathcal{D}\,b$ by Theorem 2.6. ■

Lemma 2.10 *Let ρ be a relation such that all ρ-classes are finite and only finitely many of them have size at least 2. Then for all $a, b \in T(X, \rho, R)$, if $a\,\mathcal{J}\,b$ then $a\,\mathcal{D}\,b$.*

Proof: Let $a, b \in T(X, \rho, R)$ be such that $a\,\mathcal{J}\,b$. Then, by Theorem 2.8, there are functions $f : \nabla a \to \nabla b$ and $g : \nabla b \to \nabla a$ such that $f(R \cap \nabla a) \subseteq R$, $\blacktriangledown b \hookrightarrow f(\blacktriangledown a) \hookrightarrow X/\rho$, $g(R \cap \nabla b) \subseteq R$, and $\blacktriangledown a \hookrightarrow g(\blacktriangledown b) \hookrightarrow X/\rho$.

Since only finitely many ρ-classes have size at least 2, there are finitely many ρ-classes $r\rho$ such that $|r\rho \cap \nabla a| \geq 2$, say $r_1\rho, \ldots, r_k\rho$ ($r_i \in R, k \geq 0$), and finitely many ρ-classes $s\rho$ such that $|s\rho \cap \nabla b| \geq 2$, say $s_1\rho, \ldots, s_m\rho$ ($s_j \in R, m \geq 0$).

Note that for every $i \in \{1, \ldots, k\}$ there is $j \in \{1, \ldots, m\}$ such that $g(s_j\rho \cap \nabla b) \subseteq r_i\rho$. (Indeed, otherwise, since g preserves ρ-classes, there would be an i such that $r_i\rho \cap g(s_j\rho \cap \nabla b) = \emptyset$ for all $j \in \{1, \ldots, m\}$. But then, since $g(R \cap \nabla b) \subseteq R$, we would have $r_i\rho \cap g(\nabla b) \subseteq \{r_i\}$, which would contradict the fact that g is onto.) Since $r_1\rho, \ldots, r_k\rho$ are pairwise disjoint, it follows that $k \leq m$. By a similar argument, we have $m \leq k$, and so $k = m$.

Suppose $k = 0$. Then $\blacktriangledown a = \{\{r\} : r \in \nabla a\}$ and $\blacktriangledown b = \{\{s\} : s \in \nabla b\}$. Since $|\nabla a| = |\nabla b|$, there is a bijection $h : \nabla a \to \nabla b$. Note that $h(R \cap \nabla a) \subseteq R$ (since $\nabla a, \nabla b \subseteq R$) and $h(\blacktriangledown a) = \blacktriangledown b$. Thus $a\,\mathcal{D}\,b$ by Theorem 2.6.

Suppose $k \geq 1$. Let $\{t_1\rho, \ldots, t_p\rho\}$ be the set of all ρ-classes of size at least 2 ($t_i \in R$). (Note that each $r_i\rho$ and each $s_i\rho$ ($i = 1, \ldots, k$) is an element of this set.) Let

$$X_0 = t_1\rho \cup \ldots \cup t_p\rho, \;\; Y = r_1\rho \cup \ldots \cup r_k\rho, \text{ and } Z = s_1\rho \cup \ldots \cup s_k\rho.$$

Note that X_0 is finite, $Y \cup Z \subseteq X_0$, $f(Y \cap \nabla a) \subseteq Z$, and $g(Z \cap \nabla b) \subseteq Y$. (Indeed, if, say, $g(Z \cap \nabla b) \not\subseteq Y$ then for at least one $i \in \{1, \ldots, k\}$ there would be no $j \in \{1, \ldots, k\}$ such that $g(s_j\rho \cap \nabla b) \subseteq r_i\rho$, which would contradict the observation made in the third paragraph of the proof.)

Consider the semigroup $T(X_0, \rho_0, R_0)$, where X_0 is the set defined above, ρ_0 is the equivalence relation on X_0 with the partition $\{t_1\rho, \ldots, t_p\rho\}$, and $R_0 = \{t_1, \ldots, t_p\}$. Define $a_0, b_0 \in T(X_0)$ by:

$$xa_0 = \begin{cases} xa & \text{if } xa \in Y \\ r_1 & \text{otherwise} \end{cases} \quad \text{and} \quad xb_0 = \begin{cases} xb & \text{if } xb \in Z \\ s_1 & \text{otherwise.} \end{cases}$$

Then $a_0, b_0 \in T(X_0, \rho_0, R_0)$, $\nabla a_0 = Y \cap \nabla a$, and $\nabla b_0 = Z \cap \nabla b$. Moreover, there are sets (possibly empty) $R' \subseteq \{r_1, \ldots, r_k\}$ and $S' \subseteq \{s_1, \ldots, s_k\}$ such that

$$\begin{aligned} \blacktriangledown a_0 &= \{A \in \blacktriangledown a : |A| \geq 2\} \cup \{\{r\} : r \in R'\} \text{ and} \\ \blacktriangledown b_0 &= \{B \in \blacktriangledown b : |B| \geq 2\} \cup \{\{s\} : s \in S'\}. \end{aligned}$$

Define $f_0 : \nabla a_0 \to \nabla b_0$ and $g_0 : \nabla b_0 \to \nabla a_0$ by: $f_0 = f\,|\,(Y \cap \nabla a)$ and $g_0 = g\,|\,(Z \cap \nabla b)$. Then $f_0(R_0 \cap \nabla a_0) \subseteq R_0$, $\blacktriangledown b_0 \hookrightarrow f_0(\blacktriangledown a_0) \hookrightarrow X_0/\rho_0$, $g_0(R_0 \cap \nabla b_0) \subseteq R_0$, and $\blacktriangledown a_0 \hookrightarrow g_0(\blacktriangledown b_0) \hookrightarrow X_0/\rho_0$. Thus $a_0 \mathcal{J} b_0$ in $T(X_0, \rho_0, R_0)$ by Theorem 2.8. Hence $a_0 \mathcal{D} b_0$ in $T(X_0, \rho_0, R_0)$ (since $T(X_0, \rho_0, R_0)$ is finite and $\mathcal{D} = \mathcal{J}$ in any finite semigroup).

