INDEPENDENT AXIOM SYSTEMS FOR NEARLATTICES

JOÃO ARAÚJO* AND MICHAEL KINYON

ABSTRACT. A *nearlattice* is a join semilattice such that every principal filter is a lattice with respect to the induced order. Hickman and later Chajda *et al* independently showed that nearlattices can be treated as varieties of algebras with a ternary operation satisfying certain axioms. Our main result is that the variety of nearlattices is 2-based, and we exhibit an explicit system of two independent identities. We also show that the original axiom systems of Hickman and of Chajda *et al* are respectively dependent.

1. INTRODUCTION

A *nearlattice* $(L, \vee, \{\wedge_a\}_{a\in L})$ is a join semilattice $(L, \vee)$ such that for each $a \in L$, the principal filter $(a] = \{x \in L \,|\, a \leq x\}$ is a lattice with respect to the induced order. Each $x, y \in (a]$ has a meet, denoted by $x \wedge_a y$. Define $m : L \times L \times L \to L$ by

$$m(x, y, z) = (x \vee z) \wedge_z (y \vee z)\,, \tag{1.1}$$

for $x, y, z \in L$. Sometimes the literature on nearlattices considers instead the dual object consisting of a meet semilattice such that each principal ideal is a lattice.

Hickman [4] and, independently, Chajda and Halaš [1] and Chajda and Kolařík [3] characterized nearlattices in terms of the ternary operation m. Part (1) of the following proposition gives Hickman's axioms and part (2) gives the axioms of Chajda *et al* as found in, for instance, [2], §2.6.

Proposition 1.1. *(1) Let $(L, \vee, \{\wedge_a\}_{a\in L})$ be a nearlattice. Then $m : L \times L \times L \to L$ defined by (1.1) satisfies the following identities:*

$$m(x, x, x) = x \tag{H1}$$
$$m(x, x, y) = m(y, y, x) \tag{H2}$$
$$m(m(x, x, y), m(x, x, y), z) = m(x, x, m(y, y, z)) \tag{H3}$$
$$m(x, y, z) = m(y, x, z) \tag{H4}$$
$$m(m(x, y, z), m(x, y, z), m(x, x, z)) = m(x, x, z) \tag{H5}$$
$$m(m(x, x, y), z, y) = m(x, z, y) \tag{H6}$$
$$m(x, m(x, x, y), z) = m(x, x, z) \tag{H7}$$
$$m(m(x, m(y, z, u), u), m(x, m(y, z, u), u), m(x, z, u)) = m(x, z, u)\,. \tag{H8}$$

Conversely, let (L, m) be a ternary algebra satisfying the identities (P1)–(P8). Define $x \vee y = m(x, x, y)$. Then $(L, \vee)$ is a semilattice. For each $a \in L$ and all $x, y \in (a]$, define $x \wedge_a y = m(x, y, a)$. Then $(L, \vee, \{\wedge_a\}_{a\in L})$ is a nearlattice.

*Partially supported by FCT and FEDER, Project POCTI-ISFL-1-143 of Centro de Algebra da Universidade de Lisboa, and by FCT and PIDDAC through the project PTDC/MAT/69514/2006.

(2) Let $(L, \vee, \{\wedge_a\}_{a\in L})$ be a nearlattice. Then $m : L\times L\times L \to L$ defined by (1.1) *satisfies the following identities:*

$$m(x, y, x) = x \tag{P1}$$

$$m(x, x, y) = m(y, y, x) \tag{P2}$$

$$m(m(x, x, y), m(x, x, y), z) = m(x, x, m(y, y, z)) \tag{P3}$$

$$m(x, y, z) = m(y, x, z) \tag{P4}$$

$$m(m(x, y, z), w, z) = m(x, m(y, w, z), z) \tag{P5}$$

$$m(x, m(y, y, x), z) = m(x, x, z) \tag{P6}$$

$$m(x, x, m(x, y, z)) = m(x, x, z) \tag{P7}$$

$$m(m(x, x, z), m(y, y, z), z) = m(x, y, z)\,. \tag{P8}$$

Conversely, let (L, m) be a ternary algebra satisfying the identities (P1)–(P8). Define $x \vee y = m(x, x, y)$. Then $(L, \vee)$ is a semilattice. For each $a \in L$ and all $x, y \in (a]$, define $x \wedge_a y = m(x, y, a)$. Then $(L, \vee, \{\wedge_a\}_{a\in L})$ is a nearlattice.

(In fact, Hickman worked with the dual notion of nearlattice, and used a different convention for ordering the variables. Our $m(x, y, z)$ is Hickman's $j(x, z, y)$.)

Thus nearlattices can be treated as varieties of algebras, and from now on, we will refer to the ternary structures (L, m) themselves as nearlattices. The main result of this paper is that the variety of nearlattices (L, m) is 2-based.

Theorem 1.2. *The following is a basis of identities for nearlattices.*

$$m(x, y, x) = x \tag{N1}$$

$$m(m(x, y, z), m(y, m(u, x, z), z), w) = m(w, w, m(y, m(x, u, z), z))\,. \tag{N2}$$

Left open in both the investigations of Hickman and of Chajda *et al* was the independence of their respective axiom systems. In fact, three of Hickman's axioms are dependent upon the others, and one of the axioms of Chajda *et al* is dependent upon the rest.

