Dense Relations Are Determined by Their Endomorphism Monoids

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Abstract

We introduce the class of dense relations on a set X and prove that for any finitary or infinitary dense relation ρ on X, the relational system (X,ρ) is determined up to semi-isomorphism by the monoid $End\,(X,\rho)$ of endomorphisms of (X,ρ). In the case of binary relations, a semi-isomorphism is an isomorphism or an anti-isomorphism.

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1 Introduction

For a mathematical structure M, let $End(M)$ denote the endomorphism monoid of M. A general problem, which attracted a considerable attention, can be stated as follows: Let M_1, M_2 be two mathematical structures. Given that $End(M_1) \cong End(M_2)$, what can we say about the relation between the structures M_1 and M_2 themselves? For example, Schein [8] proved that if M_1 and M_2 are two partially ordered sets, semilattices, distributive lattices, or Boolean algebras, then $End(M_1) \cong End(M_2)$ if and only if M_1 and M_2 are isomorphic or anti-isomorphic. For other results of this kind, see [2], [4], and [5].

The aim of this note is to prove a similar result for the class of dense relations. These relations include partial orders, binary relations that are reflexive and symmetric, and generalized equivalence relations.

Let I be an arbitrary non-empty index set. An *I-tuple* of elements of a set X is a mapping $f : I \to X$. If I is finite with $|I| = n$, we shall assume that $I = \{1, 2, \ldots, n\}$, denote $f : I \to X$ by $(1f, 2f, \ldots, nf)$, and refer to f as an n-tuple. An *I-relation* ρ on X is any set of I-tuples of elements of X. If $|I| = n$, an I-relation ρ is an n-ary relation on X, that is, a set of n-tuples of elements of X. By a *relation* on X, we shall mean an I-relation on X for some index set I.

Let ρ be an I-relation on X. An *endomorphism* of a relational system (X,ρ) is a mapping $a : X \to X$ that preserves ρ, that is, $fa \in \rho$ for every $f \in \rho$, where $fa : I \to X$ is the composition of $f : I \to X$ and $a : X \to X$. (We compose from left to right, that is, $i(fa) = (if)a$ for $i \in I$.) If ρ is an n-ary relation, we can use the n-tuple notation. With that notation, we have that $a : X \to X$ is an endomorphism of (X,ρ) if and only if $(x_1a, \ldots, x_na) \in \rho$ for every $(x_1, \ldots, x_n) \in \rho$. We denote by $End\,(X,\rho)$ the monoid of endomorphisms of (X,ρ).

Turning to the definition of a dense relation, we denote by ρ^* the set of all mappings $f : I \to X$ such that $\sigma f \notin \rho$ for every $\sigma \in S(I)$, where $S(I)$ is the symmetric group of I.

That is,

$$\rho^* = \{f : I \to X : (\forall \sigma \in S(I))\ \sigma f \notin \rho\}.$$

A reflexive I-relation ρ on X is said to be *dense* if it satisfies the following two properties:

(D_1) For every injective $f_1 \in \rho \cup \rho^*$ and every $f \in \rho$, there is $a \in End\,(X, \rho)$ such that $f_1 a = f$.

(D_2) There is an injective f_1 in ρ.

Examples of dense relations (for details see [1]) are partial orders, binary relations that are reflexive and symmetric, generalized equivalence relations (see [6]), relations defined by families of sets intersecting in at most one element (see [7]).

Let ρ_1, ρ_2 be I-relations on X_1, X_2, respectively. We say that a bijection $g : X_1 \to X_2$ is a *semi-isomorphism* of (X_1, ρ_1) to (X_2, ρ_2) if there is a permutation $\sigma \in S(I)$ such that for all $f : I \to X_1$,

$$f \in \rho_1 \Leftrightarrow \sigma f g \in \rho_2.$$

We say that relational systems (X_1, ρ_1) and (X_2, ρ_2) are *semi-isomorphic* if there is a semi-isomorphism from (X_1, ρ_1) to (X_2, ρ_2).

Note that if $I = \{1, 2\}$ then the only elements of $S(I)$ are id_I (the identity permutation of I) and the transposition $(1\ 2)$. It follows that if ρ_1 and ρ_2 are binary relations then any semi-isomorphism $g : X_1 \to X_2$ is either an isomorphism ($(x, y) \in \rho_1 \Leftrightarrow (xg, yg) \in \rho_2$) or an anti-isomorphism ($(x, y) \in \rho_1 \Leftrightarrow (yg, xg) \in \rho_2$).