Thus, by Theorem 2.6, there is a bijection $h_0 : \nabla a_0 \to \nabla b_0$ such that $h_0(R_0 \cap \nabla a_0) \subseteq R_0$ and $h_0(\blacktriangledown a_0) \leftrightarrow \blacktriangledown b_0$. It follows that $|Y \cap \nabla a| = |Z \cap \nabla b|$ (since $Y \cap \nabla a = \nabla a_0$ and $Z \cap \nabla b = \nabla b_0$). Thus, since $|\nabla a| = |\nabla b|$ and $Y \cap \nabla a$ is finite, we have $|\nabla a - Y| = |\nabla b - Z|$. Hence there is a bijection $h_1 : \nabla a - Y \to \nabla b - Z$. Define $h : \nabla a \to \nabla b$ by:

$$h(x) = \begin{cases} h_0(x) & \text{if } x \in Y \cap \nabla a \\ h_1(x) & \text{if } x \in \nabla a - Y. \end{cases}$$

Since $h_0 : Y \cap \nabla a \to Z \cap \nabla b$ and $h_1 : \nabla a - Y \to \nabla b - Z$ are bijections, we have that h is a bijection. Note that

$$\begin{aligned} \blacktriangledown a &= \{A \in \blacktriangledown a : A \subseteq Y\} \cup \{\{r\} : r \in \nabla a - Y\} \text{ and} \\ \blacktriangledown b &= \{B \in \blacktriangledown b : B \subseteq Z\} \cup \{\{s\} : s \in \nabla b - Z\}. \end{aligned}$$

Let $r \in R \cap \nabla a$. If $r \in Y \cap \nabla a$ then $h(r) = h_0(r) \in R_0 \subseteq R$. If $r \in \nabla a - Y$ then $h(r) = h_1(r) \in R$ (since $h_1 : \nabla a - Y \to \nabla b - Z$ and $\nabla b - Z \subseteq R$). Thus $h(R \cap \nabla a) \subseteq R$.

Now, $h_0(\blacktriangledown a_0) \leftrightarrow \blacktriangledown b_0$ implies $h(\{A \in \blacktriangledown a : A \subseteq Y\}) \leftrightarrow \{B \in \blacktriangledown b : B \subseteq Z\}$. Indeed, let $B \in \blacktriangledown b$ and $B \subseteq Z$. Then $B \subseteq s_i\rho$ for some $i \in \{1, \ldots, k\}$. Since $|s_i\rho \cap \nabla b| \geq 2$, there is $B_0 \in \blacktriangledown b$ such that $|B_0| \geq 2$ and $B \subseteq B_0 \subseteq s_i\rho$. Then $B_0 \in \blacktriangledown b_0$, and so $B_0 \subseteq h_0(A)$ for some $A \in \blacktriangledown a_0$. Note that $|A| \geq 2$, and so $A \in \blacktriangledown a$, $A \subseteq Y$, and $h_0(A) = h(A)$. Thus $B \subseteq B_0 \subseteq h_0(A) = h(A)$, and so $\{B \in \blacktriangledown b : B \subseteq Z\} \hookrightarrow h(\{A \in \blacktriangledown a : A \subseteq Y\})$. By a similar argument, we have $h(\{A \in \blacktriangledown a : A \subseteq Y\}) \hookrightarrow \{B \in \blacktriangledown b : B \subseteq Z\}$.

Finally, since $h_1(\{\{r\} : r \in \nabla a - Y\}) = \{\{s\} : s \in \nabla b - Z\}$, it follows that $h(\blacktriangledown a) \leftrightarrow \blacktriangledown b$. Hence $a \mathcal{D} b$ by Theorem 2.6. ■

With the previous two lemmas, we are ready to describe the equivalence relations ρ on X for which $\mathcal{D} = \mathcal{J}$ in $T(X, \rho, R)$.

Theorem 2.11 *In the semigroup $T(X, \rho, R)$, $\mathcal{D} = \mathcal{J}$ if and only if ρ satisfies one of the following conditions:*

(1) *Exactly one ρ-class is infinite and all other ρ-classes (if any) have size* 1*; or*

(2) *All ρ-classes are finite and only finitely many of them have size at least* 2.

Proof: If (1) or (2) holds then $\mathcal{J} \subseteq \mathcal{D}$ by Lemma 2.9 and Lemma 2.10, and so $\mathcal{D} = \mathcal{J}$ (since $\mathcal{D} \subseteq \mathcal{J}$ in any semigroup). Conversely, suppose that neither (1) nor (2) holds. Then there are two possible cases to consider.

Case 1. There is an infinite ρ-class and another ρ-class of size at least 2.