Theorem 1.3. *The system $\mathcal{H}$ = {*(H1), (H2), (H4), (H7), (H8)*} is a basis of independent identities for the variety of nearlattices. In particular, the identities $\mathcal{H}$ imply* (H3), (H5) *and* (H6).

Theorem 1.4. *The system $\mathcal{C}$ = {*(P1), (P2)–(P8)*} is a basis of independent identities for the variety of nearlattices. In particular, the identities $\mathcal{C}$ imply* (P3).

In §2, §3 and §4, we prove Theorems 1.3, 1.4 and 1.2, respectively. We conclude in §5 with some open problems.

2. Hickman's Axioms

In this section we prove Theorem 1.3. Assume now that (A, m) is an algebra satisfying (H1), (H2), (H4), (H7) and (H8).

The identity (H8) is taken directly from Hickman's paper [4] after appropriate change of notation. There is, however, a somewhat more useful equivalent form.

Lemma 2.1. *In the presence of (H2) and (H4), the identity (H8) is equivalent to*

$$m(m(x, y, z), m(x, y, z), m(x, m(y, u, z), z)) = m(x, y, z)\,. \tag{H8'}$$

Proof. Starting with the left of (H8'), we have

$$\begin{aligned}
&m(m(x,y,z),m(x,y,z),m(x,m(y,u,z),z)) \\
&\overset{(H4)}{=} m(m(x,y,z),m(x,y,z),m(x,m(u,y,z),z)) \\
&\overset{(H2)}{=} m(m(x,m(u,y,z),z),m(x,m(u,y,z),z),m(x,y,z))\,,
\end{aligned}$$

which is the left side of (H8). Since the right sides of (H8) and (H8') coincide, this proves the desired equivalence. □

Next we need a pair of useful identities.

Lemma 2.2. *For all x, y,*

$$m(x,y,x) = x \tag{2.1}$$
$$m(x,y,y) = y \tag{2.2}$$

Proof. First, we have

$$\begin{aligned}
m(x,x,m(x,y,x)) &\overset{(H2)}{=} m(m(x,y,x),m(x,y,x),\underbrace{x}) \\
&\overset{(H1)}{=} m(m(x,y,x),m(x,y,x),m(x,x,x)) \\
&\overset{(H7)}{=} m(m(x,y,x),m(x,y,x),m(x,m(x,x,y),x)) \\
&\overset{(H2)}{=} m(m(x,y,x),m(x,y,x),m(x,m(y,y,x),x)) \\
&\overset{(H8')}{=} m(x,y,x)\,,
\end{aligned}$$

that is,

$$m(x,x,m(x,y,x)) = m(x,y,x)\,. \tag{2.3}$$

Next,

$$\begin{aligned}
m(x,m(x,y,x),x) &\overset{(2.3)}{=} m(\underbrace{x},\underbrace{x},m(x,m(x,y,x),x)) \\
&\overset{(H1)}{=} m(m(x,x,x),m(x,x,x),m(x,m(x,y,x),x)) \\
&\overset{(H8')}{=} m(x,x,x) \\
&\overset{(H1)}{=} x\,,
\end{aligned}$$

that is,

$$m(x,m(x,y,x),x) = x\,. \tag{2.4}$$

In (H8'), take $z = y$ and $x = m(y,v,y)$. The left side of (H8') becomes

$$\begin{aligned}
&m(\underbrace{m(m(y,v,y),y,y)},\underbrace{m(m(y,v,y),y,y)},m(m(y,v,y),m(y,u,y),y)) \\
&\overset{(H4)}{=} m(\underbrace{m(y,m(y,v,y),y)},\underbrace{m(y,m(y,v,y),y)},m(m(y,v,y),m(y,u,y),y)) \\
&\overset{(2.4)}{=} m(y,y,m(m(y,v,y),m(y,u,y),y))\,.
\end{aligned}$$

The right side of (H8') becomes

$$m(m(y,v,y),y,y) \stackrel{(H4)}{=} m(y,m(y,v,y),y) \stackrel{(2.4)}{=} y\,.$$

Putting this together, replace y with x, v with y and u with z to get

$$m(x,x,m(m(x,y,x),m(x,z,x),x)) = x\,. \tag{2.5}$$

Now we can verify (2.1):

$$\begin{aligned} m(x,y,x) &\stackrel{(2.3)}{=} m(x,x,\underbrace{m(x,y,x)}) \\ &\stackrel{(2.3)}{=} m(x,x,\underbrace{m(x,x,m(x,y,x))}) \\ &\stackrel{(H2)}{=} m(x,x,m(m(x,y,x),m(x,y,x),x)) \\ &\stackrel{(2.5)}{=} x\,. \end{aligned}$$

Finally, (2.2) follows easily: $m(x,y,y) \stackrel{(H4)}{=} m(y,x,y) \stackrel{(2.1)}{=} y$. ☐

Next we work toward (H5). Key to this are the following identities.