In the next section, we prove that if ρ is a dense I-relation on X then $End\,(X, \rho)$ determines ρ up to a semi-isomorphism.

2 Main Theorem and Its Application to Binary Relations

The following lemma belongs to the folklore (see [2], for example) and we include a proof just for the sake of completeness.

Lemma 2.1 *Let ρ_1, ρ_2 be reflexive I-relations on X_1, X_2, respectively, and suppose that $\phi : End\,(X_1, \rho_1) \to End\,(X_2, \rho_2)$ is an isomorphism. Then there exists a bijection $g : X_1 \to X_2$ such that $a\phi = g^{-1} a g$ for every $a \in End\,(X_1, \rho_1)$.*

Proof: For a set X and $x \in X$, we denote by X_x the constant mapping from X to X defined by: $yX_x = x$ for every $y \in X$. Let S be a semigroup of mappings from X to X with $X_x \in S$ for some $x \in X$. It is easy to see that $\{X_x\}$ is a minimal left ideal of S. In fact, all minimal left ideals of S are of the form $\{X_y\}$ where $y \in X$. For suppose J is a minimal left ideal of S and let $a \in J$. Then $X_x a \in J$ and $X_x a = X_{xa}$. Since $\{X_{xa}\} \subseteq J$ and J is a minimal left ideal, it follows that $\{X_{xa}\} = J$. Denote by $\mathcal{M}(S)$ the set of minimal left ideals of S. We proved that $\mathcal{M}(S) = \{\{X_x\} : X_x \in S\}$.

Since $End\,(X_1, \rho_1)$ and $End\,(X_2, \rho_2)$ contain all constant mappings on X_1 and X_2, respectively, we have $\mathcal{M}(End\,(X_1, \rho_1)) = \{\{(X_1)_x\} : x \in X_1\}$ and $\mathcal{M}(End\,(X_2, \rho_2)) = \{\{(X_2)_y\} : y \in X_2\}$. Since an isomorphism must map minimal left ideals to minimal left ideals, it follows that $\{(X_1)_x \phi : x \in X_1\} = \{(X_2)_y : y \in X_2\}$, so that $|X_1| = |X_2|$. Define $g : X_1 \to X_2$ by: $xg = y$ if $(X_1)_x \phi = (X_2)_y$. Then g is a bijection and $(X_1)_x \phi = (X_2)_{xg}$ for every $x \in X_1$.

Let $x \in X_1$ and $a \in End(X_1, \rho_1)$. We claim that $(xg)(a\phi) = (xa)g$. Indeed, $(xg)(X_2)_{xg}(a\phi) = (xg)((X_1)_x\phi)(a\phi) = (xg)((X_1)_x a)\phi = (xg)((X_1)_{xa}\phi) = (xg)(X_2)_{(xa)g}$. Therefore $(xg)(a\phi) = (xg)(X_2)_{xg}(a\phi) = (xg)(X_2)_{(xa)g} = (xa)g$. It follows that $g(a\phi) = ag$ and hence $a\phi = g^{-1}ag$. ■

Note that if g is as in the statement of Lemma 2.1 then for all $a \in End(X_1, \rho_1)$ and $b \in End(X_2, \rho_2)$, $g^{-1}ag \in End(X_2, \rho_2)$ and $gbg^{-1} \in End(X_1, \rho_1)$.

Lemma 2.2 *Let ρ_1, ρ_2 be dense I-relations on X_1, X_2, respectively, and let $g : X_1 \to X_2$ be a bijection such that $g^{-1}ag \in End(X_2, \rho_2)$ for every $a \in End(X_1, \rho_1)$. Then for every $f \in \rho_1$ there is a permutation $\sigma \in S(I)$ such that $\sigma fg \in \rho_2$.*

Proof: Fix an injective $f_2 \in \rho_2$ (which exists by (D_2)) and let $f_1 = f_2 g^{-1}$. Note that f_1 is injective and $f_2 = f_1 g$. Let f be an arbitrary element of ρ_1.