Let $r\rho$ be infinite and $s\rho$ be of size at least 2, where $r, s \in R$ and $r \neq s$. Select $x_0 \in r\rho$ such that $x_0 \neq r$, and $y_0 \in s\rho$ such that $y_0 \neq s$. Consider the mappings

$$a = \begin{pmatrix} x & y \\ x & r \end{pmatrix} \quad \text{and} \quad b = \begin{pmatrix} z & x_0 & y_0 & w \\ z & r & x_0 & r \end{pmatrix},$$

where x is an arbitrary element of $r\rho$, y is an arbitrary element of $X - r\rho$, z is an arbitrary element of $r\rho - \{x_0\}$, and w is an arbitrary element of $X - (r\rho \cup \{y_0\})$. Then $a, b \in T(X, \rho, R)$ with $\nabla a = \nabla b = r\rho$,

$$\blacktriangledown a = \{r\rho, \{r\}\} \quad \text{and} \quad \blacktriangledown b = \{r\rho - \{x_0\}, \{r, x_0\}\} \text{ or } \{r\rho - \{x_0\}, \{r, x_0\}, \{r\}\}.$$

Define $f : r\rho \to r\rho$ by $f(x) = x$. Since $r\rho$ is infinite, there is $g : r\rho \to r\rho$ such that $g(r) = r$ and $g(r\rho - \{x_0\}) = r\rho$. Then f and g satisfy the conditions from the statement of Theorem 2.8, and so $a \mathcal{J} b$.

Let $h : r\rho \to r\rho$ be a bijection. Then $h(r\rho) = r\rho$. Note that $r\rho$ is not included in $r\rho - \{x_0\}$ or $\{r, x_0\}$ or $\{r\}$. Since $r\rho \in \blacktriangledown a$, it follows that it is not true that $h(\blacktriangledown a) \hookrightarrow \blacktriangledown b$. Thus a and b are not $\mathcal{D}$-related by Theorem 2.6, and so $\mathcal{D} \neq \mathcal{J}$.

Case 2. There are infinitely many ρ-classes of size at least 2.

Let $r_1\rho = \{r_1, x_1, \dots\}$, $r_2\rho = \{r_2, x_2, \dots\}, \dots$ be an infinite sequence of ρ-classes of size at least 2 ($r_i \in R$, $x_i \neq r_i$, $i = 1, 2, \dots$). We may assume that there is a ρ-class not in the sequence. Consider the mappings

$$a = \begin{pmatrix} r_i & x_i & y_i & y \\ r_i & x_i & r_i & r_1 \end{pmatrix} \quad \text{and} \quad b = \begin{pmatrix} r_i & x_1 & x_j & y_i & y \\ r_i & r_1 & x_j & r_i & r_1 \end{pmatrix},$$

where $i \geq 1$, $j \geq 2$, y_i is an arbitrary element of $r_i\rho - \{r_i, x_i\}$, and y is an arbitrary element of $X - (r_1\rho \cup r_2\rho \cup \dots)$. Then $a, b \in T(X, \rho, R)$ with

$$\nabla a = \{r_1, x_1, r_2, x_2, r_3, x_3, \dots\}, \quad \nabla b = \{r_1, r_2, x_2, r_3, x_3, \dots\},$$

$$\blacktriangledown a = \{\{r_1\}, \{r_1, x_1\}, \{r_2, x_2\}, \{r_3, x_3\}, \dots\} \quad \text{and} \quad \blacktriangledown b = \{\{r_1\}, \{r_2, x_2\}, \{r_3, x_3\}, \dots\}.$$

Define $f : \nabla a \to \nabla b$ and $g : \nabla b \to \nabla a$ by:

$$f = \begin{pmatrix} r_1 & x_1 & r_j & x_j \\ r_1 & r_1 & r_j & x_j \end{pmatrix} \quad \text{and} \quad g = \begin{pmatrix} r_1 & r_j & x_j \\ r_1 & r_{j-1} & x_{j-1} \end{pmatrix},$$

where $j \geq 2$. Then $f(\blacktriangledown a), g(\blacktriangledown b) \hookrightarrow X/\rho$, $f(\blacktriangledown a) = \blacktriangledown b$, and $g(\blacktriangledown b) = \blacktriangledown a$. Thus $a \mathcal{J} b$ by Theorem 2.8.

Let $h : \nabla a \to \nabla b$ be such that $h(\blacktriangledown a) \hookrightarrow \blacktriangledown b$. Then, since $\{r_1\} \in \blacktriangledown b$, there is $A \in \blacktriangledown a$ such that $h(A) \subseteq \{r_1\}$. If $A = \{r_i, x_i\}$ for some i then h is not one-to-one. Suppose $A = \{r_1\}$. Then $h(r_1) = r_1$. Since $\{r_1, x_1\} \in \blacktriangledown a$, we have $h(\{r_1, x_1\}) \subseteq B$ for some $B \in \blacktriangledown b$. Since $h(r_1) = r_1$ and $\{r_1\}$ is the only element of $\blacktriangledown b$ containing r_1, B must be $\{r_1\}$. Then $h(x_1) = r_1$, and so again h is not one-to-one. It follows from Theorem 2.6 that a and b are not $\mathcal{D}$-related, and so $\mathcal{D} \neq \mathcal{J}$. ■

Recall that the $\mathcal{J}$-classes of any semigroup S are partially ordered by the relation $\leq$ defined by: $J_a \leq J_b$ if $S^1aS^1 \subseteq S^1bS^1$. The poset $(S/\mathcal{J}, \leq)$ of $\mathcal{J}$-classes of S is isomorphic to the poset $\{S^1aS^1 : a \in S\}, \subseteq)$ of principal ideals of S.

In the semigroup $T(X)$ of full transformations on X, the poset of $\mathcal{J}$-classes is a chain [9, Proposition 4.1]. We find that this never happens in the semigroup $T(X,\rho,R)$ except in the two extreme cases when ρ is the identity relation on X or the universal relation on X.

Theorem 2.12 *The partially ordered set of $\mathcal{J}$-classes of $T(X,\rho,R)$ is a chain if and only if $\rho = \{(x,x) : x \in X\}$ or $\rho = X \times X$.*

Proof: Suppose $\rho = \{(x,x) : x \in X\}$. Then $T(X,\rho,R) = T(X)$ and it is well known that the $\mathcal{J}$-classes of $T(X)$ form a chain [9, Proposition 4.1].