Lemma 2.3. *For all x,y,z,*

$$m(m(x,y,z),m(x,y,z),z) = m(x,y,z) \tag{2.6}$$

$$m(x,m(x,y,z),z) = m(x,y,z) \tag{2.7}$$

Proof. For (2.6), we have

$$\begin{aligned} m(m(x,y,z),m(x,y,z),\underbrace{z}) &\stackrel{(2.2)}{=} m(m(x,y,z),m(x,y,z),m(x,\underbrace{z},z)) \\ &\stackrel{(2.2)}{=} m(m(x,y,z),m(x,y,z),m(x,m(y,z,z),z)) \\ &\stackrel{(H8')}{=} m(x,y,z)\,. \end{aligned}$$

Then we compute

$$\begin{aligned} m(x,m(x,y,z),z) &\stackrel{(H4)}{=} m(m(x,y,z),x,z) \\ &\stackrel{(H8')}{=} m(m(m(x,y,z),x,z),m(m(x,y,z),x,z),m(m(x,y,z),m(x,y,z),z)) \\ &\stackrel{(2.6)}{=} m(m(m(x,y,z),x,z),m(m(x,y,z),x,z),m(x,y,z)) \\ &\stackrel{(H2)}{=} m(m(x,y,z),m(x,y,z),\underbrace{m(m(x,y,z),x,z)}) \\ &\stackrel{(H4)}{=} m(m(x,y,z),m(x,y,z),m(x,\underbrace{m(x,y,z)},z)) \\ &\stackrel{(H4)}{=} m(m(x,y,z),m(x,y,z),m(x,m(y,x,z),z)) \\ &\stackrel{(H8')}{=} m(x,y,z)\,. \end{aligned}$$

This establishes (2.7). ☐

Lemma 2.4. *If (A, m) is an algebra satisfying (H1), (H2), (H4), (H7) and (H8), then (H5) holds.*

Proof. We compute

$$\begin{aligned} m(m(x,y,z), m(x,y,z), m(x,x,z)) &\overset{(H2)}{=} m(m(x,x,z), m(x,x,z), \underbrace{m(x,y,z)}) \\ &\overset{(2.7)}{=} m(m(x,x,z), m(x,x,z), m(x, m(x,y,z), z)) \\ &\overset{(H8')}{=} m(x,x,z)\,, \end{aligned}$$

which proves the desired result. □

We continue with the assumptions of this section that we have an algebra (A, m) satisfying (H1), (H2), (H4), (H7) and (H8). By Lemma 2.4, we may now freely use (H5). Our next goal is to establish (H6).

Lemma 2.5. *For all x, y, z,*

$$m(m(x,x,y), m(z,x,y), y) = m(z,x,y)\,. \tag{2.8}$$

Proof. We compute

$$\begin{aligned} m(m(x,x,y), m(z,x,y), y) &\overset{(H4)}{=} m(\underbrace{m(z,x,y)}, m(x,x,y), y) \\ &\overset{(H4)}{=} m(m(x,z,y), \underbrace{m(x,x,y)}, y) \\ &\overset{(H5)}{=} m(m(x,z,y), m(m(x,z,y), m(x,z,y), m(x,x,y)), y) \\ &\overset{(H7)}{=} m(m(x,z,y), m(x,z,y), y) \\ &\overset{(H2)}{=} m(y,y,m(x,z,y)) \\ &\overset{(2.7)}{=} m(x,z,y) \\ &\overset{(H4)}{=} m(z,x,y)\,, \end{aligned}$$

which establishes the desired result. □

Lemma 2.6. *If (A, m) is an algebra satisfying (H1), (H2), (H4), (H7) and (H8), then (H6) holds.*

Proof. We compute

$$\begin{aligned}
m(m(x,x,y),z,y) &\overset{(H8')}{=} m(m(m(x,x,y),z,y), m(m(x,x,y),z,y), \underbrace{m(m(x,x,y),m(z,x,y),y)}) \\
&\overset{(2.8)}{=} m(m(m(x,x,y),z,y), m(m(x,x,y),z,y), m(z,x,y)) \\
&\overset{(H2)}{=} m(m(z,x,y), m(z,x,y), \underbrace{m(m(x,x,y),z,y)}) \\
&\overset{(H4)}{=} m(m(z,x,y), m(z,x,y), m(z,m(x,x,y),y)) \\
&\overset{(H8')}{=} m(z,x,y) \\
&\overset{(H4)}{=} m(x,z,y)\,,
\end{aligned}$$

which establishes (H6). □

The next goal is to verify (H3). We may now use (H5) and (H6) freely in calculations.