Suppose $f_1 \in \rho_1^*$. Then, by (D_1), there is $a \in End(X_1, \rho_1)$ such that $f_1 a = f$. Thus, since $g^{-1}ag \in End(X_2, \rho_2)$, we have

$$f_2 \in \rho_2 \Rightarrow f_2(g^{-1}ag) \in \rho_2 \Rightarrow f_1 ag \in \rho_2 \Rightarrow fg \in \rho_2 \Rightarrow id_I fg \in \rho.$$

Suppose $f_1 \notin \rho_1^*$. Then $\sigma f_1 \in \rho_1$ for some $\sigma \in S(I)$. By (D_1), there is $a \in End(X_1, \rho_1)$ such that $\sigma f_1 a = f$. Then $f_1 a = \sigma^{-1} f$ and we have

$$f_2 \in \rho_2 \Rightarrow f_2(g^{-1}ag) \in \rho_2 \Rightarrow f_1 ag \in \rho_2 \Rightarrow \sigma^{-1} fg \in \rho_2.$$

The lemma follows. ■

Lemma 2.3 *With the hypothesis of Lemma 2.2, let $f_1 \in \rho_1$ be injective and let $\sigma \in S(I)$. If $\sigma f_1 g \in \rho_2$ then $\sigma fg \in \rho_2$ for every $f \in \rho_1$.*

Proof: Suppose $\sigma f_1 g \in \rho_2$ and let $f \in \rho_1$. By (D_1), there is $a \in End(X_1, \rho_1)$ such that $f_1 a = f$. Since $g^{-1}ag \in End(X_2, \rho_2)$, we have

$$\sigma f_1 g \in \rho_2 \Rightarrow \sigma f_1 g(g^{-1}ag) \in \rho_2 \Rightarrow \sigma f_1 ag \in \rho_2 \Rightarrow \sigma fg \in \rho_2,$$

which concludes the proof. ■

We can now prove the main theorem of this note.

Theorem 2.4 *Let ρ_1, ρ_2 be dense I-relations on X_1, X_2, respectively. Then the endomorphism monoids $End(X_1, \rho_1)$ and $End(X_2, \rho_2)$ are isomorphic if and only if the relational systems (X_1, ρ_1) and (X_2, ρ_2) are semi-isomorphic.*

Proof: Suppose $\phi : End(X_1, \rho_1) \to End(X_2, \rho_2)$ is an isomorphism. By Lemma 2.1, there is a bijection $g : X_1 \to X_2$ such that $a\phi = g^{-1}ag$ for every $a \in End(X_1, \rho_1)$. We claim that g is a semi-isomorphism from (X_1, ρ_1) to (X_2, ρ_2). Since ρ_1 and ρ_2 are dense, there are $f_1 \in \rho_1$ and $f_2 \in \rho_2$ such that f_1 and f_2 are injective. By Lemma 2.2, there is $\sigma \in S(I)$ such that $\sigma f_1 g \in \rho_2$. By Lemma 2.3, $\sigma fg \in \rho_2$ for every $f \in \rho_1$. Now, note that $f_2' = \sigma f_1 g$ is an injective element of ρ_2 and $\sigma^{-1} f_2' g^{-1} = f_1 \in \rho_1$. Thus, by

Lemma 2.3, $\sigma^{-1}f'g^{-1} \in \rho_1$ for every $f' \in \rho_2$. It follows that for every $f : I \to X_1$, $f \in \rho_1 \Leftrightarrow \sigma f g \in \rho_2$, and so g is a semi-isomorphism.

Conversely, suppose that $g : X_1 \to X_2$ is a semi-isomorphism, and let $\sigma \in S(I)$ be such that for every $f : I \to X_1$, $f \in \rho_1 \Leftrightarrow \sigma f g \in \rho_2$. Then for all $a \in End\,(X_1, \rho_1)$ and $f' : I \to X_2$,

$$f' \in \rho_2 \Rightarrow \sigma^{-1}f'g^{-1} \in \rho_1 \Rightarrow \sigma^{-1}f'g^{-1}a \in \rho_1 \Rightarrow \sigma\sigma^{-1}f'g^{-1}ag \in \rho_2 \Rightarrow f'g^{-1}ag \in \rho_2.$$

It follows that $\phi : End\,(X_1, \rho_1) \to End\,(X_2, \rho_2)$ defined by: $a\phi = g^{-1}ag$ is an isomorphism. ■

Applying Theorem 2.4 to binary relations, we obtain the following corollary.

Corollary 2.5 *Let ρ_1, ρ_2 be binary dense relations on X_1, X_2, respectively. Then the endomorphism monoids $End\,(X_1, \rho_1)$ and $End\,(X_2, \rho_2)$ are isomorphic if and only if the relational systems (X_1, ρ_1) and (X_2, ρ_2) are isomorphic or anti-isomorphic.*

Problem Classify the n-ary dense relations, for $n \geq 2$.

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