Suppose $\rho = X \times X$. Then R is a one-element set, say $R = \{r\}$, and $r\rho = X$. Let $a, b \in T(X,\rho,R)$. Note that $r \in \nabla a \cap \nabla b$. Suppose $|\nabla a| \leq |\nabla b|$. Then there is a function $g : \nabla b \to \nabla a$ such that $g(r) = r$ and $g(\nabla b) = \nabla a$. We have $g(R \cap \nabla b) = g(\{r\}) = \{r\}$ and $\blacktriangledown a \hookrightarrow g(\blacktriangledown b) \hookrightarrow X/\rho$ (since $\blacktriangledown a = \{\nabla a\}$, $g(\blacktriangledown b) = g(\{\nabla b\}) = \{\nabla a\}$, and $X/\rho = \{X\}$). Thus $J_a \leq J_b$ by Theorem 2.7. Similarly, if $|\nabla b| \leq |\nabla a|$ then $J_b \leq J_a$. It follows that the $\mathcal{J}$-classes of $T(X,\rho,R)$ form a chain.

Conversely, suppose that $\rho \neq \{(x,x) : x \in X\}$ and $\rho \neq X \times X$. Then there are two distinct ρ-classes of which at least one has size at least 2, say $r\rho = \{r, x, \dots\}$ and $s\rho = \{s, \dots\}$ $(r, s \in R)$. Consider the mappings

$$a = \begin{pmatrix} x & y \\ x & r \end{pmatrix} \quad \text{and} \quad b = \begin{pmatrix} z & w \\ r & s \end{pmatrix},$$

where y is an arbitrary element of $X - \{x\}$, z is an arbitrary element of $r\rho$, and w is an arbitrary element of $X - r\rho$. Then $a, b \in T(X,\rho,R)$ with

$$\nabla a = \{r,x\}, \quad \nabla b = \{r,s\}, \quad \blacktriangledown a = \{\{r,x\},\{r\}\}, \quad \text{and} \quad \blacktriangledown b = \{\{r\},\{s\}\}.$$

Thus there is no $f : \nabla a \to \nabla b$ such that $\blacktriangledown b \hookrightarrow f(\blacktriangledown a) \hookrightarrow X/\rho$ (since otherwise we would have $f(\{r,x\}) = \{r\}$ and $f(\{r\}) = \{s\}$ or $f(\{r,x\}) = \{s\}$ and $f(\{r\}) = \{r\}$, which is impossible). Similarly, there is no $g : \nabla b \to \nabla a$ such that $\blacktriangledown a \hookrightarrow g(\blacktriangledown b)$ (since $\{r,x\} \in \blacktriangledown a$ has size 2 and, since $\blacktriangledown b = \{\{r\},\{s\}\}$, every element of $g(\blacktriangledown b)$ would have size 1).

It follows from Theorem 2.7 that $J_a \not\leq J_b$ and $J_b \not\leq J_a$. Hence the partially ordered set of $\mathcal{J}$-classes of $T(X,\rho,R)$ is not a chain. ■

We recall that if $\rho = X \times X$ and $|X| \geq 2$ then the semigroup $T(X,\rho,R)$ is isomorphic to the semigroup $PT(X')$ of partial transformations on X', where X' is the set X with one element removed. Thus, the poset of $\mathcal{J}$-classes of $T(X,\rho,R)$ is a chain if and only if $T(X,\rho,R) = T(X)$ or $T(X,\rho,R)$ is isomorphic to $PT(X')$.

3 Regular $T(X,\rho,R)$

An element a of a semigroup S is called *regular* if $a = axa$ for some x in S. If all elements of S are regular, we say that S is a *regular semigroup*. If a $\mathcal{D}$-class D in S contains a regular element then all elements in D are regular [5, Proposition 2.3.1], and we call D a *regular $\mathcal{D}$-class*. In a regular $\mathcal{D}$-class, every $\mathcal{L}$-class and every $\mathcal{R}$-class contains an idempotent [5, Proposition 2.3.2]. It follows that an element of S is regular if and only if it is $\mathcal{L}$-related ($\mathcal{R}$-related) to an idempotent in S.

Let A be a nonempty subset of X. An equivalence relation ρ on X induces a partition A/ρ of A:

$$A/\rho = \{x\rho \cap A : x \in X \text{ and } x\rho \cap A \neq \emptyset\}.$$

The semigroup $T(X)$ is regular [5, Exercise 15, p. 63]. This is not true in general about $T(X, \rho, R)$. The next theorem characterizes the regular elements of $T(X, \rho, R)$.

Theorem 3.1 *Let $a \in T(X, \rho, R)$. Then a is regular if and only if $\nabla a/\rho \subseteq \blacktriangledown a$.*

Proof: Suppose a is regular, that is, $a = aba$ for some $b \in T(X, \rho, R)$. Let $r\rho \cap \nabla a \in \nabla a/\rho$ $(r \in R)$. Since $b \in T(X, \rho, R)$, there is $p \in R$ such that $(r\rho)b \subseteq p\rho$ and $rb = p$. Since $r\rho \cap \nabla a \neq \emptyset$, $r = ta$ for some $t \in R$, and so $r = ta = ((ta)b)a = (rb)a = pa$. It follows that $(p\rho)a \subseteq r\rho \cap \nabla a$. For the reverse inclusion, if $xa \in r\rho \cap \nabla a$ then $(xa)b \in p\rho$ and so $xa = ((xa)b)a \in (p\rho)a$. It follows that $r\rho \cap \nabla a = (p\rho)a \in \blacktriangledown a$, and so $\nabla a/\rho \subseteq \blacktriangledown a$.