Lemma 2.7. *For all* x, y, z, u,

$$m(x, m(y,y,z), y) = m(x,z,y) \tag{2.9}$$
$$m(x,x,m(x,y,z)) = m(z,z,x) \tag{2.10}$$
$$m(x, m(y,z,u), z) = m(x,u,z) \tag{2.11}$$
$$m(x, m(y,y,z), m(y,y,u)) = m(x,z,m(y,y,u)) \tag{2.12}$$

Proof. For (2.9), we have

$$\begin{aligned}
m(x,m(y,y,z),y) &\overset{(H2)}{=} m(x,m(z,z,y),y) \\
&\overset{(H4)}{=} m(m(z,z,y),x,y) \\
&\overset{(H6)}{=} m(z,x,y) \\
&\overset{(H4)}{=} m(x,z,y)\,.
\end{aligned}$$

For (2.10), we compute

$$
\begin{aligned}
m(x,x,m(x,y,z)) &\overset{(H2)}{=} m(m(x,y,z),m(x,y,z),x) \\
&\overset{(H7)}{=} m(m(x,y,z),\underbrace{m(m(x,y,z),m(x,y,z),m(x,x,z))},x) \\
&\overset{(H5)}{=} m(m(x,y,z),m(x,x,z),x) \\
&\overset{(2.9)}{=} m(m(x,y,z),z,x) \\
&\overset{(H4)}{=} m(z,\underbrace{m(x,y,z)},x) \\
&\overset{(2.6)}{=} m(z,\underbrace{m(m(x,y,z),m(x,y,z),z)},x) \\
&\overset{(H2)}{=} m(z,m(z,z,m(x,y,z)),x) \\
&\overset{(H7)}{=} m(z,z,x)\,.
\end{aligned}
$$

For (2.11), we have

$$
\begin{aligned}
m(x,\underbrace{m(y,z,u)},z) &\overset{(H4)}{=} m(x,\underbrace{m(z,y,u)},z) \\
&\overset{(2.9)}{=} m(x,\underbrace{m(z,z,m(z,y,u))},z) \\
&\overset{(2.10)}{=} m(x,m(u,u,z),z) \\
&\overset{(H4)}{=} m(m(u,u,z),x,z) \\
&\overset{(H6)}{=} m(u,x,z) \\
&\overset{(H4)}{=} m(x,u,z)\,.
\end{aligned}
$$

Finally, for (2.12), we have

$$
\begin{aligned}
m(x,\underbrace{m(y,y,z)},m(y,y,u)) &\overset{(H7)}{=} m(x,m(y,m(y,y,u),z),m(y,y,u)) \\
&\overset{(2.11)}{=} m(x,z,m(y,y,u))\,.
\end{aligned}
$$

This completes the proof. □

Lemma 2.8. *If (A,m) is an algebra satisfying (H1), (H2), (H4), (H7) and (H8), then (H3) holds.*

Proof. We compute

$$\begin{aligned}
m(x,x,m(y,y,z)) &\overset{(2.12)}{=} m(x,m(y,y,x),m(y,y,z)) \\
&\overset{(H4)}{=} m(m(y,y,x),x,m(y,y,z)) \\
&\overset{(2.12)}{=} m(m(y,y,x),m(y,y,x),m(y,y,z)) \\
&\overset{(H2)}{=} m(m(y,y,z),m(y,y,z),m(y,y,x)) \\
&\overset{(2.12)}{=} m(m(y,y,z),z,m(y,y,x)) \\
&\overset{(H4)}{=} m(z,m(y,y,z),m(y,y,x)) \\
&\overset{(2.12)}{=} m(z,z,\underbrace{m(y,y,x)}) \\
&\overset{(H2)}{=} m(z,z,m(x,x,y)) \\
&\overset{(H2)}{=} m(m(x,x,y),m(x,x,y),z)\,.
\end{aligned}$$

This establishes (H3) as claimed. □

Next, we check the independence of the axioms (H1), (H2), (H4), (H7) and (H8). We found these models using Mace4 [5]. We simply state the models and leave the verification that each model has its claimed properties to the reader.

Example 2.9. An example of an algebra (S,m) satisfying (H1), (H2), (H4) and (H7), but not (H8) is given by $S=\{0,1,2\}$ with $m : S\times S\times S\to S$ defined by the following tables.

$m(0,\cdot,\cdot)$	0	1	2
0	0	2	2
1	0	0	0
2	0	2	2

$m(1,\cdot,\cdot)$	0	1	2
0	0	0	0
1	2	1	2
2	2	1	2

$m(2,\cdot,\cdot)$	0	1	2
0	0	2	2
1	2	1	2
2	2	2	2

Example 2.10. An example of an algebra (S,m) satisfying (H1), (H2), (H4) and (H8), but not (H7) is given by $S=\{0,1,2\}$ with $m : S\times S\times S\to S$ defined by the following tables.

$m(0,\cdot,\cdot)$	0	1	2
0	0	1	0
1	0	1	2
2	0	1	2

$m(1,\cdot,\cdot)$	0	1	2
0	0	1	2
1	1	1	1
2	0	1	2

$m(2,\cdot,\cdot)$	0	1	2
0	0	1	2
1	0	1	2
2	0	1	2

Example 2.11. An example of an algebra (S,m) satisfying (H1), (H2), (H7) and (H8), but not (H4) is given by $S=\{0,1\}$ with $m : S\times S\times S\to S$ defined by the following tables.

$m(0,\cdot,\cdot)$	0	1
0	0	1
1	0	1

$m(1,\cdot,\cdot)$	0	1
0	1	1
1	1	1

$m(0,\cdot,\cdot)$	0	1
0	0	1
1	0	1

$m(1,\cdot,\cdot)$	0	1
0	0	1
1	0	1

Example 2.12. An example of an algebra (S, m) satisfying (H1), (H4), (H7) and (H8), but not (H2) is given by $S = \{0, 1\}$ with $m : S \times S \times S \to S$ defined by the following tables.

Example 2.13. An example of an algebra (S, m) satisfying (H2), (H4), (H7) and (H8), but not (H1) is given by $S = \{0, 1\}$ with $m : S \times S \times S \to S$ defined by the following tables.

$m(0,\cdot,\cdot)$	0	1
0	1	1
1	1	1

$m(1,\cdot,\cdot)$	0	1
0	1	1
1	1	1

Putting together Lemmas 2.4, 2.6 and 2.8, along with Examples 2.9, 2.10, 2.11, 2.12 and 2.13, we have completed the proof of Theorem 1.3.

3. Axioms of Chajda *et al*

In this section we prove Theorem 1.4. Assume now that we have an algebra (A, m) satisfying (P1), (P2) and (P4)–(P8).

Lemma 3.1. *For all x, y, z,*

$$m(m(x,y,z), m(x,x,z), y) = m(y,y,z)\,. \tag{3.1}$$

Proof. We compute

$$\begin{aligned} m(m(x,y,z), \underbrace{m(x,x,z)}, y) &\overset{(P7)}{=} m(m(x,y,z), m(x,x,m(x,y,z)), y) \\ &\overset{(P6)}{=} m(m(x,y,z), m(x,y,z), y) \\ &\overset{(P2)}{=} m(y,y,\underbrace{m(x,y,z)}) \\ &\overset{(P4)}{=} m(y,y,m(y,x,z)) \\ &\overset{(P7)}{=} m(y,y,z)\,, \end{aligned}$$

which establishes the claim. ☐

Lemma 3.2. *Let (A, m) be an algebra satisfying (P1), (P2) and (P4)–(P8). Then (A, m) satisfies (P3).*

Proof. First, we have

$$\begin{aligned} m(x,x,m(y,y,z)) &\overset{(P2)}{=} m(m(y,y,z), m(y,y,z), x) \\ &\overset{(P7)}{=} m(m(y,y,z), m(y,y,z), \underbrace{m(m(y,y,z), m(y,x,z), x)}) \\ &\overset{(P4)}{=} m(m(y,y,z), m(y,y,z), \underbrace{m(m(y,x,z), m(y,y,z), x)}) \\ &\overset{(3.1)}{=} m(m(y,y,z), m(y,y,z), m(x,x,y))\,, \end{aligned}$$

that is,

$$m(x,x,m(y,y,z)) = m(m(y,y,z),m(y,y,z),m(x,x,y))\,. \tag{3.2}$$

Now

$$\begin{aligned}
m(x,x,m(y,y,z)) &\overset{(3.2)}{=} m(m(y,y,z),m(y,y,z),m(x,x,y))\\
&\overset{(P2)}{=} m(\underbrace{m(x,x,y)},\underbrace{m(x,x,y)},\underbrace{m(y,y,z)})\\
&\overset{(P2)}{=} m(m(y,y,x),m(y,y,x),m(z,z,y))\\
&\overset{(3.2)}{=} m(z,z,\underbrace{m(y,y,x)})\\
&\overset{(P2)}{=} m(z,z,m(x,x,y))\\
&\overset{(P2)}{=} m(m(x,x,y),m(x,x,y),z)\,,
\end{aligned}$$

which establishes (P3). ☐

Note that we used only (P2), (P4), (P6) and (P7) in the proof of (P3).

Next we consider the independence of the axioms (P1), (P2) and (P4)–(P8). As in the previous section, we simply give the models and leave the verification that each model has its claimed properties to the reader.

Example 3.3. An example of an algebra (S,m) satisfying (P1), (P2), (P4)–(P7), but not (P8) is given by $S=\{0,1,2,3\}$ with $m: S\times S\times S\to S$ defined by the following tables.

$m(0,\cdot,\cdot)$	0	1	2	3
0	0	1	1	1
1	0	1	1	1
2	0	3	2	1
3	0	1	1	3

$m(1,\cdot,\cdot)$	0	1	2	3
0	0	1	1	1
1	1	1	1	1
2	1	1	2	1
3	1	1	1	3

$m(2,\cdot,\cdot)$	0	1	2	3
0	0	3	2	1
1	1	1	2	1
2	1	1	2	1
3	1	1	2	3

$m(3,\cdot,\cdot)$	0	1	2	3
0	0	1	1	3
1	1	1	1	3
2	1	1	2	3
3	1	1	1	3

Example 3.4. An example of an algebra (S,m) satisfying (P1), (P2), (P4)–(P6), (P8), but not (P7) is given by $S=\{0,1,2,3\}$ with $m: S\times S\times S\to S$ defined by the following tables.

Example 3.5. An example of an algebra (S,m) satisfying (P1), (P2), (P4), (P5), (P7), (P8), but not (P6) is given by $S=\{0,1,2\}$ with $m: S\times S\times S\to S$ defined by the following tables.

Example 3.6. An example of an algebra (S,m) satisfying (P1), (P2), (P4), (P6)–(P8), but not (P5) is given by $S=\{0,1,2,3,4\}$ with $m: S\times S\times S\to S$ defined by the following tables.