Conversely, suppose $\nabla a/\rho \subseteq \blacktriangledown a$. We shall construct $b \in T(X, \rho, R)$ such that $a = aba$. Consider $r\rho$ $(r \in R)$. If $r\rho \cap \nabla a = \emptyset$, define $xb = x$ for every $x \in r\rho$. Suppose $r\rho \cap \nabla a \neq \emptyset$. Then $r\rho \cap \nabla a \in \nabla a/\rho \subseteq \blacktriangledown a$ and so there is $p \in R$ such that $r\rho \cap \nabla a = (p\rho)a$. Let $x \in r\rho$. If $x \in \nabla a$ then $x = wa$ for some $w \in p\rho$ (if $x = r$, we may assume $w = p$), and we define $xb = w$. If $x \notin \nabla a$, we define $xb = p$.

By the construction of b, $b \in T(X, \rho, R)$ and $a = aba$. Thus a is regular. ■

Using the fact that $a \in T(X, \rho, R)$ is regular if and only if $a \mathcal{R} e$ for some idempotent $e \in T(X, \rho, R)$, we can obtain another characterization of the regular elements of $T(X, \rho, R)$.

For $a \in T(X, \rho, R)$, let $\rho_a = \rho \vee \operatorname{Ker}(a)$, that is, ρ_a is the smallest equivalence relation on X that contains both ρ and $\operatorname{Ker}(a)$. Note that every ρ_a-class $x\rho_a$ is a union of ρ-classes and a union of $\operatorname{Ker}(a)$-classes.

Lemma 3.2 *For every $x \in X$, $(x\rho_a)a \subseteq r\rho$ for some $r \in R$.*

Proof: Let $x \in X$. Then $xa \in r\rho$ for some $r \in R$. We claim that $(x\rho_a)a \subseteq r\rho$. Let $y \in x\rho_a$. Since $\rho_a = \rho \vee \operatorname{Ker}(a)$, there are $z_1, z_2, \ldots, z_{2n-1} \in X$ $(n \geq 1)$ such that

$$(x, z_1) \in \rho,\ (z_1, z_2) \in \operatorname{Ker}(a),\ (z_2, z_3) \in \rho,\ \ldots,\ (z_{2n-1}, y) \in \operatorname{Ker}(a).$$

Since $a \in T(X, \rho, R)$, $(xa, z_1a) \in \rho$ and so $z_1a \in r\rho$. Thus, since $z_1a = z_2a$, $z_2a \in r\rho$. It follows by induction on n that $ya \in r\rho$, and so $(x\rho_a)a \subseteq r\rho$. ■

We say that an element $a \in T(X, \rho, R)$ is *normal* if for every ρ_a-class $x\rho_a$ there is a ρ-class $r\rho$ that intersects all $\operatorname{Ker}(a)$-classes included in $x\rho_a$. (Note that such a ρ-class $r\rho$ must be included in $x\rho_a$.)

Lemma 3.3 *Let $a \in T(X, \rho, R)$ with $X/\rho_a = \{E_i : i \in I\}$. Then a is normal if and only if for every $i \in I$ there is $x_i \in E_i$ such that for every $y \in E_i$, $(x_i, y) \in \rho \circ \operatorname{Ker}(a)$.*

Proof: Suppose a is normal and let $i \in I$. Then there is $x_i \in E_i$ such that $(x_i\rho) \cap K \neq \emptyset$ for every $\operatorname{Ker}(a)$-class K included in E_i. Let $y \in E_i$. Then $y \in K$ for some $\operatorname{Ker}(a)$-class K. Let $z \in (x_i\rho) \cap K$. Then $(x_i, z) \in \rho$ and $(z, y) \in \operatorname{Ker}(a)$. Thus $(x_i, y) \in \rho \circ \operatorname{Ker}(a)$.

Conversely, suppose that the given condition holds and let $i \in I$. Then there is $x_i \in E_i$ such that $(x_i, y) \in \rho \circ \mathrm{Ker}(a)$ for every $y \in E_i$. Let K be a $\mathrm{Ker}(a)$-class included in E_i and let $y \in K$. Then, since $(x_i, y) \in \rho \circ \mathrm{Ker}(a)$, there is $z \in x_i\rho$ such that $(z, y) \in \mathrm{Ker}(a)$. Thus $z \in (x_i\rho) \cap K$. It follows that $x_i\rho$ intersects all $\mathrm{Ker}(a)$-classes included in E_i, and so a is normal. ■

Lemma 3.4 *Let $e \in T(X, \rho, R)$ be an idempotent. Then e is normal.*

Proof: Consider $x\rho_e$ $(x \in X)$. By Lemma 3.2, $(x\rho_e)e \subseteq r\rho$ for some $r \in R$. We claim that $r\rho$ intersects all $\mathrm{Ker}(e)$-classes included in $x\rho_e$. Let K be a $\mathrm{Ker}(e)$-class included in $x\rho_e$. Then $Ke = \{y\}$ for some $y \in r\rho$. Since e is an idempotent and $y \in \nabla e$, $y = ye$. Thus $y \in K$ and so $r\rho \cap K \neq \emptyset$. It follows that e is normal. ■

Corollary 3.5 *Let $a, e \in T(X, \rho, R)$ such that e is an idempotent and* $\mathrm{Ker}(a) = \mathrm{Ker}(e)$. *Then a is normal.*

Proof: Since $\mathrm{Ker}(a) = \mathrm{Ker}(e)$, $\rho_a = \rho_e$. Thus the result follows from Lemma 3.4 and the definition of normal elements. ■

Theorem 3.6 *Let $a \in T(X, \rho, R)$ with $X/\rho_a = \{E_i : i \in I\}$. Then the following are equivalent:*

(1) *a is regular.*

(2) *a is normal.*

(3) $(\forall i \in I)(\exists x_i \in E_i)(\forall y \in E_i)\ (x_i, y) \in \rho \circ \mathrm{Ker}(a)$.