$m(0,\cdot,\cdot)$	0	1	2	3
0	0	3	0	3
1	0	1	1	3
2	0	1	2	3
3	0	3	0	3

$m(1,\cdot,\cdot)$	0	1	2	3
0	0	1	1	3
1	3	1	1	3
2	0	1	2	3
3	3	1	1	3

$m(2,\cdot,\cdot)$	0	1	2	3
0	0	1	2	3
1	0	1	2	3
2	0	1	2	3
3	0	1	2	3

$m(3,\cdot,\cdot)$	0	1	2	3
0	0	3	0	3
1	3	1	1	3
2	0	1	2	3
3	3	3	3	3

$m(0,\cdot,\cdot)$	0	1	2
0	0	1	0
1	0	1	2
2	0	1	2

$m(1,\cdot,\cdot)$	0	1	2
0	0	1	2
1	1	1	1
2	0	1	2

$m(2,\cdot,\cdot)$	0	1	2
0	0	1	2
1	0	1	2
2	0	1	2

$m(0,\cdot,\cdot)$	0	1	2	3	4
0	0	0	3	3	0
1	0	1	2	3	1
2	0	4	2	3	1
3	0	0	3	3	0
4	0	1	2	3	4

$m(1,\cdot,\cdot)$	0	1	2	3	4
0	0	1	2	3	1
1	0	1	2	3	1
2	0	1	2	3	1
3	0	1	2	3	1
4	0	1	2	3	4

$m(2,\cdot,\cdot)$	0	1	2	3	4
0	0	4	2	3	1
1	0	1	2	3	1
2	3	2	2	3	2
3	3	2	2	3	2
4	0	1	2	3	4

$m(3,\cdot,\cdot)$	0	1	2	3	4
0	0	0	3	3	0
1	0	1	2	3	1
2	3	2	2	3	2
3	3	3	3	3	3
4	0	1	2	3	4

$m(4,\cdot,\cdot)$	0	1	2	3	4
0	0	1	2	3	4
1	0	1	2	3	4
2	0	1	2	3	4
3	0	1	2	3	4
4	0	1	2	3	4

$m(0,\cdot,\cdot)$	0	1
0	0	1
1	0	1

$m(1,\cdot,\cdot)$	0	1
0	1	1
1	1	1

Example 3.7. An example of an algebra (S, m) satisfying (P1), (P2), (P5)–(P8), but not (P4) is given by $S = \{0, 1\}$ with $m : S \times S \times S \to S$ defined by the following tables.

Example 3.8. An example of an algebra (S, m) satisfying (P1), (P4)–(P8), but not (P2) is given by $S = \{0, 1\}$ with $m : S \times S \times S \to S$ defined by the following tables.

Example 3.9. An example of an algebra (S, m) satisfying (P2), (P4)–(P8), but not (P1) is given by $S = \{0, 1\}$ with $m : S \times S \times S \to S$ defined by the following tables.

Putting together Lemma 3.2 with Examples 3.3, 3.4, 3.5, 3.6, 3.7, 3.8 and 3.9, we have completed the proof of Theorem 1.4.

$m(0,\cdot,\cdot)$	0	1
0	0	1
1	0	1

$m(1,\cdot,\cdot)$	0	1
0	0	1
1	0	1

$m(0,\cdot,\cdot)$	0	1
0	1	1
1	1	1

$m(1,\cdot,\cdot)$	0	1
0	1	1
1	1	1

4. A 2-Base For Nearlattices

In this section we prove Theorem 1.2. We first prove the easy direction.

Lemma 4.1. *Every nearlattice satisfies (N1) and (N2).*

Proof. We are, of course, free to use Hickman's axioms and the axioms of Chajda *et al*, together with any derived properties from the previous two sections. The identity (N1) is just (P1), so (N2) is the only identity requiring proof:

$$
\begin{aligned}
& m(m(x,y,z), m(y, \underbrace{m(u,x,z)}, z), w) \\
& \overset{(H4)}{=} m(\underbrace{m(x,y,z)}, m(y, m(x,u,z), z), w) \\
& \overset{(H4)}{=} m(m(y,x,z), m(y, m(x,u,z), z), w) \\
& \overset{(H4)}{=} m(m(y, m(x,u,z), z), \underbrace{m(y,x,z)}, w) \\
& \overset{(H8')}{=} m(m(y, m(x,u,z), z), \underbrace{m(m(y,x,z), m(y,x,z), m(y, m(x,u,z), z))}, w) \\
& \overset{(P6)}{=} m(m(y, m(x,u,z), z), m(y, m(x,u,z), z), w) \\
& \overset{(H2)}{=} m(w,w,m(y,m(x,u,z),z))\,.
\end{aligned}
$$

This completes the proof of the lemma. ☐

Lemma 4.2. *Every algebra* (A, m) *satisfying (N1) and (N2) satisfies the identities (H2),*

$$m(m(x,y,y),y,z) = m(z,z,y) \tag{4.1}$$

$$m(x,y,y) = y\,. \tag{4.2}$$

Proof. First, set $z = y$ in (N2). The left side becomes

$$m(m(x,y,y), \underbrace{m(y, m(u,x,y), y)}, w) \overset{(N1)}{=} m(m(x,y,y), y, w)\,.$$

The right side becomes

$$m(w,w,\underbrace{m(y,m(x,u,y),y)}) = m(w,w,y)\,.$$

The equality of the two sides proves (4.1).

Now (H2) follows easily: $m(x,x,y) \overset{(4.1)}{=} m(\underbrace{m(y,y,y)}, y, x) \overset{(N1)}{=} m(y,y,x)$.

Next, set $z = x$ and $w = m(x, y, x)$ in (N2). The left side becomes

$$m(m(x,y,x), m(y, m(u,x,x), x), m(x,y,x)) \overset{(N1)}{=} m(x,y,x) \overset{(N1)}{=} x\,.$$

The right side becomes

$$\begin{aligned} m(\underbrace{m(x,y,x)},\underbrace{m(x,y,x)}, m(y,\underbrace{m(x,u,x)}, x)) &\overset{(N1)}{=} m(x,x,m(y,x,x)) \\ &\overset{(H2)}{=} m(m(y,x,x), m(y,x,x), x) \\ &\overset{(4.1)}{=} m(m(y,x,x), x, m(y,x,x)) \\ &\overset{(N1)}{=} m(y,x,x)\,. \end{aligned}$$

The equality of the two sides proves (4.2). □

Lemma 4.3. *For all x, y, z, u,*

$$m(x, m(y,z,u), u) = m(x, m(z,y,u), u) \tag{4.3}$$
$$m(m(x,y,z), z, u) = m(u,u,z) \tag{4.4}$$
$$m(x,x,m(y,z,x)) = m(y,z,x) \tag{4.5}$$
$$m(m(x,y,z), m(y,x,z), z) = m(x,y,z) \tag{4.6}$$
$$m(m(x, m(y,x,z), z), m(y,x,z), z) = m(y,x,z) \tag{4.7}$$

Proof. We compute

$$\begin{aligned} m(x, m(y,z,u), u) &\overset{(4.2)}{=} m(m(z,x,u), m(x,m(y,z,u),u), m(x,m(y,z,u),u)) \\ &\overset{(N2)}{=} m(m(x,m(y,z,u),u), m(x,m(y,z,u),u), m(x,m(z,y,u),u)) \\ &\overset{(H2)}{=} m(m(x,m(z,y,u),u), m(x,m(z,y,u),u), m(x,m(y,z,u),u)) \\ &\overset{(N2)}{=} m(m(y,x,u), m(x,m(z,y,u),u), m(x,m(z,y,u),u)) \\ &\overset{(4.2)}{=} m(x, m(z,y,u), u)\,, \end{aligned}$$

which establishes (4.3)

Next, in (N2), set $u = z$. The left side becomes

$$\begin{aligned} m(m(x,y,z), m(y,\underbrace{m(z,x,z)}, z), w) &\overset{(N1)}{=} m(m(x,y,z), \underbrace{m(y,z,z)}, w) \\ &\overset{(4.2)}{=} m(m(x,y,z), z, w)\,. \end{aligned}$$

The right side becomes

$$\begin{aligned} m(w,w,m(y,\underbrace{m(x,z,z)}, z)) &\overset{(4.2)}{=} m(w,w,\underbrace{m(y,z,z)}) \\ &\overset{(4.2)}{=} m(w,w,z)\,. \end{aligned}$$

The equality of the two sides establishes (4.4).

Now we have

$$\begin{aligned} m(x,x,m(y,z,x)) &\overset{(H2)}{=} m(m(y,z,x),m(y,z,x),x) \\ &\overset{(4.4)}{=} m(m(y,z,x),x,m(y,z,x)) \\ &\overset{(N1)}{=} m(y,z,x)\,, \end{aligned}$$

which gives (4.5)

Next,

$$\begin{aligned} m(m(x,y,z),m(y,x,z),z) &\overset{(4.3)}{=} m(m(x,y,z),m(x,y,z),z) \\ &\overset{(H2)}{=} m(z,z,m(x,y,z)) \\ &\overset{(4.5)}{=} m(x,y,z)\,, \end{aligned}$$

which yields (4.6).

Now in (N2), set $y = m(u,x,z)$ and $w = z$. The left side of (N2) becomes

$$\begin{aligned} & m(m(x,m(u,x,z),z),\underbrace{m(m(u,x,z),m(u,x,z),z)},z) \\ &\overset{(H2)}{=} m(m(x,m(u,x,z),z),\underbrace{m(z,z,m(u,x,z))},z) \\ &\overset{(4.5)}{=} m(m(x,m(u,x,z),z),m(u,x,z),z)\,. \end{aligned}$$

The right side becomes

$$\begin{aligned} m(z,z,\underbrace{m(m(u,x,z),m(x,u,z),z)}) &\overset{(4.6)}{=} m(z,z,m(u,x,z)) \\ &\overset{(4.5)}{=} m(u,x,z)\,. \end{aligned}$$