Proof: (2) is equivalent to (3) by Lemma 3.3. Suppose a is regular. Then $a\,\mathcal{R}\,e$ for some idempotent $e \in T(X, \rho, R)$. By Theorem 2.3, $\mathrm{Ker}(a) = \mathrm{Ker}(e)$. Thus a is normal by Corollary 3.5. Hence (1) implies (2).

It remains to show that (2) implies (1). Suppose a is normal. Then for every $i \in I$ there is $x_i \in E_i$ such that $(x_i\rho) \cap K \neq \emptyset$ for every $\mathrm{Ker}(a)$-class K included in E_i. We shall construct an idempotent $e \in T(X, \rho, R)$ such that $\mathrm{Ker}(e) = \mathrm{Ker}(a)$. Let K be a $\mathrm{Ker}(a)$-class. Then there is a unique $i \in I$ such that $K \subseteq E_i$. Select $y_i \in x_i\rho \cap K$ in such a way that $y_i = r_{x_i}$ if $r_{x_i} \in x_i\rho \cap K$. Define $e \in T(X)$ by $Ke = \{y_i\}$.

It is clear that $\mathrm{Ker}(e) = \mathrm{Ker}(a)$ and that e preserves ρ (since it maps all ρ-classes included in E_i to $x_i\rho$). By Lemma 3.2, all elements of R contained in E_i are in the same $\mathrm{Ker}(a)$-class. Thus e maps all such elements to r_{x_i} and so it preserves R. Hence $e \in T(X, \rho, R)$. By Theorem 2.3, $a\,\mathcal{R}\,e$ and so a is regular. ■

Let ρ be an equivalence relation on X. We say that ρ is a *T-relation* if there is at most one ρ-class containing two or more elements. If there is $n \geq 1$ such that each ρ-class has at most n elements, we say that ρ is *n-bounded*.

The following theorem characterizes the equivalence relations ρ on X for which the semigroup $T(X, \rho, R)$ is regular.

Theorem 3.7 *The semigroup $T(X, \rho, R)$ is regular if and only if ρ is 2-bounded or a T-relation.*

Proof: Suppose ρ is neither 2-bounded nor a T-relation. Then there are $r, s \in R$ such that $r \neq s$ and

$$r\rho = \{r, x_1, x_2, \dots\} \quad \text{and} \quad s\rho = \{s, y_1, \dots\}.$$

Consider the mapping

$$a = \begin{pmatrix} r & x_1 & x_2 & s & y_1 & z \\ r & x_1 & x_1 & r & x_2 & r \end{pmatrix},$$

where z denotes an arbitrary element in $X - \{r, x_1, x_2, s, y_1\}$. Then $a \in T(X, \rho, R)$ with $\nabla a/\rho = \{\{r, x_1, x_2\}\}$ and either $\blacktriangledown a = \{\{r, x_1\}, \{r, x_2\}\}$ or $\blacktriangledown a = \{\{r, x_1\}, \{r, x_2\}, \{r\}\}$. In either case, $\nabla a/\rho$ is not included in $\blacktriangledown a$, which implies that a is not regular (by Theorem 3.1).

Conversely, suppose that ρ is 2-bounded or a T-relation and let $a \in T(X, \rho, R)$. We shall prove that $\nabla a/\rho \subseteq \blacktriangledown a$. Let $r\rho \cap \nabla a \in \nabla a/\rho$. Then there is $p \in R$ such that $r = pa$ and $(p\rho)a \subseteq r\rho$.

Suppose $r\rho$ has at least 3 elements. Then ρ is not 2-bounded and so it must be a T-relation. Thus every ρ-class except $r\rho$ has 1 element. Hence $r\rho \cap \nabla a = (r\rho)a$ (if $(r\rho)a \subseteq r\rho$) or $r\rho \cap \nabla a = \{r\} = (p\rho)a$ (if $(r\rho)a$ is not included in $r\rho$). Suppose $r\rho = \{r, x\}$ has 2 elements. If $x \in \nabla a$ then $x \in (s\rho)a$ for some $s \in R$, and so $r\rho \cap \nabla a = \{r, x\} = (s\rho)a$. If $x \notin \nabla a$ then $r\rho \cap \nabla a = \{r\} = (p\rho)a$. Finally, if $r\rho$ has 1 element then $r\rho \cap \nabla a = \{r\} = (p\rho)a$.

It follows that $r\rho \cap \nabla a \in \blacktriangledown a$, and so a is regular by Theorem 3.1. ■

There is an asymmetry between the relations $\mathcal{R}$ and $\mathcal{L}$ in $T(X, \rho, R)$: while the $\mathcal{R}$-relation is simply the restriction of the $\mathcal{R}$-relation in $T(X)$ to $T(X, \rho, R) \times T(X, \rho, R)$, the corresponding result is not true in general for the $\mathcal{L}$-relation. The following theorem determines the semigroups $T(X, \rho, R)$ in which the $\mathcal{L}$-relation is the restriction of the $\mathcal{L}$-relation in $T(X)$.