The equality of the two sides, along with replacing u with y, gives (4.7). ☐

Lemma 4.4. *Every algebra (A,m) satisfying (N1) and (N2) satisfies (H4).*

Proof. In (4.7), set $x = m(u,v,z)$ and $y = m(v,u,z)$. The left hand side of (4.7) becomes

$$\begin{aligned} & m(m(m(u,v,z),\underbrace{m(m(v,u,z),m(u,v,z),z)},z),\underbrace{m(m(v,u,z),m(u,v,z),z)},z) \\ &\overset{(4.6)}{=} m(\underbrace{m(m(u,v,z),m(v,u,z),z)},m(v,u,z),z) \\ &\overset{(4.6)}{=} m(m(u,v,z),m(v,u,z),z) \\ &\overset{(4.6)}{=} m(u,v,z)\,. \end{aligned}$$

The right side of (4.7) becomes

$$m(m(v,u,z),m(u,v,z),z) \overset{(4.6)}{=} m(v,u,z)\,.$$

The equality of the two sides, along with replacing u with x and v with y, gives (H4). ☐

Lemma 4.5. *Every algebra (A,m) satisfying (N1) and (N2) satisfies (H7).*

Proof. We compute

$$\begin{aligned} m(x, m(x,x,y), z) &\stackrel{(H4)}{=} m(\underbrace{m(x,x,y)}, x, z) \\ &\stackrel{(H2)}{=} m(m(y,y,x), x, z) \\ &\stackrel{(4.4)}{=} m(z,z,x) \\ &\stackrel{(H2)}{=} m(x,x,z)\,, \end{aligned}$$

which establishes (H7). □

Lemma 4.6. *Every algebra (A, m) satisfying (N1) and (N2) satisfies (H8').*

Proof. In (H8'), set $w = m(x,y,z)$. The left side is

$$m(m(x,y,z), m(y, m(u,x,z), z), m(x,y,z)) \stackrel{(H2)}{=} m(x,y,z) \stackrel{(H4)}{=} m(y,x,z)\,.$$

The right side becomes

$$m(\underbrace{m(x,y,z)}, \underbrace{m(x,y,z)}, m(y, m(x,u,z), z)) \stackrel{(H4)}{=} m(m(y,x,z), m(y,x,z), m(y, m(x,u,z), z))\,.$$

The equality of the two sides, along with exchanging the roles of x and y, gives (H8'). □

Now we consider the independence of the axioms (N1) and (N2).

Example 4.7. An example of an algebra (S, m) satisfying (N1) but not (N2) is given by $S = \{0, 1\}$ with $m : S \times S \times S \to S$ defined by the following tables.

$m(0,\cdot,\cdot)$	0	1
0	0	0
1	0	0

$m(1,\cdot,\cdot)$	0	1
0	0	1
1	1	1

Example 4.8. An example of an algebra (S, m) satisfying (N2) but not (N1) is given by $S = \{0, 1\}$ with $m : S \times S \times S \to S$ defined by the following tables.

$m(0,\cdot,\cdot)$	0	1
0	1	1
1	1	1

$m(1,\cdot,\cdot)$	0	1
0	1	1
1	1	1

Putting together Lemmas 4.1, 4.2, 4.4, 4.5 and 4.6, together with Examples 4.7 and 4.8, we have proved Theorem 1.2.

5. Problems

The following questions arise rather naturally from this investigation.

Problem 5.1.
(1) Is there a 2-base for nearlattices with one axiom no longer than (N1) and the other shorter than (N2)?
(2) Is there a 2-base for nearlattices involving fewer than five variables?
(3) Is the variety of nearlattices 1-based?

Acknowledgment. The investigations in this paper were aided by the automated deduction tool Prover9 and the finite model builder Mace4, both developed by McCune [5].

The first author was partially supported by FCT and FEDER, Project POCTI-ISFL-1-143 of Centro de Algebra da Universidade de Lisboa, by FCT and PIDDAC through the project PTDC/MAT/69514/2006, by PTDC/MAT/69514/2006 Semigroups and Languages, and by PTDC/MAT/101993/2008 Computations in groups and semigroups.

References

[1] I. Chajda and R. Halaš, An example of a congruence distributive variety having no near-unanimity term. *Acta Univ. M. Belii Ser. Math.* No. 13 (2006), 29–31.

[2] I. Chajda, R. Halaš and J Kühr, *Semilattice structures.* Research and Exposition in Mathematics **30**, Heldermann Verlag, Lemgo, 2007.

[3] I. Chajda and M. Kolařík, Nearlattices, *Discrete Math.* **308** (2008), 4906–4913.

[4] R. Hickman, Join algebras. *Comm. Algebra* **8** (1980), 1653–1685.

[5] W. McCune, Prover9 and Mace4, version 2009-11A, (`http://www.cs.unm.edu/~mccune/prover9/`)

(Araújo) Universidade Aberta and Centro de Álgebra, Universidade de Lisboa, 1649-003 Lisboa, Portugal

E-mail address: `jaraujo@ptmat.fc.ul.pt`

(Kinyon) Department of Mathematics, University of Denver, 2360 S Gaylord St, Denver, Colorado 80208 USA

E-mail address: `mkinyon@math.du.edu`