Theorem 3.8 *The $\mathcal{L}$-relation in $T(X, \rho, R)$ is the restriction of the $\mathcal{L}$-relation in $T(X)$ to $T(X, \rho, R) \times T(X, \rho, R)$ if and only if $T(X, \rho, R)$ is regular.*

Proof: Suppose $T(X, \rho, R)$ is not regular. Then, by Theorem 3.7, there are $r, s \in R$ such that $r \neq s$ and

$$r\rho = \{r, x_1, x_2, \dots\} \quad \text{and} \quad s\rho = \{s, y_1, \dots\}.$$

Consider the mappings

$$a = \begin{pmatrix} r & x_1 & x_2 & s & y_1 & z \\ r & x_1 & x_1 & r & x_2 & r \end{pmatrix} \quad \text{and} \quad b = \begin{pmatrix} r & x_1 & x_2 & s & y_1 & z \\ r & x_1 & x_2 & r & r & r \end{pmatrix},$$

where z is an arbitrary element in $X - \{r, x_1, x_2, s, y_1\}$. Then $a, b \in T(X, \rho, R)$ with $\nabla a = \nabla b = \{r, x_1, x_2\}$. Thus, by Lemma 2.1, $a \mathcal{L} b$ in $T(X)$. However, $\{r, x_1, x_2\} \in \blacktriangledown b$ and $\{r, x_1, x_2\}$ is not included in any $A \in \blacktriangledown a$ (since $\blacktriangledown a = \{\{r, x_1\}, \{r, x_2\}\}$ or $\blacktriangledown a = \{\{r, x_1\}, \{r, x_2\}, \{r\}\}$). Thus, by Theorem 2.5, a and b are not $\mathcal{L}$-related in $T(X, \rho, R)$.

The converse follows from a general result saying that if T is a regular subsemigroup of a semigroup S then the relations $\mathcal{L}$ and $\mathcal{R}$ in T are the restrictions of the relations $\mathcal{L}$ and $\mathcal{R}$, respectively, in S to $T \times T$ [5, Proposition 2.4.2]. ■

4 Abundant $T(X, \rho, R)$

Let S be a semigroup. We say that $a, b \in S$ are $\mathcal{L}^*$-related if they are $\mathcal{L}$-related in a semigroup T such that S is a subsemigroup of T. We have the dual definition of the $\mathcal{R}^*$-relation on S [3]. The relations $\mathcal{L}^*$ and $\mathcal{R}^*$ are equivalence relations. They have been studied by J. Fountain [2], [3] and others. A semigroup S is called *abundant* if every $\mathcal{L}^*$-class and every $\mathcal{R}^*$-class of S contains an idempotent [3]. As stated in [3], where the concept was introduced, the word "abundant" comes from the fact that such semigroups have a plentiful supply of idempotents.

It is clear from the definition of $\mathcal{L}^*$ and $\mathcal{R}^*$ that $\mathcal{L} \subseteq \mathcal{L}^*$ and $\mathcal{R} \subseteq \mathcal{R}^*$ in any semigroup S. Since in a regular semigroup, every $\mathcal{L}$-class and every $\mathcal{R}$-class contains an idempotent, we have that regular semigroups are abundant. Of course, the converse is not true. For example, A. Umar [12] proved that the semigroup of non-bijective, order-decreasing transformations on the set $X = \{1, \ldots, n\}$ is abundant but not regular.

We first note that every $\mathcal{L}^*$-class of $T(X, \rho, R)$ contains an idempotent.

Proposition 4.1 *Let $a \in T(X, \rho, R)$. Then there is an idempotent $e \in T(X, \rho, R)$ such that $\nabla a = \nabla e$.*

Proof: Select $r_0 \in \nabla a \cap R$ and define $e \in T(X)$ as follows:

$$xe = \begin{cases} x & \text{if } x \in \nabla a \\ r_x & \text{if } x \notin \nabla a \text{ but } x\rho \cap \nabla a \neq \emptyset \\ r_0 & \text{if } x\rho \cap \nabla a = \emptyset. \end{cases}$$

By the definition of e and the fact that $a \in T(X, \rho, R)$, we have that $e \in T(X, \rho, R)$, e is an idempotent, and $\nabla e = \nabla a$. ■

The statement preceding Proposition 4.1 follows since, by Lemma 2.1, the elements a and e are $\mathcal{L}$-related in $T(X)$. The corresponding statement for $\mathcal{R}^*$-classes of $T(X, \rho, R)$ is not true, and so not every semigroup $T(X, \rho, R)$ is abundant. Similar results have been obtained by A. Umar [13] for the semigroup of order-decreasing transformations on an infinite totally ordered set X. In contrast with Umar [13], who showed that in the class he studied the abundant semigroups are not regular, we find that abundant semigroups $T(X, \rho, R)$ are precisely those that are regular.

Theorem 4.2 *A semigroup $T(X, \rho, R)$ is abundant if and only if it is regular.*

Proof: If $T(X, \rho, R)$ is regular then it is abundant (since every regular semigroup is abundant). Conversely, suppose that $T(X, \rho, R)$ is abundant, and let $a \in T(X, \rho, R)$. Then there is an idempotent $e \in T(X, \rho, R)$ such that $a\mathcal{R}^*e$. By [3, the dual of Corollary 1.2], $ea = a$ and for all $c, d \in T(X, \rho, R)$, $ca = da$ implies $ce = de$. We claim that $\mathrm{Ker}(a) = \mathrm{Ker}(e)$.

The inclusion $\mathrm{Ker}(e) \subseteq \mathrm{Ker}(a)$ follows immediately from $ea = a$. Suppose $(x, y) \in \mathrm{Ker}(a)$, that is, $xa = ya$. Since $(x, r_x) \in \rho$, $(y, r_y) \in \rho$, and $a \in T(X, \rho, R)$, we have $(xa, r_x a) \in \rho$ and $(ya, r_y a) \in \rho$. Thus, since $xa = ya$, we have $(r_x a, r_y a) \in \rho$, which implies $r_x a = r_y a$ (since $r_x a, r_y a \in R$ and R is a cross-section of X/ρ).

Define $c, d \in T(X)$ by : $(X - R)c = \{x\}$, $Rc = \{r_x\}$, $(X - R)d = \{y\}$, and $Rd = \{r_y\}$. It is clear that $c, d \in T(X, \rho, R)$, and that there is $z_0 \in X$ such that $z_0 c = x$ and $z_0 d = y$. Let $z \in X$. If $z \in X - R$ then $z(ca) = xa = ya = z(da)$. If $z \in R$ then $z(ca) = r_x a = r_y a = z(da)$. Hence $ca = da$, which implies $ce = de$. In particular, $z_0(ce) = z_0(de)$, which implies $xe = ye$ (since $z_0 c = x$ and $z_0 d = y$). Hence $\mathrm{Ker}(a) \subseteq \mathrm{Ker}(e)$, and so $\mathrm{Ker}(a) = \mathrm{Ker}(e)$.

Thus a is normal (by Corollary 3.5), and so a is regular (by Theorem 3.6). It follows that $T(X, \rho, R)$ is a regular semigroup. ■

5 Inverse $T(X, \rho, R)$ and Completely Regular $T(X, \rho, R)$

An element a' in a semigroup S is called an *inverse* of $a \in S$ if $a = aa'a$ and $a' = a'aa'$. If every element of S has exactly one inverse then S is called an *inverse semigroup*. An alternative definition is that S is an inverse semigroup if it is regular and its idempotents commute [5, Theorem 5.1.1]. If every element of S is in some subgroup of S then S is called a *completely regular semigroup*. Of course, both inverse semigroups and completely regular semigroups are regular semigroups.

Theorem 5.1 *Suppose $|X| \geq 3$. Then $T(X, \rho, R)$ is not an inverse semigroup.*

Proof: We shall construct idempotents $e, f \in T(X, \rho, R)$ such that $ef \neq fe$.

Suppose there are at least two ρ-classes, that is, there are $r\rho$ and $s\rho$ ($r, s \in R$) such that $r \neq s$. Define $e, f \in T(X)$ by:

$$e = \begin{pmatrix} y & z \\ r & z \end{pmatrix} \quad \text{and} \quad f = \begin{pmatrix} y & z \\ s & z \end{pmatrix},$$

where y is an arbitrary element in $r\rho \cup s\rho$ and z is an arbitrary element in $X - (r\rho \cup s\rho)$. Note that $r(ef) = s$ and $r(fe) = r$.

Suppose there is only one ρ-class, say $r\rho$. Since $|X| \geq 3$, $r\rho = \{r, x_1, x_2, \dots\}$. Define $e, f \in T(X)$ by:

$$e = \begin{pmatrix} x_1 & y \\ x_2 & y \end{pmatrix} \quad \text{and} \quad f = \begin{pmatrix} x_2 & z \\ x_1 & z \end{pmatrix},$$

where y is an arbitrary element in $X - \{x_1\}$ and z is an arbitrary element in $X - \{x_2\}$. Note that $x_1(ef) = x_1$ and $x_1(fe) = x_2$.

In both cases we have: $e, f \in T(X, \rho, R)$, e, f are idempotents, and $ef \neq fe$. It follows that $T(X, \rho, R)$ is not an inverse semigroup (since idempotents in an inverse semigroup commute). ■

When $|X| = 2$, $T(X, \rho, R)$ is an inverse semigroup if $X/\rho = \{\{r, x\}\}$ and $T(X, \rho, R)$ is not inverse if $X/\rho = \{\{r\}, \{s\}\}$.

Theorem 5.2 *Suppose $|X| \geq 4$. Then $T(X,\rho,R)$ is not a completely regular semigroup.*

Proof: We shall construct $a \in T(X,\rho,R)$ such that $\nabla a \neq \nabla a^2$.

Suppose there is a ρ-class with at least three elements, say $r\rho = \{r, x_1, x_2, \dots\}$ $(r \in R)$. Define $a \in T(X)$ by:

$$a = \begin{pmatrix} x_1 & z \\ x_2 & r \end{pmatrix},$$

where z is an arbitrary element in $X - \{x_1\}$. Note that $\nabla a = \{r, x_2\}$ and $\nabla a^2 = \{r\}$.

Suppose there are at least three ρ-classes, that is, there are $r\rho$, $s\rho$, and $t\rho$ with $r, s, t \in R$ pairwise distinct. Define $a \in T(X)$ by:

$$a = \begin{pmatrix} y & z \\ s & t \end{pmatrix},$$

where y is an arbitrary element in $r\rho$ and z is an arbitrary element in $X - r\rho$. Note that $\nabla a = \{s, t\}$ and $\nabla a^2 = \{t\}$.

Since $|X| \geq 4$, the only remaining case to consider is when there are exactly two ρ-classes with two elements each, say $r\rho = \{r, x\}$ and $s\rho = \{s, y\}$ $(r, s \in R)$. Define $a \in T(X)$ by:

$$a = \begin{pmatrix} x & z \\ y & s \end{pmatrix},$$

where z is an arbitrary element in $X - \{x\}$. Note that $\nabla a = \{s, y\}$ and $\nabla a^2 = \{s\}$.

In all cases we have: $a \in T(X,\rho,R)$ and $\nabla a \neq \nabla a^2$. By Lemma 2.1, $\nabla a \neq \nabla a^2$ implies that a and a^2 are not $\mathcal{H}$-related in $T(X)$ (not even $\mathcal{L}$-related in $T(X)$), and so they are not $\mathcal{H}$-related in $T(X,\rho,R)$. It follows that $T(X,\rho,R)$ is not completely regular (since for every element a in a completely regular semigroup, a and a^2 are $\mathcal{H}$-related [5, Proposition 4.1.1]). ■

When $|X| = 3$, $T(X,\rho,R)$ is completely regular if $X/\rho = \{\{r,x\},\{s\}\}$, and $T(X,\rho,R)$ is not completely regular if $X/\rho = \{\{r\},\{s\},\{t\}\}$ or $\{\{r,x_1,x_2\}\}$. When $|X| = 2$, $T(X,\rho,R)$ is completely regular.

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